อัตราการลู่เข้าของฟังก์ชันการแจกแจงของผลบวกของส่วนกลับของฟังก์ชันไซน์ และฟังก์ชันลอการิทึม ของตัวแปรสุ่มต่อเนื่องที่อิสระต่อกัน สู่ฟังก์ชันการแจกแจงโคชี

นางสาวอังคณา สันทัดการ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2545

ISBN 974-17-1029-1

RATES OF CONVERGENCES OF DISTRIBUTION FUNCTIONS OF SUMS OF THE RECIPROCALS OF SINE AND LOGARITHM OF INDEPENDENT CONTINUOUS RANDOM VARIABLES TO CAUCHY DISTRIBUTION FUNCTIONS

Miss Angkana Suntadkarn

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics

Department of Mathematics
Faculty of Science
Chulalongkorn University

Academic Year 2002
ISBN 974-17-1029-1

Thesis Title	Rates of Convergences of Ditribution Functions of Sums of The Reciprocals of Sine and Logarithm of Independent Continuous Random Variables to Cauchy Distribution Functions
By	Miss Angkana Suntadkarn
Field of study	Mathematics
Thesis Advisor	Associate Professor Kritsana Neammanee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master 's Degree

.... Dean of Faculty of Science

(Associate Professor Wanchai Phothiphichitr, Ph.D.)

Thesis Committee

.... Chairman

(Assistant Professor Imchit Termwuttipong, Ph.D.)

.... Thesis Advisor

(Associate Professor Kritsana Neammanee, Ph.D.)

.... Member

(Wicharn Lewkeeratiyutkul, Ph.D.)

อังคณา สันทัดการ : อัตราการลู่เข้าของฟังก์ชันการแจกแจงของผลบวกของส่วนกลับของฟังก์ชันไซน์ และฟังก์ชันลอการิทึมของตัวแปรสุ่มต่อเนื่องที่อิสระต่อกัน สู่ฟังก์ชันการแจกแจงโคชี (RATES OF - CONVERGENCES OF DISTRIBUTION FUNCTIONS OF SUMS OF RECIPROCALS OF SINE AND LOGARITHM OF INDEPENDENT CONTINUOUS RANDOM VARIABLES TO CAUCHY DISTRIBUTION FUNCTIONS) อ. ที่ปรึกษา : รองศาสตราจารย์ ดร.กฤษณะ เนียมมณี., 46 หน้า. ISBN 974-17-1029-1

กำหนดให้ $X_1, X_2, \ldots$ เป็นลำดับของตัวแปรสุ่มต่อเนื่องที่อิสระต่อกัน และมีฟังก์ชันการแจกแจงเดียวกัน

ในวิทยานิพนธ์นี้ เราได้หาอัตราการลู่เข้าของฟังก์ชันการแจกแจง ของ $\frac{1}{n}\left(\frac{1}{\sin X_1}+\frac{1}{\sin X_2}+\cdots+\frac{1}{\sin X_n}\right)$ สู่ฟังก์ชันการแจกแจงโคชี $F(x)=\frac{1}{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{x}{\pi f(0)}\right)\right)$ และอัตราการลู่เข้าของฟังก์ชันการแจกแจงของ $\frac{1}{n}\left(\frac{1}{\ln|X_1|}+\frac{1}{\ln|X_2|}+\cdots+\frac{1}{\ln|X_n|}\right)$ สู่ฟังก์ชันการแจกแจงโคชี $F(x)=\frac{1}{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{x}{\pi(f(-1)+f(1))}\right)\right)$ โดยที่ขอบเขตความคลาดเคลื่อนอยู่ในรูป $\frac{C}{n^d}$ เมื่อ $0<d<\frac{1}{9}$

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..
สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา.......................................
ปีการศึกษา 2545 ลายมือชื่ออาจารย์ที่ปรึกษาร่วม -

4372488923 : MAJOR MATHEMATICS

KEY WORD: CAUCHY DISTRIBUTION / LEVY-KHINCHINE FORMULA / INFINITELY DIVISIBLE

ANGKANA SUNTADKARN : RATES OF CONVERGENCES OF DISTRIBUTION FUNCTIONS OF SUMS OF THE RECIPROCALS OF SINE AND LOGARITHM OF INDEPENDENT CONTINUOUS RANDOM VARIABLES TO CAUCHY DISTRIBUTION FUNCTIONS. THESIS ADVISOR:ASSOC.PROF. KRITSANA NEAMMANEE, Ph.D., 46 pp. ISBN 974-17-1029-1

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables.

In this study, we find the rate of convergence of the sequence of distribution functions of $\frac{1}{n}\left(\frac{1}{\sin X_1}+\frac{1}{\sin X_2}+\cdots+\frac{1}{\sin X_n}\right)$ to the Cauchy distribution function $F(x)=\frac{1}{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{x}{\pi f(0)}\right)\right)$ and the rate of convergence of the sequence of distribution functions of $\frac{1}{n}\left(\frac{1}{\ln|X_1|}+\frac{1}{\ln|X_2|}+\cdots+\frac{1}{\ln|X_n|}\right)$ to the Cauchy distribution function $F(x)=\frac{1}{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{x}{\pi(f(-1)+f(1))}\right)\right)$. The bound of errors is of the form $\frac{C}{n^d}$ with $0<d<\frac{1}{9}$.

Department **Mathematics** Student's signature..

Field of study **Mathematics** Advisor's signature..

Academic year **2002** Co-advisor's signature -

ACKNOWLEDGMENTS

I am greatly indebted to Associate Professor Dr. Kritsana Neammanee, my thesis advisor, for his suggestions and helpful advice in preparing and writing this thesis. I would like to thank Assistant Professor Dr. Imchit Termwuttipong and Dr. Wicharn Lewkeeratiyutkul, my thesis committee, for their suggestions to this thesis. I also would like to thank all teachers, who have taught me, for my knowledge and skill.

In particular, I would like to express my deep gratitude to my father and my mother for their encouragement throughout my graduate study.

CONTENTS

page

CHAPTER I

Introduction

Let (X_n) be a sequence of independent continuous random variables. Many authors (for examples, Shapiro (1975, 1977, 1988), Termwuttipong (1986), Neammanee(1998, 2001, 2002) and Siricheon and Neammanee (2001)) investigated the limit distribution of reciprocals of the random variables. Siricheon and Neammanee (2001) and Neammanee (2001) considered the sequences of distribution functions of $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin X_k}$ and $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\ln|X_k|}$, respectively. They found that both of the sequences converge to Cauchy distribution functions, which are stated in the followings theorems (Theorem 1.1 and Theorem 1.2) .

Theorem 1.1 Let (X_n), $n = 1, 2, 3, \ldots$ be a sequence of independent continuous random variables such that $\text{Im}X_n \subseteq \mathbb{R} - \{j\pi \mid j \in \mathbb{Z}\}$ and let f_k and F_k be the probability density function and the distribution function of X_k, respectively. Assume that

1. $\{j^2 f_k : k \in \mathbb{N}, j \in \mathbb{Z}\}$ is uniformly equicontinuous, i.e. for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for real numbers x and y if $|x - y| < \delta$ then $|j^2 f_k(x) - j^2 f_k(y)| < \varepsilon$ for every integer j and positive integer k,

2. $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j=-\infty}^{\infty} f_k((2j-1)\pi) = L_1$ for some $L_1 > 0$ and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j=-\infty}^{\infty} f_k(2j\pi) = L_2$ for some $L_2 > 0$.

Then there exists a sequence of real constants (A_n) such that the sequence of distribution functions of $\frac{1}{n}\Big(\sum_{k=1}^{n}\frac{1}{\sin X_k}\Big) - A_n$ converges to the Cauchy distribution function

$$F(x) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan\frac{x}{\pi(L_1+L_2)}\Big)$$

where the constants A_n are given by $A_n = \frac{1}{n}\sum_{k=1}^{n}\int_{|x|>1} xdF_{nk}(x)$ and $F_{nk}(x)$ is the distribution function of X_{nk}.

Theorem 1.2 Let $(X_n), n = 1,2,3,\ldots$ be a sequence of independent continuous random variables such that $\mathrm{Im}X_n \subseteq \mathbb{R}-\{0,\pm1\}$. Let f_k be the probability density function of X_k. Assume that

1. $(f_k(x))$ is equicontinuous at ±1,
2. $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_k(1) = L_1$ and $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_k(-1) = L_2$ where $L_1, L_2 > 0$.

Then there exists a sequence of real constants (A_n) such that the sequence of distribution functions of $\frac{1}{n}\Big(\sum_{k=1}^{n}\frac{1}{\ln|X_k|}\Big) - A_n$ converges weakly to the Cauchy distribution function

$$F(x) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \arctan\frac{x}{\pi(L_1+L_2)}\Big)$$

where the constants A_n are given by $A_n = \frac{1}{n}\sum_{k=1}^{n}\int_{|x|>1} xdF_{nk}(x)$ and $F_{nk}(x)$ is the distribution function of X_{nk}.

In this work, we will find the rates of convergences of the sequence of distribution functions of random variables in theorem 1.1 and theorem 1.2.

The followings are our main results.

Theorem 1.3 Let $(X_n), n = 1,2,3,\ldots$ be a sequence of independent identically distributed continuous random variables with common symmetric probability density function f. Assume that

1. $f(0) > 0$ and there exists a positive constant B such that $|f(x)| < B$ for all x,
2. f' exists and is bounded in some neighborhood of the origin,
3. $ImX_k \subseteq \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]$ and $0 \notin ImX_k$.

Then for any fixed $0 < d < \frac{1}{9}$ there exists a constant C such that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{C}{n^d},$$

where F_n is the distribution function of $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sin X_k}$ and F is the Cauchy distribution function defined by

$$F(x) = \frac{1}{\pi}\Big[\frac{\pi}{2} + \arctan\Big(\frac{x}{\pi f(0)}\Big)\Big].$$

Theorem 1.4 Let $(X_n), n = 1, 2, 3, \ldots$ be a sequence of independent identically distributed continuous random variables with common symmetric probability density function f and distribution function H. Assume that

1. $f(-1)$ and $f(1) > 0$ and there exists a positive constant B such that $|f(x)| < B$ for all x,
2. f' exists and is bounded in some neighborhood of 1,
3. $(H \circ exp)'$ is symmetric.

Then for any fixed $0 < d < \frac{1}{9}$ there exists a constant C such that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{C}{n^d},$$

where F_n is the distribution function of $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{\ln |X_k|}$ and F is the Cauchy distribution function defined by

$$F(x) = \frac{1}{\pi}\Big[\frac{\pi}{2} + \arctan\Big(\frac{x}{\pi(f(-1) + f(1))}\Big)\Big].$$

In this thesis we organize as follows. In chapter II, some important preliminary results and notations which are necessary for this work are presented. Chapter III and chapter IV contain our main results.

CHAPTER II

Preliminaries

In this chapter, we present some basic concepts and facts of theory of probability that are needed in this work. The proofs of the statements are omitted. They can be found in Cramer(1945), Laha and Rohatgi(1979), Gnedenko(1962), Gnedenko and Kolmogorov(1954), and Lukacs(1960).

2.1 Random Variables, Distribution Functions and Characteristic Functions

A **probability space** is a measure space $(\Omega, \Im, P)$ in which P is a positive measure such that $P(\Omega) = 1$. The set Ω will be referred to as a **sample space.** The elements of $\Im$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A.

A function X from a probability space $(\Omega, \Im, P)$ to the set of complex numbers $\mathbb{C}$ is said to be a **complex-valued random variable** if for every Borel set B in $\mathbb{C}$, $X^{-1}(B)$ belongs to $\Im$. If X is real-valued, we say that it is a **real-valued random variable**, or simply a **random variable**.

Let X be a random variable defined on a probability space $(\Omega, \Im, P)$. We will use the notation $P(X \leq x)$, $P(X \geq x)$ and $P(|X| \geq x)$ to denote $P(\{\omega \in \Omega | X(\omega) \leq x\}), P(\{\omega \in \Omega | X(\omega) \geq x\})$ and $P(\{\omega \in \Omega | |X(\omega)| \geq x\})$, respectively.

A function F from $\mathbb{R}$ to $\mathbb{R}$ is said to be a **distribution function** if it is non-decreasing, right-continuous, $F(-\infty) = 0$ and $F(+\infty) = 1$.

For any random variable X, the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = P(X \leq x)$$

is a distribution function. It is said to be the **distribution function of the random variable** X.

We shall say that a random variable X has a **Cauchy distribution with parameter** θ if its distribution function is given by

$$F(x) = \frac{1}{\pi}\left[\frac{\pi}{2} + \arctan\left(\frac{x}{\theta}\right)\right] \text{where } \theta > 0.$$

Let $F, F_1, F_2, \ldots$ be bounded non-decreasing functions. A sequence (F_n) **converges weakly** to F if

(i) for every continuity point x of F, $F_n(x) \to F(x)$ and

(ii) $F_n(+\infty) \to F(+\infty)$ and $F_n(-\infty) \to F(-\infty)$.

We will write

$$F_n \xrightarrow{w} F$$

if (F_n) converges weakly to F. Note that the weak limit of the sequence (F_n), if it exists, is unique.

We define the **expectation** of a complex-valued random variable X to be

$$\int_\Omega X\,dP$$

provided that the integral $\int_\Omega X\,dP$ exists. It will be denoted by $E[X]$.

Proposition 2.1.1(Cramer(1945), p.174) Let $X_1, X_2, \ldots, X_n$ be random variables. Then

$$E[X_1 + X_2 + \cdots + X_n] = \sum_{j=1}^{n} E[X_j],$$

provided that the sums on the right hand side is meaningful.

Proposition 2.1.2(Laha and Rohatgi(1979), p.28) Let X be a random variable with the distribution function F. If $E[X]$ exists, then

$$E[X] = \int_{-\infty}^{\infty} x\,dF(x).$$

The expectation of a random variable X is also known as the **mean** of X. The expectation of $(X - E[X])^2$ is known as the **variance** of X and is denoted by $\sigma^2(X)$. Note that mean and variance of random variable may be infinite.

Let F be a distribution function. The function $\varphi : \mathbb{R} \to \mathbb{C}$ defined by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

is called the **characteristic function of the distribution function** F. If F is the distribution function of a random variable X, then φ is also called the **characteristic function of** X.

Proposition 2.1.3(Gnedenko and Kolmogorov (1954), p.45) For any characteristic function φ, we have

(i) $\varphi(0) = 1$,

(ii) $|\varphi(t)| \leq 1$ for every t,

(iii) φ is continuous.

Proposition 2.1.4(Lukacs(1960), p.45)

(i) The product of two characteristic function is a characteristic function,

(ii) If φ is a characteristic function, then $|\varphi|^2$ is also a characteristic function.

The random variables $X_1, X_2, \ldots, X_n$ are called **independent** if

$$P(\bigcap_{j=1}^{n}\{\omega | X_j(\omega) \leq x_j\}) = \prod_{j=1}^{n} P(X_j \leq x_j)$$

hold for every real number $x_1, x_2, \ldots, x_n$.

A sequence of random variables (X_n) is said to be **independent** if $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ are independent for all distinct $i_1, i_2, \ldots, i_k$, for every $k \in \mathbb{N}$.

Theorem 2.1.5(Cramer(1945), p.188,191) Let $X_1, X_2, \ldots, X_n$ be random variables with the characteristic functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively. Assume that $X_1, X_2, \ldots, X_n$ are independent. Then the followings hold,

(i) the characteristic function φ of $X_1 + X_2 + \cdots + X_n$ is given by

$$\varphi(t) = \varphi_1(t)\varphi_2(t)\cdots\varphi_n(t) \quad \text{for all } t \in \mathbb{R},$$

(ii) $\sigma^2(X_1 + X_2 + \cdots + X_n) = \sigma^2(X_1) + \sigma^2(X_2) + \cdots + \sigma^2(X_n)$ if $\sigma^2(X_i) < \infty$ for $i = 1, 2, \ldots, n$.

2.2 Infinitely Divisible Distribution Functions

A distribution function F with the characteristic function φ is said to be **infinitely divisible** if for every natural number n, there exists a characteristic function φ_n such that for every t,

$$\varphi(t) = \{\varphi_n(t)\}^n.$$

The characteristic function of any infinitely divisible function is also said to be **infinitely divisible.** A random variable is said to be **infinitely divisible** if its distribution function is infinitely divisible.

Theorem 2.2.1(Lukacs(1960), p.89) The characteristic function $\varphi(t)$ is an infinitely divisible if and only if it can be written in the canonical form

$$\log \varphi(t) = i\gamma t + \int_{-\infty}^{\infty} f(t,u)dG(u) \tag{2.2.1}$$

where γ is a real constant, the function f is given by

$$f(t,u) = \begin{cases} \left(e^{itu} - 1 - \dfrac{itu}{1+u^2}\right)\left(\dfrac{1+u^2}{u^2}\right) & \text{if } u \neq 0 \\ -\dfrac{t^2}{2} & \text{if } u = 0 \end{cases}$$

and $G(u)$ is a bounded non-decreasing function which is continuous from the right and $G(-\infty) = 0$. The representation (2.2.1) is unique and it is called **Lévy-Khinchine's formula.**

There is another representation of the logarithm of an infinitely divisible characteristic function φ, known as **Lévy's formula**:

$$\log \varphi(t) = i\gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0} g(t,u)dM(u) + \int_{0}^{+\infty} g(t,u)dN(u) \qquad (2.2.2)$$

the function g is given by

$$g(t,u) = e^{itu} - 1 - \frac{itu}{1+u^2}, \quad u \neq 0$$

where $\sigma^2 \geq 0$ and γ are real constants, the functions M and N are non-decreasing functions define on $(-\infty, 0)$ and $(0, +\infty)$ respectively with $M(-\infty) = 0$, $N(+\infty) = 0$ and

$$\int_{-\varepsilon}^{0^-} x^2 dM(x) + \int_{0+}^{\varepsilon} x^2 dN(x) < +\infty$$

for every positive real number ε.

The characteristic function φ is infinitely divisible if and only if its logarithm can be represented by Lévy's formula.

For an infinitely divisible characteristic function φ, representation (2.2.1) and (2.2.2) are related by

$$\begin{aligned} M(x) &= \int_{-\infty}^{x} \frac{1+u^2}{u^2} dG(u), \text{ for } x < 0, \\ N(x) &= -\int_{x}^{+\infty} \frac{1+u^2}{u^2} dG(u), \text{ for } x > 0, \\ \sigma^2 &= G(+\infty) - G(-\infty). \end{aligned}$$

It is well-known that Cauchy random variable is infinitely divisible with Lévy-Khinchine's formula and Lévy's formula as follows:

Characteristic Function	Lévy-Khinchine's formula		Lévy's formula			
	γ	$G(u)$	γ	σ	$M(u)$ $u<0$	$N(u)$ $u>0$
$e^{-\theta\|t\|}, \theta > 0$	0	$(\frac{\theta}{\pi}) \arctan x + \frac{\theta}{2}$	0	0	$-\frac{\theta}{\pi u}$	$-\frac{\theta}{\pi u}$

Theorem 2.2.2 (Gnedenko(1962), p.307) In order that a distribution function F with finite variance is infinitely divisible it is necessary and sufficient that there exist a unique constant μ and a non-decreasing, right-continuous function of bounded variation K such that $K(-\infty) = 0$ and the logarithm of its characteristic function φ is given by

$$\log \varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)\, dK(x) \tag{2.2.3}$$

where

$$f(t,x) = \begin{cases} \left(e^{itx} - 1 - itx\right)\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ -\frac{t^2}{2} & \text{if } x = 0. \end{cases}$$

The formula (2.2.3) is known as **Kolmogorov formula.**

Theorem 2.2.3 (Gnedenko and Kolmogorov (1954)) Let X be an infinitely divisible random variable with finite variance. Let the constant μ and the function K be given as in the Kolmogorov formula of the characteristic function of X. Then

(i) $E[X] = \mu$,

(ii) $\sigma^2(X) = K(+\infty)$.

CHAPTER III

Rate of convergence of distribution function of sums of the reciprocals of sine of independent continuous random variables to the Cauchy distribution function

The purpose of this chapter is to find the rate of convergence of the sequence of distribution function of $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin X_k}$ to the Cauchy distribution function. The main theorem is the following.

Theorem *Let $(X_n), n = 1, 2, 3, \ldots$ be a sequence of independent identically distributed continuous random variables with common symmetric probability density function f. Assume that*

1. $f(0) > 0$ and there exists a positive constant B such that $|f(x)| < B$ for all x,

2. f' exists and is bounded in some neighborhood of the origin,

3. $ImX_k \subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $0 \notin ImX_k$.

Then for any fixed $0 < d < \frac{1}{9}$ there exists a constant C such that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{C}{n^d},$$

where F_n is the distribution function of $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin X_k}$ and F is the Cauchy distribution function defined by

$$F(x) = \frac{1}{\pi}\left[\frac{\pi}{2} + \arctan\left(\frac{x}{\pi f(0)}\right)\right].$$

The proof of this theorem is based on the results of Shapiro(1955) and Boonyasombut and Shapiro(1970).

Throughout this work, C stands for a positive constant with possibly different values in different places.

For $k = 1, 2, \ldots, n, n = 1, 2, \ldots$, let $X_{nk} = \dfrac{1}{n \sin X_k}$ and let F_{nk} be the distribution function of X_{nk}. To proof the main theorem we need the following 4 lemmas and 1 theorem (Lemma 3.1, Lemma 3.2, Lemma 3.4, Lemma 3.7 and Theorem 3.6).

Lemma 3.1 *Let H be the common distribution function of X_k.*

$$F_{nk}(x) = \begin{cases} H(0) - H(\sin^{-1}\frac{1}{nx}) & if\ -1 < \frac{1}{nx} < 0 \\ H(0) & if\ x = 0\ or\ \left|\frac{1}{nx}\right| \geq 1, \\ H(0) + 1 - H(\sin^{-1}\frac{1}{nx}) & if\ 0 < \frac{1}{nx} < 1. \end{cases}$$

Proof. Let x be any real number.

Case 1. $x = 0$ or $\left|\frac{1}{nx}\right| \geq 1$.

If $x = 0$, then

$$\begin{aligned} F_{nk}(0) &= P(X_{nk} \leq 0) \\ &= P(\frac{1}{n \sin X_k} \leq 0) \\ &= P(\sin X_k < 0) \\ &= P(-\frac{\pi}{2} \leq X_k < 0) \\ &= H(0) - H(-\frac{\pi}{2}) \\ &= H(0). \end{aligned}$$

If $\frac{1}{nx} \leq -1$, then

$$\begin{aligned} F_{nk}(x) &= P(\frac{1}{n\sin X_k} \leq x) \\ &= P(\frac{1}{nx} \leq \sin X_k < 0) \\ &= P(-1 \leq \sin X_k < 0) \\ &= P(-\frac{\pi}{2} \leq X_k < 0) \\ &= H(0). \end{aligned}$$

If $\frac{1}{nx} \geq 1$, then

$$\begin{aligned} F_{nk}(x) &= F_{nk}(0) + P(0 < \frac{1}{n\sin X_k} \leq x) \\ &= F_{nk}(0) + P(\sin X_k \geq \frac{1}{nx}) \\ &= F_{nk}(0) + P(\sin X_k = 1) \\ &= F_{nk}(0) \\ &= H(0). \end{aligned}$$

Case 2. $-1 < \frac{1}{nx} < 0.$

$$\begin{aligned} F_{nk}(x) &= P(\frac{1}{n\sin X_k} \leq x) \\ &= P(\frac{1}{nx} \leq \sin X_k < 0) \\ &= P(\sin^{-1}\frac{1}{nx} \leq X_k < 0) \\ &= H(0) - H(\sin^{-1}\frac{1}{nx}). \end{aligned}$$

Case 3. $0 < \dfrac{1}{nx} < 1.$

$$
\begin{aligned}
F_{nk}(x) &= F_{nk}(0) + P(0 < \frac{1}{n\sin X_k} \le x) \\
&= F_{nk}(0) + P(\sin X_k \ge \frac{1}{nx}) \\
&= F_{nk}(0) + 1 - P(\sin X_k < \frac{1}{nx}) \\
&= H(0) + 1 - P(X_k < \sin^{-1}\frac{1}{nx}) \\
&= H(0) + 1 - H(\sin^{-1}\frac{1}{nx}).
\end{aligned}
$$

□

For $a \ge 1$ we define $X^a_{nk} = X_{nk}$ if $-a < X_{nk} \le a$ and otherwise let $X^a_{nk} = 0$. That is

$$
X^a_{nk} = \begin{cases} \dfrac{1}{n\sin X_k} & \text{if } X_k < \sin^{-1}\left(-\dfrac{1}{na}\right) \text{ or } \sin^{-1}\left(\dfrac{1}{na}\right) \le X_k \\ 0 & \text{if } \sin^{-1}\left(-\dfrac{1}{na}\right) \le X_k < \sin^{-1}\left(\dfrac{1}{na}\right). \end{cases}
$$

Let F^a_{nk}, φ^a_{nk}, $\mu_{nk}(a)$ and $\sigma^2_{nk}(a)$ be the distribution function, characteristic function, mean and variance of X^a_{nk} respectively.

Lemma 3.2

$$
F^a_{nk}(x) = \begin{cases} 0 & if\ x \le -a \\ H\Big(\sin^{-1}\left(-\dfrac{1}{na}\right)\Big) - H\Big(\sin^{-1}\left(\dfrac{1}{nx}\right)\Big) & if\ -a < x < 0 \\ H\Big(\sin^{-1}\left(\dfrac{1}{na}\right)\Big) & if\ x = 0 \\ H\Big(\sin^{-1}\left(\dfrac{1}{na}\right)\Big) + 1 - H\Big(\sin^{-1}\left(\dfrac{1}{nx}\right)\Big) & if\ 0 < x \le a \\ 1 & if\ x > a. \end{cases}
$$

Proof.

Case 1. $x \le -a$.

$$
\begin{aligned}
F_{nk}^a(x) &= P(X_{nk}^a \le x) \\
&= P\Big[(\frac{1}{n\sin X_k} \le x) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big] \\
&= P\Big[(\frac{1}{nx} \le \sin X_k < 0) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big] \\
&= P(\phi) \\
&= 0.
\end{aligned}
$$

Case 2. $-a < x < 0$.

Note that $\quad F_{nk}^a(x) = P(X_{nk}^a \le -a) + P(-a < X_{nk}^a \le x)$

$$= 0 + P(-a < X_{nk}^a \le x) \text{ and}$$

$$
\begin{aligned}
P(-a < X_{nk}^a \le x) &= P\Big[(-na < \frac{1}{\sin X_k} \le nx) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \\
&\qquad \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big] \\
&= P\Big[\sin^{-1}\Big(\frac{1}{nx}\Big) \le X_k < \sin^{-1}\Big(-\frac{1}{na}\Big)\Big] \\
&= H\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) - H\Big(\sin^{-1}\Big(\frac{1}{nx}\Big)\Big).
\end{aligned}
$$

Hence $\quad F_{nk}^a(x) = H\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) - H\Big(\sin^{-1}\Big(\frac{1}{nx}\Big)\Big).$

Case 3. $x = 0$.

$$
\begin{aligned}
F_{nk}^a(x) &= P(X_{nk}^a \le 0) \\
&= P(X_{nk}^a \le -a) + P(-a < X_{nk}^a < 0) + P(X_{nk}^a = 0) \\
&= 0 + P(-a < X_{nk}^a < 0) + P(\sin^{-1}\Big(-\frac{1}{na}\Big) \le X_k < \sin^{-1}\Big(\frac{1}{na}\Big)) \\
&= P(-a < X_{nk}^a < 0) + H\Big(\sin^{-1}\Big(\frac{1}{na}\Big)\Big) - H\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) \\
&= H\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) + H\Big(\sin^{-1}\Big(\frac{1}{na}\Big)\Big) - H\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) \\
&= H\Big(\sin^{-1}\Big(\frac{1}{na}\Big)\Big).
\end{aligned}
$$

Case 4. $0 < x \le a$.

$$
\begin{aligned}
&F^a_{nk}(x)\\
&= F^a_{nk}(0) + P(0 < X^a_{nk} \le x)\\
&= F^a_{nk}(0) + P\Big[(0 < \frac{1}{n\sin X_k} \le x) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big]\\
&= F^a_{nk}(0) + P\Big[(\frac{1}{nx} \le \sin X_k) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big]\\
&= H\Big(\sin^{-1}\Big(\frac{1}{na}\Big)\Big) + P\Big(X_k \ge \sin^{-1}\Big(\frac{1}{nx}\Big)\Big)\\
&= H\Big(\sin^{-1}\Big(\frac{1}{na}\Big)\Big) + 1 - H\Big(\sin^{-1}\Big(\frac{1}{nx}\Big)\Big).
\end{aligned}
$$

Case 5. $x > a$.

$$
\begin{aligned}
&F^a_{nk}(x)\\
&= P(X^a_{nk} \le a) + P(a < X^a_{nk} \le x)\\
&= 1 + P\Big[(na < \frac{1}{\sin X_k} \le nx) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big]\\
&= 1 + P\Big[(\frac{1}{nx} \le \sin X_k < \frac{1}{na}) \cap (X_k < \sin^{-1}\Big(-\frac{1}{na}\Big) \text{ or } \sin^{-1}\Big(\frac{1}{na}\Big) \le X_k)\Big]\\
&= 1 + P(\phi)\\
&= 1.
\end{aligned}
$$

□

Let $S_n = X_{n1} + X_{n2} + \cdots + X_{nn}$ and $S^a_n = X^a_{n1} + X^a_{n2} + \cdots + X^a_{nn}$. Let φ_n be the characteristic function of S_n and F^a_n, φ^a_n, $\mu_n(a)$ and $\sigma^2_n(a)$ denote respectively the distribution function, characteristic function, mean and variance of S^a_n.

To bound $|F_n(x) - F(x)|$, we note that

$$|F_n(x) - F(x)| \le |F_n(x) - F^a_n(x)| + |F^a_n(x) - F(x)|. \qquad (3.1)$$

Lemma 3.3 $|F_n(x) - F_n^a(x)| \leq \sum_{k=1}^{n} \Big\{ F_{nk}(-a) + 1 - F_{nk}(a) \Big\}.$

Proof. See Boonyasombut and Shapiro(1970), p.240. □

In what following, we give a bound of each term on the right of (3.1). The following lemma gives a bound of the first term.

Lemma 3.4 $\sup_{-\infty<x<\infty} |F_n(x) - F_n^a(x)| \leq \frac{C}{a}.$

Proof. By Lemma 3.1 and Lemma 3.3 we see that

$$\begin{aligned}
|F_n(x) - F_n^a(x)| &\leq n(F_{nk}(-a) + 1 - F_{nk}(a)) \\
&= n\Big(H\Big(\sin^{-1}\Big(\frac{1}{na}\Big)\Big) - H\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big)\Big) \\
&= n\int_{\sin^{-1}(-\frac{1}{na})}^{\sin^{-1}(\frac{1}{na})} f(x)\,dx \\
&= n\int_{-\frac{1}{na}}^{\frac{1}{na}} \frac{f(\sin^{-1}u)}{\sqrt{1-u^2}}\,du \\
&\leq \frac{nB}{\sqrt{1-\left(\frac{1}{na}\right)^2}}\int_{-\frac{1}{na}}^{\frac{1}{na}} du \\
&= \frac{2nB}{\sqrt{(na)^2-1}} \\
&< \frac{2nB}{\sqrt{(na)^2-\frac{(na)^2}{2}}} \\
&= \frac{C}{a}.
\end{aligned}$$

□

To give a bound of the second term on the right of (3.1) we need the following construction.

According to chapter II, we know that Levy-Khinchine's formula of the characteristic function φ of F is defined by

$$\log\varphi(t) = \int_{-\infty}^{\infty}\Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big)\frac{1+u^2}{u^2}dG(u)$$

where $G(u) = f(0)\tan^{-1} u + \dfrac{\pi f(0)}{2}$. We define

$$G^a(u) = \begin{cases} 0 & \text{if} \quad u \le -a \\ G(u) - G(-a) & \text{if} \quad -a < u \le a \\ G(a) - G(-a) & \text{if} \quad u > a\,, \end{cases}$$

$$\gamma^a = -\int_{|u|>a} \frac{1}{u}\, dG(u),$$

$$\mu(a) = \gamma^a + \int_{-\infty}^{\infty} u\, dG^a(u)$$

and
$$K^a(u) = \int_{-\infty}^{u} (1+x^2)\, dG^a(x).$$

We also let F^a be an infinitely divisible distribution function whose logarithm of its characteristic function is given by

$$\log \varphi^a(t) = i\mu(a)t + \int_{-\infty}^{\infty} f(t,x)\, dK^a(x)$$

where

$$f(t,x) = \begin{cases} (e^{itx} - 1 - itx)\dfrac{1}{x^2} & \text{if} \ \ x \neq 0 \\ -\dfrac{t^2}{2} & \text{if} \ \ x = 0. \end{cases}$$

and let

$$K_n^a(u) = n\int_{-\infty}^{u} x^2 dF_{nk}^a(x + \mu_{nk}(a))$$

For $0 < \delta \le 2a$, define

$$m \equiv m(a,\delta) = \left[\frac{2a}{\delta}\right] + 1$$

where $[x]$ is the greatest integer less than or equal to x.

Let $-a = x_0 < x_1 < \ldots < x_m = a$ be such that

$$\max_{1\le i\le m} (x_i - x_{i-1}) < \delta.$$

Theorem 3.5 $\sup_{-\infty<x<\infty} |F_n^a(x) - F(x)| \le Cg^a(n, m(a,\delta), r)$ (3.2) *when* $0 \le \sigma^2_{nk}(a) \le 1$ *for large* n *and* $k = 1, 2, ..., n$ *and*

$$g^a(n, m(a,\delta), r) = \left[\frac{1}{3}\sigma^2_n(a) \max_{1\le k\le n} \sigma^2_{nk}(a)\right]^{\frac{1}{5}} + \left[\frac{1}{2}\sum_{i=0}^{m} |K^a_n(x_i) - K^a(x_i)|\right]^{\frac{1}{3}}$$

$$+ \left[\frac{5}{6}\delta(\sigma^2_n(a) + \sigma^2(a))\right]^{\frac{1}{4}}$$

$$+ \left\{\frac{4}{a}[K^a_n(\infty) - K^a_n(a) + K^a(\infty) - K^a(a) + K^a_n(-a)\right.$$

$$\left. + K^a(-a)] + 2|\mu_n(a) - \mu(a)|\right\}^{\frac{1}{2}} + \left[\frac{8}{r}\int_{|u|>a} |u|^r dG(u)\right]^{\frac{1}{1+r}}$$

for any positive $r \in (0, 1)$.

Proof. See Boonyasombut and Shapiro(1970), p.240. □

Theorem 3.6 *For large* n *we have a constant* C *such that*

$$\sup_{-\infty<x<\infty} |F^a_n(x) - F(x)| \le Ch^a(n, m(a,\delta), r)$$

where

$$h^a(n, m(a,\delta), r) = \left[\frac{4}{3n}B^2a^2\right]^{\frac{1}{5}} + \left[\frac{1}{2}\sum_{i=0}^{m} |K^a_n(x_i) - K^a(x_i)|\right]^{\frac{1}{3}} + \left[\frac{5}{3}\delta a(B + f(0))\right]^{\frac{1}{4}}$$

$$+ \left\{\frac{4}{a}[K^a_n(\infty) - K^a_n(a) + K^a_n(-a)]\right\}^{\frac{1}{2}} + \left[\frac{16f(0)}{r(1-r)a^{1-r}}\right]^{\frac{1}{1+r}}.$$

Proof.

First, we will show that (3.2) holds. To show this it suffices to prove that $0 \le \sigma^2_{nk}(a) \le 1$ for large n and $k = 1, 2, ..., n$.

Note that

$$\begin{aligned} \mu_{nk}(a) &= \int_{-\frac{\pi}{2}}^{\sin^{-1}(-\frac{1}{na})} \frac{1}{n \sin x} f(x)\, dx + \int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{1}{n\sin x} f(x)\, dx \\ &= -\int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{f(-x)}{n \sin x}\, dx + \int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{f(x)}{n \sin x}\, dx \\ &= 0 \end{aligned}$$

where we have used the fact that f is symmetric in the last equality.

Hence

$$\begin{aligned}
\sigma_{nk}^2(a) &= E((X_{nk}^a)^2) - (\mu_{nk}(a))^2 \\
&= \int_{-\frac{\pi}{2}}^{\sin^{-1}(-\frac{1}{na})} \frac{1}{(n\sin x)^2} f(x)\,dx + \int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{1}{(n\sin x)^2} f(x)\,dx \\
&= \int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{f(-x)}{n^2(-\sin x)^2}\,dx + \int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{f(x)}{n^2(\sin x)^2}\,dx \\
&= 2\int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \frac{f(x)}{n^2(\sin x)^2}\,dx \\
&\le \frac{2B}{n^2}\int_{\sin^{-1}(\frac{1}{na})}^{\frac{\pi}{2}} \csc^2 x\,dx \\
&= \frac{2Ba}{n}\cos\left(\sin^{-1}\left(\frac{1}{na}\right)\right) \\
&\le \frac{2Ba}{n}
\end{aligned}$$

i.e, $0 \le \sigma_{nk}^2(a) \le 1$ for large n and $k = 1, 2, ..., n$. So (3.2) holds.

In our case, we see that

$$\sigma_n^2(a) = \sigma_{n1}^2(a) + \cdots + \sigma_{nn}^2(a) \le 2Ba,$$

$$\gamma^a = -\int_{|u|>a} \frac{1}{u}\,dG(u) = -\int_{|u|>a} \frac{f(0)}{u(1+u^2)}\,du = 0,$$

$$\begin{aligned}
\mu(a) &= \gamma^a + \int_{-\infty}^{\infty} u\,dG^a(u) \\
&= \int_{-\infty}^{-a} u\,dG^a(u) + \int_{-a}^{a} u\,dG^a(u) + \int_{a}^{\infty} u\,dG^a(u) \\
&= \int_{-a}^{a} \frac{uf(0)}{(1+u^2)}\,du \\
&= 0,
\end{aligned}$$

$$\mu_n(a) = \mu_{n1}(a) + \cdots + \mu_{nn}(a) = 0,$$

$$K^a(u) = \int_{-\infty}^{u} (1+x^2)\, dG^a(x)$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{u} (1+x^2)\, dG^a(x) & \text{if } \ u < -a \\[2ex] \displaystyle\int_{-\infty}^{-a} (1+x^2)\, dG^a(x) + \int_{-a}^{u} (1+x^2)\, dG^a(x) & \text{if } -a \le u < a \\[2ex] \displaystyle\int_{-\infty}^{-a} (1+x^2)\, dG^a(x) + \int_{-a}^{a} (1+x^2)\, dG^a(x) & \\ \displaystyle + \int_{a}^{u} (1+x^2)\, dG^a(x) & \text{if } u \ge a, \end{cases}$$

$$= \begin{cases} 0 & \text{if } \ u < -a \\[2ex] \displaystyle\int_{-a}^{u} f(0)\, dx & \text{if } -a \le u < a \\[2ex] \displaystyle\int_{-a}^{a} f(0)\, dx & \text{if } u \ge a, \end{cases}$$

$$= \begin{cases} 0 & \text{if } \ u, -a \\ f(0)(u+a) & \text{if } \ -a \le u < a \\ 2f(0)a & \text{if } \ u \ge a, \end{cases}$$

$\sigma^2(a) = K^a(\infty) = 2f(0)a$ and

$$\int_{|u|>a} |u|^r dG(u) = 2f(0)\int_a^{\infty} \frac{u^r}{1+u^2}\, du \le 2f(0)\int_a^{\infty} u^{r-2}\, du = \frac{2f(0)}{(1-r)a^{1-r}}.$$

Then we have,

$$
\begin{aligned}
g^a(n,m(a,\delta),r) &= \Big[\frac{1}{3}\sigma_n^2(a)\max_{1\le k\le n}\sigma_{nk}^2(a)\Big]^{\frac{1}{5}} + \Big[\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i)-K^a(x_i)|\Big]^{\frac{1}{3}} \\
&\quad + \Big[\frac{5}{6}\delta(\sigma_n^2(a)+\sigma^2(a))\Big]^{\frac{1}{4}} \\
&\quad + \Big\{\frac{4}{a}[K_n^a(\infty)-K_n^a(a)+K^a(\infty)-K^a(a)+K_n^a(-a) \\
&\quad + K^a(-a)] + 2|\mu_n(a)-\mu(a)|\Big\}^{\frac{1}{2}} + \Big[\frac{8}{r}\int_{|u|>a}|u|^r dG(u)\Big]^{\frac{1}{1+r}} \\
&\le \Big[\frac{1}{3}(2Ba)\Big(\frac{2Ba}{n}\Big)\Big]^{\frac{1}{5}} + \Big[\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i)-K^a(x_i)|\Big]^{\frac{1}{3}} \\
&\quad + \Big[\frac{5}{6}\,\delta(2Ba+2f(0)a)\Big]^{\frac{1}{4}} \\
&\quad + \Big\{\frac{4}{a}[K_n^a(\infty)-K_n^a(a)+2f(0)a-2f(0)a+K_n^a(-a)\Big\}^{\frac{1}{2}} \\
&\quad + \Big[\frac{16f(0)}{r(1-r)a^{1-r}}\Big]^{\frac{1}{1+r}} \\
&= \Big[\frac{4}{3n}B^2a^2\Big]^{\frac{1}{5}} + \Big[\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i)-K^a(x_i)|\Big]^{\frac{1}{3}} + \Big[\frac{5}{3}\delta a(B+f(0))\Big]^{\frac{1}{4}} \\
&\quad + \Big\{\frac{4}{a}[K_n^a(\infty)-K_n^a(a)+K_n^a(-a)]\Big\}^{\frac{1}{2}} + \Big[\frac{16f(0)}{r(1-r)a^{1-r}}\Big]^{\frac{1}{1+r}}. \\
&= h^a(n,m(a,\delta),r)
\end{aligned}
$$

Hence $\sup_{-\infty<x<\infty}|F_n^a(x)-F(x)| \le Ch^a(n,m(a,\delta),r)$ □

Next lemma, we will bound the second and the forth terms of $h^a(n,m(a,\delta),r)$.

Lemma 3.7

1. $K_n^a(\infty)-K_n^a(a)+K_n^a(-a)=0$ *and*

2. $|K_n^a(x_i)-K^a(x_i)| \le \frac{C}{n}\ln na$ *for any i=0,1,2,...,m.*

Proof.

By Lemma 3.2 and the fact that $\mu_{nk}(a)=0$ we see that

$$K_n^a(u) = n\int_{-\infty}^{u} x^2 dF_{nk}^a(x)$$

$$= \begin{cases} 0 & \text{if} \quad u \le -a \\ \displaystyle\int_{-a}^{u} \frac{f(\sin^{-1}\left(\frac{1}{nx}\right))}{\sqrt{1-\left(\frac{1}{nx}\right)^2}}\,dx & \text{if} \quad -a < u < a \\ \displaystyle\int_{-a}^{a} \frac{f(\sin^{-1}\left(\frac{1}{nx}\right))}{\sqrt{1-\left(\frac{1}{nx}\right)^2}}\,dx & \text{if} \quad u \ge a. \end{cases}$$

Hence (1) follows immediately from the formula of K_n^a. Next we will prove (2). By the hypothesis of the main theorem, there exist positive constants ε and C such that $|f'(x)| < C$ for $|x| \le \varepsilon$.

Case 1. $\quad x_i \le -\dfrac{1}{n\sin\varepsilon}$.

By Mean-Value Theorem, for all $u \in \left[\sin^{-1}\left(\frac{1}{nx_i}\right), \sin^{-1}\left(-\frac{1}{na}\right)\right]$ there exists $\xi_u \in (u, 0)$ such that $|f(u) - f(0)| = |f'(\xi_u)(u)| \le C|u|$. Hence

$|K_n^a(x_i) - K^a(x_i)|$

$$= \left|\int_{-a}^{x_i} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\right|$$

$$= \left|\int_{\sin^{-1}(-\frac{1}{na})}^{\sin^{-1}(\frac{1}{nx_i})} \left(\frac{f(u)}{\cos u} - f(0)\right)\left(-\frac{\cos u}{n(\sin u)^2}\right) du\right|$$

$$= \left|\int_{\sin^{-1}(-\frac{1}{na})}^{\sin^{-1}(\frac{1}{nx_i})} -\frac{f(u)}{n(\sin u)^2}\,du + \int_{\sin^{-1}(-\frac{1}{na})}^{\sin^{-1}(\frac{1}{nx_i})} \frac{f(0)\cos u}{n(\sin u)^2}\,du\right|$$

$$= \left|\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{f(u) - f(0)}{n(\sin u)^2}\,du + \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{f(0) - f(0)\cos u}{n(\sin u)^2}\,du\right|$$

$$\le \Big|\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{f(u)-f(0)}{n(\sin u)^2}\,du\Big| + \Big|\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{f(0)(\cos 0-\cos u)}{n(\sin u)^2}\,du\Big|$$

$$\le \frac{C}{n}\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{-u}{(\sin u)^2}\,du + \frac{f(0)}{n}\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{|-\sin\xi||-u|}{(\sin u)^2}\,du$$

for some $\xi \in (u,0)$

$$\le \frac{C}{n}\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{-u}{(\sin u)^2}\,du$$

$$= \frac{C}{n}\Big[-na\sin^{-1}\Big(-\frac{1}{na}\Big)\cos\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) - nx_i\sin^{-1}\Big(\frac{1}{nx_i}\Big)\cos\Big(\sin^{-1}\Big(\frac{1}{nx_i}\Big)\Big)$$

$$-\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \cot u\,du\Big]$$

$$= \frac{C}{n}\Big[-na\sin^{-1}\Big(-\frac{1}{na}\Big)\cos\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big) - nx_i\sin^{-1}\Big(\frac{1}{nx_i}\Big)\cos\Big(\sin^{-1}\Big(\frac{1}{nx_i}\Big)\Big)$$

$$+\ln na - \ln|nx_i|\Big]$$

$$\le \frac{C}{n}\Big[\ln na - na\Big(\sin^{-1}\Big(-\frac{1}{na}\Big)\Big)\Big].$$

From this fact and the fact that

$$\begin{aligned} -na\sin^{-1}\Big(-\frac{1}{na}\Big) &= na\Big(\sin^{-1}(0) - \sin^{-1}\Big(-\frac{1}{na}\Big)\Big) \\ &= \frac{1}{\sqrt{1-\xi^2}} \quad \text{for some } \xi \in (-\frac{1}{na}, 0) \\ &\le 2 \end{aligned}$$

we have

$$|K_n^a(x_i) - K^a(x_i)| < \frac{C}{n}\ln na.$$

Case 2. $\quad -\dfrac{1}{n\sin\varepsilon} < x_i \le \dfrac{1}{n\sin\varepsilon}.$

From the first case we see that

$$\left|\int_{-a}^{-\frac{1}{n\sin\varepsilon}} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\right| \le \frac{C}{n}\ln na$$

and by the fact that $|x_i| < \dfrac{1}{n\sin\varepsilon}$

$$\begin{aligned}
\left|\int_{-\frac{1}{n\sin\varepsilon}}^{x_i} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\right| &\le \left|\int_{-\frac{1}{n\sin\varepsilon}}^{\frac{1}{n\sin\varepsilon}} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}}\,dx\right| \\
&\quad + \left|f(0)\left(\frac{2}{n\sin\varepsilon}\right)\right| \\
&\le B\int_{-\frac{1}{n\sin\varepsilon}}^{\frac{1}{n\sin\varepsilon}} \frac{1}{\sqrt{1-\left(\frac{1}{nx}\right)^2}}\,dx + \frac{2f(0)}{n\sin\varepsilon} \\
&< \frac{4B}{n\sin\varepsilon} + \frac{2f(0)}{n\sin\varepsilon}.
\end{aligned}$$

Hence

$$\begin{aligned}
|K_n^a(x_i) - K^a(x_i)| &\le \left|\int_{-a}^{-\frac{1}{n\sin\varepsilon}} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\right| \\
&\quad + \left|\int_{-\frac{1}{n\sin\varepsilon}}^{x_i} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\right| \\
&< \frac{C}{n}\ln na.
\end{aligned}$$

Case 3. $\quad x_i > \dfrac{1}{n\sin\varepsilon}.$

From the second case we see that

$$\left|\int_{-a}^{\frac{1}{n\sin\varepsilon}} \frac{f\left(\sin^{-1}\left(\frac{1}{nx}\right)\right)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\right| \le \frac{C}{n}\ln na$$

and by Mean-Value Theorem, for all $u \in \left[\sin^{-1}\left(\frac{1}{nx_i}\right), \varepsilon\right]$ there exists $\xi_u \in (0,u)$ such that $|f(u) - f(0)| = |f'(\xi_u)(u)| \le C|u|$, we have

$$\Big|\int_{\frac{1}{n\sin\varepsilon}}^{x_i} \frac{f\Big(\sin^{-1}\left(\frac{1}{nx}\right)\Big)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\Big| = \Big|\int_{\varepsilon}^{\sin^{-1}(\frac{1}{nx_i})} \Big(\frac{f(u)}{\cos u} - f(0)\Big)\Big(-\frac{\cos u}{n(\sin u)^2}\Big)\,du\Big|$$

$$\leq \Big|\int_{\sin^{-1}(\frac{1}{nx_i})}^{\varepsilon} \frac{f(u)-f(0)}{n(\sin u^2)}\,du\Big|$$

$$+ \Big|\int_{\sin^{-1}(\frac{1}{nx_i})}^{\varepsilon} \frac{f(0)-f(0)\cos u}{n(\sin u)^2}\,du\Big|$$

$$\leq \frac{C}{n}\int_{\sin^{-1}(\frac{1}{nx_i})}^{\varepsilon} \frac{u}{(\sin u)^2}\,du$$

$$= \frac{C}{n}\Big[nx_i \sin^{-1}\left(\frac{1}{nx_i}\right)\cos\left(\sin^{-1}\left(\frac{1}{nx_i}\right)\right)$$

$$-\varepsilon\cot(\varepsilon) + \int_{\sin^{-1}(\frac{1}{nx_i})}^{\varepsilon} \cot u\,du\Big]$$

$$= \frac{C}{n}\Big[nx_i \sin^{-1}\left(\frac{1}{nx_i}\right)\cos\left(\sin^{-1}\left(\frac{1}{nx_i}\right)\right)$$

$$-\varepsilon\cot(\varepsilon) + \ln|\sin\varepsilon| + \ln nx_i\Big]$$

$$\leq \frac{C}{n}\Big[nx_i \sin^{-1}\left(\frac{1}{nx_i}\right) - \varepsilon\cot(\varepsilon) + \ln|\sin\varepsilon|$$

$$+\ln na\Big].$$

From this and the fact that

$$nx_i \sin^{-1}\left(\frac{1}{nx_i}\right) = nx_i\Big(\sin^{-1}\left(\frac{1}{nx_i}\right) - \sin^{-1}(0)\Big)$$

$$= \frac{1}{\sqrt{1-\xi^2}} \text{ for some } \xi \in (0, \frac{1}{nx_i})$$

$$< \frac{1}{\sqrt{1-\left(\frac{1}{nx_i}\right)^2}}$$

$$\leq \frac{1}{\cos\varepsilon},$$

we have

$$\Big|\int_{\frac{1}{n\sin\varepsilon}}^{x_i} \frac{f\Big(\sin^{-1}\left(\frac{1}{nx}\right)\Big)}{\sqrt{1-\left(\frac{1}{nx}\right)^2}} - f(0)\,dx\Big| < \frac{C}{n}\ln na.$$

Hence

$$|K_n^a(x_i) - K^a(x_i)| \leq \Big|\int_{-a}^{\frac{1}{n\sin\varepsilon}} \frac{f\Big(\sin^{-1}\Big(\frac{1}{nx}\Big)\Big)}{\sqrt{1-\Big(\frac{1}{nx}\Big)^2}} - f(0)\,dx\Big|$$

$$+\Big|\int_{\frac{1}{n\sin\varepsilon}}^{x_i} \frac{f\Big(\sin^{-1}\Big(\frac{1}{nx}\Big)\Big)}{\sqrt{1-\Big(\frac{1}{nx}\Big)^2}} - f(0)\,dx\Big|$$

$$\leq \frac{C}{n}\ln na.$$

So we have the lemma in this case. □

Proof of the Main Theorem

From Theorem 3.6 and Lemma 3.7 we see that

$$h^a(n, m(a,\delta), r) \leq C\Big[\Big(\frac{a^2}{n}\Big)^{\frac{1}{5}} + \Big\{\Big(\frac{m+1}{n}\Big)\ln na\Big\}^{\frac{1}{3}} + (\delta a)^{\frac{1}{4}} + C(r)\Big(\frac{1}{a}\Big)^{\frac{1-r}{1+r}}\Big]$$

which implies

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leq C\Big[\frac{1}{a} + \Big(\frac{a^2}{n}\Big)^{\frac{1}{5}} + \Big\{\Big(\frac{m+1}{n}\Big)\ln na\Big\}^{\frac{1}{3}} + (\delta a)^{\frac{1}{4}}$$

$$+C(r)\Big(\frac{1}{a}\Big)^{\frac{1-r}{1+r}}\Big] \text{ for } a \geq 1 \text{ and } r \in (0,1) \tag{3.3}$$

where $C(r)$ is a constants depending on r.

We set $a = n^{k_1}$ and $\delta = n^{-k_2}$ where k_1 and k_2 are positive numbers, so

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leq C\Big[\frac{1}{n^{k_1}} + \Big(\frac{1}{n^{1-2k_1}}\Big)^{\frac{1}{5}} + \Big(\frac{(1+k_1)\ln n}{n^{1-k_1-k_2}}\Big)^{\frac{1}{3}} + \Big(\frac{1}{n^{k_2-k_1}}\Big)^{\frac{1}{4}}$$

$$+ C(r)\Big(\frac{1}{n^{k_1}}\Big)^{\frac{1-r}{1+r}}\Big].$$

Then we see that the order of the bounds in (3.3), are

$$\frac{1}{n^{k_1}}, \frac{1}{n^{\frac{1-2k_1}{5}}}, \frac{(1+k_1)\ln n}{n^{(\frac{1-k_1-k_2}{3})}}, \frac{1}{n^{\frac{k_2-k_1}{4}}}, C(r)\frac{1}{n^{\frac{k_1(1-r)}{1+r}}}. \tag{3.4}$$

Let $k_1 = \frac{1}{9}$ and $k_2 = \frac{5}{9}$ then we have that they gives the maximum value of $\frac{1}{9}$ for the first of four terms in (3.4). For $0 < \alpha < \frac{1}{9}$ the choice of $r = \frac{9\alpha}{2-9\alpha}$ implies that $k_1 \cdot \dfrac{1-r}{1+r} = \dfrac{1}{9} - \alpha$. We have that for any constant $0 < d < \dfrac{1}{9}$, there exists a constant C such that

$$\sup_{-\infty<x<\infty} [F_n(x) - F(x)] < \frac{C}{n^d}.$$

CHAPTER IV

Rate of convergence of distribution function of sums of the reciprocals of logarithm of absolutely independent continuous random variables to the Cauchy Distribution function

The purpose of this chapter is to find the rate of convergence of the sequence of distribution function of $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\ln|X_k|}$ to the Cauchy distribution function. The main theorem is the following.

Theorem *Let* $(X_n), n = 1, 2, 3, \ldots$ *be a sequence of independent identically distributed continuous random variables with common symmetric probability density function* f *and distribution function* H*. Assume that*

1. $f(-1)$ *and* $f(1) > 0$ *and there exists a positive constant* B *such that* $|f(x)| < B$ *for every* x*,*

2. f' *exists and is bounded in some neighborhood of* 1*,*

3. $(H \circ exp)'$ *is symmetric.*

Then for any fixed $0 < d < \frac{1}{9}$ *there exists a constant* C *such that*

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{C}{n^d}$$

where F_n *is the distribution function of* $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\ln|X_k|}$ *and* F *is the Cauchy distribution function defined by*

$$F(x) = \frac{1}{\pi}\Big[\frac{\pi}{2} + \arctan\Big(\frac{x}{\pi(f(-1)+f(1))}\Big)\Big].$$

Since X_k is continuous, we may assume that $ImX_k \subseteq \mathbb{R} - \{0, \pm 1\}$. To proof the theorem, we use the same idea of the main theorem in chapter III.

For $k = 1, 2, \ldots, n, n = 1, 2, \ldots$, let $X_{nk} = \dfrac{1}{n \ln |X_k|}$ and let F_{nk} be the distribution function of X_{nk}. To proof the main theorem we need the following 4 lemmas and 1 theorem (Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.5 and Theorem 4.4).

Lemma 4.1

$$F_{nk}(x) = \begin{cases} H(1) - H(-1) + H(-e^{\frac{1}{nx}}) - H(e^{\frac{1}{nx}}) & if\ x < 0 \\ H(1) - H(-1) & if\ x = 0 \\ H(1) - H(-1) + 1 + H(-e^{\frac{1}{nx}}) - H(e^{\frac{1}{nx}}) & if\ x > 0. \end{cases}$$

Proof. Let x be any real number.

Case 1. $x = 0$.

$$\begin{aligned} F_{nk}(0) &= P(X_{nk} \leq 0) \\ &= P(\ln |X_k| < 0) \\ &= P(|X_k| < 1) \\ &= P(-1 < X_k < 1) \\ &= H(1) - H(-1). \end{aligned}$$

Case 2. $x < 0$.

$$\begin{aligned} F_{nk}(x) &= P(\frac{1}{n \ln |X_k|} \leq x) \\ &= P(\frac{1}{nx} \leq \ln |X_k| < 0) \\ &= P(e^{\frac{1}{nx}} \leq |X_k| < 1) \\ &= P(e^{\frac{1}{nx}} \leq X_k < 1) + P(-1 < X_k \leq -e^{\frac{1}{nx}}) \\ &= H(1) - H(e^{\frac{1}{nx}}) + H(-e^{\frac{1}{nx}}) - H(-1). \end{aligned}$$

Case 3. $x > 0$.

$$\begin{aligned} F_{nk}(x) &= F_{nk}(0) + P(0 < \frac{1}{n\ln|X_k|} \le x) \\ &= F_{nk}(0) + P(\frac{1}{nx} \le \ln|X_k|) \\ &= F_{nk}(0) + P(e^{\frac{1}{nx}} \le |X_k|) \\ &= F_{nk}(0) + P(X_k \ge e^{\frac{1}{nx}}) + P(X_k \le -e^{\frac{1}{nx}}) \\ &= H(1) - H(-1) + 1 - H(e^{\frac{1}{nx}}) + H(-e^{\frac{1}{nx}}). \end{aligned}$$

□

For $a \ge 1$ we define $X^a_{nk} = X_{nk}$ if $-a < X_{nk} \le a$ and otherwise let $X^a_{nk} = 0$. That is

$$X^a_{nk} = \begin{cases} \dfrac{1}{n\ln|X_k|} & \text{if} \quad X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k \\ 0 & \text{if} \quad -e^{\frac{1}{na}} < X_k \le -e^{-\frac{1}{na}} \text{ or } e^{-\frac{1}{na}} \le X_k < e^{\frac{1}{na}}. \end{cases}$$

Let F^a_{nk}, φ^a_{nk}, $\mu_{nk}(a)$ and $\sigma^2_{nk}(a)$ be the distribution function, characteristic function, mean and variance of X^a_{nk} respectively.

Lemma 4.2

$$F^a_{nk}(x) = \begin{cases} 0 & if \;\; x \le -a \\ H(-e^{\frac{1}{nx}}) - H(-e^{-\frac{1}{na}}) + H(e^{-\frac{1}{na}}) - H(e^{\frac{1}{nx}}) & if \;\; -a < x < 0 \\ H(e^{\frac{1}{na}}) - H(-e^{\frac{1}{na}}) & if \;\; x = 0 \\ H(e^{\frac{1}{na}}) - H(-e^{\frac{1}{na}}) + H(-e^{\frac{1}{nx}}) + 1 - H(e^{\frac{1}{nx}}) & if \;\; 0 < x \le a \\ 1 & if \;\; x > a. \end{cases}$$

Proof.

Case 1. $x \le -a$.

$F^a_{nk}(x)$

$= P(X^a_{nk} \le x)$

$= P\Big((\dfrac{1}{n \ln |X_k|} \le x) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P\Big((\dfrac{1}{nx} \le \ln |X_k| < 0) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P\Big((e^{\frac{1}{nx}} \le |X_k| < 1) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P\Big((e^{\frac{1}{nx}} \le X_k < 1) \cup (-1 < X_k \le -e^{\frac{1}{nx}}) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}}$

$\text{or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P(\phi)$

$= 0.$

Case 2. $-a < x < 0$.

$F^a_{nk}(x)$

$= P(X^a_{nk} \le -a) + P(-a < X^a_{nk} \le x)$

$= 0 + P(-a < X^a_{nk} \le x)$

$= P\Big((-na < \dfrac{1}{\ln |X_k|} \le nx) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P\Big((e^{\frac{1}{nx}} \le |X_k| < e^{-\frac{1}{na}}) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P\Big((e^{\frac{1}{nx}} \le X_k < e^{-\frac{1}{na}}) \cup (-e^{-\frac{1}{na}} < X_k \le -e^{\frac{1}{nx}}) \cap (X_k \le -e^{\frac{1}{na}} \text{ or }$

$-e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k) \Big)$

$= P(-e^{-\frac{1}{na}} < X_k \le -e^{\frac{1}{nx}}) + P(e^{\frac{1}{nx}} \le X_k < e^{-\frac{1}{na}})$

$= H(-e^{\frac{1}{nx}}) - H(-e^{-\frac{1}{na}}) + H(e^{-\frac{1}{na}}) - H(e^{\frac{1}{nx}}).$

Case 3. $x = 0$.

$$
\begin{aligned}
&F^a_{nk}(x)\\
&= P(X^a_{nk} \leq 0)\\
&= P(X^a_{nk} \leq -a) + P(-a < X^a_{nk} < 0) + P(X^a_{nk} = 0)\\
&= 0 + P(-a < X^a_{nk} < 0) + P(-e^{\frac{1}{na}} < X_k \leq -e^{-\frac{1}{na}}) + P(e^{-\frac{1}{na}} \leq X_k < e^{\frac{1}{na}})\\
&= P(-a < X^a_{nk} < 0) + H(-e^{-\frac{1}{na}}) - H(-e^{\frac{1}{na}}) + H(e^{\frac{1}{na}}) - H(e^{-\frac{1}{na}})\\
&= H(e^{-\frac{1}{na}}) - H(-e^{-\frac{1}{na}}) + H(-e^{-\frac{1}{na}}) - H(-e^{\frac{1}{na}}) + H(e^{\frac{1}{na}}) - H(e^{-\frac{1}{na}})\\
&= H(e^{\frac{1}{na}}) - H(-e^{\frac{1}{na}}).
\end{aligned}
$$

Case 4. $0 < x \leq a$.

$$
\begin{aligned}
&F^a_{nk}(x)\\
&= F^a_{nk}(0) + P(0 < X^a_{nk} \leq x)\\
&= F^a_{nk}(0) + P\Big((0 < \frac{1}{n \ln |X_k|} \leq x) \cap (X_k \leq -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}}\\
&\quad \text{or } e^{\frac{1}{na}} \leq X_k)\Big)\\
&= F^a_{nk}(0) + P\Big((|X_k| \geq e^{\frac{1}{nx}}) \cap (X_k \leq -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \leq X_k)\Big)\\
&= F^a_{nk}(0) + P\Big((X_k \leq -e^{\frac{1}{nx}}) \cup (X_k \geq e^{\frac{1}{nx}}) \cap (X_k \leq -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}}\\
&\quad \text{or } e^{\frac{1}{na}} \leq X_k)\Big)\\
&= F^a_{nk}(0) + P(X_k \leq -e^{\frac{1}{nx}}) + P(X_k \geq e^{\frac{1}{nx}})\\
&= H(e^{\frac{1}{na}}) - H(-e^{\frac{1}{na}}) + H(-e^{\frac{1}{nx}}) + 1 - H(e^{\frac{1}{nx}}).
\end{aligned}
$$

Case 5. $x > a$

$$
\begin{aligned}
&F_{nk}^a(x)\\
&= P(X_{nk}^a \le a) + P(a < X_{nk}^a \le x)\\
&= 1 + P\Big((na < \frac{1}{\ln|X_k|} \le nx) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or }\\
&\quad e^{\frac{1}{na}} \le X_k)\Big)\\
&= 1 + P\Big((e^{\frac{1}{nx}} \le |X_k| < e^{\frac{1}{na}}) \cap (X_k \le -e^{\frac{1}{na}} \text{ or } -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k)\Big)\\
&= 1 + P\Big((e^{\frac{1}{nx}} \le X_k < e^{\frac{1}{na}}) \cup (-e^{\frac{1}{na}} < X_k \le -e^{\frac{1}{nx}}) \cap (X_k \le -e^{\frac{1}{na}} \text{ or }\\
&\quad -e^{-\frac{1}{na}} < X_k < e^{-\frac{1}{na}} \text{ or } e^{\frac{1}{na}} \le X_k)\Big)\\
&= 1 + P(\phi)\\
&= 1.
\end{aligned}
$$

□

Let $S_n = X_{n1} + X_{n2} + \cdots + X_{nn}$ and $S_n^a = X_{n1}^a + X_{n2}^a + \cdots + X_{nn}^a$. Let φ_n be the characteristic function of S_n and F_n^a, φ_n^a, $\mu_n(a)$ and $\sigma_n^2(a)$ denote respectively the distribution function, characteristic function, mean and variance of S_n^a.
To bound $|F_n(x) - F(x)|$, we note that

$$|F_n(x) - F(x)| \le |F_n(x) - F_n^a(x)| + |F_n^a(x) - F(x)|. \tag{4.1}$$

In what following, we give a bound of each term on the right of (4.1). The following lemma gives a bound of the first term.

Lemma 4.3 $\sup\limits_{-\infty<x<\infty} |F_n(x) - F_n^a(x)| \le \dfrac{C}{a}$.

Proof. By Lemma 3.3 and Lemma 4.1 we see that

$$
\begin{aligned}
|F_n(x) - F_n^a(x)| &\le n(F_{nk}(-a) + 1 - F_{nk}(a)) \\
&= n\Big(H(e^{\frac{1}{na}}) - H(e^{-\frac{1}{na}}) + H(-e^{-\frac{1}{na}}) - H(-e^{\frac{1}{na}})\Big) \\
&= n\Big(\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{na}}} f(x)\,dx + \int_{-e^{\frac{1}{na}}}^{-e^{-\frac{1}{na}}} f(x)\,dx\Big) \\
&= n\Big(\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{na}}} f(x)\,dx + \int_{e^{\frac{1}{na}}}^{e^{-\frac{1}{na}}} -f(-x)\,dx\Big) \\
&= 2n\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{na}}} f(x)\,dx \\
&= 2n\int_{\frac{-1}{na}}^{\frac{1}{na}} f(e^u)e^u\,du \\
&\le 2nBe\int_{\frac{-1}{na}}^{\frac{1}{na}} du \\
&= \frac{C}{a}.
\end{aligned}
$$

□

To give a bound of the second term on the right of (4.1) we have used the same technique of chapter 3 where $G(u) = (f(1) + f(-1))\tan^{-1} u + \frac{\pi}{2}(f(1) + f(-1))$.

Theorem 4.4 *For large n we have a constant C such that*

$$\sup_{-\infty<x<\infty} |F_n^a(x) - F(x)| \le Ch^a(n, m(a,\delta), r)$$

where

$$
\begin{aligned}
h^a(n, m(a,\delta), r) = {}& \Big[\frac{16}{3n}(a(eB+1))^2\Big]^{\frac{1}{5}} + \Big[\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i) - K^a(x_i)|\Big]^{\frac{1}{3}} \\
&+ \Big[\frac{5}{3}\delta a(2(eB+1) + (f(-1) + f(1)))\Big]^{\frac{1}{4}} \\
&+ \Big\{\frac{4}{a}[K_n^a(\infty) - K_n^a(a) + K_n^a(-a)]\Big\}^{\frac{1}{2}} + \Big[\frac{16(f(-1) + f(1))}{r(1-r)a^{1-r}}\Big]^{\frac{1}{1+r}}.
\end{aligned}
$$

Proof.

First, we will show that Theorem 3.5 holds. To show this it suffices to prove that $0 \le \sigma_{nk}^2(a) \le 1$ for large n and $k = 1, 2, ..., n$. Note that

$$\mu_{nk}(a) = \int_{-\infty}^{-e^{\frac{1}{na}}} \frac{1}{n\ln|x|} f(x)\,dx + \int_{-e^{-\frac{1}{na}}}^{e^{-\frac{1}{na}}} \frac{1}{n\ln|x|} f(x)\,dx + \int_{e^{\frac{1}{na}}}^{\infty} \frac{1}{n\ln|x|} f(x)\,dx$$

$$= \int_{\infty}^{e^{\frac{1}{na}}} -\frac{f(-x)}{n\ln x}\,dx + \int_{e^{-\frac{1}{na}}}^{0} -\frac{f(-x)}{n\ln x}\,dx + \int_{0}^{e^{-\frac{1}{na}}} \frac{f(x)}{n\ln x}\,dx + \int_{e^{\frac{1}{na}}}^{\infty} \frac{f(x)}{n\ln x}\,dx$$

$$= 2\int_{e^{\frac{1}{na}}}^{\infty} \frac{f(x)}{n\ln x}\,dx + 2\int_{0}^{e^{-\frac{1}{na}}} \frac{f(x)}{n\ln x}\,dx$$

$$= 2\Big(\int_{\frac{1}{na}}^{\infty} \frac{f(e^u)e^u}{nu}\,du + \int_{-\infty}^{-\frac{1}{na}} \frac{f(e^u)e^u}{nu}\,du\Big)$$

$$= 2\Big(\int_{\frac{1}{na}}^{\infty} \frac{f(e^u)e^u}{nu}\,du + \int_{\infty}^{\frac{1}{na}} \frac{f(e^{-u})e^{-u}}{nu}\,du\Big)$$

$$= 2\Big(\int_{\frac{1}{na}}^{\infty} \frac{f(e^u)e^u}{nu}\,du - \int_{\frac{1}{na}}^{\infty} \frac{f(e^{-u})e^{-u}}{nu}\,du\Big)$$

$$=0$$

where we have used the fact that $(H \circ exp)'$ is symmetric in the last equality. Hence

$$\sigma_{nk}^2(a) = E((X_{nk}^a)^2) - (\mu_{nk}(a))^2$$

$$= \int_{-\infty}^{-e^{\frac{1}{na}}} \frac{f(x)}{(n\ln|x|)^2}\,dx + \int_{-e^{-\frac{1}{na}}}^{e^{-\frac{1}{na}}} \frac{f(x)}{(n\ln|x|)^2}\,dx + \int_{e^{\frac{1}{na}}}^{\infty} \frac{f(x)}{(n\ln|x|)^2}\,dx$$

$$= 2\int_{e^{\frac{1}{na}}}^{\infty} \frac{f(x)}{n^2(\ln x)^2}\,dx + 2\int_{0}^{e^{-\frac{1}{na}}} \frac{f(x)}{n^2(\ln x)^2}\,dx$$

$$= 2\Big(\int_{\frac{1}{na}}^{\infty} \frac{f(e^u)e^u}{n^2u^2}\,du + \int_{-\infty}^{-\frac{1}{na}} \frac{f(e^u)e^u}{n^2u^2}\,du\Big)$$

$$= 2\Big(\int_{\frac{1}{na}}^{\infty} \frac{f(e^u)e^u}{n^2u^2}\,du + \int_{\frac{1}{na}}^{\infty} \frac{f(e^{-u})e^{-u}}{n^2u^2}\,du\Big)$$

$$= \frac{4}{n^2}\int_{\frac{1}{na}}^{\infty} \frac{f(e^u)e^u}{u^2}\,du$$

$$= \frac{4}{n^2}\Big(\int_{\frac{1}{na}}^{1} \frac{f(e^u)e^u}{u^2}\,du + \int_{1}^{\infty} \frac{f(e^u)e^u}{u^2}\,du\Big)$$

$$= \frac{4}{n^2}\Big(\int_{\frac{1}{na}}^{1} \frac{f(e^u)e^u}{u^2}\,du + \int_{e}^{\infty} \frac{f(x)}{(\ln x)^2}\,dx\Big)$$

$$\leq \frac{4Be}{n^2}\int_{\frac{1}{na}}^{1}\frac{1}{u^2}\,du+\frac{4}{n^2}\int_{-\infty}^{\infty}f(x)\,dx$$

$$=\frac{4Be}{n^2}(-1+na)+\frac{4}{n^2}$$

$$\leq \frac{4a}{n}(Be+1)$$

i.e, $0\leq\sigma_{nk}^2(a)\leq 1$ for large n and $k=1,2,...,n$. So Theorem 3.5 holds.

In our case, we see that

$$\sigma_n^2(a)=\sigma_{n1}^2(a)+\cdots+\sigma_{nn}^2(a)\leq 4a(Be+1),$$

$$\gamma^a=-\int_{|u|>a}\frac{1}{u}\,dG(u)=-\int_{|u|>a}\frac{f(1)+f(-1)}{u(1+u^2)}\,du=0,$$

$$\begin{aligned}\mu(a)&=\gamma^a+\int_{-\infty}^{\infty}u\,dG^a(u)\\&=\int_{-\infty}^{-a}u\,dG^a(u)+\int_{-a}^{a}u\,dG^a(u)+\int_{a}^{\infty}u\,dG^a(u)\\&=\int_{-a}^{a}\frac{u(f(1)+f(-1))}{(1+u^2)}\,du\\&=0,\end{aligned}$$

$$\mu_n(a)=\mu_{n1}(a)+\cdots+\mu_{nn}(a)=0,$$

$$K^a(u)=\int_{-\infty}^{u}(1+x^2)\,dG^a(x)$$

$$=\begin{cases}\displaystyle\int_{-\infty}^{u}(1+x^2)\,dG^a(x) & \text{if } u<-a\\[2ex] \displaystyle\int_{-\infty}^{-a}(1+x^2)\,dG^a(x)+\int_{-a}^{u}(1+x^2)\,dG^a(x) & \text{if } -a\leq u<a\\[2ex] \displaystyle\int_{-\infty}^{-a}(1+x^2)\,dG^a(x)+\int_{-a}^{a}(1+x^2)\,dG^a(x)\\ \displaystyle\quad+\int_{a}^{u}(1+x^2)\,dG^a(x) & \text{if } u\geq a,\end{cases}$$

$$= \begin{cases} 0 & \text{if } u < -a \\ \int_{-a}^{u} f(1) + f(-1)\,dx & \text{if } -a \le u < a \\ \int_{-a}^{a} f(1) + f(-1)\,dx & \text{if } u \ge a, \end{cases}$$

$$= \begin{cases} 0 & \text{if } u, -a \\ (f(1) + f(-1))(u + a) & \text{if } -a \le u < a \\ 2(f(1) + f(-1))a & \text{if } u \ge a, \end{cases}$$

$\sigma^2(a) = K^a(\infty) = 2(f(-1) + f(1))a$ and

$$\begin{aligned} \int_{|u|>a} |u|^r dG(u) &= 2(f(-1) + f(1)) \int_a^\infty \frac{u^r}{1 + u^2}\,du \\ &\le 2(f(-1) + f(1)) \int_a^\infty u^{r-2}\,du \\ &= \frac{2(f(-1) + f(1))}{(1 - r)a^{1-r}}. \end{aligned}$$

Then we have

$$\begin{aligned} g^a(n, m(a,\delta), r) &= \Big[\frac{1}{3}\sigma_n^2(a) \max_{!\le k \le n} \sigma_{nk}^2(a)\Big]^{\frac{1}{5}} + \Big[\frac{1}{2}\sum_{i=0}^{m} |K_n^a(x_i) - K^a(x_i)|\Big]^{\frac{1}{3}} \\ &\quad + \Big[\frac{5}{6}\delta(\sigma_n^2(a) + \sigma^2(a))\Big]^{\frac{1}{4}} \\ &\quad + \Big\{\frac{4}{a}[K_n^a(\infty) - K_n^a(a) + K^a(\infty) - K^a(a) + K_n^a(-a) \\ &\quad + K^a(-a)] + 2|\mu_n(a) - \mu(a)|\Big\}^{\frac{1}{2}} + \Big[\frac{8}{r}\int_{|u|>a} |u|^r dG(u)\Big]^{\frac{1}{1+r}} \\ &\le \Big[\frac{1}{3}(4a(Be{+}1))\Big(\frac{4a(Be + 1)}{n}\Big)\Big]^{\frac{1}{5}} + \Big[\frac{1}{2}\sum_{i=0}^{m} |K_n^a(x_i) - K^a(x_i)|\Big]^{\frac{1}{3}} \\ &\quad + \Big[\frac{5}{6}\,\delta(4a(Be + 1) + 2a(f(1) + f(-1)))\Big]^{\frac{1}{4}} \end{aligned}$$

$$+\left\{\frac{4}{a}[K_n^a(\infty)-K_n^a(a)+2f(0)a-2f(0)a+K_n^a(-a)\right\}^{\frac{1}{2}}$$

$$+\left[\frac{16(f(1)+f(-1))}{r(1-r)a^{1-r}}\right]^{\frac{1}{1+r}}$$

$$=\left[\frac{16}{3n}(a(Be+1))^2)\right]^{\frac{1}{5}}+\left[\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i)-K^a(x_i)|\right]^{\frac{1}{3}}$$

$$+\left[\frac{5}{6}\delta(4a(Be+1)+2a(f(1)+f(-1)))\right]^{\frac{1}{4}}$$

$$+\left\{\frac{4}{a}[K_n^a(\infty)-K_n^a(a)+K_n^a(-a)]\right\}^{\frac{1}{2}}+\left[\frac{16(f(-1)+f(1))}{r(1-r)a^{1-r}}\right]^{\frac{1}{1+r}}.$$

$$=\left[\frac{16}{3n}(a(Be+1))^2\right]^{\frac{1}{5}}+\left[\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i)-K^a(x_i)|\right]^{\frac{1}{3}}$$

$$+\left[\frac{5}{3}\delta a(2(eB+1)+(f(-1)+f(1)))\right]^{\frac{1}{4}}$$

$$+\left\{\frac{4}{a}[K_n^a(\infty)-K_n^a(a)+K_n^a(-a)]\right\}^{\frac{1}{2}}+\left[\frac{16(f(-1)+f(1))}{r(1-r)a^{1-r}}\right]^{\frac{1}{1+r}}$$

$$=h^a(n,m(a,\delta),r)$$

Hence $g^a(n,m(a,\delta),r)\le h^a(n,m(a,\delta),r)$. □

Next lemma, we will bound the second and the forth terms of $h^a(n,m(a,\delta),r)$.

Lemma 4.5

1. $K_n^a(\infty)-K_n^a(a)+K_n^a(-a)=0$ *and*
2. $|K_n^a(x_i)-K^a(x_i)|\le\frac{C}{n}\ln na$ *for any* $i=0,1,2,...,m$.

Proof.

By Lemma 4.2 and the fact that $\mu_{nk}(a)=0$ we see that

$$K_n^a(u) = n\int_{-\infty}^{u} x^2 dF_{nk}^a(x)$$

$$= \begin{cases} 0 & \text{if} \quad u \le -a \\ \displaystyle\int_{-a}^{u} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}}))\,dx & \text{if} \quad -a < u < a \\ \displaystyle\int_{-a}^{a} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}}))\,dx & \text{if} \quad u \ge a. \end{cases}$$

Hence (1) follows immediately from the formula of K_n^a. Next we will prove (2).

By the hypothesis of main theorem, there exist positive constants ε and C such that $|f'(x)| < C$ for $|x-1| \le \varepsilon$.

Case 1. $\quad x_i \le \dfrac{-1}{n\ln(1+\varepsilon)}$.

By Mean-Value Theorem, for all $u \in [e^{\frac{1}{nx_i}}, e^{-\frac{1}{na}}]$ there exists $\xi_u \in (u,1)$ such that $|f(1) - f(u)| = |f'(\xi_u)(1-u)| \le C(1-u)$.

Hence

$$\begin{aligned} |K_n^a(x_i) - K^a(x_i)| &= \Big|\int_{-a}^{x_i} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}})) - (f(-1) + f(1))\,dx\Big| \\ &\le \Big|\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{nx_i}}} (uf(-u) - f(-1))\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big| \\ &\quad + \Big|\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{nx_i}}} (uf(u) - f(1))\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big| \\ &= 2\Big|\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{nx_i}}} (uf(u) - f(1))\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big| \\ &= 2\Big|\int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{nx_i}}} -\frac{f(u)}{n(\ln u)^2}\,du + \int_{e^{-\frac{1}{na}}}^{e^{\frac{1}{nx_i}}} \frac{f(1)}{nu(\ln u)^2}\,du\Big| \\ &= 2\Big|\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{f(u) - f(1)}{n(\ln u)^2}\,du + \int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{f(1)u - f(1)}{nu(\ln u)^2}\,du\Big| \\ &\le 2\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{|f(u) - f(1)|}{n(\ln u)^2}\,du + 2\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{f(1)|u-1|}{nu(\ln u)^2}\,du \end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{n}\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{1-u}{(\ln u)^2}\,du + \frac{2f(1)}{n}\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{1-u}{u(\ln u)^2}\,du \\
&\le \frac{C}{n}\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{1-u}{(\ln u)^2}\,du + \frac{2f(1)(1+\varepsilon)}{n}\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{1-u}{(\ln u)^2}\,du \\
&\le \frac{C}{n}\int_{e^{\frac{1}{nx_i}}}^{e^{-\frac{1}{na}}} \frac{1-u}{(\ln u)^2}\,du \\
&= \frac{C}{n}\int_{\frac{1}{nx_i}}^{-\frac{1}{na}} \frac{(1-e^x)e^x}{x^2}\,dx \\
&= \frac{C}{n}\int_{\frac{1}{nx_i}}^{-\frac{1}{na}} \left|\frac{(1-e^x)e^x}{x}\right|\left|\frac{1}{x}\right|\,dx \\
&\le \frac{C}{n}\int_{\frac{1}{nx_i}}^{-\frac{1}{na}} -\frac{1}{x}\,dx \\
&\le \frac{C}{n}\ln na
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\left|\frac{(1-e^x)e^x}{x}\right| &= \frac{(1-e^x)e^x}{-x} \\
&< \frac{(1-e^x)}{-x} \\
&= \frac{e^{\xi}(-x)}{-x} \; ; \; \xi \in (x,0) \\
&< 1
\end{aligned}
$$

is bounded on $[\frac{1}{nx_i}, -\frac{1}{na}]$.

Case 2. $-\dfrac{1}{n\ln(1+\varepsilon)} < x_i \le \dfrac{1}{n\ln(1+\varepsilon)}$.

Noting from case 1 that

$$
\begin{aligned}
|K_n^a(x_i) - K^a(x_i)| &\le \left|\int_{-a}^{-\frac{1}{n\ln(1+\varepsilon)}} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}})) - (f(-1)+f(1))\,dx\right| \\
&\quad + \left|\int_{-\frac{1}{n\ln(1+\varepsilon)}}^{x_i} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}})) - (f(-1)+f(1))\,dx\right| \\
&\le \frac{C}{n}\ln na \\
&\quad + \left|\int_{-\frac{1}{n\ln(1+\varepsilon)}}^{x_i} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}})) - (f(-1)+f(1))\,dx\right|
\end{aligned}
$$

and

$$\left|\int_{-\frac{1}{n\ln(1+\varepsilon)}}^{x_i} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}})+f(e^{\frac{1}{nx}}))-(f(-1)+f(1))\,dx\right|$$

$$\leq\left|\int_{-\frac{1}{n\ln(1+\varepsilon)}}^{\frac{1}{n\ln(1+\varepsilon)}} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}})+f(e^{\frac{1}{nx}}))\,dx\right|+\frac{2(f(-1)+f(1))}{n\ln(1+\varepsilon)}$$

$$=\left|\int_{-\frac{1}{n\ln(1+\varepsilon)}}^{\frac{1}{n\ln(1+\varepsilon)}} 2e^{\frac{1}{nx}}f(e^{\frac{1}{nx}})\,dx\right|+\frac{2(f(-1)+f(1))}{n\ln(1+\varepsilon)}$$

$$=\left|\int_{-\frac{1}{n\ln(1+\varepsilon)}}^{0} 2e^{\frac{1}{nx}}f(e^{\frac{1}{nx}})\,dx+\int_{0}^{\frac{1}{n\ln(1+\varepsilon)}} 2e^{\frac{1}{nx}}f(e^{\frac{1}{nx}})\,dx\right|+\frac{2(f(-1)+f(1))}{n\ln(1+\varepsilon)}$$

$$\leq\frac{2B}{n\ln(1+\varepsilon)}+2\int_{0}^{\frac{1}{n\ln(1+\varepsilon)}}\left|e^{\frac{1}{nx}}f(e^{\frac{1}{nx}})\right|dx+\frac{2(f(-1)+f(1))}{n\ln(1+\varepsilon)}$$

$$=\frac{C}{n}+2\int_{\infty}^{1+\varepsilon}-\frac{f(u)}{n(\ln u)^2}\,du$$

$$=\frac{C}{n}+\int_{1+\varepsilon}^{\infty}\frac{2f(u)}{n(\ln u)^2}\,du$$

$$\leq\frac{C}{n}+\frac{2}{n(\ln(1+\varepsilon))^2}\int_{-\infty}^{\infty}f(u)\,du$$

$$=\frac{C}{n}.$$

Hence $\qquad |K_n^a(x_i)-K^a(x_i)|\leq\frac{C}{n}\ln na.$

Case 3. $\qquad x_i>\frac{1}{n\ln(1+\varepsilon)}.$

From the second case we see that

$$\left|\int_{-a}^{\frac{1}{n\ln(1+\varepsilon)}}(f(-e^{\frac{1}{nx}})+f(e^{\frac{1}{nx}}))e^{\frac{1}{nx}}-(f(-1)+f(1))\,dx\right|\leq\frac{C}{n}\ln na$$

and by Mean-Value Theorem, for all $u\in[e^{\frac{1}{nx_i}},1+\varepsilon]$ there exists $\xi_u\in(1,u)$ such that $|f(1)-f(u)|=|f'(\xi_u)(u-1)|\leq C(u-1)$, we have

$$\Big|\int_{\frac{1}{\ln(1+\varepsilon)}}^{x_i}\left(f(-e^{\frac{1}{nx}})+f(e^{\frac{1}{nx}})e^{\frac{1}{nx}}-(f(-1)+f(1))\right)dx\Big|$$

$$=\Big|\int_{1+\varepsilon}^{e^{\frac{1}{nx_i}}}\Big[(f(-u)+f(u))u-(f(-1)+f(1))\Big]\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big|$$

$$\le\Big|\int_{1+\varepsilon}^{e^{\frac{1}{nx_i}}}(f(-u)u-f(-1))\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big|$$

$$+\Big|\int_{1+\varepsilon}^{e^{\frac{1}{nx_i}}}(f(u)u-f(1))\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big|$$

$$=2\Big|\int_{1+\varepsilon}^{e^{\frac{1}{nx_i}}}(f(u)u-f(1))\Big(-\frac{1}{nu(\ln u)^2}\Big)\,du\Big|$$

$$=2\Big|\int_{1+\varepsilon}^{e^{\frac{1}{nx_i}}}-\frac{f(u)}{n(\ln u)^2}\,du+\int_{1+\varepsilon}^{e^{\frac{1}{nx_i}}}\frac{f(1)}{nu(\ln u)^2}\,du\Big|$$

$$=2\Big|\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{f(u)-f(1)}{n(\ln u)^2}\,du+\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{f(1)u-f(1)}{nu(\ln u)^2}\,du\Big|$$

$$\le 2\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{|f(u)-f(1)|}{n(\ln u)^2}\,du+2\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{f(1)|u-1|}{nu(\ln u)^2}\,du$$

$$\le\frac{C}{n}\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{u-1}{(\ln u)^2}\,du+\frac{2f(1)}{n}\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{u-1}{(\ln u)^2}\,du$$

$$=\frac{C}{n}\int_{e^{\frac{1}{nx_i}}}^{1+\varepsilon}\frac{u-1}{(\ln u)^2}\,du$$

$$=\frac{C}{n}\int_{\frac{1}{nx_i}}^{\ln(1+\varepsilon)}\frac{(e^x-1)e^x}{x^2}\,dx$$

$$\le\frac{C}{n}\int_{\frac{1}{nx_i}}^{\ln(1+\varepsilon)}\frac{(e^x-1)}{x^2}\,dx$$

$$<\frac{C}{n}\int_{\frac{1}{nx_i}}^{\ln(1+\varepsilon)}\frac{e^x}{x}\,dx$$

$$<\frac{C}{n}\int_{\frac{1}{nx_i}}^{\ln(1+\varepsilon)}\frac{1}{x}\,dx$$

$$\le\frac{C}{n}\Big[\ln(\ln(1+\varepsilon))+\ln(na)\Big]$$

Hence

$$|K_n^a(x_i) - K^a(x_i)| \leq \left| \int_{-a}^{\frac{1}{n\ln(1+\varepsilon)}} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}})) - (f(-1) + f(1))\, dx \right|$$
$$+ \left| \int_{\frac{1}{n\ln(1+\varepsilon)}}^{x_i} e^{\frac{1}{nx}}(f(-e^{\frac{1}{nx}}) + f(e^{\frac{1}{nx}})) - (f(-1) + f(1))\, dx \right|$$
$$\leq \frac{C}{n}\ln na$$

So we have the lemma in this case. □

Proof of Main Theorem

From Theorem 4.4 and Lemma 4.5 we see that

$$h^a(n, m(a,\delta), r) \leq C\Big[\Big(\frac{a^2}{n}\Big)^{\frac{1}{5}} + \Big\{\Big(\frac{m+1}{n}\Big)\ln na\Big\}^{\frac{1}{3}} + (\delta a)^{\frac{1}{4}} + C(r)\Big(\frac{1}{a}\Big)^{\frac{1-r}{1+r}}\Big]$$

which implies

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leq C\Big[\frac{1}{a} + \Big(\frac{a^2}{n}\Big)^{\frac{1}{5}} + \Big\{\Big(\frac{m+1}{n}\Big)\ln na\Big\}^{\frac{1}{3}} + (\delta a)^{\frac{1}{4}}$$
$$+ C(r)\Big(\frac{1}{a}\Big)^{\frac{1-r}{1+r}}\Big] \text{ for } a \geq 1 \text{ and } r \in (0,1) \tag{4.2}$$

where $C(r)$ are constants depend on r.

We set $a = n^{k_1}$ and $\delta = n^{-k_2}$ where k_1 and k_2 are positive numbers, so

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leq C\Big[\frac{1}{n^{k_1}} + \Big(\frac{1}{n^{1-2k_1}}\Big)^{\frac{1}{5}} + \Big(\frac{(1+k_1)\ln n}{n^{1-k_1-k_2}}\Big)^{\frac{1}{3}} + \Big(\frac{1}{n^{k_2-k_1}}\Big)^{\frac{1}{4}}$$
$$+ C(r)\Big(\frac{1}{n^{k_1}}\Big)^{\frac{1-r}{1+r}}\Big].$$

Then we see that the order of the bounds in (4.2), are

$$\frac{1}{n^{k_1}}, \frac{1}{n^{\frac{1-2k_1}{5}}}, \frac{(1+k_1)\ln n}{n^{(\frac{1-k_1-k_2}{3})}}, \frac{1}{n^{\frac{k_2-k_1}{4}}}, C(r)\frac{1}{n^{\frac{k_1(1-r)}{1+r}}}. \tag{4.3}$$

By the same argument as in Chapter III, we have that for any constant $0 < d < \frac{1}{9}$, there exists a constant C such that

$$\sup_{-\infty<x<\infty} [F_n(x) - F(x)] < \frac{C}{n^d}.$$

REFERENCES

1. Boonyasombat,V. and Shapiro,J.M. The accuracy of infinitely divisible approximations to sums of independent variables with application to stable laws, *Annals of Math.Stat.*,41(1970), 237-250.
2. Feller,W. *An introduction to probability theory and its applications*, Vol.2, Wiley, New York(1966).
3. Cramer,H. *Mathematical method of statistics. Uppsala: Almqvist and Wilesells*, 1945.
4. Gnedenko,B.V. *Theory of probability. New York: Chelsca Publishing Company*, 1962.
5. Gnedenko,B.V. and Kolmogorov,A.N. *Limit distributions for sums of independent random variables.* Cambridge:Addison-Wesley, 1954.
6. Laha,R.G. and Rohatgi,V.K. *Probability theory.* New York: John Wiley and Sons, 1979.
7. Lukacs,E. *Characteristic Functions*, Griffin, London, 1960.
8. Neammanee,K. Limit distributions for random sums of reciprocals of logarithms of independent continuous random variables,*J.Sci.Res.Chula Univ.*, Vol.23, No.2(1998), 79-100.
9. Neammanee,K. Limit distributions for sums of reciprocals of logarithms of absolute random variables, *J.Sci.Res.Chula Univ.*, Vol.29, No.2(2001), 71-84.
10. Neammanee,K. Limit distributions for sums of reciprocals of independent random variables, *East-West Journal,* to be appear.
11. Shapiro,J.M. Error estimates for certain probability limit theorems, *Annals of Math.Stat.*, 26(1955), 617-630.

12. Shapiro,J.M. Domain of attraction reciprocals of power of random variables, *SIAM Journal Appl.Math,* 29(1975), 734-739.

13. Shapiro,J.M. On domains of normal attraction to stable distributions,*Houston J.Math,* 3(1977), 539-542.

14. Shapiro,J.M. Limit distributions for sums of reciprocals of independent random variables, *Houston J.Math,* 14(1988), 281-290.

15. Siricheon,R. and Neammanee,K. *Limit distribution for sums of the reciprocals of sine of random variables,* to be appear.

16. Termwuttipong,I. *Limit Distributions for Sums of the Reciprocal of a Positive Power of Independent Random Variables.* Ph.D.Thesis, Chulalongkorn Univ., 1986.

VITA

Miss Angkana Suntadkarn was born on 2 December, 1976 in Chumporn, Thailand. She graduated with a Bachelor degree of Science in Mathematics from Khon Kaen University in 1998. In 2000, she was admitted into the Master Degree program in Mathematics, Department of Mathematics, Faculty of Science, Chulalongkorn University. During the study for her Master's degree, she received a financial support from University Development Commission for 2 years.