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COMPLETE BLOW-UP FOR A SEMILINEAR PARABOLIC EQUATION

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กำหนด $T \leq \infty,\ a$ และ x_0 เป็นค่าคงตัว โดยที่ $a > 0$ และ $0 < x_0 < a$

$$u_t(x,t) - u_{xx}(x,t) = f(u(x_0,t)) \text{ สำหรับ } 0 < x < a,\ 0 < t < T,$$

$$u(x,0) = \phi(x) \geq 0 \text{ สำหรับ } 0 \leq x \leq a,$$

$$u(0,t) = u_x(a,t) = 0 \text{ สำหรับ } 0 < t < T,$$

โดยที่ f และ ϕ เป็นฟังก์ชันที่กำหนดให้ เราต้องการแสดงว่าภายใต้เงื่อนไขบางประการ u โบลว์อัพในเวลาจำกัด T และเซตของจุดโบลว์อัพคือช่วง $[0, a]$

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..

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Let $T \le \infty$, a and x_0 be constants with $a > 0$ and $0 < x_0 < a.$ We establish the unique solution u for the following semilinear parabolic initial-boundary value problem:

$$u_t(x,t) - u_{xx}(x,t) = f(u(x_0,t)) \text{ for } 0 < x < a,\ 0 < t < T,$$

$$u(x,0) = \phi(x) \ge 0 \text{ for } 0 \le x \le a,$$

$$u(0,t) = u_x(a,t) = 0 \text{ for } 0 < t < T,$$

where f and ϕ are given functions. We also show that under certain conditions, u blows up in a finite time, and the set of the blow-up points is the entire interval $[0, a]$.

Department **Mathematics** Student's signature......................................

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CONTENTS

Chapter I

Introduction

The problem which appears some variables tending to infinity in a finite time $T > 0$ is called a blow-up phenomenon. In the theory of ordinary differential equations, the simplest example is the initial-value problem

$$u_t = u^2, \ t > 0,$$

$$u(0) = b.$$

For $b > 0$ it is immediate that the unique solution exists in the time interval $0 < t < T = 1/b$. Solving the problem, we find that $u(t) = 1/(T - t)$, one sees that $u(t) \to \infty$ as $t \to T^-$. We say that the solution blows up at $t = T$ and also that $u(t)$ has a blow-up at a finite time. Starting from this example, the concept of blow-up can be widely generalized. Thus we consider the more general form

$$u_t = f(u),$$

where f is a positive and continuous function satisfying the condition

$$\int_1^\infty \frac{1}{f(s)} ds < \infty.$$

This Osgood's condition in the theory of ordinary differential equations established in 1898 is necessary and sufficient for the occurrence of a blow-up in a finite time for any solutions with positive initial data. Further details about blow-up phenomena can be found in [10]. In this work, we are interested in a blow-up phenomenon in a semilinear parabolic equation.

Previously, there were many mathematicians studied blow-up phenomenon. For instance:

In 1989, M.S. Floater [6] studied degenerate semilinear parabolic equation:

$$u_t(x,t) = u_{xx}(x,t) + u^p(x,t) \ \text{ in } (0,1)\times(0,\infty),$$

$$u(0,t) = u(1,t) = 0 \ \text{ for } t > 0,$$

$$u(x,0) = u_0(x) \geq 0 \ \text{ on } [0,1].$$

Under certain conditions, it is shown that the solution may blow up at the boundary in a finite time.

In 1991, Z. Lin and M. Wang [10] studied the semilinear parabolic equation:

$$u_t(x,t) = u_{xx}(x,t) + u^p(x,t) \ \text{ in } (0,1)\times(0,\infty),$$

$$u_x(0,t) = 0, u_x(1,t) = u^q(x,t) \ \text{ for } t > 0,$$

$$u(x,0) = u_0(x) \geq 0 \ \text{ on } [0,1].$$

Again, under certain conditions, they proved that the blow-up would occur only at the boundary $x = 1$.

In 2000, C.Y. Chan and H.Y. Tian [2] showed that, under certain conditions, a degenerate semilinear parabolic equation with initial-boundary value became a single point blow-up problem. In addition, C.Y. Chan and J. Yang [4] proved that the degenerate semilinear parabolic problem under the certain conditions is a complete.

Based on the above results, we will show that, under certain conditions, the following semilinear parabolic equation blows up in a finite time.

Let $T \leq \infty$, and a and x_0 be constants with $a > 0$ and $0 < x_0 < a$. We would like to study the following semilinear parabolic initial-boundary value problem,

$$\left.\begin{aligned} u_t(x,t) - u_{xx}(x,t) = f(u(x_0,t)) \ \text{ for } \ 0 < x < a,\ 0 < t < T, \\ u(x,0) = \phi(x) \ \text{ on } 0 \leq x \leq a, \\ u(0,t) = u_x(a,t) = 0 \ \text{ for } \ 0 < t < T, \end{aligned}\right\} \quad (1)$$

where $T \leq \infty,\ a$ and x_0 be constants with $a > 0,\ 0 < x_0 < a$, and f, ϕ are given functions. We will also show that under certain conditions, u blows up in a finite time, and the set of the blow-up points is the enire interval $[0,a]$.

Similarly, a solution $u(x,t)$ is said to blow up at the point $(\overline{x},T)$ if there exists a sequence $\{(x_n,t_n)\}$ such that $\lim_{n\to\infty} u(x_n,t_n) \to \infty$ as $(x_n,t_n) \to (\overline{x},T)$. Furthermore, if $u(x,t)$ blows up at every point $x \in [0,a]$, then the complete blow-up occurs.

The complete blow-up of the solution of a degenerate semilinear problem with $u_x(a,t) = 0$ replaced by $u(a,t) = 0$ was studied by Chan and Yang [4]. Baras and Cohen [1] and Lacey and Tzanetis [9] studied the problem of a complete blow-up with $f(u(x_0,t))$ being replaced by $f(u(x,t))$.

In chapter 2, we transform the problem from $[0,a]$ to $[0,1]$. In chapter 3, we show that the transformed solution satisfies a nonlinear integral equation, and establish the existence of a unique continuous solution u to this integral equation. In chapter 4, we show that u blows up in a finite time if the initial data are sufficiently large in some neighborhood of x_0. In chapter 5, we prove that the set of blow-up points is the entire interval $[0,1]$.

Chapter II

Transformation

Let $\widetilde{T} \leq \infty$, and a and $\tilde{x}_0$ be constants with $a > 0$ and $0 < \tilde{x}_0 < a$. We consider the following semilinear parabolic initial-boundary value problem,

$$\left.\begin{aligned} u_{\tilde{t}}(\tilde{x},\tilde{t}) - u_{\tilde{x}\tilde{x}}(\tilde{x},\tilde{t}) &= F(u(\tilde{x}_0,\tilde{t})) \text{ in } (0,a)\times(0,\widetilde{T}), \\ u(\widetilde{x},0) &= \phi(\widetilde{x}) \text{ on } [0,a], \\ u(0,\widetilde{t}) = u_{\widetilde{x}}(a,\widetilde{t}) &= 0 \text{ for } 0<\widetilde{t}<\widetilde{T}, \end{aligned}\right\} \tag{2}$$

where F and ϕ are given functions. Let $\widetilde{x} = ax$, $\widetilde{t} = a^2 t$, $\widetilde{T} = a^2 T$, $Lu = u_t - u_{xx}$, $f(u(x_0,t)) = F(u(\tilde{x}_0,\widetilde{t}))$, $D = (0,1)$, $\overline{D} = [0,1]$ and $\Omega = D \times (0,T)$. We have,

$$\begin{aligned} u_{\widetilde{t}} &= u_t \frac{dt}{d\widetilde{t}} = \frac{1}{a^2} u_t, \\ u_{\widetilde{x}} &= u_x \frac{dx}{d\widetilde{x}} = \frac{1}{a} u_x, \\ u_{\widetilde{x}\widetilde{x}} &= (\frac{1}{a} u_x)_{\widetilde{x}} = \frac{1}{a^2} u_{xx}. \end{aligned}$$

Then the above system (2) is transformed into the following problem,

$$\left.\begin{aligned} Lu(x,t) &= a^2 f(u(x_0,t)) \text{ in } \Omega, \\ u(x,0) &= \phi(x) \text{ on } \overline{D}, \\ u(0,t) = u_x(1,t) &= 0 \text{ for } 0<t<T \end{aligned}\right\} \tag{3}$$

with $T = \widetilde{T}/a^2$. We assume that $f \in C^2([0,\infty))$, $f(0) \geq 0$, $f'(s) > 0$ and $f''(s) > 0$ for $s > 0$, $\int\limits_{z_0}^{\infty} \frac{1}{f(s)} ds < \infty$ for some $z_0 > 0$, and $\phi(x)$ is nontrivial, nonnegative and continuous such that $\phi(0) = \phi'(1) = 0$, and

$$\phi''(x) + a^2 f(\phi(x_0)) \geq 0 \text{ in } D. \tag{4}$$

We note that the last condition is used to show that before u blows up, u is a nondecreasing function of t.

Chapter III

Existence of a unique solution

Let us construct Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (3). It is determined by the following system: for x and ξ in D and t and τ in $(0,T)$,

$$\left.\begin{aligned} LG(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau), \\ G(x,t;\xi,\tau) &= 0 \quad \text{for } t<\tau, \\ G(0,t;\xi,\tau) &= G_x(1,t;\xi,\tau)=0, \end{aligned}\right\} \tag{5}$$

where $\delta\left(x\right)$ is the Dirac delta function. By the method of eigenfunction expansion,

$$G(x,t;\xi,\tau)=\sum_{n=1}^{\infty} a_n\left(t\right) g_n\left(x\right), \tag{6}$$

where

$$g_n\left(x\right)=\sqrt{2}\sin\sqrt{\lambda_n}x,\ \lambda_n=\left[\left(\frac{2n-1}{2}\right)\pi\right]^2, n=1,2,3,...$$

are the n^{th} orthonormal eigenfunction and eigenvalue of the Sturm-Liouville problem,

$$g''\left(x\right)+\lambda g\left(x\right)=0,\ \ g\left(0\right)=g'\left(1\right)=0.$$

Substituting (6) into (5), we find that

$$\sum_{n=1}^{\infty} a_n'\left(t\right) g_n\left(x\right)-\sum_{n=1}^{\infty} a_n\left(t\right) g_n''\left(x\right)=\delta(x-\xi)\delta(t-\tau).$$

Since $g_n''\left(x\right)+\lambda_n g_n\left(x\right)=0$, we have

$$\sum_{n=1}^{\infty} a_n'\left(t\right) g_n\left(x\right)+\sum_{n=1}^{\infty} a_n\left(t\right)\lambda_n g_n\left(x\right)=\delta(x-\xi)\delta(t-\tau).$$

Therefore,

$$\sum_{n=1}^{\infty}\left[a_n'\left(t\right)+\lambda_n a_n\left(t\right)\right] g_n\left(x\right)=\delta(x-\xi)\delta(t-\tau).$$

Multiplying both sides by $g_m(x)$ and integrating from 0 to 1 with respect to x, we formally obtain that

$$\int_0^1 g_m(x)\sum_{n=1}^{\infty}\left[a_n'(t)+\lambda_n a_n(t)\right]g_n(x)\,dx=\int_0^1 g_m(x)\,\delta(x-\xi)\delta(t-\tau)dx.$$

Thus

$$a_n'(t)+\lambda_n a_n(t)=g_n(\xi)\,\delta(t-\tau).$$

Multiplying both sides by $\exp(\lambda_n t)$, we get

$$\frac{d}{dt}\left[a_n(t)\exp(\lambda_n t)\right]=g_n(\xi)\,\delta(t-\tau)\exp(\lambda_n t).$$

By integrating from τ^- to u with respect to t and then replacing u by t, we have

$$a_n(t)\exp(\lambda_n t)-a_n\left(\tau^-\right)\exp\left(\lambda_n\tau^-\right)=g_n(\xi).$$

Since $G(x,t;\xi,\tau)=\sum\limits_{n=1}^{\infty}a_n(t)\,g_n(x)=0$, for $t<\tau$ and $g_n(x)\neq 0$, we have $a_n(t)=0$ for $t<\tau$. This implies that,

$$a_n(t)\exp(\lambda_n t)=g_n(\xi)\exp(\lambda_n\tau).$$

Thus,

$$a_n(t)=g_n(\xi)\exp[-\lambda_n(t-\tau)].$$

Therefore,

$$G(x,t;\xi,\tau)=\sum_{n=1}^{\infty}g_n(x)\,g_n(\xi)\exp[-\lambda_n(t-\tau)]\text{ for }t>\tau. \tag{7}$$

Let us show that $G(x,t;\xi,\tau)$ exists. We have

$$\begin{aligned}\left|\sum_{n=1}^{\infty}g_n(x)\,g_n(\xi)\exp[-\lambda_n(t-\tau)]\right| &\leq\sum_{n=1}^{\infty}|g_n(x)|\,|g_n(\xi)|\exp[-\lambda_n(t-\tau)]\\ &\leq 2\sum_{n=1}^{\infty}\exp[-\lambda_n(t-\tau)].\end{aligned}$$

Using the Ratio test, we see that $\sum\limits_{n=1}^{\infty}\exp\left[-\lambda_n(t-\tau)\right]$ converges. By the Weierstrass M-test, the series $\sum\limits_{n=1}^{\infty}g_n(x)\,g_n(\xi)\exp[-\lambda_n(t-\tau)]$ converges uniformly. Hence $G(x,t;\xi,\tau)$ exists.

Let us now verify that (7) is indeed the solution to (5). We begin by computing

$$\frac{\partial G}{\partial t} = -\left[\sum_{n=1}^{\infty} \lambda_n g_n(x)\, g_n(\xi) \exp[-\lambda_n(t-\tau)]\right] H(t-\tau) + \left[\sum_{n=1}^{\infty} g_n(x)\, g_n(\xi) \exp[-\lambda_n(t-\tau)]\right] \delta(t-\tau),$$

where H is the Heaviside unit-step function. Using $f(t)\,\delta(t-\tau) = f(\tau)\,\delta(t-\tau)$, we have

$$\frac{\partial G}{\partial t} = -\left[\sum_{n=1}^{\infty} \lambda_n g_n(x)\, g_n(\xi) \exp[-\lambda_n(t-\tau)]\right] H(t-\tau) + \left[\sum_{n=1}^{\infty} g_n(x)\, g_n(\xi)\right] \delta(t-\tau).$$

From appendix B, $\sum_{n=1}^{\infty} g_n(x)\, g_n(\xi) = \delta(x-\xi)$. Therefore,

$$\frac{\partial G}{\partial t} = -\left[\sum_{n=1}^{\infty} \lambda_n g_n(x)\, g_n(\xi) \exp[-\lambda_n(t-\tau)]\right] H(t-\tau) + \delta(x-\xi)\delta(t-\tau).$$

Hence,

$$LG = -\left\{\sum_{n=1}^{\infty} g_n(\xi) \left\{g_n''(x) + \lambda_n g_n(x)\right\} \exp[-\lambda_n(t-\tau)]\right\} H(t-\tau) + \delta(x-\xi)\delta(t-\tau).$$

Since $g_n''(x) + \lambda_n g_n(x) = 0$, we have

$$LG = \delta(x-\xi)\delta(t-\tau).$$

By direct computation, $G(0,t;\xi,\tau) = G_x(1,t;\xi,\tau) = 0$.

To obtain the integral equation,

$$u(x,t) = a^2 \int_0^t \int_0^1 G(x,t;\xi,\tau) f\left(u\left(x_0,\tau\right)\right) d\xi d\tau + \int_0^1 G(x,t;\xi,0)\phi(\xi)\, d\xi, \qquad (8)$$

corresponding to the problem (3), let us show that $L^* u = -u_t - u_{xx}$, where L^* denote the adjoint operator of L:

$$\begin{aligned} v\frac{\partial^2 u}{\partial x^2} &= (vu_x)_x - v_x u_x \\ &= (vu_x)_x - (v_x u)_x + v_{xx} u, \\ v\frac{\partial u}{\partial t} &= (vu)_t - v_t u, \end{aligned}$$

$$\begin{aligned}vLu &= v\frac{\partial u}{\partial t} - v\frac{\partial^2 u}{\partial x^2}\\ &= \left[(vu)_t - v_t u\right] - \left[(vu_x)_x - (v_x u)_x + v_{xx}u\right]\\ &= (vu)_t - v_t u - (vu_x)_x + (v_x u)_x - v_{xx}u,\end{aligned}$$

which gives

$$vLu - uL^*v = (v_x u - vu_x)_x + (vu)_t\,,$$

where $L^*u \equiv -u_t - u_{xx}$.

Next, we show that a solution of the problem (3) is also a solution of the integral equation (8). Using $G^*(\xi,\tau;x,t) = G(x,t;\xi,\tau)$, and Green′s theorem, which states that $\iint\limits_D (P_x + Q_y)\,dxdy = \int\limits_{\partial D} Pdy - Qdx$, we obtain

$$\begin{aligned}\iint\limits_{\Omega} (GLu - uL^*G^*)\,d\xi d\tau &= \iint\limits_{\Omega} \left[(G_\xi u - Gu_\xi)_\xi + (Gu)_\tau\right] d\xi d\tau\\ &= \int\limits_{\partial\Omega} (G_\xi u - Gu_\xi) d\tau - Gud\xi. \end{aligned} \qquad (9)$$

On $\{0\} \times (0,T)$,

$$\begin{aligned}\int\limits_{\partial\Omega} (G_\xi u - Gu_\xi) d\tau - Gud\xi &= \int\limits_0^T \left[G_\xi(x,t;0,\tau)\,u(0,\tau) - G(x,t;0,\tau)\,u_\xi(0,\tau)\right] d\tau\\ &= 0\end{aligned}$$

since $u(0,\tau) = 0$ and $G(x,t;0,\tau) = 0$. On $\{1\} \times (0,T)$,

$$\begin{aligned}\int\limits_{\partial\Omega} (G_\xi u - Gu_\xi) d\tau - Gud\xi &= \int\limits_0^T \left[G_\xi(x,t;1,\tau)\,u(1,\tau) - G(x,t;1,\tau)\,u_\xi(1,\tau)\right] d\tau\\ &= 0\end{aligned}$$

since $u_\xi(1,\tau) = 0$ and $G_\xi(x,t;1,\tau) = 0$. On $\overline{D} \times \{0\}$,

$$\begin{aligned}\int\limits_{\partial\Omega} (G_\xi u - Gu_\xi) d\tau - Gud\xi &= -\int\limits_0^1 G(x,t;\xi,0)u(\xi,0)\,d\xi\\ &= -\int\limits_0^1 G(x,t;\xi,0)\phi(\xi)\,d\xi\end{aligned}$$

since $u(\xi,0) = \phi(\xi)$. On the other hand, let us consider the left-hand side of (9).

$$\iint\limits_{\Omega} (GLu - uL^{*}G)\, d\xi d\tau$$
$$= a^2 \int_0^T \int_0^1 G(x,t;\xi,\tau)\, f(u(x_0,\tau)) d\xi d\tau - \int_0^T \int_0^1 u(\xi,\tau)\, \delta(x-\xi)\, \delta(t-\tau)\, d\xi d\tau$$
$$= a^2 \int_0^T \int_0^1 G(x,t;\xi,\tau)\, f(u(x_0,\tau)) d\xi d\tau - u(x,t)\,.$$

From (9),

$$a^2 \int_0^T \int_0^1 G(x,t;\xi,\tau)\, f(u(x_0,\tau)) d\xi d\tau - u(x,t) = -\int_0^1 G(x,t;\xi,0) \phi(\xi)\, d\xi.$$

Therefore, we have (8).

Next, we will prove some properties of Green's function.

LEMMA 1. In the set $\{(x,t;\xi,\tau) : x \text{ and } \xi \text{ are in } D,\ 0 \le \tau < t \le T\}$, $G(x,t;\xi,\tau) > 0$.

Proof. Let $D_1 = \{(x,t;\xi,\tau) : x \text{ and } \xi \text{ are in } D,\ 0 \le \tau < t \le T\}$. Suppose that there exists a point $(x_1,t_1;\xi_1,\tau_1)$ in D_1 such that $G(x,t;\xi,\tau) < 0$. Since $G(x,t;\xi,\tau)$ is continuous in D_1, there exists a positive number ε such that $G(x,t;\xi,\tau) < 0$ in the set,

$$W_0 = (x_1 - \varepsilon, x_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \times (\xi_1 - \varepsilon, \xi_1 + \varepsilon) \times (\tau_1 - \varepsilon, \tau_1 + \varepsilon)$$

which is contained in D_1. Let

$$W_1 = (\xi_1 - \varepsilon, \xi_1 + \varepsilon) \times (\tau_1 - \varepsilon, \tau_1 + \varepsilon),$$
$$W_2 = (\xi_1 - \frac{\varepsilon}{2}, \xi_1 + \frac{\varepsilon}{2}) \times (\tau_1 - \frac{\varepsilon}{2}, \tau_1 + \frac{\varepsilon}{2}).$$

We would like to show that there exists a function $h(x,t)$ in $C^2(\mathbb{R}^2)$ such that $h \equiv 1$ on $\overline{W_2}$, $h \equiv 0$ outside W_1, and $0 \le h \le 1$ in $W_1 \backslash W_2$. We construct the desired function explicitly in a sequence of steps:

Step one: the function f_1 defined by

$$f_1(s) = \begin{cases} 0, & s \leq 0, \\ \exp(-s^{-1}), & s > 0, \end{cases}$$

belongs to $C^2(\mathbb{R})$, vanishes for $s \leq 0$, is positive for $s > 0$, and is monotone increasing.

Step two: the function f_2 defined by

$$f_2(s) = f_1(s) f_1(1-s)$$

belongs to $C^2(\mathbb{R})$, vanishes for $s \leq 0$ and $s \geq 1$, and is positive for $0 < s < 1$.

Step three: the function f_3 in $C^\infty(\mathbb{R})$ defined by

$$f_3(s) = \frac{\int\limits_0^s f_2(t)\,dt}{\int\limits_0^1 f_2(t)\,dt}$$

vanishes for $s \leq 0$, is monotone increasing, equals one for $s \geq 1$, and satisfies $0 < f_3(s) < 1$ for all $s \in D$.

Step four: the function $h(x,t)$ defined by

$$h(x,t) = f_3\left(\frac{\varepsilon - |x - x_1|}{\varepsilon/2}\right) f_3\left(\frac{\varepsilon - |t - t_1|}{\varepsilon/2}\right)$$

is in $C^2(\mathbb{R}^2)$ and has $h(x,t) = 1$ on $\overline{W_2}$, $h(x,t) = 0$ outside W_1, and $0 \leq h(x,t) \leq 1$ in $W_1 \backslash W_2$. Hence, the solution of the problem, $Lu(x,t) = h(x,t)$ in $D \times (0, \alpha]$, $t_1 < \alpha$ with u satisfying zero initial and $u(0,t) = 0 = u_x(1,t)$, is given by

$$u(x,t) = \int\limits_{\tau_1-\varepsilon}^{\tau_1+\varepsilon} \int\limits_{\xi_1-\varepsilon}^{\xi_1+\varepsilon} G(x,t;\xi,\tau) h(\xi,\tau) d\xi d\tau.$$

Since $G(x,t;\xi,\tau) < 0$ in W_0, $h(\xi,\tau) \geq 0$ in W_1, and $h \equiv 1$ on $\overline{W}_2$, it follows that $u(x,t) < 0$ for (x,t) in $(x_1 - \varepsilon, x_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon)$. On the other hand, $h(x,t) \geq 0$ in $D \times (0, \alpha]$ implies that $u(x,t) \geq 0$ by weak maximum principle and Holf's Lemma. We have a contradiction, and therefore, $G(x,t;\xi,\tau) \geq 0$ in D_1.

Next, we show that $G(x,t;\xi,\tau) \neq 0$ in D_1. Suppose that there exists a point $(x_2,t_2;\xi_2,\tau_2)$ in D_1 such that $G(x,t;\xi,\tau)=0$. Using the strong maximum principle, we have $G(x,t;\xi_2,\tau_2)=0$ in $D_1 \cap \{(x,t;\xi_2,\tau_2) : 0<x<1,\ t \leq t_2\}$. On the other hand, $G(\xi_2,t_2,\xi_2,\tau_2) = 2\sum\limits_{n=1}^{\infty} \sin^2 \sqrt{\lambda_n}\xi_2 \exp[-\lambda_n(t_2-\tau_2)]$, which is positive. We again have a contradiction. This shows that $G(x,t;\xi,\tau)$ is positive in D_1. □

Lemma 2. For any function $\gamma \in C([0,T])$, $\int\limits_0^t \int\limits_0^1 G(x,t;\xi,\tau)\gamma(\tau)d\xi d\tau$ is continuous on $\overline{\Omega}$.

Proof. Let ε be any positive number such that $t-\varepsilon>0$. For $x \in \overline{D}$ and $\tau \in [0,t-\varepsilon]$, we multiply

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]$$

by $\gamma(\tau)$, to get

$$G(x,t;\xi,\tau)\gamma(\tau) = \sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau).$$

Since $g_n(x) = \sqrt{2}\sin\sqrt{\lambda_n}x$, we have

$$\sum_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau) \leq 2\left[\max_{0\leq\tau\leq T}\gamma(\tau)\right]\sum_{n=1}^{\infty}\exp\left[-\lambda_n(t-\tau)\right],$$

which converges. By the Weierstrass M-test, $\sum\limits_{n=1}^{\infty} g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau)$ converges uniformly, and we have

$$\int_0^{t-\varepsilon}\int_0^1 G(x,t;\xi,\tau)\gamma(\tau)d\xi d\tau = \sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau)d\xi d\tau.$$

Since

$$\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau)d\xi d\tau$$

$$\leq 2\left[\max_{0\leq\tau\leq T}\gamma(\tau)\right]\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 \exp\left[-\lambda_n(t-\tau)\right]d\xi d\tau$$

$$= 2\left[\max_{0\leq\tau\leq T}\gamma(\tau)\right]\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\exp\left[-\lambda_n(t-\tau)\right]d\tau$$

$$= 2\left[\max_{0\leq\tau\leq T}\gamma(\tau)\right]\sum_{n=1}^{\infty}\lambda_n^{-1}\left[\exp(-\lambda_n\varepsilon)-\exp(-\lambda_n t)\right]$$

$$\leq 2\left[\max_{0\leq\tau\leq T}\gamma(\tau)\right]\sum_{n=1}^{\infty}\lambda_n^{-1},$$

which converges. Furthermore, it follows from the Weierstrass M-test that

$$\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau)d\xi d\tau$$

converges uniformly with respect to x, t and ε. Since the uniform convergence also holds for $\varepsilon = 0$, it follows that

$$\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau)d\xi d\tau$$

is a continuous function of x, t and $\varepsilon \geq 0$. Therefore,

$$\int_0^t\int_0^1 G(x,t;\xi,\tau)\gamma(\tau)d\xi d\tau = \lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\int_0^{t-\varepsilon}\int_0^1 g_n(x)g_n(\xi)\exp\left[-\lambda_n(t-\tau)\right]\gamma(\tau)d\xi d\tau$$

is a continuous function of x and t. □

Based on Theorem 2 of Chan and Tian [2], we will prove the following theorem.

THEOREM 3. There exists some t_0 such that for $0 \leq t \leq t_0$, the integral equation (8) has the unique continuous solution $u \geq \phi(x)$ and u is a nondecreasing function of t. Let t_b be the supremum of the interval for which the integral equation (8) has the unique continuous solution u. If t_b is finite, then $u(x_0,t)$ is unbounded in $[0,t_b)$.

Proof. We construct a sequence $\{u_n\}$ by $u_0(x,t) = \phi(x)$, and for $n = 0, 1, 2, ...,$

$$Lu_{n+1}(x,t) = a^2 f(u_n(x_0,t)) \text{ in } \Omega = D \times (0,T),$$

$$u_{n+1}(x,0) = \phi(x) \text{ on } [0,1],$$

$$u_{n+1}(0,t) = (u_{n+1})_x(1,t) = 0 \text{ for } 0 < t < T.$$

To show that the sequence $u_n(x,t) \geq \phi(x)$ for all $n = 0, 1, 2, ...,$ we use the condition (4) to obtain that

$$\begin{aligned} L(u_1 - u_0)(x,t) &= a^2 f(u_0(x_0,t)) + \phi''(x) \\ &\geq a^2[f(u_0(x_0,t)) - f(\phi(x_0))] \\ &= a^2[f(\phi(x_0)) - f(\phi(x_0))] = 0 \text{ in } \Omega. \end{aligned}$$

Since

$$(u_1 - u_0)(x,0) = 0 \text{ on } [0,1],$$

$$(u_1 - u_0)(0,t) = 0 = (u_1 - u_0)_x(1,t) = 0 \text{ for } 0 < t < T,$$

it follows from (8) and $G(x,t;\xi,\tau)$ being positive that $u_1(x,t) \geq u_0(x,t)$ in Ω. Let us assume that for some positive integer j,

$$\phi \leq u_1 \leq u_2 \leq .... \leq u_{j-1} \leq u_j \text{ in } \Omega.$$

Since f is an increasing function, and $u_j \geq u_{j-1}$, we have

$$L(u_{j+1} - u_j) = a^2[f(u_j) - f(u_{j-1})] \geq 0 \text{ in } \Omega,$$

$$(u_{j+1} - u_j)(x,0) = 0 \quad \text{on } [0,1],$$

$$(u_{j+1} - u_j)(0,t) = 0 = (u_{j+1} - u_j)_x(1,t) \text{ for } 0 < t < T.$$

From (8),

$$(u_{j+1} - u_j)(x,t) = a^2 \int_0^t \int_0^1 G(x,t;\xi,\tau)[f(u_j) - f(u_{j-1})]d\xi d\tau \geq 0.$$

Thus, $u_{j+1} \geq u_j$. By the principle of mathematical induction,

$$\phi \leq u_1 \leq u_2 \leq .... \leq u_{n-1} \leq u_n \text{ in } \Omega \text{ for all positive integer } n. \tag{10}$$

Next, let us show that the sequence $\{u_n\}$ is a nondecreasing function of t. Let $w_n(x,t) = u_n(x,t+h) - u_n(x,t)$ for $n = 0,1,2,...,$ where h is any positive number less than $T-t$. It follows that

$$\begin{aligned} w_0(x,t) &= u_0(x,t+h) - u_0(x,t) \\ &= \phi(x) - \phi(x) \\ &= 0 \end{aligned}$$

In $D \times (0, T-h)$,

$$\begin{aligned} Lw_1(x,t) &= a^2 f(u_0(x_0,t+h)) - a^2 f(u_0(x_0,t)) \\ &= a^2 \left[f(\phi(x_0)) - f(\phi(x_0))\right] \\ &= 0. \end{aligned}$$

By (10) and the construction of u_1, we get that

$$\begin{aligned} w_1(x,0) &= u_1(x,h) - u_1(x,0) \\ &= u_1\,(x,h) - \phi\,(x) \geq 0 \text{ on } \overline{D}, \end{aligned}$$
$$w_1(0,t) = u_1(0,t+h) - u_1(0,t) = 0,\ (w_1(1,t))_x = 0,\ 0 < t < T-h.$$

By (8), $w_1 \geq 0$ for $0 < t < T-h$. Let us assume that for some positive integer j, $w_j \geq 0$ for $0 < t < T-h$. Using the Mean Value Theorem, we get

$$\begin{aligned} Lw_{j+1}(x,t) &= a^2 \left[f(u_j(x_0,t+h)) - f(u_j(x_0,t))\right] \\ &= a^2 \left[f'(u_j(x_0,t_1))w_j(x,t)\right] \end{aligned}$$

in $D \times (0, T-h)$ for some t_1 in $(t, t+h)$. Also,

$$w_{j+1}(x,0) = 0 \text{ on } [0,1],$$

$$w_{j+1}(0,t) = (w_{j+1}(1,t))_x = 0 \text{ for } 0 < t \leq T-h.$$

From (8) and $G(x,t;\xi,\tau)$ being positive, we get that for $0 < t < T-h$,

$$w_{j+1}(x,t) = a^2 \int_0^t \int_0^1 G(x,t;\xi,\tau) f'(u_j(x_0,t_1)) w_j(x,t)\, d\xi d\tau \geq 0.$$

By the principle of mathematical induction, $w_n \geq 0$ for all positive integer n. This shows that u_n is a nondecreasing function of t.

Next, we would like to show that there exists some $\widehat{t}$ such that the integral equation (8) has a unique continuous solution u for $0 \leq t \leq \widehat{t}$. We consider the problem

$$Lv(x,t) = 0 \quad \text{in } \Omega,$$

$$v(x,0) = \phi(x) \quad \text{on } \overline{D},$$

$$v(0,t) = v_x(1,t) = 0 \quad \text{for } 0 < t < T.$$

From (8), the solution of the problem is

$$v(x,t) = \int_0^1 G(x,t;\xi,0)\phi(\xi)d\xi.$$

We know that $G(x,t;\xi,\tau)$ is positive and $\phi(x)$ is nontrivial, nonnegative and continuous. Thus, $v > 0$ in Ω. By the weak maximum principle and the parabolic version of Hopf's lemma, v attains its maximum $k_0 = \max\limits_{x\in[0,1]} \phi(x)$ in $D \times \{0\}$.

Next, we show that for some given positive constant $M > k_0$, there exists some t_2 such that $u_i \leq M$ for $0 \leq t \leq t_2$. By Lemma 2, $G(x,t;\xi,\tau)$ is integrable. Let us consider

$$u_i(x,t) = a^2 \int_0^t \int_0^1 G(x,t;\xi,\tau) f(u_{i-1}(x_0,\tau)) d\xi d\tau + \int_0^1 G(x,t;\xi,0)\phi(\xi)d\xi. \quad (11)$$

As $t \to 0$, we see that

$$\begin{aligned}
\lim_{t\to 0} u_i(x,t) &= \lim_{t\to 0} \int_0^1 G(x,t;\xi,0)\phi(\xi)d\xi \\
&= \int_0^1 \lim_{t\to 0} G(x,t;\xi,0)\phi(\xi)d\xi \\
&= \int_0^1 \sum_{n=1}^{\infty} g_n(x)\, g_n(\xi)\, \phi(\xi)d\xi \\
&= \int_0^1 \delta(x-\xi)\phi(\xi)d\xi \\
&= \phi(x).
\end{aligned}$$

This shows that there exists t_2 such that $u_i(x,t) \le M$ for $0 \le t \le t_2$. Let u denote $\lim_{i\to\infty} u_i$. From (11), we have (8) for $0 \le t \le t_2$.

Next, we show that $\{u_i\}$ converges uniformly to u for $0 \le t \le t_2$. From (11),

$$u_{i+1}(x,t)-u_i(x,t) = a^2 \int_0^t \int_0^1 G(x,t;\xi,\tau)[f(u_i(x_0,\tau))-f(u_{i-1}(x_0,\tau))]d\xi d\tau. \quad (12)$$

Let $S_i = \max_{[0,1]\times[0,t_2]} (u_i - u_{i-1})$. Using the Mean Value Theorem, we have

$$f(u_i(x_0,\tau)) - f(u_{i-1}(x_0,\tau)) = f'(\mu)\,(u_i(x_0,\tau) - u_{i-1}(x_0,\tau))$$

for some μ between $u_{i-1}(x_0,\tau)$ and $u_i(x_0,\tau)$. Since $u_i \le M$ for all i and $f''(s) > 0$ for $s > 0$, we get

$$\begin{aligned} f(u_i(x_0,\tau)) - f(u_{i-1}(x_0,\tau)) &\le f'(M)\,(u_i(x_0,\tau) - u_{i-1}(x_0,\tau)) \\ &\le f'(M)\,S_i. \end{aligned}$$

From (12), we have

$$\begin{aligned} S_{i+1} &\le 2a^2 f'(M) S_i \sum_{n=1}^{\infty} \int_0^t \int_0^1 \exp\left[-\lambda_n(t-\tau)\right] d\xi d\tau \\ &= 2a^2 f'(M) S_i \sum_{n=1}^{\infty} \int_0^t \exp\left[-\lambda_n(t-\tau)\right] d\tau \\ &= 2a^2 f'(M) \left[\sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t))\right] S_i. \end{aligned} \quad (13)$$

We also know that $\sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t)) \le \sum_{n=1}^{\infty} \lambda_n^{-1}$, which converges. Therefore, by the Weierstrass M-test, $\sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t))$ converges uniformly.

We would like to show that there exists some $\sigma_1\,(> 0)$ such that

$$2a^2 f'(M) \sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t)) < 1 \quad \text{for } t \in [0,\sigma_1].$$

Since $\lim_{t\to 0} \sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t)) = \sum_{n=1}^{\infty} \lim_{t\to 0} \lambda_n^{-1}(1-\exp(-\lambda_n t)) = 0$, there exists some $\sigma_1\,(> 0)$ such that

$$\left|\sum_{n=1}^{\infty} \lambda_n^{-1}(1-\exp(-\lambda_n t))\right| < \frac{1}{2a^2 f'(M)} \quad \text{for } t \in [0,\sigma_1],$$

that is,

$$2a^2 f'(M) \sum_{n=1}^{\infty} \lambda_n^{-1}(1 - \exp(-\lambda_n t)) < 1 \text{ for } t \in [0, \sigma_1]. \tag{14}$$

From (13) and (14), it implies that $\{u_i\}$ converges uniformly to $u(x,t)$ for $0 \leq t \leq \sigma_1$.

Similarly for $\sigma_1 \leq t \leq t_2$, we use $u(\xi, \sigma_1)$ in place of $\phi(\xi)$ in (11), and obtain that

$$u_i(x,t) = a^2 \int_{\sigma_1}^{t} \int_0^1 G(x,t;\xi,\tau) f(u_{i-1}(x_0,\tau)) d\xi d\tau + \int_0^1 G(x,t;\xi,0) u(\xi,\sigma_1)\, d\xi.$$

Furthermore,

$$u_{i+1}(x,t) - u_i(x,t) = a^2 \int_{\sigma_1}^{t} \int_0^1 G(x,t;\xi,\tau) \left[f(u_i(x_0,\tau)) - f(u_{i-1}(x_0,\tau))\right] d\xi d\tau.$$

Since $S_i = \max\limits_{[0,1]\times[0,t_2]} (u_i - u_{i-1})$, it follows from the Mean Value Theorem that

$$f(u_i(x_0,\tau)) - f(u_{i-1}(x_0,\tau)) \leq f'(M)\, S_i$$

From (12), we have

$$\begin{aligned} S_{i+1} &\leq 2a^2 f'(M) S_i \sum_{n=1}^{\infty} \int_{\sigma_1}^{t} \int_0^1 \exp\left[-\lambda_n (t-\tau)\right] d\xi d\tau \\ &= 2a^2 f'(M) S_i \sum_{n=1}^{\infty} \int_{\sigma_1}^{t} \exp\left[-\lambda_n (t-\tau)\right] d\tau \\ &= 2a^2 f'(M) \left[\sum_{n=1}^{\infty} \lambda_n^{-1}(1 - \exp(-\lambda_n (t-\sigma_1)))\right] S_i. \end{aligned} \tag{15}$$

Thus, there exists $\sigma_2 = \min\{\sigma_1, t_2 - \sigma_1\} > 0$ such that

$$2a^2 f'(M) \sum_{n=1}^{\infty} \lambda_n^{-1}(1 - \exp(-\lambda_n (t-\sigma_1))) < 1, \text{ for } t \in [\sigma_1, \min\{2\sigma_1, t_2\}]. \tag{16}$$

Hence, $\{u_i\}$ converges uniformly to u for $t \in [\sigma_1, \min\{2\sigma_1, t_2\}]$.

By proceeding in this way the sequence $\{u_i\}$ converges uniformly for $0 \leq t \leq t_2$. Therefore, the integral equation (8) has a continuous solution u for $0 \leq t \leq t_2$.

To show that the solution u is unique, let us suppose that the integral equation (8) has two distinct solutions u and $\widetilde{u}$ on the interval $[0,t_2]$. Also, let $\Phi = \max\limits_{\overline{D}\times[0,t_2]} |u-\widetilde{u}| > 0$. From (8),

$$u(x,t)-\widetilde{u}(x,t) = a^2\int\limits_0^t\int\limits_0^1 G(x,t;\xi,\tau)\left[f(u(x_0,\tau))-f(\widetilde{u}(x_0,\tau))\right]d\xi d\tau$$

As in the derivation of (13), we obtain that

$$\Phi \le 2a^2f'(M)\left[\sum_{n=1}^{\infty}\lambda_n^{-1}(1-\exp(-\lambda_n t))\right]\Phi \text{ for } t\in[0,\sigma_1]\,.$$

This implies that

$$2a^2f'(M)\left[\sum_{n=1}^{\infty}\lambda_n^{-1}(1-\exp(-\lambda_n t))\right]\ge 1 \text{ for } t\in[0,\sigma_1]\,.$$

For $t\in[0,\sigma_1]$, it follows from (14) that we have a contradiction. Hence, the solution is unique for $0\le t\le\sigma_1$.

As in the derivation of (15), we obtain that

$$\Phi \le 2a^2f'(M)\left[\sum_{n=1}^{\infty}\lambda_n^{-1}(1-\exp(-\lambda_n(t-\sigma_1)))\right]\Phi \text{ for } t\in[\sigma_1,\min\{2\sigma_{1,}t_2\}]\,.$$

This shows that

$$2a^2f'(M)\left[\sum_{n=1}^{\infty}\lambda_n^{-1}(1-\exp(-\lambda_n(t-\sigma_1)))\right]\ge 1 \text{ for } t\in[\sigma_1,\min\{2\sigma_{1,}t_2\}]\,.$$

For $t\in[\sigma_1,\min\{2\sigma_{1,}t_2\}]$, it follows from (16) that we have a contradiction. Hence, the solution is unique for $\sigma_1\le t\le\min\{2\sigma_{1,}t_2\}$. By proceeding in this way, the integral (8) has the unique continuous solution u for $0\le t\le t_2$.

Let t_b be the supremum of the interval for which the integral equation (8) has the unique continuous solution u. We would like to show that if t_b is finite, then $u(x_0,t)$ is unbounded in $[0,t_b)$. Suppose that $u(x_0,t)$ is bounded in $[0,t_b)$. We consider (8) for $t\in[t_b,T)$ with the initial condition $u(x,0)$ replaced by $u(x,t_b)$.

$$u(x_0,t) = a^2\int\limits_{t_b}^t\int\limits_0^1 G(x_0,t;\xi,\tau)f\left(u\left(x_0,\tau\right)\right)d\xi d\tau + \int\limits_0^1 G(x_0,t;\xi,t_b)u(\xi,t_b)d\xi$$

For any positive constant $N > u(x_0, t_b)$, an argument as before shows that there exists t_3 such that the integral equation (8) has the unique continuous solution u on $[t_b, t_3]$. This contradicts the definition of t_b. Hence, if t_b is finite, then $u\,(x_0, t)$ is unbounded in $[0, t_b)$.

Since u_i is also a nondecreasing function of t, u is a nondecreasing function of t. $\square$

Chapter IV

A sufficient condition for blow-up in a finite time

In this chapter, we will give a sufficient condition for the solution u to blow-up in a finite time.

LEMMA 4. Let $u(x,t)$ be a solution of the following problem:

$$Lu = b(x,t)\,u(x_0,t) \text{ in } \Omega,$$

$$u(x,0) \geq 0 \text{ on } \overline{D},$$

$$u(0,t) = 0 = u_x(1,t) \quad \text{for } 0 < t < T,$$

where $b(x,t)$ is nonnegative and bounded, then $u(x,t) \geq 0$ in Ω.

Proof. Case 1: $b(x,t) \equiv 0$.

If $u < 0$ in Ω, then by the weak maximum principle, u attains its negative minimum somewhere at $x = 1$. By the parabolic version of Hopf's lemma, $u_x < 0$ at this point. This contradiction shows that $u(x,t) \geq 0$ in Ω.

Case 2: $b(x,t)$ being nonnegative and nontrivial.

Let η be a positive constant, and

$$V(x,t) = u(x,t) + \eta\left(1 + x^{1/2}\right)e^{ct},$$

where c is a positive constant to be determined. Also, we obtain that $V(x,0) > 0$ on $\overline{D}$ and $V(0,t) > 0$ for $0 < t < T$. Then we have

$$\begin{aligned}
&LV(x,t) - b(x,t)\,V(x_0,t) \\
&= L\left[u(x,t) + \eta\left(1 + x^{1/2}\right)e^{ct}\right] - b(x,t)\left[u(x_0,t) + \eta\left(1 + x_0^{1/2}\right)e^{ct}\right] \\
&= b(x,t)u(x_0,t) + L\left[\eta\left(1 + x^{1/2}\right)e^{ct}\right] - b(x,t)u(x_0,t) - b(x,t)\,\eta\left(1 + x_0^{1/2}\right)e^{ct} \\
&= L\left[\eta\left(1 + x^{1/2}\right)e^{ct}\right] - b(x,t)\,\eta\left(1 + x_0^{1/2}\right)e^{ct} \\
&= \eta e^{ct}\left[c\left(1 + x^{1/2}\right) + \frac{1}{4x^{3/2}} - b(x,t)\left(1 + x_0^{1/2}\right)\right] \\
&\geq \eta e^{ct}\left[c\left(1 + x^{1/2}\right) + \frac{1}{4x^{3/2}} - \left(1 + x_0^{1/2}\right)\max_{(x,t)\in\Omega_T} b(x,t)\right].
\end{aligned}$$

Let $M = \max_{(x,t)\in\Omega} b(x,t)$, and we choose $c \geq \left(1 + x_0^{1/2}\right) M$. Then,

$$\begin{aligned} &LV(x,t) - b(x,t)\,V(x_0,t) \\ &\geq \eta e^{t\left(1+x_0^{1/2}\right)M}\left[\left(1+x^{1/2}\right)\left(1+x_0^{1/2}\right)M + \frac{1}{4x^{3/2}} - \left(1+x_0^{1/2}\right)M\right] \\ &\geq \eta e^{t\left(1+x_0^{1/2}\right)M}\left[\left(1+x_0^{1/2}\right)M\left[\left(1+x^{1/2}\right)-1\right] + \frac{1}{4x^{3/2}}\right] \end{aligned}$$

Therefore,

$$LV(x,t) - b(x,t)\,V(x_0,t) > 0 \text{ in } \Omega.$$

To show that $V(x,t) > 0$ in Ω, let us suppose that there exists some point in Ω such that $V(x,t) \leq 0$. Since $V(x,0) > 0$ and $V(x,t)$ is continuous, the set

$$\{t : V(x,t) \leq 0 \text{ for some } x \in D\}$$

is nonempty. Let $\bar{t}$ denote its infimum. Then, there exists some $x_1 \in D$ such that $V\left(x_1,\bar{t}\right) = 0$ and $V_t\left(x_1,\bar{t}\right) \leq 0$. For $t < \bar{t}$, we have $V(x,t) > 0$ for all x. Since $V(x,t)$ is continuous, we have $V\left(x,\bar{t}\right) \geq 0$ for all x. Because $V\left(x_1,\bar{t}\right) = 0$, $V\left(x_1,\bar{t}\right)$ is a local minimum. Thus, $V\left(x_0,\bar{t}\right) \geq 0$ and $V_{xx}\left(x_1,\bar{t}\right) \geq 0$. We have

$$0 \geq V_t\left(x_1,\bar{t}\right) \geq LV\left(x_1,\bar{t}\right) - b\left(x_1,\bar{t}\right)V\left(x_0,\bar{t}\right) > 0.$$

We have a contradiction. This show that $V(x,t) > 0$ in Ω. Since V is continuous, it follows that $V(1,t) \geq 0$ for $0 < t < T$ As $\eta \to 0^+$, we also have that $u(x,t) \geq 0$ in Ω. □

The following theorem gives a sufficient condition for the solution u to blow-up in a finite time.

THEOREM 5. If $\phi(x)$ is sufficiently large in a neighborhood of x_0, then u blows up in a finite time.

Proof. Let us consider following problem,

$$\left.\begin{aligned} Lv(x,t) = a^2 f(v(x_0,t)) \;\text{ in } (x_0-\delta,x_0)\times(0,T), \\ v(x,0) = v_0(x) \geq 0 \;\text{ on } [x_0-\delta,x_0], \\ v(x_0-\delta,t) = v_x(x_0,t) = 0 \;\text{ for } 0<t<T, \end{aligned}\right\} \tag{17}$$

where $v_0(x)$ is nondecreasing on $[x_0-\delta, x_0]$ and $v_0(x_0-\delta) = 0 = v_0'(x_0)$. We would like to show that $\lim_{x\to\infty}(f(x)/x) = \infty$. Suppose that $\lim_{x\to\infty}(f(x)/x) = N$ for some positive number N. Then, there exists some positive number $z_0 > 0$ such that

$$\left|\frac{f(x)}{x} - N\right| < 1 \text{ for } x > z_0.$$

Thus $f(x)/x < 1+N$. We have

$$\frac{1}{f(x)} > \frac{1}{(1+N)x} \text{ for } x > z_0.$$

This implies that

$$\int_{z_0}^{\infty} \frac{1}{f(x)}dx > \frac{1}{1+N}\int_{z_0}^{\infty}\frac{1}{x}dx = \infty,$$

which contradicts the assumption $\int_{z_0}^{\infty}\frac{1}{f(s)}ds < \infty$ for some z_0. Thus, $\lim_{x\to\infty}(f(x)/x)$ $= \infty$.

Let λ_1 be the principal eigenvalue of the problem,

$$g''(x) = -\lambda_1 g(x)$$
$$g(x_0-\delta) = 0 = g'(x_0).$$

Since $\lambda_1 > 0$, there exists a positive constant $k_1 > z_0$ such that

$$\frac{f(x)}{x} \geq \max\{2\lambda_1, \frac{2}{\delta^2 a^2}\} \text{ for } x \geq k_1. \tag{18}$$

From $f(x)/x \geq 2\lambda_1$, we have $f(x)/2 \geq \lambda_1 x$. Therefore,

$$f(x) > f(x) - \lambda_1 x \geq f(x) - \frac{f(x)}{2} = \frac{f(x)}{2},$$

which gives

$$\frac{1}{f(x)} < \frac{1}{f(x)-\lambda_1 x} \leq \frac{2}{f(x)}.$$

From $\int_{z_0}^{\infty}\frac{1}{f(s)}ds < \infty$, we have

$$\int_{k_1}^{\infty}\frac{1}{f(x)-\lambda_1 x}dx < \infty.$$

From appendix C, the solution of the problem (17) blows up in a finite time at $x = x_0$, provided that $v_0(x)$ is large enough.

Next, we choose a positive constant $k_2 \geq k_1/\left(\delta^2 a^2\right)$ big enough such that

$$w_0(x) = a^2 k_2[x - (x_0 - \delta)][(x_0 + \delta) - x] \geq v_0(x) \text{ in } [x_0 - \delta, x_0].$$

We see that

$$w_0(x_0 - \delta) = 0 \text{ and } w_0'(x_0) = 0.$$

By (18), we see that $f(x) \geq 2x/\left(\delta^2 a^2\right)$. Then,

$$\begin{aligned} w_0''(x) + a^2 f(w_0(x_0)) &= -2a^2 k_2 + a^2 f(a^2 \delta^2 k_2) \\ &\geq -2a^2 k_2 + a^2 a^2 \delta^2 k_2 \left(\frac{2}{a^2 \delta^2}\right) \\ &= 0. \end{aligned}$$

Let us consider the following problem,

$$Lw(x,t) = a^2 f(w(x_0,t)) \text{ in } (x_0 - \delta, x_0) \times (0,T),$$

$$w(x,0) = w_0(x) \text{ on } [x_0 - \delta, x_0],$$

$$w(x_0 - \delta, t) = w_x(x_0, t) = 0 \text{ for } 0 < t < T.$$

In $(x_0 - \delta, x_0) \times (0,T)$,

$$L(w - v)(x,t) = a^2 f'(\beta)\left[w(x_0,t) - v(x_0,t)\right]$$

for some β between $w(x_0,t)$ and $v(x_0,t)$. Also,

$$w(x,0) - v(x,0) \geq 0 \text{ on } [x_0 - \delta, x_0],$$

$$w(x_0 - \delta, t) - v(x_0 - \delta, t) = 0,\ w_x(x_0,t) - v_x(x_0,t) = 0 \text{ for } 0 < t < T.$$

From Lemma 4, $w(x,t) \geq v(x,t)$ in $[x_0 - \delta, x_0] \times [0,T)$, and $w(x,t)$ blows up in a finite time.

By choosing $\phi(x) \geq w_0(x)$ in $[x_0 - \delta, x_0] \times [0,T)$ and using Lemma 4, $u(x,t) \geq w(x,t)$. Therefore, $u(x,t)$ blows up in a finite time, provided that $\phi(x)$ is sufficiently large in some neighborhood of x_0. □

Chapter V

Complete blow-up

In this chapter, we will show the complete blow-up of the solution u.

Lemma 6. Given any $x \in D$ and any finite T, there exists positive constants C_1 (depending on x and T) and C_2 (depending on T) such that

$$\int_0^1 G(x,t;\xi,0)d\xi > C_1 \quad \text{for} \;\; 0 \le t \le T,$$

$$\int_0^1 G(x_0,t;\xi,0)d\xi < C_2 \quad \text{for} \;\; 0 \le t \le T.$$

Proof. Let us consider the following auxiliary problem,

$$\left.\begin{aligned} Lv(x,t) = a^2 \text{ in } D\times(0,T)\,, \\ v(x,0) = 0 \text{ on } \overline{D}, \\ v(0,t) = v_x(1,t) = 0 \text{ for } 0<t<T \end{aligned}\right\} \tag{19}$$

The problem (19) has a unique solution v given by

$$\begin{aligned} v(x,t) &= a^2 \int_0^t \int_0^1 G(x,t-\tau;\xi,0)d\xi d\tau \\ &= a^2 \int_0^t \int_0^1 G(x,\tau;\xi,0)d\xi d\tau, \end{aligned}$$

which gives

$$v_t(x,t) = a^2 \int_0^1 G(x,t;\xi,0)d\xi.$$

It follows from Lemma 1 that $v_t(x,t) > 0$ for any $x \in D$ and any $t > 0$. Since for any $x \in D$,

$$v_t(x,0) = a^2 \int_0^1 G(x,0;\xi,0)d\xi$$
$$= a^2 \int_0^1 \sum_{n=1}^{\infty} g_n(x)\, g_n(\xi)\, d\xi$$
$$= a^2 \int_0^1 \delta(x-\xi)d\xi$$
$$= a^2,$$

it follows that for any $x \in D$ and for any finite T, there exists a positive C_1 (depending on x and T) such that

$$\int_0^1 G(x,t;\xi,0)d\xi > C_1 \text{ for } 0 \le t \le T.$$

Since $\int_0^1 G(x,t;\xi,0)d\xi$ exists, there exists a positive C_2 (depending on T) such that

$$\int_0^1 G(x_0,t;\xi,0)d\xi < C_2 \text{ for } 0 \le t \le T.$$

which completes the proof. □

THEOREM 7. If the solution of the problem (8) blows up in a finite time T, then the blow-up set is $\overline{D}$.

Proof. For any $t < T$,

$$u(x,t) = a^2 \int_0^t \int_0^1 G(x,t;\xi,\tau)f(u(x_0,\tau))d\xi d\tau + \int_0^1 G(x,t;\xi,0)\phi(\xi)d\xi$$
$$= a^2 \int_0^t \int_0^1 G(x,t-\tau;\xi,0)f(u(x_0,\tau))d\xi d\tau + \int_0^1 G(x,t;\xi,0)\phi(\xi)d\xi \quad (20)$$

If $u(x,t)$ blows up in a finite time T, we know that u blows up at least at $x = x_0$ by Theorem 3.

From (20) and Lemma 6,

$$\begin{aligned}
u(x_0,t) &= a^2 \int_0^t \int_0^1 G(x_0,t;\xi,0) f(u(x_0,t-\tau))d\xi d\tau + \int_0^1 G(x_0,t;\xi,0)\phi(\xi)d\xi \\
&= a^2 \int_0^t f(u(x_0,t-\tau))d\tau \int_0^1 G(x_0,t;\xi,0)d\xi + \int_0^1 G(x_0,t;\xi,0)\phi(\xi)d\xi \\
&\leq C_2 a^2 \int_0^t f(u(x_0,t-\tau))d\tau + C_2 \max_{x\in\overline{D}} \phi(x)
\end{aligned}$$

Since $u(x_0,t) \to \infty$ as $t \to T$, we also have $\int_0^T f(u(x_0,T-\tau))d\tau = \infty$.

For any $(x,t) \in \Omega$

$$\begin{aligned}
u(x,t) &\geq C_1 a^2 \int_0^t f(u(x_0,t-\tau))d\tau + \int_0^1 G(x,t;\xi,0)\phi(\xi)d\xi \\
&\geq C_1 a^2 \int_0^t f(u(x_0,t-\tau))d\tau.
\end{aligned}$$

As t approaches T, it follows from $\int_0^T f(u(x_0,T-\tau))d\tau \to \infty$ that $u(x,t)$ tends to infinity. Thus, the blow-up set is D. For $\widetilde{x} \in \{0,1\}$, we can always find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (\widetilde{x},T)$ and $\lim_{n\to\infty} u(x_n,t_n) \to \infty$. Therefore the blow-up set is $\overline{D}$. $\square$

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The parabolic boundary of $S(P)$ is

$$([0,1]\times\{0\})\cup(\{0\}\times(0,1])\cup(\{1\}\times(0,1])$$

Then the positive maximum or negative minimum is attained on the parabolic boundary.

Hopf's Lemma (Parabolic version).

Let $c(x,t)$ be a continuous function in T with $c \leq 0$. If $(L+c)u \geq 0$ in T, the maximum M (minimum m) of u is attained at a point $P \in \partial T$, and a sphere through P, having its interior lying in T such that $u < M$ $(u > m)$ there, can be constructed, then $\dfrac{\partial u}{\partial \eta} > 0$ (< 0) at P, provided that the radial direction from the centre of the sphere to P is not parallel to the t-axis.

Appendices

Appendix A

Maximum principle and Hopf's lemma

We outline briefly on strong and weak maximum. Hopf's lemma is also included. Interested readers may consult [11].

Let T be a $(n+1)$-dimensional domain in E^{n+1} and

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} - u_t,$$

where $a_{ij} = a_{ji}$. The operator L is parabolic at (x,t) if there is a number $\mu > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2$$

for all n-tuple $(\xi_1, \xi_2, ..., \xi_n)$. The operator L is uniformly parabolic in T if the above inequality holds with the same number μ for all $(x,t) \in T$.

Let us assume that L is uniformly parabolic, and a_{ij} and b_i are continuous in T. For each $P \in T$, denote by $S(P)$ the set of points Q which may be connected to P by a simple curve in T along which the coordinate t is nondecreasing from Q to P.

Strong Maximum Principle

Let $c(x,t)$ be a continuous function in T such that $c(x,t) \leq 0$.

If $(L+c)u \geq 0$ and u achieves its positive maximum at a point $P_0 \in T$, then $u \equiv u(P_0)$ in $S(P_0)$.

If $(L+c)u \geq 0$ and u achieves its negative minimum at a point $P_0 \in T$, then $u \equiv u(P_0)$ in $S(P_0)$.

Weak Maximum Principle

If $(L+c)u \geq 0$ and u is continuous on $\overline{T}$, then for any point $P \in T$, the positive maximum of u in $\overline{S}(P)$ is attained at a point on the complement of $S(P)$.

If $(L+c)u \geq 0$ and u is continuous on $\overline{T}$, then for any point $P \in T$, the negative of u in $\overline{S}(P)$ is attained at a point on the complement of $S(P)$.

the solution of the problem is unbounded and exists till time

$$T_0 \leq T_* = \int\limits_{E_0}^{\infty} \frac{d\eta}{Q(\eta) - \lambda_1 \eta} < \infty.$$

Proof. Let

$$E(t) = \int\limits_0^1 u(x,t)\psi_1(x)dx.$$

Then $E(0) = E_0$ and furthermore, as follows from (C1), $E(t)$ satisfies the equality

$$\frac{dE}{dt} = \int\limits_0^1 u_{xx}(x,t)\psi_1(x)dx + \int\limits_0^1 Q(u(x,t))\psi_1(x)dx. \qquad \text{(C4)}$$

Integrating by parts and taking into account (C1) and (C2), we obtain

$$\begin{aligned}\int\limits_0^1 u_{xx}(x,t)\psi_1(x)dx &= \int\limits_0^1 u(x,t)\psi_1''(x)dx \\ &= -\lambda_1 \int\limits_0^1 u(x,t)\psi_1(x)dx \\ &= -\lambda_1 E(t).\end{aligned}$$

Furthermore, from Jensen's inequality for convex functions, we obtain

$$\int\limits_0^1 Q(u)\psi_1(x)dx \geq Q(\int\limits_0^1 u(x,t)\psi_1(x)dx) = Q(E),$$

from (C4), we have the inequality

$$\frac{dE}{dt} \geq Q(E) - \lambda_1 E > 0,\ \ t > 0,$$

$$E(0) = E_0 \geq \delta_0.$$

Hence under assumptions we have that $E(t) > E_0$ for all $t > 0$, and consequently

$$\int\limits_{E_0}^{E(t)} \frac{d\eta}{Q(\eta) - \lambda_1(\eta)} \geq t,\ \ t > 0.$$

Therefore, by (C3), $E(t) \to \infty$ as $t \to T_1^- \leq T_*$, and since $E(t) \leq \sup u(x,t)$, the solution $u(x,t)$ is unbounded. $\square$

Appendix B

Orthogonality of eigenfunctions

The following lemma gives the relation between eigenfunctions and δ-function, for further reading, see [5].

LEMMA.

$$\sum_{n=1}^{\infty} g_n(\xi) g_n(x) = \delta(x-\xi),$$

where $g_n(x)$ is an orthonormal eigenfunction of the Sturm-Liouville problem

$$g''(x) + \lambda g(x) = 0,$$

and the boundary conditions

$$g(0) = g'(1) = 0.$$

Proof. Let us expand $\delta(x-\xi)$ in term of $g_n(x)$. From the Sturm-Liouville theorem

$$\delta(x-\xi) = \sum_{n=1}^{\infty} c_n g_n(x),$$

where

$$c_n = \frac{\int_0^1 \delta(x-\xi) g_n(x) dx}{\int_0^1 g_n^2(x) dx} = g_n(\xi).$$

Hence

$$\sum_{n=1}^{\infty} g_n(\xi) g_n(x) = \delta(x-\xi),$$

which completes the proof. $\square$

Appendix C

Blowing up problem

We will show that the following problem blows up in a finite time under certain conditions. The generalized problem is contained in [12]

THEOREM. Let us consider a boundary value problem for a semilinear

$$\left.\begin{aligned} u_t(x,t) = u_{xx}(x,t) + Q(u(x,t)) \ \text{ for } t>0,\ x\in(0,1), \\ u(x,0) = u_0(x) \geq 0 \ \text{ on } x\in[0,1], \\ u(0,t) = u_x(1,t) = 0 \ \text{ for } t>0, \end{aligned}\right\} \tag{C1}$$

where $Q \in C^2$ is a convex function: $Q''(u) \geq 0,\ u > 0$.
Let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$\left.\begin{aligned} \psi''(x) + \lambda\psi(x) = 0, \\ \psi(0) = \psi'(1) = 0, \end{aligned}\right\} \tag{C2}$$

and by $\psi_1(x)$ the first eigenfunction. Let $\psi_1(x) > 0$ and

$$\int_0^1 \psi_1(x)dx = 1.$$

If $Q(u) - \lambda_1 u > 0$ for all $u \geq \delta_0$, where δ_0 is a positive constant, and

$$\int_{\delta_0}^{\infty} \frac{d\eta}{Q(\eta) - \lambda_1\eta} < \infty, \tag{C3}$$

then for any initial function $u_0(x) \geq 0$ such that

$$E_0 = \int_0^1 u_0(x)\psi_1(x)dx \geq \delta_0,$$

[12] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov. *Blow-up in Quasilinear Parabolic Equations (translated from Russian by M. Grinfeld).* New York, NY: Walter de Gruyter, 1995.

Appendix D

An example of initial value data

In this appendix, we will give an example of initial value data of the problem (3) that satisfies all the needed conditions and is guarunteed to blow up in a finite time.

Let $f \in C^2([0,\infty))$, $f(0) \geq 0$, $f'(s) > 0$ and $f''(s) > 0$ for $s > 0$, $\int\limits_{z_0}^{\infty} \frac{1}{f(s)} ds < \infty$ for some $z_0 > 0$. It is easy to see that $\lim\limits_{x\to\infty} (f(x)/x) = \infty$. Then there exists $\beta > 0$ such that

$$f(x) \geq \frac{2x}{x_0 a^2 (2 - x_0)} \quad \text{for } x \geq \beta \tag{D1}$$

(see the proof of Theorem 5 for details). Let

$$\phi(x) = -a^2 k (x-1)^2 + a^2 k \quad \text{for } 0 \leq x \leq 1$$

where $k \geq \dfrac{\beta}{x_0 a^2 (2 - x_0)}$.

We can see that ϕ is nontrivial, nonnegative and continuous such that $\phi(0) = \phi'(1) = 0$. Moreover, its second derivative with respect to x is given by

$$\phi''(x) = -2a^2 k \quad \text{for } 0 < x < 1.$$

Using (D1), we obtain that

$$\begin{aligned}
\phi''(x) + a^2 f(\phi(x_0)) &= -2a^2k + a^2 f(-a^2k(x_0-1)^2 + a^2k) \\
&= -2a^2k + a^2 f(kx_0a^2(2-x_0)) \\
&\geq -2a^2k + a^2 \frac{2kx_0a^2(2-x_0)}{x_0a^2(2-x_0)} \\
&= 0.
\end{aligned}$$

Hence $\phi''(x) + a^2 f(\phi(x_0)) \geq 0 \quad$ for $\quad 0 < x < 1$.

Remark: Since $\phi(x_0 = kx_0a^2(2-x_0)$, $\phi(x_0)$ depends on the positive constant k. We can always choose a positive constant k big enough to meet the required condition in Theorem 5. Consequently, the blow-up phenomenon occurs in a finite time.

VITA

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