เศษส่วนต่อเนื่องเหนือสนามที่มีค่าลักษณะเฉพาะเป็นบวก

นางสาวตวงรัตน์ ไชยชนะ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2544

ISBN 974-03-0683-7

CONTINUED FRACTIONS IN FIELDS OF POSITIVE CHARACTERISTIC

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2001

ISBN 974-03-0683-7

Thesis title	Continued Fractions in Fields of Positive Characteristic
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Field of study	Mathematics
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ตวงรัตน์ ไชยชนะ : เศษส่วนต่อเนื่องเหนือสนามที่มีค่าลักษณะเฉพาะเป็นบวก (CONTINUED FRACTIONS IN FIELDS OF POSITIVE CHARACTERISTIC) อ. ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร.อัจฉรา หาญชูวงศ์, อ.ที่ปรึกษาร่วม : รองศาสตราจารย์ ดร. วิเชียร เลาหโกศล, 45 หน้า
ISBN 974-03-0683-7

ให้ $\oplus_q[x]$ เป็นวงของพหุนามเหนือ $\oplus_q$ เมื่อ $\oplus_q$ เป็นสนามจำกัดที่มีจำนวนสมาชิก q ตัว

$\oplus_q(x)$ เป็นสนามผลหารของ $\oplus_q[x]$

$\oplus_q((\frac{1}{x}))$ เป็นสนามบริบูรณ์ของ $\oplus_q(x)$ เทียบกับ แวลูเอชันอนันต์

$\oplus_q((x))$ เป็นสนามบริบูรณ์ของ $\oplus_q(x)$ เทียบกับ เอกซ์-แอดิกแวลูเอชัน

วิทยานิพนธ์ฉบับนี้ทำเกี่ยวกับเศษส่วนต่อเนื่องและสมบัติลักษณะเฉพาะต่าง ๆ ใน $\oplus_q((\frac{1}{x}))$ และ $\oplus_q((x))$ ซึ่งต่อไปจะเรียกว่า สนามฟังก์ชัน การสร้างเศษส่วนต่อเนื่องเหนือสนามเฉพาะที่ มีหลายแบบ เช่น ในสนามจำนวนพี-แอดิค มีเศษส่วนต่อเนื่อง 2 แบบที่สำคัญซึ่งสร้างโดยรูบันและชไนเดอร์ในช่วง 1970-1979

เศษส่วนต่อเนื่องแบบรูบันซึ่งมีวิธีสร้างล้อกับเศษส่วนต่อเนื่องแบบฉบับสำหรับจำนวนจริงถูกพัฒนาครั้งแรกใน $\oplus_2((\frac{1}{x}))$ โดยบาวม์และสวีท ขณะที่เศษส่วนต่อเนื่องแบบชไนเดอร์ยังไม่เคยถูกพิจารณาอย่างจริงจังในสนามเฉพาะที่ เราจะแสดงวิธีสร้างเศษส่วนต่อเนื่องทั้ง 2 แบบนี้ใน $\oplus_q((\frac{1}{x}))$ และ $\oplus_q((x))$ พร้อมทั้งแสดงสมบัติพื้นฐานของเศษส่วนต่อเนื่องเหล่านั้น

ต่อไปเราจะแสดงเช่นเดียวกับในกรณีแบบฉบับว่า เศษส่วนต่อเนื่องทั้ง 2 แบบจะรู้จบ ก็ต่อเมื่อ เป็นเศษส่วนต่อเนื่องที่แทนจำนวนตรรกยะ สำหรับการให้ลักษณะเฉพาะของจำนวนอตรรกยะกำลังสอง เป็นที่รู้กันว่าจำนวนจริงจะเป็นจำนวนอตรรกยะกำลังสอง ก็ต่อเมื่อ เศษส่วนต่อเนื่องแบบฉบับของมันเป็นคาบ ในกรณีของสนามฟังก์ชัน ผลนี้ยังเป็นจริงสำหรับเศษส่วนต่อเนื่องแบบรูบัน ในขณะที่เศษส่วนต่อเนื่องแบบชไนเดอร์ เราเพียงแสดงได้ว่าจำนวนอตรรกยะกำลังสองส่วนใหญ่มีเศษส่วนต่อเนื่องแบบชไนเดอร์เป็นคาบ

ในส่วนสุดท้าย เราสร้างเศษส่วนต่อเนื่องในสนามฟังก์ชันโดยใช้เกณฑ์การประมาณค่าที่ดีที่สุดซึ่งในที่สุดแล้วเป็นเศษส่วนต่อเนื่องแบบรูบัน

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต……………………………………
สาขาวิชา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา…………………………..
ปีการศึกษา **2544**	ลายมือชื่ออาจารย์ที่ปรึกษาร่วม……………………...

4272282823 MAJOR : MATHEMATICS
KEYWORD : CONTINUED FRACTION / VALUATION / LOCAL FIELD / FUNCTION FIELD

TUANGRAT CHAICHANA: CONTINUED FRACTIONS IN FIELDS OF POSITIVE CHARACTERISTIC. THESIS ADVISOR: ASSIST.PROF. AJCHARA HARNCHOOWONG, Ph.D. THESIS CO-ADVISOR: ASSOC.PROF. VICHIAN LAOHAKOSOL, Ph.D. 45pp. ISBN 974-03-0683-7

Let $\oplus_q[x]$ be the ring of polynomials over $\oplus_q$, the finite field of q elements,

$\oplus_q(x)$ its field of quotients,

$\oplus_q((\frac{1}{x}))$ the completion of $\oplus_q(x)$ with respect to the infinite valuation,

and $\oplus_q((x))$ the completion of $\oplus_q(x)$ with respect to the x–adic valuation.

This thesis deals with continued fractions in $\oplus_q((\frac{1}{x}))$ and $\oplus_q((x))$, which we shall refer to as function fields, and their characterization properties. There have been different kinds of continued fractions constructed over local fields, such as the p-adic number field; the two notable ones being due to Ruban and Schneider in the seventies.

The Ruban type continued fraction, which mimics the classical continued fraction in the reals, was first developed in $\oplus_2((\frac{1}{x}))$ by Baum & Sweet, while the Schneider type continued fraction has never been seriously considered in function fields. Here we present the constructions of both types of continued fractions (Ruban and Schneider) in $\oplus_q((\frac{1}{x}))$ and $\oplus_q((x))$ and derive their basic properties.

Next, it is shown that as in the classical case both continued fractions terminate if and only if they represent rational elements. As to the characterization of quadratic irrationals, it is well known that a real number is a quadratic irrational if and only if its classical continued fraction is periodic. In the function fields case, this result remains true for Ruban continued fraction, while for Schneider continued fraction, we can only show that a quadratic irrational belonging to a large class does indeed have periodic Schneider continued fraction.

In the last part, we prove that should one try to construct continued fraction in function fields using the best approximation criteria, one will inevitably end up with Ruban continued fraction.

Department **Mathematics** Student's signature……………………………..
Field of Study **Mathematics** Advisor's signature…………………………….
Academic year **2001** Co-advisor's signature…………………………

ACKNOWLEDGEMENTS

I am greatly indebted to Associate Professor Dr. Vichian Laohakosol and Assistant Professor Dr. Ajchara Harnchoowong, my thesis advisors, for their untired offering me some thoughtful and helpful advice in preparing and writing this thesis. I would like to thank Assistant Professor Dr. Amorn Wasanawichit and Assistant Professor Dr. Patanee Udomkavanich, my thesis committee, for their suggestions to this thesis. I also would like to thank all of the teachers who have taught me, for the knowledge and skill I have gained from them. Thanks to all my friends for giving me warmth and happiness.

In particular, I would like to express my highest gratitude to my family for their love and encouragement throughout my graduate study.

CONTENTS

CHAPTER I

Introduction

There are two well-known continued fractions for p-adic numbers, namely the one due to Ruban [15] and the other due to Schneider [16]. As seen from their algorithms, both kinds of continued fractions can be constructed in any local field. Indeed, as pointed out by Browkin [4], by choosing different sets of representatives for the residue class field, many more similar, yet with certain different properties, continued fractions can be derived. In the classical case, real numbers are rational if and only if their continued fractions are finite. In the p-adic case, the situation, though already settled, is more complicated for there are rational numbers whose p-adic continued fractions are infinite periodic, see e.g. Bundchuh [5], Laohakosol [7], Lianxiang [8], de Weger [6], and Browkin [4]. A beautiful characterization of quadratic irrationals, due to Lagrange, with periodic continued fractions in the classical case leads one to ask whether there is such a characterization in other fields. The situation in the p-adic case is much more difficult, for example there are p-adic quadratic irrationals whose continued fractions are not periodic. To date this is not completely settled, though there have been various investigations, see e.g. de Weger [6], and Browkin [4]. In this thesis, we consider analogous questions in the case of function field, K, i.e. completions of $\mathbb{F}_q(x)$, where $\mathbb{F}_q$ denotes the finite field of q elements, with respect to two main valuations, namely the infinite valuation $|\cdot|_\infty$, and the π-adic valuation $|\cdot|_\pi$, where $\pi := \pi(x)$ is a non-constant irreducible element in $\mathbb{F}_q(x)$.

In Chapter II, we collect definitions and results, mainly without proofs, to be used throughout the entire thesis.

In Chapter III, we describe the constructions of the so-called Ruban continued fraction, henceforth called **RCF**, in any local field K, and specialize K to be $\mathbb{F}_q((\frac{1}{x}))$, or $\mathbb{F}_q((x))$ the completions of $\mathbb{F}_q(x)$ with respect to $|\cdot|_\infty$, and $|\cdot|_\pi$ with $\pi = x$, respectively. We show that rational elements in both fields are precisely those with finite continued fractions. As for quadratic irrationals, it is not difficult to see that infinite periodic continued fractions of any kind represent a quadratic irrational. However, we can only establish that a large class of quadratic irrationals has periodic continued fractions.

In Chapter IV, we describe the constructions of the so-called Schneider continued fraction, henceforth called **SCF**, in any local field K, and specialize K to be $\mathbb{F}_q((\frac{1}{x}))$, or $\mathbb{F}_q((x))$. Rationality and quadratic irrationality characterization are considered with similar results as those in Chapter III.

In the final chapter, Chapter V, we prove that should one start constructing continued fractions via the concept of best approximations, one will end up with **RCF**.

CHAPTER II

Basic Definitions and Results

In this chapter, we collect definitions and results, mainly without proofs, to be used throughout the entire thesis. The first section deals with valuations and related concepts. Details and proofs can be found in McCarthy [10] or Bachman [1]. The second section deals with continued fractions and their properties. Details and proofs can be found in Lorentzen and Waadeland [9] or Niven, Zuckerman and Montgomery [14] for the classical case, and in Ruban [15], Schneider [16], Bundchuh [5], Laohakosol [7], Browkin [4], de Weger [6], and Lianxiang [8] for the p-adic case.

2.1 Valuations

Definition 2.1. A *valuation* on a field K is a real-valued function $a \mapsto |a|$ defined on K which satisfies the following conditions:

(i) $\forall a \in K,\ |a| \geq 0$ and $|a| = 0 \Leftrightarrow a = 0$

(ii) $\forall a, b \in K,\ |ab| = |a||b|$

(iii) $\forall a, b \in K,\ |a + b| \leq |a| + |b|$.

There is always at least one valuation on K, namely, that given by setting $|a| = 1$ if $a \in K - \{0\}$ and $|0| = 0$. This valuation is called the *trivial valuation* on K.

Definition 2.2. A valuation $|\cdot|$ on K is called *non-Archimedean* if the condition (iii) in Definition 2.1 is replaced by a stronger condition, called the *strong triangle* *inequality*

$$|a+b| \leq max(|a|,|b|) \quad (\forall a,\ b \in K).$$

Any other valuation on K is called *Archimedean*.

A *valuated field* $(K,\ |\cdot|)$ is a field K together with a prescribed valuation $|\cdot|$. If the valuation is non-Archimedean, then K is called a *non-Archimedean valuated field*.

Examples 2.3. 1) For $K = \mathbb{Q}$, the ordinary absolute value $|\cdot|$ is an Archimedean valuation on K.

2) For $K = \mathbb{Q}$, let p be a prime number. Each $a \in \mathbb{Q} - \{0\}$ can be written uniquely in the form

$$a = p^n \left(\frac{u}{v}\right),$$

where $u,\ v \in \mathbb{Z},\ (v > 0),\ (u, v) = 1,\ n \in \mathbb{Z},\ p \nmid u$ and $p \nmid v$. Define

$$|a|_p = p^{-n} \ and \ |0|_p = 0.$$

Then $|\cdot|_p$ is a non-Archimedean valuation on $\mathbb{Q}$ and called the *p-adic valuation*.

3) Consider the field $\mathbb{F}_q(x)$ of rational functions over a finite field $\mathbb{F}_q$ of q elements. Let $\dfrac{f(x)}{g(x)} \in \mathbb{F}_q(x) - \{0\}$. Define

$$\left|\frac{f(x)}{g(x)}\right|_\infty = 2^{deg\, f - deg\, g} \ and \ |0|_\infty = 0.$$

Then $|\cdot|_\infty$ is a non-Archimedean valuation on $\mathbb{F}_q(x)$.

4) Let $\pi(x)$ be an irreducible polynomial in $\mathbb{F}_q[x]$.
If $\dfrac{f(x)}{g(x)} \in \mathbb{F}_q(x) - \{0\}$, we can write uniquely as

$$\frac{f(x)}{g(x)} = \pi(x)^n \frac{u(x)}{v(x)},$$

where $u(x)$ and $v(x)$ are relatively prime elements of $\mathbb{F}_q[x]$, neither of which is divisible by $\pi(x)$. Define

$$\left|\frac{f(x)}{g(x)}\right|_\pi = 2^{-n} \ and \ |0|_\pi = 0.$$

Then $|\cdot|_\pi$ is a non-Archimedean valuation on $\mathbb{F}_q(x)$. We will consider mostly the case where $\pi(x) = x$, and write $|\cdot|_x$ instead of $|\cdot|_\pi$.

Since valuation gives rise to a metric on any valuated field K, the usual completion process is applicable. In the case of $\mathbb{Q}$, with the usual absolute value, its completion is the field $\mathbb{R}$ of real numbers and in the case of $(\mathbb{Q},\ |\cdot|_p)$, its completion is the p-adic number field $(\mathbb{Q}_p,\ |\cdot|_p)$, while in the cases of $(\mathbb{F}_q(x),\ |\cdot|_\infty)$ and $(\mathbb{F}_q(x),\ |\cdot|_\pi)$ the completions are $(\mathbb{F}_q((\frac{1}{x})),\ |\cdot|_\infty)$ and $(\mathbb{F}_q((\pi(x))),\ |\cdot|_\pi)$ the fields of formal Laurent series in $\dfrac{1}{x}$ and $\pi(x)$, respectively.

Definition 2.4. Let $(K,\ |\cdot|)$ be a valuated field. i) The set

$$V = \{|a|; a \in K - \{0\}\}$$

is easily checked to be a group and is called the *value group* of $(K,\ |\cdot|)$.

ii) If V is an infinite cyclic group, then $(K,\ |\cdot|)$ is called a *discrete* valuated field.

iii) A *local field* is a complete, discrete non-Archimedean valuated field.

iv) The set $\omega = \{a \in K : |a| \leq 1\}$ is a ring, called the *valuation ring* of $(K,\ |\cdot|)$.

v) The set $\wp = \{a \in K : |a| < 1\}$ is the unique maximal ideal of ω.

vi) The field $\omega/\wp$ is called the *residue class field* of $(K,\ |\cdot|)$.

Examples 2.5. 1) $(\mathbb{Q}_p,\ |\cdot|_p)$ is a local field with $\{0,\ 1,\ 2, \ldots,\ p-1\}$ as a set of representatives of its residue class field.

2) $(\mathbb{F}_q((\frac{1}{x})),\ |\cdot|_\infty)$ is a local field with $\mathbb{F}_q$ as a set of representatives of its residue class field.

3) $(\mathbb{F}_q((x)),\ |\cdot|_x)$ is a local field with $\mathbb{F}_q$ as a set of representatives of its residue class field.

In a local field $(K,\ |\cdot|)$ with R being the set of representatives of its residue class field, each element $\alpha \in K$ can be uniquely represented as

$$\alpha = \sum_{i=r}^{\infty} c_i \pi^i,$$

where $c_i \in R,\ r \in \mathbb{Z}$, and $\pi \in K$ is called a prime element which is usually normalized so that $|\pi| = 2^{-1}$. Thus $|\alpha| = |\pi|^r = 2^{-r}$. Sometimes, it is convenient to use the *ordinal function* which is defined by $ord_\pi(\alpha) = r$, and so $ord_\pi(\pi) = 1$.

2.2 Classical continued fractions

The expansion

$$b_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{\ddots}}}}}$$

is called a *continued fraction.*

It is more convenient to use the notation

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n;\ldots] \tag{2.1}$$

for the above continued fractions. The elements $a_1, a_2, a_3, \ldots$ are called its *partial numerators*; $b_1, b_2, b_3, \ldots$ its *partial denominators.* When all $a_i = 1$ we use

$$[b_0,\ b_1,\ b_2, \ldots]$$

for $[b_0;\ 1,\ b_1;\ 1,\ b_2;\ldots;\ 1,\ b_n;\ldots]$. We assume that all partial denominators are not equal to zero.

The *terminating* or *finite* continued fraction

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n] := \frac{p_n}{q_n}$$

is called the n^{th} *convergent* of the continued fraction (2.1) .

In $\mathbb{R}$, it is known that any real number can be represented as a continued fraction of the form

$$[b_0,\ b_1,\ b_2, \ldots]$$

where $b_0 \in \mathbb{Z},\ b_i \in \mathbb{N}$. This is called a *simple* continued fraction and the b_i are called its *partial quotients.* Such representation is unique for real irrationals, but for real rationals, we have the following characterization.

Theorem 2.6. Any finite simple continued fraction represents a rational number. Conversely, any rational number can be expressed as a finite simple continued fraction, and in exactly two ways,

$$[b_0,\ b_1,\ b_2, \ldots,\ b_n] = [b_0,\ b_1,\ b_2, \ldots,\ b_n - 1,\ 1].$$

An infinite simple continued fraction

$$[b_0,\ b_1,\ b_2, \ldots]$$

is said to be *periodic* if there is an integer r such that $b_n = b_{n+r}$ for all sufficiently large integers n. A well known theorem of Lagrange characterizing infinite, periodic, simple continued fractions states that:

Theorem 2.7. An infinite, periodic, simple continued fraction is a quadratic irrational number, and conversely.

2.3 p-adic Continued fractions

There are many p-adic continued fractions constructed by various authors. We shall consider only two types, namely, *Ruban Continued Fraction* first developed by Ruban [15] and *Schneider Continued Fraction* first developed by Schneider [16].

The process for the expansion of the p-adic Ruban continued fraction, denoted by p-adic **RCF**, was described by Ruban [15] and Laohakosol [7] as follows:

Let $\xi \in \mathbb{Q}_p$. As usual, ξ can be represented uniquely as

$$\xi = \sum_{i=r}^{\infty} c_i p^i$$

where $r \in \mathbb{Z}$, $c_i \in \{0,\ 1, \ldots,\ p-1\} := \mathbb{F}_p\ \ (i \geq r)$. Define

$$[\xi] := \sum_{i=r}^{0} c_i p^i \in \mathbb{F}_p \left[\frac{1}{p}\right], \quad (\xi) := \sum_{i=1}^{\infty} c_i p^i$$

and we call $[\xi]$ and (ξ) the *head* part and the *tail* part of ξ, respectively. The head and tail parts of ξ are uniquely determined, and so uniquely write $\xi = [\xi] + (\xi)$.

Let $b_0 = [\xi] \in \mathbb{F}_p[\frac{1}{p}]$. Hence $|b_0|_p \geq 1$.

If $(\xi) = 0$, the process stops.

Otherwise, write ξ in the form $\xi = b_0 + \dfrac{1}{\xi_1}$, where $\xi_1^{-1} = (\xi)$ with $|\xi_1|_p > 1$. As above, we can uniquely write $\xi_1 = [\xi_1] + (\xi_1)$. Let $b_1 = [\xi_1] \in \mathbb{F}_p[\frac{1}{p}] - \{0\}$.

If $(\xi_1) = 0$, the process stops.

Otherwise, write ξ_1 in the form $\xi_1 = b_1 + \dfrac{1}{\xi_2}$, where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2|_p > 1$. As above, we can uniquely write $\xi_2 = [\xi_2] + (\xi_2)$. Let $b_2 = [\xi_2] \in \mathbb{F}_p[\frac{1}{p}] - \{0\}$.

Again, if $(\xi_2) = 0$, the process stops.

Otherwise proceed in the same manner.

Therefore ξ has a unique p-adic **RCF** of the form

$$[b_0,\ b_1,\ b_2, \ldots]$$

where all $b_i \in \mathbb{F}_p[\frac{1}{p}] - \{0\}\ \ (i \geq 1)$.

It is quite trivial that a finite p-adic **RCF** always represents a rational number. However, there exist infinitely many rational numbers with infinite periodic p-adic **RCF**'s. Laohakosol [7] gave a characterization of rational numbers via p-adic **RCF** as follows:

Theorem 2.8. Let $\xi \in \mathbb{Q}_p - \{0\}$. Then ξ is a rational number if and only if its p-adic **RCF** is either finite or periodic from a certain fraction onwards with the shape

$$[(p-1)p^{-1} + (p-1),\ (p-1)p^{-1} + (p-1), \ldots].$$

Schneider [15] constructed another p-adic continued fraction, denoted henceforth by p-adic **SCF**, as follows:

Let $\xi \in \mathbb{Q}_p - \{0\}$. It can be assumed without loss of generality that $|\xi|_p = 1$. Then ξ can be represented uniquely as

$$\xi = \sum_{i=0}^{\infty} c_i p^i$$

where $c_i \in \mathbb{F}_p \ \ (i \geq 0),\ c_0 \neq 0$. Let $b_0 = c_0$ and write ξ in the form $\xi = b_0 + \dfrac{a_1}{\xi_1}$ with $|\xi_1|_p = 1 = |b_0|_p, a_1 = p^{\alpha_1} \ \ (\alpha_1 \in \mathbb{N})$. Let

$$\xi_1 = \sum_{i=0}^{\infty} d_i p^i$$

where $d_i \in \mathbb{F}_p \ \ (i \geq 0),\ d_0 \neq 0$. Let $b_1 = d_0$ and write ξ_1 in the form $\xi_1 = b_1 + \dfrac{a_2}{\xi_2}$ with $|\xi_2|_p = 1 = |b_1|_p,\ a_2 = p^{\alpha_2} \ \ (\alpha_2 \in \mathbb{N})$. Continuing in the same manner, we have generally

$$\xi_n = b_n + \frac{a_{n+1}}{\xi_{n+1}} \ \ (n \geq 0)$$

where $b_n \in \mathbb{F}_p - \{0\},\ a_{n+1} = p^{\alpha_{n+1}}$ with $|b_n|_p = 1 = |\xi_{n+1}|_p$. Therefore ξ has a unique p-adic **SCF** of the form

$$\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n; \ldots]$$

where $a_n = p^{\alpha_n},\ \alpha_n \in \mathbb{N},\ b_n \in \mathbb{F}_p - \{0\}$.

The expansion into p-adic **SCF** is unique. The following theorem (see [5]) contains a necessary and sufficient condition for rationality of p-adic numbers.

Theorem 2.9. Let $\xi \in \mathbb{Q}_p - \{0\}$. Then ξ is rational if and only if its p-adic **SCF** is either finite or periodic with period length 1 and $a_n = p,\ b_n = p - 1$ for sufficiently large n.

CHAPTER III

RCF

As seen in Chapter II, the underlying idea of p-adic **RCF** algorithm is exactly the same as in the classical case, i.e. separate the p-adic expansion of each number $\xi = [\xi] + (\xi)$ with $|[\xi]|_p \geq 1,\ |(\xi)|_p < 1$ into the head part , $[\xi]$, which is kept as partial quotient, and the tail part, (ξ), which is then inverted. The p-adic expansion of $\dfrac{1}{(\xi)}$, provided $(\xi) \neq 0$, is again separated into head and tail parts, and the process repeats. Browkin [4] observed that the same construction can be done in any local field. It is to be noted that almost all continued fractions considered in function fields are of this type, see e.g. Baum and Sweet [2], [3], Mills and Robbins [12], Mesirov and Sweet [11], Niederreiter and Wien [13], Thakur [17], [18], [19] and we shall refer to them throughout as **RCF**. In the first section of this chapter, a brief description of **RCF** in local field and its basic properties are given. In the last two sections, our main concerns are the two function field cases of $(\mathbb{F}_q((\frac{1}{x})),\ |\cdot|_\infty)$ and $(\mathbb{F}_q((x)),\ |\cdot|_x)$. Section 3.2 deals with complete characterization of rationals, while Section 3.3 does the same for quadratic irrationals but with less complete characterization.

3.1 Construction and basic properties

Let $(K,\ |\cdot|)$ be a local field, R the set of representatives of its residue class field. Every element $\xi \in K - \{0\}$ can be uniquely written in the form

$$\xi = \sum_{n=r}^{\infty} c_n \pi^n$$

with prime element π so normalized that $|\pi| = 2^{-ord_\pi \pi} = 2^{-1},\ r \in \mathbb{Z},\ c_i \in R$ and $c_r \neq 0$. We assume that $0 \in R$. Define

$$[\xi] := \sum_{n=r}^{0} c_n \pi^n,\ (\xi) := \sum_{n=1}^{\infty} c_n \pi^n.$$

We call $[\xi]$ and (ξ) the *head* part and the *tail* part of ξ, respectively. Then

$$R[\frac{1}{\pi}] = \{\alpha \in K; \alpha = \sum_{n=r}^{0} c_n \pi^n,\ r \in \mathbb{Z},\ r \leq 0\},$$

the set of all head parts of elements in K. The head part and tail part of ξ are uniquely determined, and so we can uniquely write $\xi = [\xi] + (\xi)$.

Let $b_0 = [\xi] \in R[\frac{1}{\pi}]$. Hence $|b_0| \geq 1$.

If $(\xi) = 0$, then the process stops.

If $(\xi) \neq 0$, then write $\xi = b_0 + \dfrac{1}{\xi_1}$, where $\xi_1^{-1} = (\xi)$ with $|\xi_1| > 1$. Next write $\xi_1 = [\xi_1] + (\xi_1)$. Let $b_1 = [\xi_1] \in R[\frac{1}{\pi}] - R$, then $|b_1| > 1$.

If $(\xi_1) = 0$, then the process stops.

If $(\xi_1) \neq 0$, then write $\xi_1 = b_1 + \dfrac{1}{\xi_2}$, where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2| > 1$. Let $b_2 = [\xi_2] \in R[\frac{1}{\pi}] - R$, then $|b_2| > 1$.

Again, if $(\xi_2) = 0$, then the process stops.

If $(\xi_2) \neq 0$, then we proceed in the same manner.

Therefore ξ has a unique **RCF** of the form

$$\xi = [b_0,\ b_1,\ b_2, \ldots,\ b_{n-1},\ \xi_n],$$

where all $b_i \in R[\frac{1}{\pi}] - \{0\}\ \ (i \geq 1)$, $\xi_n \in K,\ |\xi_n| > 1$ if exists and ξ_n is referred to as the n^{th} *complete quotient*. The sequence (b_n) so obtained is uniquely determined and we call b_n the partial quotients of ξ.

In order to establish convergence, we define two sequences (A_n), (B_n) as follows:

$$A_{-1} = 1, \qquad A_0 = b_0, \quad A_{n+1} = b_{n+1}A_n + A_{n-1} \ (n \geq 0) \tag{3.1}$$

$$B_{-1} = 0, \qquad B_0 = 1, \quad B_{n+1} = b_{n+1}B_n + B_{n-1} \ (n \geq 0) \tag{3.2}$$

Proposition 3.1. For any $n \geq 0,\ \alpha \in K - \{0\}$, we have

$$\frac{\alpha A_n + A_{n-1}}{\alpha B_n + B_{n-1}} = [b_0,\ b_1,\ b_2, \ldots,\ b_n,\ \alpha].$$

Proof. Let $P(n):\ \dfrac{\alpha A_n + A_{n-1}}{\alpha B_n + B_{n-1}} = [b_0,\ b_1,\ b_2, \ldots,\ b_n,\ \alpha].$
Since $\dfrac{\alpha A_0 + A_{-1}}{\alpha B_0 + B_{-1}} = \dfrac{\alpha b_0 + 1}{\alpha + 0} = b_0 + \dfrac{1}{\alpha}$, then $P(0)$ is true.
Suppose that $P(n-1)$ holds. Consider

$$\begin{aligned}
[b_0,\ b_1,\ b_2, \ldots,\ b_n,\ \alpha] =& [b_0,\ b_1,\ b_2, \ldots,\ b_n + \frac{1}{\alpha}] \\
=& \frac{(b_n + \frac{1}{\alpha})A_{n-1} + A_{n-2}}{(b_n + \frac{1}{\alpha})B_{n-1} + B_{n-2}}, \quad \text{(by induction hypothesis)} \\
=& \frac{\alpha(b_nA_{n-1} + A_{n-2}) + A_{n-1}}{\alpha(b_nB_{n-1} + B_{n-2}) + B_{n-1}} = \frac{\alpha A_n + A_{n-1}}{\alpha B_n + B_{n-1}},
\end{aligned}$$

which gives the truth of $P(n)$. □

From the above proposition, we have

$$\frac{A_n}{B_n} = \frac{b_nA_{n-1} + A_{n-2}}{b_nB_{n-1} + B_{n-2}} = [b_0,\ b_1,\ b_2, \ldots,\ b_n] \ \ (n \geq 1).$$

We call $\dfrac{A_n}{B_n}$ the n^{th} convergent of the **RCF** to ξ.

If $(\xi_n) = 0$ for some n, then $\xi = [b_0,\ b_1,\ b_2, \ldots,\ b_{n-1}]$ i.e. the **RCF** of ξ is finite. If $(\xi_n) \neq 0$ for all n, we will show that its **RCF** converges.

Proposition 3.2. $A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}$ $(n \geq 0)$.

Proof. Let $P(n):\ A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}$.

Since $A_0B_{-1} - A_{-1}B_0 = 0 - 1 = -1 = (-1)^{0-1}$, then $P(0)$ is true.

Suppose that $P(n-1)$ holds. Consider

$$A_nB_{n-1} - A_{n-1}B_n = (b_nA_{n-1} + A_{n-2})B_{n-1} - A_{n-1}(b_nB_{n-1} + B_{n-2})$$
$$= A_{n-2}B_{n-1} - B_{n-2}A_{n-1} = (-1)^{n-1},$$

and so $P(n)$ holds. □

Proposition 3.3. $|B_n| > |B_{n-1}| \quad (n \geq 0)$.

Proof. We have $|B_0| = 1 > 0 = |B_{-1}|$. Suppose $|B_{n-1}| > |B_{n-2}|$. Since $|b_n| > 1$, for $n \geq 1$, then by strong triangle inequality

$$|B_n| = |b_nB_{n-1} + B_{n-2}| = |b_nB_{n-1}| > |B_{n-1}|.$$

□

Proposition 3.4. $|B_n| \geq 2^n \quad (n \geq 1)$ and so $B_n \neq 0 \quad (n \geq 1)$.

Proof. Let $P(n): \ |B_n| \geq 2^n$.
Since $|B_1| = |b_1B_0 + B_{-1}| = |b_1B_0| = |b_1| \geq 2^1$, then $P(1)$ is true. Suppose that $P(k)$ holds. Consider $P(k+1)$. Since $B_{k+1} = b_{k+1}B_k + B_{k-1}$ and $|b_{k+1}B_k| > |B_{k-1}|$, then $|B_{k+1}| = |b_{k+1}B_k| \geq 2^{k+1}$.

□

Proposition 3.5. $\xi - \dfrac{A_n}{B_n} = \dfrac{(-1)^n}{B_n(\xi_{n+1}B_n + B_{n-1})} \quad (n \geq 1)$.

Proof. By Proposition 3.1

$$\xi = [b_0,\ b_1,\ b_2, \ldots,\ b_n,\ \xi_{n+1}] = \frac{\xi_{n+1}A_n + A_{n-1}}{\xi_{n+1}B_n + B_{n-1}},$$

and so by Proposition 3.2,

$$\xi - \frac{A_n}{B_n} = \frac{\xi_{n+1}A_n + A_{n-1}}{\xi_{n+1}B_n + B_{n-1}} - \frac{A_n}{B_n}$$
$$= \frac{B_n(\xi_{n+1}A_n + A_{n-1}) - A_n(\xi_{n+1}B_n + B_{n-1})}{B_n(\xi_{n+1}B_n + B_{n-1})}$$
$$= \frac{-(A_nB_{n-1} - A_{n-1}B_n)}{B_n(\xi_{n+1}B_n + B_{n-1})} = \frac{(-1)^n}{B_n(\xi_{n+1}B_n + B_{n-1})}.$$

□

Since $|\xi_n| = |b_n| \geq 2^1$ and $|B_n| > |B_{n-1}|$, then by Proposition 3.4 $|B_n(\xi_{n+1}B_n + B_{n-1})| = |B_n|^2|b_{n+1}| \geq 2^{2n+1}$. It follows that

$$|\xi - \frac{A_n}{B_n}| = \frac{1}{|B_n(\xi_{n+1}B_n + B_{n-1})|} \leq \frac{1}{2^{2n+1}} \to 0 \quad (n \to \infty)$$

and so $\dfrac{A_n}{B_n}$ converges to ξ which enables us to write $\xi = [b_0,\ b_1,\ b_2,\ b_3, \ldots]$.

Example 3.6. Case of $\mathbb{F}_q((\frac{1}{x}))$

Take $K = \mathbb{F}_q((\frac{1}{x}))$, the completion of $\mathbb{F}_q(x)$ with respect to the infinite non-Archimedean valuation $|\cdot|_\infty$ so normalized that $|x^{-1}|_\infty = 2^{-1}$.

Each $\xi \in \mathbb{F}_q((\frac{1}{x}))$ can be uniquely written as

$$\xi = f_m x^m + f_{m-1}x^{m-1} + \cdots + f_0 + f_{-1}x^{-1} + \cdots$$

where $f_i \in \mathbb{F}_q, \quad f_m \neq 0, \quad m \in \mathbb{Z}$. Specializing the construction in Section 3.1, we have $[\xi] := f_m x^m + f_{m-1}x^{m-1} + \cdots + f_0 \in \mathbb{F}_q[x], \quad (\xi) := f_{-1}x^{-1} + f_{-2}x^{-2} + \cdots$ and so ξ has a unique **RCF** of the form

$$\xi = [b_0,\ b_1,\ b_2,\ b_3, \ldots],$$

where $b_0 = f_m x^m + f_{m-1}x^{m-1} + \cdots + f_0 = [\xi] \in \mathbb{F}_q[x],\ b_i = g_{m_i}x^{m_i} + g_{m_i-1}x^{m_i-1} + \ldots + g_0 = [\xi_i] \in \mathbb{F}_q[x] - \mathbb{F}_q,\ g_{m_i} \neq 0,\ |b_i|_\infty > 1 \ \ (i \geq 1)$.

Example 3.7. Case of $\mathbb{F}_q((x))$

Take $K = \mathbb{F}_q((x))$, the completion of $\mathbb{F}_q(x)$ with respect to the x-adic non-Archimedean valuation $|\cdot|_x$ so normalized that $|x|_x = 2^{-1}$. Each $\xi \in \mathbb{F}_q((x))$ can be uniquely written as

$$\xi = f_{-m}x^{-m} + f_{-m+1}x^{-m+1} + \cdots + f_0 + f_1 x^1 + \cdots$$

where $f_i \in \mathbb{F}_q, \quad f_{-m} \neq 0, \quad m \in \mathbb{Z}$. Specializing the construction in Section 3.1, we have $[\xi] := f_{-m}x^{-m} + f_{-m+1}x^{-m+1} + \cdots + f_0 \in \mathbb{F}_q[\frac{1}{x}], \quad (\xi) := f_1 x^1 + f_2 x^2 + \cdots$

and so ξ has a unique **RCF** of the form

$$\xi = [b_0,\ b_1,\ b_2,\ b_3, \ldots],$$

where $b_0 = f_{-m}x^{-m} + f_{-m+1}x^{-m+1} + \cdots + f_0 = [\xi] \in \mathbb{F}_q[\frac{1}{x}],\ b_i = g_{-m_i}x^{-m_i} + g_{-m_i+1}x^{-m_i+1} + \ldots + g_0 = [\xi_i] \in \mathbb{F}_q[\frac{1}{x}] - \mathbb{F}_q,\ g_{-m_i} \neq 0,\ |b_i|_x > 1 \ \ (i \geq 1)$.

3.2 Characterization of rationals

In this section the word "rational" refers to elements of $\mathbb{F}_q(x)$.

Theorem 3.8. Let $\xi \in \mathbb{F}_q((\frac{1}{x}))$. Then ξ is rational $\Leftrightarrow$ its **RCF** is finite.

Proof. It is easy to see that if the **RCF** of $\xi \in \mathbb{F}_q((\frac{1}{x}))$ is finite, then ξ is rational. Assume $\xi \in \mathbb{F}_q((\frac{1}{x}))$ is rational and using the notation of Example 3.6, let its **RCF** be $[b_0,\ b_1,\ b_2, \ldots,\ b_n, \ldots]$.

Writing $\xi = [b_0,\ b_1,\ b_2, \ldots,\ b_{n-1},\ \xi_n]$. Since ξ is rational, then ξ_n is rational and $|\xi_n|_\infty = |b_n|_\infty > 1$. Writing ξ_n as fraction

$$\xi_n = \frac{x_n}{x_{n+1}} = b_n + \frac{x_{n+2}}{x_{n+1}} \quad \text{with } x_n,\ x_{n+1},\ x_{n+2} \in \mathbb{F}_q[x].$$

We see that $1 \leq |x_{n+1}|_\infty < |x_n|_\infty$. It follows that (x_n) is a sequence of polynomials in $\mathbb{F}_q[x]$ with strictly decreasing degrees and must then terminate. $\square$

Remark 3.9. Theorem 3.8 can be proved using Euclid algorithm as follows: let $\xi = \dfrac{A(x)}{B(x)} \in \mathbb{F}_q(x)$. By Euclid algorithm, $\exists Q_1,\ Q_2, \ldots,\ Q_{n+1},\ R_1, \ldots,\ R_n \in \mathbb{F}_q[x]$,

$0 \leq deg\ R_n < deg\ R_{n-1} < \ldots < deg\ R_1 < deg\ B$ such that

$$\begin{aligned}
A(x) &= Q_1(x)B(x) + R_1(x) \\
B(x) &= Q_2(x)R_1(x) + R_2(x) \\
R_1(x) &= Q_3(x)R_2(x) + R_3(x) \\
&\vdots \\
R_{n-2}(x) &= Q_n(x)R_{n-1}(x) + R_n(x) \\
R_{n-1}(x) &= Q_{n+1}R_n(x).
\end{aligned}$$

Thus the **RCF** of $\xi = \dfrac{A(x)}{B(x)}$ is finite of the form

$$[Q_1,\ Q_2, \ldots,\ Q_{n+1}].$$

Theorem 3.10. Let $\xi \in \mathbb{F}_q((x))$. Then ξ is rational $\Leftrightarrow$ its **RCF** is finite.

Proof. It is easy to see that if the **RCF** of ξ is finite then ξ is rational. Assume $\xi \in \mathbb{F}_q((x))$ is rational and using the notation of Example 3.7, let its **RCF** be

$$[b_0,\ b_1,\ b_2, \ldots,\ b_{n-1}, \ldots]. \quad \text{Writing } \xi = [b_0,\ b_1,\ b_2, \ldots,\ b_{n-1},\ \xi_n].$$

Since ξ is rational, then ξ_n is rational and $|\xi_n|_x = |b_n|_x > 1$. Writing ξ_n as fraction

$$\xi_n = \frac{x_n}{x_{n+1}} = b_n + \frac{x_{n+2}}{x_{n+1}} \quad \text{with } x_n,\ x_{n+1},\ x_{n+2} \in \mathbb{F}_q[\tfrac{1}{x}].$$

Since $|\xi_n|_x > 1$, then $|x_{n+1}|_x < |x_n|_x$. Considering as polynomials in $\frac{1}{x}$, this implies that $deg_{\frac{1}{x}}(x_{n+1}) < deg_{\frac{1}{x}}(x_n)$ i.e. (x_n) is a sequence of polynomials in $\frac{1}{x}$ with strictly decreasing degree (in $\frac{1}{x}$) and must then terminate. □

3.3 Quadratic irrationals

In this section, the word "irrational" refers to elements of $\mathbb{F}_q((\frac{1}{x}))$ (or $\mathbb{F}_q((x))$) which are not in $\mathbb{F}_q(x)$.

An infinite continued fraction of the shape $[b_0,\ b_1,\ b_2, \ldots]$ is said to be periodic if there is an integer k such that $b_n = b_{n+k}$ for all sufficiently large integer n and is denoted by $[b_0,\ b_1, \ldots, b_{n-1},\ \overline{b_n,\ b_{n+1}, \ldots, b_{n+k-1}}]$.

Theorem 3.11. Let $\xi \in \mathbb{F}_q((\frac{1}{x})),\ |\xi|_\infty \leq 1$. If the continued fraction expansion of ξ is periodic, then ξ is a nonrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x],\ a \neq 0$.

Proof. Let $\xi = [b_0,\ b_1, \ldots,\ b_{n-1},\ \overline{b_n,\ b_{n+1}, \ldots, b_{n+k}}]$, and $\xi_n = [\overline{b_n,\ b_{n+1}, \ldots, b_{n+k}}]$ $= [b_n,\ b_{n+1}, \ldots, b_{n+k}, \xi_n]$ be the n^{th} complete quotient of the periodic **RCF** of ξ. Then by Proposition 3.1,

$$\xi_n = \frac{A'\xi_n + A''}{B'\xi_n + B''} \text{ where } \frac{A''}{B''} = [b_n,\ b_{n+1}, \ldots, b_{n+k-1}],\ \frac{A'}{B'} = [b_n,\ b_{n+1}, \ldots, b_{n+k}],$$

and $A',\ A'',\ B',\ B'' \in \mathbb{F}_q[x]$.

It follows that

$$B'\xi_n^2 + (B'' - A')\xi_n - A'' = 0 \tag{3.3}$$

Since $\xi = \dfrac{\xi_n A_{n-1} + A_{n-2}}{\xi_n B_{n-1} + B_{n-2}}$, and so $\xi_n = \dfrac{A_{n-2} - \xi B_{n-2}}{\xi B_{n-1} - A_{n-1}}$. We substitute for ξ_n in (3.3), and clear of fraction to obtain an equation $a\xi^2 + b\xi + c = 0$, where

$a = B'B_{n-2}^2 - A''B_{n-1}^2 - B''B_{n-2}B_{n-1} + A'B_{n-2}B_{n-1} \neq 0,$

$b = -2B'A_{n-2}B_{n-2} + 2B_{n-1}A_{n-1}A'' + B''A_{n-2}B_{n-1} + B''A_{n-1}B_{n-2} -$ $A'A_{n-2}B_{n-1} - A'B_{n-2}A_{n-1},$

$c = B'A_{n-2}^2 - B''A_{n-2}A_{n-1} + A'A_{n-2}A_{n-1} - A''A_{n-1}^2.$

Since $A_i,\ B_i\ \ (i \geq 0),\ A',\ A'',\ B',\ B'' \in \mathbb{F}_q[x]$, then $a,\ b,\ c \in \mathbb{F}_q[x]$ and $a \neq 0$ because ξ is irrational. □

Theorem 3.12. Let $\xi \in \mathbb{F}_q((\frac{1}{x})),\ |\xi|_\infty \leq 1$. If ξ is a nonrational root of a quadratic equation of the form $at^2+bt+c = 0$ where $a, b, c \in \mathbb{F}_q[x],\ a \neq 0$, then the continued fraction expansion of ξ is periodic.

Proof. Let $\xi \in \mathbb{F}_q((\frac{1}{x}))$ with $[b_0,\ b_1,\ b_2, \ldots]$ being its **RCF**. Assume that ξ is a root of a quadratic equation

$$at^2 + bt + c = 0 \tag{3.4}$$

where $a, b, c \in \mathbb{F}_q[x]$ and $a \neq 0$. Writing

$\xi = [b_0,\ b_1,\ b_2, \ldots,\ b_{n-1},\ \xi_n]$ where $\xi_n = [b_n,\ b_{n+1},\ b_{n+2}, \ldots]$.

Then by Proposition 3.1

$\xi = \dfrac{\xi_n A_{n-1} + A_{n-2}}{\xi_n B_{n-1} + B_{n-2}}$ where $\dfrac{A_n}{B_n}$ is the n^{th} convergent to the **RCF** of ξ.

Substituting into (3.4), we get

$$R_n \xi_n^2 + S_n \xi_n + T_n = 0$$

where $R_n = aA_{n-1}^2 + bA_{n-1}B_{n-1} + cB_{n-1}^2$

$S_n = 2aA_{n-1}A_{n-2} + b(A_{n-1}B_{n-2} + B_{n-1}A_{n-2}) + 2cB_{n-1}B_{n-2},$

$T_n = aA_{n-2}^2 + bA_{n-2}B_{n-2} + cB_{n-2}^2.$

Observe that $a,\ b,\ c,\ A_i$, and B_i all belong to $\mathbb{F}_q[x]$ which yields $R_n,\ S_n, T_n \in \mathbb{F}_q[x]$. If $R_n = 0$ then ξ_n is rational, contradicting the fact that ξ is irrational. Hence $R_n \neq 0$. Note that

$$S_n^2 - 4R_nT_n = (b^2 - 4ac)(A_{n-1}B_{n-2} - B_{n-1}A_{n-2})^2 = b^2 - 4ac. \tag{3.5}$$

By Proposition 3.5, $\xi - \dfrac{A_{n-1}}{B_{n-1}} = \dfrac{(-1)^{n-1}}{B_{n-1}(\xi_n B_{n-1} + B_{n-2})}$,

and so

$$\xi B_{n-1} - A_{n-1} = \frac{(-1)^{n-1} B_{n-1}}{B_{n-1}(\xi_n B_{n-1} + B_{n-2})}.$$

Therefore

$$A_{n-1} = \xi B_{n-1} + \frac{(-1)^n B_{n-1}}{B_{n-1}(\xi_n B_{n-1} + B_{n-2})} = \xi B_{n-1} + \frac{\delta_{n-1}}{B_{n-1}}$$

where $\delta_{n-1} = \dfrac{B_{n-1}}{\xi_n B_{n-1} + B_{n-2}}$. Since $|B_{n-1}|_\infty > |B_{n-2}|_\infty$ and $|\xi_n|_\infty = |b_n|_\infty > 1$, then

$$|\delta_{n-1}|_\infty = \frac{|B_{n-1}|_\infty}{|\xi_n B_{n-1} + B_{n-2}|_\infty} = \frac{|B_{n-1}|_\infty}{|b_n B_{n-1}|_\infty} < 1.$$

Next

$$\begin{aligned}
R_n &= a(\xi B_{n-1} + \frac{\delta_{n-1}}{B_{n-1}})^2 + bB_{n-1}(\xi B_{n-1} + \frac{\delta_{n-1}}{B_{n-1}}) + cB_{n-1}^2 \\
&= a(\xi^2 B_{n-1}^2 + 2\xi\delta_{n-1} + \frac{\delta_{n-1}^2}{B_{n-1}^2}) + b\xi B_{n-1}^2 + b\delta_{n-1} + cB_{n-1}^2 \\
&= (a\xi^2 + b\xi + c)B_{n-1}^2 + 2a\xi\delta_{n-1} + a\frac{\delta_{n-1}^2}{B_{n-1}^2} + b\delta_{n-1} \\
&= 2a\xi\delta_{n-1} + a\frac{\delta_{n-1}^2}{B_{n-1}^2} + b\delta_{n-1},
\end{aligned}$$

which gives $|R_n|_\infty < \max\{|2a\xi|_\infty, |a|_\infty, |b|_\infty\} := \ell$.

Since $T_n = R_{n-1}$, then $|T_n|_\infty = |R_{n-1}|_\infty < \max\{|2a\xi|_\infty, |a|_\infty, |b|_\infty\} = \ell$.

From (3.5), $|S_n^2|_\infty = |4R_nT_n + b^2 - 4ac|_\infty < \max\{4\ell^2,\ |b^2 - 4ac|_\infty\}$.

Hence $|R_n|_\infty,\ |S_n|_\infty,\ |T_n|_\infty$ are bounded by a constant independent of n. It follows that, being elements in $\mathbb{F}_q[x]$, there are only a finite number of different triplets $(R_n,\ S_n,\ T_n)$ and we can find a triplet $(R,\ S,\ T)$ which occurs at least three times, say $(R_{n_1},\ S_{n_1},\ T_{n_1})$, $(R_{n_2},\ S_{n_2},\ T_{n_2})$, $(R_{n_3},\ S_{n_3},\ T_{n_3})$. These $\xi_{n_1},\ \xi_{n_2},\ \xi_{n_3}$ are roots of

$$Rt^2 + St + T = 0$$

and at least two of them must be equal. But if, for example, $\xi_{n_1} = \xi_{n_2}$, then $b_{n_2} = b_{n_1},\ b_{n_2+1} = b_{n_1+1}, \ldots$ and the **RCF** is periodic. □

Next, we consider $(\mathbb{F}_q((x)),\ |\cdot|_x)$. By the same proof of Theorem 3.11 and Theorem 3.12, respectively, we have:

Theorem 3.13. Let $\xi \in \mathbb{F}_q((x)),\ |\xi|_x \leq 1$. If the continued fraction expansion of ξ is periodic, then ξ is a nonrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[\frac{1}{x}]$, $a \neq 0$.

Proof. Let $\xi = [b_0,\ b_1, \ldots,\ b_{n-1},\ \overline{b_n,\ b_{n+1}, \ldots, b_{n+k}}]$, and $\xi_n = [\overline{b_n,\ b_{n+1}, \ldots, b_{n+k}}]$ $= [b_n,\ b_{n+1}, \ldots, b_{n+k}, \xi_n]$ be the n^{th} complete quotient of the periodic **RCF** of ξ.

Then $\xi_n = \dfrac{A'\xi_n + A''}{B'\xi_n + B''}$ where $\dfrac{A''}{B''} = [b_n,\ b_{n+1}, \ldots, b_{n+k-1}]$,
$\dfrac{A'}{B'} = [b_n,\ b_{n+1}, \ldots, b_{n+k}]$, $A',\ A'',\ B',\ B'' \in \mathbb{F}_q[\frac{1}{x}]$. It follows that

$$B'\xi_n^2 + (B'' - A')\xi_n - A'' = 0 \tag{3.6}$$

But $\xi = \dfrac{\xi_n A_{n-1} + A_{n-2}}{\xi_n B_{n-1} + B_{n-2}}$, $\xi_n = \dfrac{A_{n-2} - \xi B_{n-2}}{\xi B_{n-1} - A_{n-1}}$. We substitute for ξ_n in (3.6), and clear of fraction to obtain an equation $a\xi^2 + b\xi + c = 0$ where

$a = B'B_{n-2}^2 - A''B_{n-1}^2 - B''B_{n-2}B_{n-1} + A'B_{n-2}B_{n-1} \neq 0$

$b = -2B'A_{n-2}B_{n-2} + 2B_{n-1}A_{n-1}A'' + B''A_{n-2}B_{n-1} + B''A_{n-1}B_{n-2} -$
$A'A_{n-2}B_{n-1} - A'B_{n-2}A_{n-1}$,

$c = B'A_{n-2}^2 - B''A_{n-2}A_{n-1} + A'A_{n-2}A_{n-1} - A''A_{n-1}^2$.

Since $A_i,\ B_i\ \ (i \geq 0),\ A',\ A'',\ B',\ B'' \in \mathbb{F}_q[\frac{1}{x}]$, then $a,\ b,\ c \in \mathbb{F}_q[\frac{1}{x}]$ and $a \neq 0$ because ξ is irrational. □

Theorem 3.14. Let $\xi \in \mathbb{F}_q((x))$, $|\xi|_x \leq 1$. If ξ is a nonrational root of a quadratic equation of the form $at^2+bt+c = 0$ where $a, b, c \in \mathbb{F}_q[\frac{1}{x}]$, $a \neq 0$, then the continued fraction expansion of ξ is periodic.

Proof. Let $\xi \in \mathbb{F}_q((x))$ with $[b_0,\ b_1,\ b_2, \ldots]$ being its **RCF**. Assume that ξ is a root of a quadratic equation

$$at^2 + bt + c = 0 \tag{3.7}$$

where $a, b, c \in \mathbb{F}_q[\frac{1}{x}]$ and $a \neq 0$. Writing $\xi = [b_0,\ b_1,\ b_2, \ldots,\ b_{n-1},\ \xi_n]$ where $\xi_n = [b_n,\ b_{n+1},\ b_{n+2}, \ldots$. Then $\xi = \dfrac{\xi_n A_{n-1} + A_{n-2}}{\xi_n B_{n-1} + B_{n-2}}$ where $\dfrac{A_n}{B_n}$ is the n^{th} convergent to the **RCF** of ξ. Substituting into (3.7), we get

$$R_n\xi_n^2 + S_n\xi_n + T_n = 0$$

where $R_n = aA_{n-1}^2 + bA_{n-1}B_{n-1} + cB_{n-1}^2$

$S_n = 2aA_{n-1}A_{n-2} + b(A_{n-1}B_{n-2} + B_{n-1}A_{n-2}) + 2cB_{n-1}B_{n-2}$,

$T_n = aA_{n-2}^2 + bA_{n-2}B_{n-2} + cB_{n-2}^2$.

Observe that $A_i,\ B_i\ \ (i \geq 0),\ a,\ b$, and c all belong to $\mathbb{F}_q[\frac{1}{x}]$ which yields $R_n,\ S_n,\ T_n \in \mathbb{F}_q[\frac{1}{x}]$. If $R_n = 0$ then ξ_n is rational, contradicting the fact that ξ is irrational. Hence $R_n \neq 0$. Note that

$$S_n^2 - 4R_nT_n = (b^2 - 4ac)(A_{n-1}B_{n-2} - B_{n-1}A_{n-2})^2 = b^2 - 4ac. \tag{3.8}$$

By Proposition 3.5, $\xi - \dfrac{A_{n-1}}{B_{n-1}} = \dfrac{(-1)^{n-1}}{B_{n-1}(\xi_n B_{n-1} + B_{n-2})}$, and so

$$\xi B_{n-1} - A_{n-1} = \frac{(-1)^{n-1}B_{n-1}}{B_{n-1}(\xi_n B_{n-1} + B_{n-2})}.$$

Therefore

$$A_{n-1} = \xi B_{n-1} + \frac{(-1)^n B_{n-1}}{B_{n-1}(\xi_n B_{n-1} + B_{n-2})} = \xi B_{n-1} + \frac{\delta_{n-1}}{B_{n-1}}$$

where $\delta_{n-1} = \dfrac{(-1)^n B_{n-1}}{\xi_n B_{n-1} + B_{n-2}}$. Since $|B_{n-1}|_x > |B_{n-2}|_x$ and $|\xi_n|_x = |b_n|_x > 1$, then

$$|\delta_{n-1}|_x = \frac{|B_{n-1}|_x}{|\xi_n B_{n-1} + B_{n-2}|_x} = \frac{|B_{n-1}|_x}{|b_n B_{n-1}|_x} < 1.$$

Next

$$\begin{aligned}
R_n &= a(\xi B_{n-1} + \frac{\delta_{n-1}}{B_{n-1}})^2 + bB_{n-1}(\xi B_{n-1} + \frac{\delta_{n-1}}{B_{n-1}}) + cB_{n-1}^2 \\
&= a(\xi^2 B_{n-1}^2 + 2\xi\delta_{n-1} + \frac{\delta_{n-1}^2}{B_{n-1}^2}) + b\xi B_{n-1}^2 + b\delta_{n-1} + cB_{n-1}^2 \\
&= (a\xi^2 + b\xi + c)B_{n-1}^2 + 2a\xi\delta_{n-1} + a\frac{\delta_{n-1}^2}{B_{n-1}^2} + b\delta_{n-1} \\
&= 2a\xi\delta_{n-1} + a\frac{\delta_{n-1}^2}{B_{n-1}^2} + b\delta_{n-1},
\end{aligned}$$

which gives $|R_n|_x < \max\{|2a\xi|_x, |a|_x, |b|_x\} := \ell$.

Since $T_n = R_{n-1}$, then $|T_n|_x = |R_{n-1}|_x < \max\{|2a\xi|_x, |a|_x, |b|_x\} = \ell$.

From(3.8), $|S_n^2|_\infty = |4R_nT_n + b^2 - 4ac|_x < \max\{4\ell^2,\ |b^2 - 4ac|_x\}$.

Hence $|R_n|_x,\ |S_n|_x,\ |T_n|_x$ are bounded by a constant independent of n. It follows that, being elements in $\mathbb{F}_q[\frac{1}{x}]$, there are only a finite number of different triplets $(R_n,\ S_n,\ T_n)$ and we can find a triplet $(R,\ S,\ T)$ which occurs at least three times, say $(R_{n_1},\ S_{n_1},\ T_{n_1})$, $(R_{n_2},\ S_{n_2},\ T_{n_2})$, $(R_{n_3},\ S_{n_3},\ T_{n_3})$. These $\xi_{n_1},\ \xi_{n_2},\ \xi_{n_3}$

are roots of

$$Rt^2 + St + T = 0$$

and at least two of them must be equal. But if, for example, $\xi_{n_1} = \xi_{n_2}$, then $b_{n_2} = b_{n_1},\ b_{n_2+1} = b_{n_1+1}, \ldots$ and the **RCF** is periodic. $\square$

CHAPTER IV

SCF

In 1970, Schneider [16] developed an algorithm to compute continued fractions for p-adic numbers, ξ, which we may assume without loss of generality that $|\xi|_p < 1$. Writing $\xi = p^{ord_p(\xi)} \cdot u$, where u is a p-adic unit. Setting $p^{ord_p(\xi)} = a$, its first partial numerator and rewriting $\dfrac{1}{u} = b + \xi_1$ with $|\xi_1|_p < 1$ and $b \in \{1,\ 2, \ldots,\ p-1\}$, we see that

$$\xi = \frac{a}{b + \xi_1}.$$

Now repeat the process with ξ_1 in place of ξ. Clearly, the steps can also be done in any local field and we shall describe more fully in the first section. The continued fractions so obtained will be referred to as Schneider continued fractions, **SCF**.

4.1 Construction and Basic Properties

Let $(K,\ |\cdot|)$ be a local field , R its set of representatives of the residue class field of K. Every element $\xi \in K - \{0\}$ can be uniquely written in the form

$$\xi = \sum_{n=r}^{\infty} c_n \pi^n$$

with prime element π so normalized that $|\pi| = 2^{-ord_\pi \pi} = 2^{-1},\ r \in \mathbb{Z}$ and $a_i \in R$, $a_r \neq 0$. We assume that $0 \in R$.

Define $b_0 = \sum_{n=r}^{0} c_n \pi^n$. Hence $|b_0| \geq 1$.

If $\xi = b_0$, the process stops.

Otherwise, write $\xi - b_0 = \sum_{n=\alpha_1}^{\infty} c_n \pi^n$ where $\alpha_1 \geq 1,\ c_{\alpha_1} \neq 0$.

Define $a_1 = \pi^{\alpha_1},\ \xi_1^{-1} = \sum_{n=\alpha_1}^{\infty} c_n \pi^{n-\alpha_1}$. Then $|a_1| = 2^{-\alpha_1}$, $|\xi_1^{-1}| = 1$, and

$$\xi = b_0 + \sum_{n=\alpha_1}^{\infty} c_n \pi^n = [b_0;\ a_1,\ \xi_1].$$

Write $\xi_1 = \sum_{n=\alpha_1}^{\infty} c_n^{(1)} \pi^{n-\alpha_1},\ c_{\alpha_1}^{(1)} \neq 0$.

Let $b_1 = c_{\alpha_1}^{(1)}$. Hence $b_1 \in R$ and $|b_1| = 1$.

If $\xi_1 = b_1$, the process stops.

Otherwise, write $\xi_1 - b_1 = \sum_{n=\alpha_2}^{\infty} c_n^{(1)} \pi^n$ where $\alpha_2 \geq 1$, $c_{\alpha_2}^{(1)} \neq 0$.

Define $a_2 = \pi^{\alpha_2},\ \xi_2^{-1} = \sum_{n=\alpha_2}^{\infty} c_n^{(1)} \pi^{n-\alpha_2}$. Then $|a_2| = 2^{-\alpha_2}$, $|\xi_2^{-1}| = 1$, and

$$\xi = [b_0;\ a_1,\ \xi_1] = [b_0;\ a_1,\ b_1;\ a_2,\ \xi_2].$$

Write $\xi_2 = \sum_{n=\alpha_2}^{\infty} c_n^{(2)} \pi^{n-\alpha_2},\ c_{\alpha_2}^{(2)} \neq 0$.

Let $b_2 = c_{\alpha_2}^{(2)}$. Hence $b_2 \in R$ and $|b_2| = 1$.

If $\xi_2 = b_2$, the process stops.

Otherwise, write $\xi_2 - b_2 = \sum_{n=\alpha_3}^{\infty} c_n^{(2)} \pi^n$ where $\alpha_3 \geq 1$, $c_{\alpha_3}^{(2)} \neq 0$.

Define $a_3 = \pi^{\alpha_3},\ \xi_3^{-1} = \sum_{n=\alpha_3}^{\infty} c_n^{(2)} \pi^{n-\alpha_3}$. Then $|a_3| = 2^{-\alpha_3}$, $|\xi_3^{-1}| = 1$, and

$$\xi = [b_0;\ a_1,\ b_1;\ a_2,\ \xi_2] = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ a_3,\ \xi_3].$$

In general if $\xi_n = b_n$, the process stops.

Otherwise, write $\xi_n - b_n = \sum_{r=\alpha_{n+1}}^{\infty} c_r^{(n)} \pi^r$ where $\alpha_{n+1} \geq 1,\ c_{\alpha_{n+1}}^{(n)} \neq 0$.

Define $a_{n+1} = \pi^{\alpha_{n+1}},\ \xi_{n+1}^{-1} = \sum_{r=\alpha_{n+1}}^{\infty} c_r^{(n)} \pi^{r-\alpha_{n+1}}$. Then $|a_{n+1}| = 2^{-\alpha_{n+1}}$, $|\xi_{n+1}^{-1}| = 1$, and

$$\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n;\ a_{n+1},\ \xi_{n+1}],$$

where $|b_0| \geq 1,\ |b_n| = 1,\ |a_n| = 2^{-\alpha_n}\ (n \geq 1)$.

We call the uniquely constructed b_n and a_n the partial denominators and numerators of the **SCF** of ξ. We also called ξ_n the n^{th} complete quotient of its **SCF**.

In order to establish convergence, we define two sequences $A_n,\ B_n$ as follows:

$$A_{-1} = 1, \qquad A_0 = b_0, \quad A_{n+1} = b_{n+1}A_n + a_{n+1}A_{n-1} \ (n \geq 0) \tag{4.1}$$

$$B_{-1} = 0, \qquad B_0 = 1, \quad B_{n+1} = b_{n+1}B_n + a_{n+1}B_{n-1} \ (n \geq 0) \tag{4.2}$$

Proposition 4.1. For any $n \geq 0,\ \alpha \in K - \{0\}$, we have

$$\frac{\alpha A_n + a_{n+1}A_{n-1}}{\alpha B_n + a_{n+1}B_{n-1}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n;\ a_{n+1},\ \alpha].$$

Proof. By induction on n,

let $P(n):\ \dfrac{\alpha A_n + a_{n+1}A_{n-1}}{\alpha B_n + a_{n+1}B_{n-1}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n;\ a_{n+1},\ \alpha].$

Since $\dfrac{\alpha A_0 + a_1 A_{-1}}{\alpha B_0 + a_1 B_{-1}} = \dfrac{\alpha b_0 + a_1}{\alpha} = [b_0;\ a_1,\ \alpha]$, $P(0)$ is true.

Suppose that $P(n-1)$ holds. Consider

$$\begin{aligned}[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n;\ a_{n+1},\ \alpha] &= \frac{(b_n + \frac{a_{n+1}}{\alpha})A_{n-1} + a_n A_{n-2}}{(b_n + \frac{a_{n+1}}{\alpha})B_{n-1} + a_n B_{n-2}} \\ &= \frac{\alpha(b_n A_{n-1} + a_n A_{n-2}) + a_{n+1}A_{n-1}}{\alpha(b_n B_{n-1} + a_n B_{n-2}) + a_{n+1}B_{n-1}} \\ &= \frac{\alpha A_n + a_{n+1}A_{n-1}}{\alpha B_n + a_{n+1}B_{n-1}},\end{aligned}$$

which gives the truth of $P(n)$. □

From the above proposition, we have

$$\frac{A_n}{B_n} = \frac{b_n A_{n-1} + a_n A_{n-2}}{b_n B_{n-1} + a_n B_{n-2}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n] \ \ (n \geq 1).$$

We call $\dfrac{A_n}{B_n}$ the n^{th} convergent of **SCF** to $\xi \quad (n \geq 0)$. If the **SCF** of ξ is finite, i.e. $\xi_n = b_n$ for some n, then the **SCF** of ξ terminates as $[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n]$

In what follows we assume that $\xi_n \neq b_n$ for all n.

Proposition 4.2. $A_n B_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_1 a_2 \cdots a_n \ (n \geq 1)$.

Proof. By induction on n,

let $P(n):\ A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_1a_2\cdots a_n$. Since

$$\begin{aligned} A_1B_0 + A_0B_1 &= b_1A_0 + a_1A_{-1} - b_0(b_1B_0 + a_1B_{-1}) \\ &= b_1b_0 + a_1 - b_0b_1 - 0 = (-1)^{1-1}a_1, \end{aligned}$$

$P(1)$ is true. Suppose that $P(n-1)$ holds. Consider

$$\begin{aligned} A_nB_{n-1} + A_{n-1}B_n &= (b_nA_{n-1} + a_nA_{n-2})B_{n-1} - A_{n-1}(b_nB_{n-1} + a_nB_{n-2}) \\ &= a_nA_{n-2}B_{n-1} - a_nB_{n-2})A_{n-1} \\ &= -a_n(-1)^{n-2}a_1a_2\cdots a_{n-1} = (-1)^{n-1}a_1a_2\cdots a_n. \end{aligned}$$

and so $P(n)$ holds. □

Proposition 4.3. $|B_n| = 1\ \ (n \geq 1)$ i.e. $B_n \neq 0 \quad (n \geq 1)$.

Proof. Let $P(n):\ |b_n| = 1$.

Since $|B_1| = |b_1B_0 + a_1B_{-1}| = |b_1B_0| = 1$, then $P(1)$ is true.

Suppose that $P(k)$ holds. Consider $P(k+1)$,

Since $|B_{k+1} = b_{k+1}B_k + a_{k+1}B_{k-1}|$ and $|b_{k+1}B_k| > |a_{k+1}B_{k-1}|$,

then $|B_{k+1}| = |b_{k+1}B_k| = 1$. □

By proposition 4.2 and proposition 4.3, we have

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{(-1)^n a_1a_2\cdots a_{n+1}}{B_nB_{n+1}}. \tag{4.3}$$

Proposition 4.4. (i) $|\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}| = 2^{-(\alpha_1+\alpha_2+\cdots+\alpha_{n+1})}\quad (n \geq 1)$

(ii) $|\frac{A_m}{B_m} - \frac{A_n}{B_n}| = |\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}|\quad (m > n \geq 1)$.

Proof. By (4.3), $\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{A_{n+1}B_n - A_nB_{n+1}}{B_nB_{n+1}} = \frac{(-1)^n a_1a_2\cdots a_{n+1}}{B_nB_{n+1}}$.

Hence $|\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}| = \frac{|(-1)^n a_1a_2\cdots a_{n+1}|}{|B_nB_{n+1}|} = 2^{-(\alpha_1+\alpha_2+\cdots+\alpha_{n+1})}\quad (n \geq 1)$.

This proves (i).

For each $m > n \geq 1$, from part (i), since $\alpha_i \in \mathbb{N}$, we get by the strong triangle inequality

$$
\begin{aligned}
|\frac{A_m}{B_m} - \frac{A_n}{B_n}| &= \max\{|\frac{A_m}{B_m} - \frac{A_{m-1}}{B_{m-1}}|, |\frac{A_{m-1}}{B_{m-1}} - \frac{A_{m-2}}{B_{m-2}}|, \cdots, |\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}|\} \\
&= |\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}|.
\end{aligned}
$$

□

The sequence $(2^{-(\alpha_1+\alpha_2+\cdots+\alpha_{n+1})})$ is decreasing and by (i), the sequence $(\frac{A_n}{B_n})$ is convergent in the complete field K.

Proposition 4.5. $\xi - \frac{A_n}{B_n} = \frac{(-1)^n a_1 a_2 \cdots a_{n+1}}{B_n(\xi_{n+1} B_n + a_{n+1} B_{n-1})}$ $(n \geq 1)$.

Proof. By Proposition 4.1 and Proposition 4.2,

$$
\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n;\ a_{n+1},\ \xi_{n+1}] = \frac{\xi_{n+1} A_n + a_{n+1} A_{n-1}}{b_{n+1} B_n + a_{n+1} B_{n-1}} ,
$$

and so

$$
\begin{aligned}
\xi - \frac{A_n}{B_n} &= \frac{\xi_{n+1} A_n + a_{n+1} A_{n-1}}{\xi_{n+1} B_n + a_{n+1} B_{n-1}} - \frac{A_n}{B_n} = \frac{a_{n+1} A_{n-1} B_n - a_{n+1} B_{n-1} A_n}{B_n(\xi_{n+1} B_n + a_{n+1} B_{n-1})} \\
&= \frac{-a_{n+1}(A_{n-1} B_n - B_{n-1} A_n)}{B_n(\xi_{n+1} B_n + a_{n+1} B_{n-1})} = \frac{(-1)^n a_1 a_2 \cdots a_{n+1}}{B_n(\xi_{n+1} B_n + a_{n+1} B_{n-1}).}
\end{aligned}
$$

□

By Proposition 4.3 and the construction, we see that $|a_{n+1} B_{n-1}| < |\xi_{n+1} B_n|$, and so $|\xi_{n+1} B_n + a_{n+1} B_{n-1}| = |\xi_{n+1} B_n| = 1$. It follows that

$$
|\xi - \frac{A_n}{B_n}| = 2^{-(\alpha_1+\alpha_2+\cdots+\alpha_{n+1})} \to 0 \ \ (n \to \infty)
$$

and so $\frac{A_n}{B_n}$ converges to ξ enabling us to write $\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots]$.

Example 4.6. Case of $\mathbb{F}_q((\frac{1}{x}))$

Take $K = \mathbb{F}_q((\frac{1}{x}))$, the completion of $\mathbb{F}_q(x)$ with respect to the infinite non-Archimedean valuation $|\cdot|_\infty$, so normalized that $|x| = 2$. Let

$$
\xi = f_m x^m + f_{m-1} x^{m-1} + \cdots + f_0 + f_{-1} x^{-1} + \cdots \in \mathbb{F}_q((\frac{1}{x}))
$$

where $f_i \in \mathbb{F}_q, \quad f_m \neq 0, \quad m \in \mathbb{Z}$. Specializing the construction in Section 4.1, we have a unique **SCF** for ξ of the form

$$\xi = [b_0;\ a_1,\ \xi_1] = [b_0;\ a_1,\ b_1;\ a_2,\ \xi_2] = \cdots = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n;\ a_{n+1},\ \xi_{n+1}],$$

where $b_0 \in \mathbb{F}_q[x],\ b_i \in \mathbb{F}_q - \{0\}$ and $a_i = \dfrac{1}{x^{\alpha_i}}, \quad \alpha_i \in \mathbb{N} \ \ (i \geq 1)$.

Example 4.7. Case of $\mathbb{F}_q((x))$

Take $K = \mathbb{F}_q((x))$, the completion of $\mathbb{F}_q(x)$ with respect to the x-adic non-Archimedean absolute valuation $|\cdot|_x$ so normalized that $|x|_x = 2^{-1}$. Let

$$\xi = f_{-m}x^{-m} + f_{-m+1}x^{-m+1} + \cdots + f_0 + f_1x^1 + \cdots \in \mathbb{F}_q((x))$$

where $f_i \in \mathbb{F}_q, \quad f_{-m} \neq 0, \quad m \in \mathbb{Z}$. Specializing the construction in Section 4.1, we have a unique **SCF** for ξ of the form

$$\xi = [b_0;\ a_1,\ \xi_1] = [b_0;\ a_1,\ b_1;\ a_2,\ \xi_2] = \cdots = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n;\ a_{n+1},\ \xi_{n+1}],$$

where $b_0 \in \mathbb{F}_q[\frac{1}{x}],\ b_i \in \mathbb{F}_q - \{0\}$ and $a_i = x^{\alpha_i}, \quad \alpha_i \in \mathbb{N} \ \ (i \geq 1)$.

4.2 Characterization of rationals

In this section the word "rational" refers to element in $\mathbb{F}_q(x)$.

Theorem 4.8. Let $\xi \in \mathbb{F}_q((\frac{1}{x}))$. Then ξ is rational $\Leftrightarrow$ its **SCF** is finite.

Proof. It is easy to see that if the **SCF** for ξ is finite, then ξ is rational. The converse also holds as we now show.

Let the **SCF** for ξ be

$$\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ \xi_n]$$

where $b_0 \in \mathbb{F}_q[x],\ b_i \in \mathbb{F}_q - \{0\},\ a_i = \dfrac{1}{x^{\alpha_i}},\ \alpha_i \in \mathbb{N} \ \ (i \geq 1)$, so that $|\xi_n|_\infty = 1$. Since ξ and b_0 are rational, ξ_n is rational. For $n \geq 1$ write $\xi_n = \dfrac{x_n}{x_{n+1}}$, where

$x_n,\ x_{n+1} \in \mathbb{F}_q[\frac{1}{x}]$ and the constant terms of both x_n and x_{n+1} are non zero. Since $\xi_n = b_n + \dfrac{a_n}{\xi_{n+1}} = \dfrac{x_n}{x_{n+1}}$ $(n \geq 1)$, then $\dfrac{x_n}{x_{n+1}} = b_n + \dfrac{a_n x_{n+2}}{x_{n+1}} = \dfrac{b_n x_{n+1} + a_n x_{n+2}}{x_{n+1}}$, and so $x_n = b_n x_{n+1} + a_n x_{n+2}$. Instead of using the infinite valuation, we estimate the size of $x_i \in \mathbb{F}_q[\frac{1}{x}]$ using x-adic valuation, using also the fact that for $i \geq 1$, $|b_i|_x = 1$, $|a_i|_x = 2^{\alpha_i} > 1$. Thus

$$\begin{aligned} |x_{n+2}|_x = |\frac{x_n - b_n x_{n+1}}{a_n}|_x &\leq \frac{\max\{|x_n|_x,\ |b_n x_{n+1}|_x\}}{|a_n|_x} \\ &< \max\{|x_n|_x,\ |x_{n+1}|_x\}. \end{aligned}$$

Thus the elements of sequences $(x_n) \subseteq \mathbb{F}_q[\frac{1}{x}]$, considered as sequence of polynomials in $\frac{1}{x}$, have bounded degree strictly decreasing after every two successive ones. This sequence must then terminate yielding a finite **SCF**. □

Theorem 4.9. Let $\xi \in \mathbb{F}_q((x))$. Then ξ is rational $\Leftrightarrow$ its **SCF** is finite.

Proof. It is easy to see that if the **SCF** to ξ is finite then ξ is rational. The converse also holds as we now show.

Let the **SCF** for ξ be

$$\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ \xi_n]$$

where $b_0 \in \mathbb{F}_q[\frac{1}{x}],\ b_i \in \mathbb{F}_q - \{0\},\ a_i = x^{\alpha_i},\ \alpha_i \in \mathbb{N}\ \ (i \geq 1)$. Since ξ is rational, ξ_n is rational and $|\xi_n|_\infty = 1$. For $n \geq 1$, we can write $\xi_n = \dfrac{x_n}{x_{n+1}}$ with $x_n,\ x_{n+1} \in \mathbb{F}_q[x]$ and are polynomials in x of the same degree.

Since $\dfrac{x_n}{x_{n+1}} = \xi_n = b_n + \dfrac{a_n}{\xi_{n+1}} = b_n + \dfrac{a_n x_{n+2}}{x_{n+1}}$, and so $x_n = b_n x_{n+1} + a_n x_{n+2}$.

In contrast to the last theorem, we use the infinite valuation to estimate the size of x_i, keeping in mind that $|b_i|_\infty = 1$, $|a_i|_\infty = 2^{\alpha_i} > 1$. Thus

$$\begin{aligned} |x_{n+2}|_\infty = |\frac{x_n - b_n x_{n+1}}{a_n}|_\infty &\leq \frac{\max\{|x_n|_\infty,\ |b_n x_{n+1}|_\infty\}}{|a_n|_\infty} \\ &< \max\{|x_n|_\infty,\ |x_{n+1}|_\infty\}, \end{aligned}$$

and so considering (x_n) as a sequence of polynomials in $\mathbb{F}_q[x]$, we observe as in the last theorem that it must terminate, yielding a finite **SCF** to ξ. □

4.3 Quadratic irrationals

In this section, the word "irrational" refers to elements in $\mathbb{F}_q((\frac{1}{x}))$ (or $\mathbb{F}_q((x))$) which are not in $\mathbb{F}_q(x)$.

An infinite continued fraction

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots]$$

is said to be periodic if there is an integer k such that $a_n = a_{n+k+1}$ and $b_n = b_{n+k+1}$ for all sufficiently large integer n and is denoted by

$$[b_0;\ a_1,\ b_1;\ldots;\ a_{n-1},\ b_{n-1};\ \overline{a_n,\ b_n;\ a_{n+1},\ b_{n+1};\ldots;a_{n+k},\ b_{n+k}}].$$

Theorem 4.10. Let $\xi \in \mathbb{F}_q((\frac{1}{x})),\ |\xi|_\infty \leq 1$. If the **SCF** of ξ is periodic, then ξ is a non rational root of a quadratic equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{F}_q[x]$, $a \neq 0$.

Proof. Let $\xi = [b_0;\ a_1,\ b_1;\ldots;\ a_{n-1},\ b_{n-1};\ \overline{a_n,\ b_n;\ a_{n+1},\ b_{n+1};\ldots;a_{n+k},\ b_{n+k}}]$, and $\xi_n = [\overline{b_n;\ a_{n+1},\ b_{n+1};\ldots;a_{n+k},\ b_{n+k};\ a_n}]$ be the n^{th} complete quotient of the periodic **SCF** of ξ. Then by Proposition 4.1 $\xi_n = \dfrac{A'\xi_n + a_nA''}{B'\xi_n + a_nB''}$ where $\dfrac{A''}{B''} = [b_n;\ a_{n+1},\ b_{n+1};\ldots;a_{n+k-1},\ b_{n+k-1}]$, $\dfrac{A'}{B'} = [b_n;\ a_{n+1},\ b_{n+1};\ldots;a_{n+k},\ b_{n+k}]$, the last two convergents to $[b_n;\ a_{n+1},\ b_{n+1};\ldots;a_{n+k},\ b_{n+k}]$.

It follows that

$$B'\xi_n^2 + (a_nB'' - A')\xi_n - a_nA'' = 0. \tag{4.4}$$

But

$$\xi = \frac{\xi_nA_{n-1} + a_nA_{n-2}}{\xi_nB_{n-1} + a_nB_{n-2}},\ \xi_n = \frac{a_n(A_{n-2} - \xi B_{n-2})}{\xi B_{n-1} - A_{n-1}}.$$

Substituting for ξ_n in (4.4), we obtain an equation $a'\xi^2 + b'\xi + c' = 0$ where

$a' = a_n^2 B' B_{n-2}^2 - a_n A'' B_{n-1}^2 - a_n^2 B'' B_{n-2} B_{n-1} + a_n B_{n-2} B_{n-1}$

$b' = -2a_n^2 B' A_{n-2} B_{n-2} + a_n^2 B'' A_{n-2} B_{n-1} - a_n A' A_{n-2} B_{n-1} + a_n^2 B'' B_{n-2} A_{n-1} -$
$a_n A' A_{n-1} B_{n-2} + 2a_n A'' B_{n-1} A_{n-1},$

$c' = a_n^2 B' A_{n-2}^2 - a_n^2 B'' A_{n-2} A_{n-1} + a_n A' A_{n-2} A_{n-1} - a_n A'' A_{n-1}^2.$

Since ξ is irrational, then $a' \neq 0$. After clearing the fraction the new coefficients $a,\ b,\ c$ are in $\mathbb{F}_q[x]$. □

Theorem 4.11. Let $\xi \in \mathbb{F}_q((\frac{1}{x})),\ |\xi|_\infty \leq 1$. Let the **SCF** of ξ be of the form

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots]$$

where $a_i = \frac{1}{x^{\alpha_i}}$ and let (γ_i) be defined by $\gamma_1 = \alpha_1,\ \gamma_2 = \alpha_2 - \alpha_1,\ \gamma_3 = \alpha_3 - \alpha_2 + \alpha_1,$ $\ldots,\ \gamma_i = \alpha_i - \alpha_{i-1} + \cdots + (-1)^{i+1}\alpha_1 (i \geq 1)$. Assume $\gamma_i \geq 0\ \ (i \geq 1)$. If ξ is a nonrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x],\ a \neq 0$, then the **SCF** of ξ is periodic.

Proof. Assume that ξ is a root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a,\ b,\ c \in \mathbb{F}_q[x],\ a \neq 0$. Let $\xi = [b_0;\ x^{-\alpha_1},\ b_1;\ x^{-\alpha_2},\ b_2;\ldots]$. Then inverting the $\frac{1}{x^{\alpha_i}}$'s, we see that

$$\begin{aligned}
\xi &= [b_0;\ x^{-\alpha_1},\ b_1;\ x^{-\alpha_2},\ b_2;\ldots] \\
&= [b_0;\ 1,\ b_1 x^{\alpha_1};\ x^{-(\alpha_2-\alpha_1)},\ b_2;\ x^{-\alpha_3},\ b_3;\ldots] \\
&= [b_0;\ 1,\ b_1 x^{\alpha_1}; 1,\ b_2 x^{\alpha_2-\alpha_1};\ x^{-(\alpha_3-(\alpha_2-\alpha_1))},\ b_3;\ldots] \\
&\ \vdots \\
&= [b_0,\ b_1 x^{\gamma_1},\ldots,\ b_i x^{\gamma_i},\ldots]
\end{aligned}$$

which is just the **RCF** of ξ. Being a quadratic irrationals, by Theorem 3.12, we deduce that this **RCF** of ξ must be periodic, say

$$[b_0, b_1 x^{\gamma_1},\ldots, b_i x^{\gamma_i}, \overline{b_{i+1} x^{\gamma_{i+1}},\ b_{i+2} x^{\gamma_{i+2}},\ldots, b_{i+r} x^{\gamma_{i+r}}}].$$

Reverting this **RCF**, we get

$$[b_0, b_1x^{\gamma_1}, \ldots, b_ix^{\gamma_i}, \overline{b_{i+1}x^{\gamma_{i+1}},\ b_{i+2}x^{\gamma_{i+2}}, \ldots, b_{i+r}x^{\gamma_{i+r}}}]$$

$$= [b_0; x^{-\alpha_1}, b_1; x^{-\alpha_1}, b_2x^{\gamma_2}; 1, b_3x^{\gamma_3}; \ldots]$$

$$= [b_0; x^{-\alpha_1}, b_1; x^{-(\alpha_1+\gamma_2)}, b_2; x^{-\gamma_2}, b_3x^{\gamma_3}; \ldots]$$

$\vdots$

$$= [b_0; x^{-\gamma_1}, b_1; x^{-(\gamma_1+\gamma_2)}, b_2; \ldots; x^{-(\gamma_{i+r-1}+\gamma_{i+r})}, b_{i+r}; \overline{x^{-(\gamma_{i+r}+\gamma_{i+1})}, b_{i+1}; \ldots; x^{-(\gamma_{i+r-1}+\gamma_{i+r})}, b_{i+r}}]$$

$$= [b_0; a_1, b_1; \ldots; a_{i+r}, b_{i+r}; \overline{a'_{i+r+1}, b_{i+r+1}; a'_{i+r+2}, b_{i+r+2}; \ldots; a'_{i+2r}, b_{i+2r}}],$$

where $a'_{i+r+1} = \frac{1}{x^{\gamma_{i+r}+\gamma_{i+1}}},\ a'_{i+r+2} = \frac{1}{x^{\gamma_{i+1}+\gamma_{i+2}}}, \ldots, a'_{i+2r} = \frac{1}{x^{\gamma_{i+r-1}+\gamma_{i+r}}}$ which is a periodic **SCF**. □

Theorem 4.12. Let $\xi \in \mathbb{F}_q((x)),\ |\xi|_x \le 1$. If the **SCF** of ξ is periodic, then ξ is a nonrational root of a quadratic equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{F}_q[\frac{1}{x}],\ a \neq 0$.

Proof. Let $\xi = [b_0;\ a_1,\ b_1; \ldots;\ a_{n-1},\ b_{n-1};\ \overline{a_n,\ b_n;\ a_{n+1},\ b_{n+1}; \ldots;\ a_{n+k},\ b_{n+k}}]$, and $\xi_n = [\overline{b_n;\ a_{n+1},\ b_{n+1}; \ldots; a_{n+k},\ b_{n+k};\ a_n}]$ be the n^{th} complete quotient of the periodic continued fraction ξ. Then $\xi_n = \dfrac{A'\xi_n + a_nA''}{B'\xi_n + a_nB''}$ where

$\frac{A''}{B''} = [b_n;\ a_{n+1},\ b_{n+1}; \ldots; a_{n+k-1},\ b_{n+k-1}]$, $\frac{A'}{B'} = [b_n;\ a_{n+1},\ b_{n+1}; \ldots; a_{n+k},\ b_{n+k}]$, the last two convergents to $[b_n;\ a_{n+1},\ b_{n+1}; \ldots; a_{n+k},\ b_{n+k}]$.

It follows that

$$B'\xi_n^2 + (a_nB'' - A')\xi_n - a_nA'' = 0 \tag{4.5}$$

But $\xi = \dfrac{\xi_nA_{n-1} + a_nA_{n-2}}{\xi_nB_{n-1} + a_nB_{n-2}}$, $\xi_n = \dfrac{a_n(A_{n-2} - \xi B_{n-2})}{\xi B_{n-1} - A_{n-1}}$. Substituting for ξ_n in (4.5), we obtain an equation $a'\xi^2 + b'\xi + c' = 0$ where

$$a' = a_n^2B'B_{n-2}^2 - a_nA''B_{n-1}^2 - a_n^2B''B_{n-2}B_{n-1} + a_nB_{n-2}B_{n-1}A'$$

$$b' = -2a_n^2B'A_{n-2}B_{n-2} + a_n^2B''A_{n-2}B_{n-1} - a_nA'A_{n-2}B_{n-1} + a_n^2B''B_{n-2}A_{n-1} -$$
$$a_nA'A_{n-1}B_{n-2} + 2a_nA''B_{n-1}A_{n-1}$$

$$c' = a_n^2 B' A_{n-2}^2 - a_n^2 B'' A_{n-2} A_{n-1} + a_n A' A_{n-2} A_{n-1} - a_n A'' A_{n-1}^2.$$

Since ξ is irrational, then $a' \neq 0$. After adjusting the fraction the new coefficients $a,\ b,\ c \in \mathbb{F}_q[\frac{1}{x}]$. □

Theorem 4.13. Let $\xi \in \mathbb{F}_q((x)), |\xi|_x \leq 1$. Let the **SCF** of ξ be $[b_0; a_1, b_1; a_2, b_2; \ldots]$ where $a_i = x^{\alpha_i}$ and let (γ_i) be defined by $\gamma_1 = \alpha_1,\ \gamma_2 = \alpha_2 - \alpha_1,\ \gamma_3 = \alpha_3 - \alpha_2 + \alpha_1$, $\ldots,\ \gamma_i = \alpha_i - \alpha_{i-1} + \cdots + (-1)^{i+1}\alpha_1 (i \geq 1)$. Assume $\gamma_i \geq 0 \quad (i \geq 1)$. If ξ is a nonrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[\frac{1}{x}],\ a \neq 0$, then the **SCF** of ξ is periodic.

Proof. Assume that ξ is a root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a,\ b,\ c \in \mathbb{F}_q[\frac{1}{x}],\ a \neq 0$. Let $\xi = [b_0; x^{\alpha_1}, b_1; x^{\alpha_2}, b_2; \ldots]$. Then inverting the x^{α_i}'s, we see that

$$\begin{aligned}
\xi &= [b_0; x^{\alpha_1}, b_1; x^{\alpha_2}, b_2; \ldots] \\
&= [b_0; 1, b_1 x^{-\alpha_1}; x^{\alpha_2 - \alpha_1}, b_2; x^{\alpha_3}, b_3; \ldots] \\
&= [b_0; 1, b_1 x^{-\alpha_1}; 1, b_2 x^{-(\alpha_2 - \alpha_1)}; x^{\alpha_3 - (\alpha_2 - \alpha_1)}, b_3; \ldots] \\
&\ \vdots \\
&= [b_0, b_1 x^{-\gamma_1}, \ldots, b_i x^{-\gamma_i}, \ldots]
\end{aligned}$$

which is just the **RCF** of ξ. Being a quadratic irrationals, by Theorem 3.14, we deduce that this **RCF** of ξ must be periodic, say

$$[b_0, b_1 x^{-\gamma_1}, \ldots, b_i x^{-\gamma_i}, \overline{b_{i+1} x^{-\gamma_{i+1}},\ b_{i+2} x^{-\gamma_{i+2}}, \ldots, b_{i+r} x^{-\gamma_{i+r}}}].$$

Reverting this **RCF**, we get

$$[b_0, b_1x^{-\gamma_1}, \ldots, b_ix^{-\gamma_i}, \overline{b_{i+1}x^{-\gamma_{i+1}},\ b_{i+2}x^{-\gamma_{i+2}}, \ldots, b_{i+r}x^{-\gamma_{i+r}}}]$$

$$= [b_0; x^{\gamma_1}, b_1; x^{\gamma_1}, b_2x^{-\gamma_2}; 1, b_3x^{-\gamma_3}; \ldots]$$

$$= [b_0; x^{\gamma_1}, b_1; x^{\gamma_1+\gamma_2}, b_2; x^{\gamma_2}, b_3x^{-\gamma_3}; \ldots]$$

$$\vdots$$

$$= [b_0; x^{\gamma_1}, b_1; x^{\gamma_1+\gamma_2}, b_2; \ldots; x^{\gamma_{i+r-1}+\gamma_{i+r}}, b_{i+r}; \overline{x^{\gamma_{i+r}+\gamma_{i+1}}, b_{i+1}; \ldots; x^{\gamma_{i+r-1}+\gamma_{i+r}}, b_{i+r}}]$$

$$= [b_0;\ a_1,\ b_1; \ldots;\ a_{i+r},\ b_{i+r};\ \overline{a'_{i+r+1},\ b_{i+r+1};\ a'_{i+r+2},\ b_{i+r+2}; \ldots; a'_{i+2r},\ b_{i+2r}}],$$

where $a'_{i+r+1} = x^{\gamma_{i+r}+\gamma_{i+1}},\ a'_{i+r+2} = x^{\gamma_{i+1}+\gamma_{i+2}}, \ldots, a'_{i+2r} = x^{\gamma_{i+r-1}+\gamma_{i+r}}$ which is a periodic **SCF**. □

CHAPTER V

Best Approximations

Baum and Sweet [2] showed how to construct continued fractions for $\mathbb{F}_2((\frac{1}{x}))$ from the sequence of best approximations. The purpose of this chapter is to generalize this to any local field.

5.1 Definition

Let $(K,\ |\cdot|)$ be a local field, R its set of representatives of the residue class field of K. Every element $\xi \in K - \{0\}$ can be uniquely written as

$$\xi = \sum_{n=r}^{\infty} a_n \pi^n$$

where $|\xi| = 2^{-r}, r \in \mathbb{Z},\ a_n \in R$ and $a_r \neq 0$. Set $[\xi] = \displaystyle\sum_{n=r}^{0} a_n \pi^n$ and $\ ||\xi|| = |\xi - [\xi]|$. Then

$$R\left[\frac{1}{\pi}\right] = \{\alpha \in K; \alpha = \sum_{n=r}^{0} a_n \pi^n\},$$

the set of the head parts of elements in K.

For any $\alpha \in R[\frac{1}{\pi}]$, $\alpha = \displaystyle\sum_{n=r}^{0} a_n \pi^n$ with $a_r \neq 0$. The leading coefficient a_r of α is denoted by $h(\alpha)$.

Lemma 5.1. Let $\xi \in K$. We have $\|\xi\| < |\xi - \beta|$ for all $\beta \in R[\frac{1}{\pi}]$ and $\beta \neq [\xi]$.

Proof. Let $\xi = \displaystyle\sum_{n=r}^{\infty} a_n \pi^n$. Then $\|\xi\| \leq 2^{-1}$. Let $\beta = \displaystyle\sum_{n=s}^{0} c_n \pi^n \in R[\frac{1}{\pi}]$ be such that $\beta \neq [\xi]$. Thus $|\xi - \beta| \geq 1 > \|\xi\|$. □

Lemma 5.2. Let $\xi = \sum_{n=r}^{\infty} a_n \pi^n \in K$. For any $d \in \mathbb{N}$, there exists a non-zero element $B = \sum_{n=-(d-1)}^{0} c_n \pi^n \in R[\frac{1}{\pi}]$ such that $\|B\xi\| \leq 2^{-d}$.

Proof. Let $d \in \mathbb{N}$. We find $B = \sum_{n=-(d-1)}^{0} c_n \pi^n \in R[\frac{1}{\pi}]$ which satisfies $\|B\xi\| \leq 2^{-d}$ by considering

$$\begin{aligned} B\xi &= a_r \pi^r B + a_{r+1} \pi^{r+1} B + \cdots \\ &= a_r c_{-(d-1)} \pi^{r-(d-1)} + a_r c_{-(d-2)} \pi^{r-(d-2)} + \cdots + a_r c_{-1} \pi^{r-1} + a_r c_0 \pi^r \\ &+ a_{r+1} c_{(d-1)} \pi^{r-(d-2)} + a_{r+1} c_{-(d-2)} \pi^{r-(d-3)} + \cdots + a_{r+1} c_{-1} \pi^r + a_{r+1} c_0 \pi^{r+1} \\ &+ \cdots . \end{aligned}$$

Equating the coefficients of $\pi^{d-1}, \ \pi^{d-2}, \ldots, \ \pi^2, \ \pi^1$ to 0, we get the system

$$\begin{aligned} &c_{-(d-1)} a_d + c_{-(d-2)} a_{d-1} + \ldots + c_0 a_1 = 0 \\ &c_{-(d-1)} a_{d+1} + c_{-(d-2)} a_d + \ldots + c_0 a_2 = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &c_{-(d-1)} a_{2d-2} + c_{-(d-2)} a_{2d-1} + \ldots + c_0 a_{d-1} = 0. \end{aligned}$$

By solving for c_i from the $d-1$ equations above, the result follows. □

Definition 5.3. A sequence $\left(\frac{p_n}{q_n}\right)$ is said to be a *sequence of best approximations* to $\xi \in K$ provided:

1. q_n and $p_n = [q_n \xi] \in R[\frac{1}{\pi}]$
2. $q_0 = 1$
3. $|q_n| < |q_{n+1}|$
4. $\|q_{n+1}\xi\| < \|q_n \xi\| < 1$
5. $\|q_{n+1}\xi\| < \|q_n \xi\| \leq \|B\xi\|$ for all $B \in R[\frac{1}{\pi}]$ satisfying $|q_n| \leq |B| < |q_{n+1}|$.

5.2 Constructing continued fractions from best approximations

Given $\xi \in K$ and starting from $q_0 = 1,\ p_0 = [\xi]$, we first construct a sequence of best approximations to ξ. Note that we need only to construct the sequence (q_n).

If $\|q_0\xi\| = 0$ then the process stops.

If not, write $\|q_0\xi\| = \|\xi\| = 2^{-d_0}$ for some $d_0 \geq 1$. By Lemma 5.2, $\exists B \in R[\frac{1}{\pi}]$ such that $|B| \leq 2^{d_0}$ and $\|B\xi\| \leq 2^{-(d_0+1)} < 2^{-d_0} = \|q_0\xi\|$. Choose q_1 from such B with $|q_1|$ least and $h(q_1) = 1$ (if $h(q_1) \neq 1$ then choose $\frac{q_1}{h(q_1)}$ in place of q_1). Now we verify that q_1 satisfies all relevant properties of Definition 5.3.

Lemma 5.4. q_1 is uniquely determined.

Proof. Suppose that there exists $\acute{q}_1 \neq q_1 \in R[\frac{1}{\pi}]$ such that $h(\acute{q}_1) = 1$, $|\acute{q}_1| = |q_1|, |\acute{q}_1| \leq 2^{d_0}$ and $\|\acute{q}_1\xi\| \leq 2^{-(d_0+1)}$. Let $q = q_1 - \acute{q}_1$.
Then $|q| < \max\{|q_1|,\ |\acute{q}_1|\}$. Hence by Lemma 5.1

$$\begin{aligned} \|q\xi\| &\leq |q\xi - ([q_1\xi] - [q_1'\xi])| = |q_1\xi - q_1'\xi - ([q_1\xi] - [q_1'\xi])| \\ &\leq \max\{\|q_1\xi\|, \|q_1'\xi\|\} \leq 2^{-d_0-1}, \end{aligned}$$

which contradicts the minimality of $|q_1|$. □

Lemma 5.5. $|q_0| < |q_1|$.

Proof. We have that $|q_1| \geq 1 = |q_0|$. If $|q_1| = 1$ and $h(q_1) = 1$ then $q_1 = 1 = q_0$, and $\|q_1\xi\| = \|q_0\xi\|$, which is a contradiction. □

Lemma 5.6. $\forall Q \in R[\frac{1}{\pi}], (|q_0| \leq |Q| < |q_1| \Rightarrow \|q_0\xi\| \leq \|Q\xi\|)$.

Proof. Let $Q \in R[\frac{1}{\pi}], |Q| < |q_1| \leq 2^{d_0}$. Without loss of generality let $h(Q_1) = 1$. If $\|Q\xi\| < \|q_0\xi\| = 2^{-d_0}$ then $\|Q\xi\| \leq 2^{-(d_0+1)}$, contradicting with the minimality of $|q_1|$. □

Having verified q_1, we now continue the construction.

If $\|q_1\xi\| = 0$, the process stops.

If not, write $\|q_1\xi\| = 2^{-d_1}$ for some $d_1 \geq 1$.Then by Lemma 5.2, $\exists B \in R[\frac{1}{\pi}]$ such that $|B| \leq 2^{d_1}$ and $\|B\xi\| \leq 2^{-(d_1+1)} < 2^{-d_1} = \|q_1\xi\|$. Choose q_2 from such B with $|q_2|$ least and $h(q_2) = 1$ (if $h(q_2) \neq 1$ then choose $\frac{q_2}{h(q_2)}$ in place of q_2). By the same proofs of as Lemma 5.4, see that, q_2 is uniquely determined.

Lemma 5.7. $|q_1| < |q_2|$.

Proof. If $|q_2| < |q_1| \leq 2^{d_0}$ and we have that $\|q_2\xi\| < \|q_1\xi\|$ then it contradicts with the minimality of $|q_1|$. Hence $|q_1| \leq |q_2|$. Suppose $|q_2| = |q_1|$. Let $q^* = q_2 - q_1$. Thus $|q^*| < |q_1|, |q_2|$ and by Lemma 5.1

$$\begin{aligned}\|q^*\xi\| &\leq |q^*\xi - ([q_2\xi] - [q_1\xi])| = |q_2\xi - [q_2\xi] - (q_1\xi - [q_1\xi])| \\ &= \max\{\|q_2\xi\|, \|q_1\xi\|\} = \|q_1\xi\|,\end{aligned}$$

which contradicts the minimality of $|q_1|$. □

By the same proof as Lemma 5.6, we see that

$$\forall Q \in R[\frac{1}{\pi}], (|q_1| \leq |Q| < |q_2| \Rightarrow \|q_1\xi\| \leq \|Q\xi\|),$$

and so $|q_2|$ possesses the relevant properties of Definition 5.3.

If $\|q_2\xi\| = 0$, the process stops.

If not, we continue the process in the same manner.

In general, write $\|q_{n-1}\xi\| = 2^{-d_{n-1}}$. Then by Lemma 5.2, $\exists\, B \in R[\frac{1}{\pi}]$ such that $|B| \leq 2^{d_{n-1}}$ and $\|B\xi\| \leq 2^{-(d_{n-1}+1)} < 2^{-d_{n-1}} = \|q_{n-1}\xi\|$. Choose q_n from such B with $|q_n|$ least and $h(q_n) = 1$. We deduce as above that, q_n is uniquely determined, $|q_{n-1}| < |q_n|$ and $\forall Q \in R[\frac{1}{\pi}], (|q_{n-1}| \leq |Q| < |q_n| \Rightarrow \|q_{n-1}\xi\| \leq \|Q\xi\|)$.

By so doing, we have a sequence of a best approximations $\dfrac{p_n}{q_n}$ to ξ possessing the relevant properties as in Definition 5.3. In order to fix notation, we collect most of the facts here.

Fact 1. $|q_0| = 1 < |q_1| < |q_2| < \cdots < |q_n| < \cdots$

Fact 2. $\|q_o\xi\| = \frac{1}{2^{d_0}} > \|q_1\xi\| = \frac{1}{2^{d_1}} > \cdots > \|q_n\xi\| = \frac{1}{2^{d_n}} > \cdots$, where $d_i \in \mathbb{N}$ are such that $d_o < d_1 < d_2 < \cdots$

Fact 3. $\|q_n\xi\| = \frac{1}{2^{d_n}} \leq |q_{n+1}|^{-1} \quad (n \geq 0)$.

Fact 4. Recalling that $p_n = [q_n\xi] \quad (n \geq 0)$, then

$$|q_n(q_{n+1}\xi - p_{n+1})| = |q_n|\|q_{n+1}\xi\| = |q_n|(2^{d_{n+1}})^{-1} < |q_{n+1}|(2^{d_n})^{-1}$$

$$|q_{n+1}(q_n\xi - p_n)| = |q_{n+1}|\|q_n\xi\| = |q_{n+1}|(2^{d_n})^{-1} \leq 1.$$

Fact 5. $|p_nq_{n+1} - p_{n+1}q_n| = 1 \quad (n \geq 0)$. This is so because by Fact 4

$$|p_nq_{n+1} - p_{n+1}q_n| = |q_n(q_{n+1}\xi - p_{n+1}) - q_{n+1}(q_n\xi - p_n)| = |q_{n+1}|\|q_n\xi\| \leq 1.$$

If $p_nq_{n+1} - p_{n+1}q_n = 0$ then $\dfrac{p_n}{q_n} = \dfrac{p_{n+1}}{q_{n+1}}$, implying that

$$\frac{\|q_n\xi\|}{|q_n|} = |\xi - \frac{p_n}{q_n}| = |\xi - \frac{p_{n+1}}{q_{n+1}}| = \frac{\|q_{n+1}\xi\|}{|q_{n+1}|},$$

which is a contradicts the description in Fact 4, and so being in $R[\frac{1}{\pi}]$, we have $|p_nq_{n+1} - p_{n+1}q_n| \geq 1$, yielding the result of Fact 5.

Fact 6. $\|q_n\xi\| = 2^{d_n} = |q_{n+1}|^{-1} \quad (n \geq 0)$. This follows from Fact 3 and the description in the proof of Fact 5.

From such a sequence of best approximations, we now proved to construct its associated continued fraction.

Since $|p_nq_{n+1} - p_{n+1}q_n| = 1$ and $p_nq_{n+1} - p_{n+1}q_n \in R[\frac{1}{\pi}]$, then $p_nq_{n+1} - p_{n+1}q_n \in R - \{0\}$, which yield $g.c.d.\ (p_{n+1}, q_{n+1}) = 1$.

Similarly, $p_{n+1}q_{n+2} - p_{n+2}q_{n+1} \in R - \{0\}$.

We can then write $-a_{n+2}(p_nq_{n+1} - p_{n+1}q_n) = p_{n+1}q_{n+2} - p_{n+2}q_{n+1}$ where $a_{n+2} \in R - \{0\}$, and so $q_{n+1}(p_{n+2} - a_{n+2}p_n) = p_{n+1}(q_{n+2} - a_{n+2}q_n)$.

Since $g.c.d.(p_{n+1}, q_{n+1}) = 1$, then $p_{n+1}|(p_{n+2} - a_{n+2}p_n)$, i.e., there exists $b_{n+2} \in R[\frac{1}{\pi}]$ such that $p_{n+2} - a_{n+2}p_n = b_{n+2}p_{n+1}$. Now

$$q_{n+1}(b_{n+2}p_{n+1}) = q_{n+1}(p_{n+2} - a_{n+2}p_n) = p_{n+1}(q_{n+2} - a_{n+2}q_n),$$

i.e. $q_{n+2} - a_{n+2}q_n = b_{n+2}q_{n+1}$.

We have thus found unique $a_{n+2} \in R - \{0\}$ and $b_{n+2} \in R[\frac{1}{\pi}]$ such that

$$p_{n+2} = b_{n+2}p_{n+1} + a_{n+2}p_n,\ q_{n+2} = b_{n+2}q_{n+1} + a_{n+2}q_n \quad (n \geq 0).$$

This result continues to hold for $n = -1$ if we put $q_{-1} = 0,\ p_{-1} = 1,\ b_0 = p_0$, $b_1 = q_1$ and $a_1 = p_1 - b_1b_0 \in R[\frac{1}{\pi}]$. Since $1 = |p_0q_1 - p_1q_0| = |b_0b_1 - p_1| = |a_1|$, then $a_1 \in R - \{0\}$.

Lemma 5.8. For $n \geq 0, \alpha \in K - \{0\}$, we have

$$\frac{\alpha p_n + a_{n+1}p_{n-1}}{\alpha q_n + a_{n+1}q_{n-1}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;a_n,\ b_n;\ a_{n+1},\ \alpha]$$

Proof. Let $P(n)$: $\dfrac{\alpha p_n + a_{n+1}p_{n-1}}{\alpha q_n + a_{n+1}q_{n-1}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;a_n,\ b_n;\ a_{n+1},\ \alpha]$.
Since $\dfrac{\alpha p_0 + a_1p_{-1}}{\alpha q_0 + a_1q_{-1}} = \dfrac{\alpha b_0 + a_1}{\alpha} = [b_0;\ a_1,\ \alpha]$, $P(0)$ is true.
Suppose that $P(n-1)$ holds. Consider $P(n)$,

$$\begin{aligned}[b_o;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;a_n,\ b_n;\ a_{n+1},\ \alpha] &= \frac{(b_n + \frac{a_{n+1}}{\alpha})p_{n-1} + a_np_{n-2}}{(b_n + \frac{a_{n+1}}{\alpha})q_{n-1} + a_nq_{n-2}} \\ &= \frac{\alpha(b_np_{n-1} + a_np_{n-2}) + a_{n+1}p_{n-1}}{\alpha(b_nq_{n-1} + a_nq_{n-2}) + a_{n+1}q_{n-1}} \\ &= \frac{\alpha p_n + a_{n+1}p_{n-1}}{\alpha q_n + a_{n+1}q_{n-1}}.\end{aligned}$$

Hence $P(n)$ holds. □

By Lemma 5.8,

$$\frac{p_n}{q_n} = \frac{b_np_{n-1} + a_np_{n-2}}{b_nq_{n-1} + a_nq_{n-2}} = [b_o;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;a_n,\ b_n].$$

It follows that $\dfrac{p_n}{q_n}$ $(n \geq 0)$ is the n^{th} convergent of a continued fraction of ξ, with b_n as partial denominators and a_n $(n \geq 1)$ as partial numerators to ξ. Since $\|q_n\xi\| = |q_{n+1}|^{-1}$ and a sequence d_i is increasing, for $n \geq 1$

$$|\xi - \frac{p_n}{q_n}| = |q_n|^{-1}\|q_n\xi\| = \|q_{n-1}\xi\|\|q_n\xi\| = 2^{-d_n-d_{n-1}} \to 0 \ (n \to \infty).$$

This convergence allows us to call $[b_o;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;a_n,\ b_n;\ldots]$ a continued fraction to ξ.

Next we will show that the above continued fraction is just the **RCF** to ξ, This relies mainly on the fact that all $a_n \in R$. Putting $\beta_0 = b_0$ and $\xi_n = [b_n;\ a_{n+1},\ b_{n+1};\ a_{n+2},\ b_{n+2};\ldots]$, then

$$\xi = \beta_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\xi_2}} = \beta_0 + \cfrac{1}{b_1/a_1 + \cfrac{a_2/a_1}{\xi_2}} = \beta_0 + \cfrac{1}{\beta_1 + \cfrac{a_2/a_1}{\xi_2}}$$
$$= \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{b_2 a_1/a_2 + \cfrac{a_3 a_1/a_2}{\xi_3}}} = \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\beta_2 + \cfrac{a_3 a_1/a_2}{\xi_3}}} = \cdots .$$

It is clear that $\forall i \geq 0,\ \beta_i \in R[\frac{1}{\pi}] - \{0\} \subset$ and $|\beta_i| > 1$. Since ξ has a unique **RCF**, the continued fraction constructed from best approximation of ξ and its **RCF** are the same.

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