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ANALYSIS OF DIRECTIONAL REGULARITY BY TRANSFORMS WITH PARABOLIC SCALING

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ในการพยายามหาลักษณะเฉพาะของความราบเรียบแบบเอกรูปและรายจุด เราได้อัตราการลู่ลงที่จำเป็นและอัตราการลู่ลงที่เพียงพอสำหรับการแปลงเชียร์เลตแบบต่อเนื่องที่มีมาตราวัดต่างๆ นอกจากนี้เราได้ด้วยว่าอัตราการลู่ลงที่จำเป็นและอัตราการลู่ลงที่เพียงพอนี้ได้ผลเช่นเดียวกันกับการใช้การแปลงฮาร์ทสมิธและการแปลงเคิร์ฟเลตแบบต่อเนื่องในการหาลักษณะเฉพาะของความราบเรียบแบบเอกรูปและรายจุด เราพิจารณาฟังก์ชันที่มีความราบเรียบบนเส้นในทิศทางที่ไม่ขนานกับเส้นน้อยกว่าความราบเรียบในทิศตามเส้นในย่านใกล้เคียง ในทำนองเดียวกับการแปลงฮาร์ทสมิธและการแปลงเคิร์ฟเลตแบบต่อเนื่อง เงื่อนไขที่จำเป็นสำหรับทิศทางของภาวะเอกรูปนี้ได้แก่การแปลงเชียร์เลตแบบต่อเนื่องต้องลู่ลงในทิศทางที่ห่างจากทิศทางตามเส้นเร็วกว่าอัตราการลู่ลงในทิศที่ใกล้เส้นอยู่ครึ่งอันดับและอัตราการลู่ลงในทิศทางใกล้เส้นยังขึ้นกับระยะห่างตามแนวนอนจากเส้นตรงถึงเส้นขนานที่ผ่านจุดกึ่งกลางของฟังก์ชันเชียร์เลต ยิ่งไปกว่านั้นเราได้นิยามความเรียบระบุทิศทางที่มีการปรับมาตราเชิงพาราโบลา และหาเงื่อนไขจำเป็นและเงื่อนไขเพียงพอเพื่อวิเคราะห์ความราบเรียบนี้โดยการแปลงเวฟเลตแบบวิยุต

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In an effort to characterize uniform and pointwise regularities, we obtain necessary decay rates and sufficient decay rates of continuous shearlet transform across scales. They are the same rates as those of the Hart Smith and continuous curvelet transforms. We then consider the situation where regularity on a line in a non-parallel direction is much lower than directional regularity along the line in a neighborhood. Similar to that of the Hart Smith and continuous curvelet transforms, a set of necessary conditions for this direction of singularity is that the continuous shearlet transform decays half an order faster in directions "away" from the direction of the line and that the decay rate in directions "near" the line depends also on the horizontal distance from the line to the parallel line containing the center of the shearlet function. Moreover, we define a new directional regularity based on parabolic scaling. Then we obtain necessary condition and sufficient condition for analysis the directional regularity by discrete wavelet transform .

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CHAPTER I

INTRODUCTION

1.1 Introduction

Since the advent of wavelet analysis in the 1980's, its application in characterizing Hölder regularity had been made apparent by the original work of Jaffard[21] and Holschneider and Tchamitchian[15]. Even though these characterizations of uniform and pointwise regularity by continuous wavelet transform is also valid in $\mathbb{R}^d$, the transform with the same scaling in all directions lacks the ability to detect directional singularity and singularity along lines. Many techniques have been introduced to overcome this shortcoming [4, 6, 7, 24]. Continuous curvelet transform by Candés and Donoho[2] is a very successful approach in compressing images with edges. Edges can be associated with singularity along curves and hence they should benefit from the same techniques. In [27], curvelet transform and its cousin, the Hart Smith transform, with true parabolic scaling were shown to be able to detect line singularity at a point through observation of two different decay rates of the transform across angles. It was also obtained that, even for uniform regularity, the necessary decay rate and the sufficient decay rate differ by $\frac{1}{2}$. The sharpness of these estimates should be investigated further. All these estimates depend heavily on the fact that the kernels of these two transforms satisfy certain directional vanishing monents, smoothness and decay properties. It

is then believed that similar theorems are true for parabolically scaled transforms satisfying these directional vanishing moments, smoothness and decay properties. More study on other examples of such a transform should be done before a general theory of transforms with parabolic scaling can be materialized. There is another reason why we seek characterizations by other transforms. Even though the Hart Smith transform uses true parabolic scaling, its reconstruction formula involves a Fourier multiplier which makes it harder to use. On the other hand, curvelet transform has a desirable reconstruction formula but lacks true parabolic scaling. Based on the theory of wavelets with composite dilations[12], shearlet transform with three parameters, namely location, parabolic scale, and shear strength, was introduced by [22] and [8]. It was shown that, unlike curvelet and Smith transforms, this parabolically scaled transform also possesses a reconstruction formula similar to that of wavelet transform. This should make it a better candidate for the application at hand.

This thesis consists of two parts. In the first part, we consider the continuous and discrete shearlet transforms. We shall investigate how the behavior of these transforms of a given function can reflect its regularity properties. Here, we study uniform and pointwise Hölder regularities. We then go on to consider the situation where a function is much less regular on a line in the perpendicular direction than on a neighborhood of the line in the parallel direction. Although these results are similar to those in [27], the shear version of the theorem on direction of singularity takes quite a different form. The two directions are not necessarily perpendicular which may give rise to some future applications. In the second part, we define the directional regularity and then analyze the directional regularity of function by the discrete wavelet transforms.

1.2 Notation

We begin with notations and definitions. Throughout this paper, we shall consider

1. points $\boldsymbol{x} \in \mathbb{R}^d$ to be column vectors, i.e., $\boldsymbol{x} = (x_1, ..., x_d)^T$ (the transpose of $(x_1, ..., x_d)$)

2. points $\boldsymbol{\xi} \in \hat{\mathbb{R}}^d$ (the frequency domain) to be row vector, i.e., $\boldsymbol{\xi} = (\xi_1, ..., \xi_d)$.

3. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and $\nu \in \mathbb{N}_0^d$,

   (a) $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \cdots, x_d + y_d)$

   (b) $a\boldsymbol{x} = (ax_1, \cdots, ay_d)$ where a in $\mathbb{R}$

   (c) $\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^d x_i \overline{x}_i}$

   (d) $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_d$

   (e) $\boldsymbol{x}^\nu = \prod_{i=1}^d x_i^{\nu_i}$

   (f) $\partial^\nu f = \partial_1^{\nu_1} \partial_2^{\nu_2} \cdots \partial_d^{\nu_d} f$ where ∂_i means the partial derivative with respect to the i^{th}-variable.

1.3 The L^p-spaces

Definition 1.3.1. If $0 < p < \infty$ and f is a complex measurable function on (X, m, μ) define

$$\|f\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}$$

and let $L^p(\mu)$ consist of all f for which $\|f\|_p < \infty$. We call $\|f\|_p$ the L^p-norm of f.

Example In $\mathbb{R}^2$;

$$f \in L^2(\mathbb{R}^2) \quad \text{iff} \quad \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} < \infty$$

Theorem 1.3.2. *(Hölder's inequality) If $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Theorem 1.3.3. *(Fubini's theorem) If $\int\int |f(x,y)| \, dydx < \infty$ with $(x, y) \in \mathbb{R}^2$, then*

$$\begin{aligned} \int\int f(x,y) \, dydx &= \int \left[\int f(x,y) \, dy \right] dx \\ &= \int \left[\int f(x,y) \, dx \right] dy, \end{aligned}$$

i.e, the oder of the integrations can be permuted.

When Hilbert spaces are used, they are usually denoted by $\mathcal{H}$, unless they already have a name. We will follow the mathematician's convention and use scalar products which are linear in the first argument

$$\langle \lambda_1 \mu_1 + \lambda_2 \mu_2, v \rangle = \lambda_1 \langle \mu_1, v \rangle + \lambda_2 \langle \mu_2, v \rangle, \quad \lambda_1, \lambda_2 \in \mathbb{C}.$$

As usual, we have $\langle u, v \rangle = \overline{\langle v, u \rangle}$ where $\overline{\alpha}$ denotes the complex conjugate of α, and $\langle u, u \rangle \geq 0$ for all $u \in \mathcal{H}$. We define the norm $\|u\|$ of u by $\|u\|^2 = \langle u, u \rangle$. A standard example of such a Hilbert space is $L^2(\mathbb{R}^2)$, with

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} \, d\boldsymbol{x}.$$

A standard inequality in a Hilbert space is the Cauchy-Schwarz inequality,

$$| \langle v, w \rangle | \leq \|v\| \, \|w\| \, .$$

1.4 Fourier Transform

In $\mathbb{R}^d$, the Fourier Transform becomes

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{\xi}\boldsymbol{x}}\, d\boldsymbol{x} \text{ for all } \boldsymbol{\xi} \in \hat{\mathbb{R}}^d.$$

So the Plancherel's formula becomes

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

and, the inversion formula is

$$f(\boldsymbol{x}) = \int_{\widehat{\mathbb{R}^d}} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi}\boldsymbol{x}}\, d\boldsymbol{\xi} \text{ for all } \boldsymbol{x} \in \mathbb{R}^d.$$

We assume that two functions f and g are each pointwise continuous and absolutely intgrable on the plane. The convolution of f and g is defined as

$$(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y}$$

Then we have

$$\widehat{(f * g)}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) \hat{g}(\boldsymbol{\xi})$$

and

$$\widehat{(fg)}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) * \hat{g}(\boldsymbol{\xi}).$$

1.5 Hölder Regularities

Pointwise, uniform, and directional Hölder regularities of functions of several variables are defined as follows.

Definition 1.5.1. Let $\alpha > 0$ and $\alpha \notin \mathbb{N}$. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to be *pointwise Hölder regular with exponent α at $\boldsymbol{u}$*, denoted by $f \in C^{\alpha}(\boldsymbol{u})$, if there exists a polynomial $P_{\boldsymbol{u}}$ of degree less than α and a constant $C = C_{\boldsymbol{u}}$ such that for all $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{u}$

$$|f(\boldsymbol{x}) - P_{\boldsymbol{u}}(\boldsymbol{x} - \boldsymbol{u})| \leq C\|\boldsymbol{x} - \boldsymbol{u}\|^{\alpha}. \tag{I.1}$$

If there exists a uniform constant C so that for all $\boldsymbol{u} \in \Omega$ there is a polynomial $P_{\boldsymbol{u}}$ of degree less than α such that (I.1) holds for all $\boldsymbol{x} \in \Omega$, then we say that f is *uniformly Hölder regular with exponent* α *on* Ω or $f \in C^{\alpha}(\Omega)$.

Definition 1.5.2. Let $\boldsymbol{v} \in \mathbb{R}^d$ be a fixed unit vector and $\boldsymbol{u}$ be a point in $\mathbb{R}^d$. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is *pointwise Hölder regular with exponent* α *at* $\boldsymbol{u}$ *in the direction* $\boldsymbol{v}$, denoted by $f \in C^{\alpha}(\boldsymbol{u}; \boldsymbol{v})$, if there exist a constant $C = C_{\boldsymbol{u},\boldsymbol{v}}$ and a polynomial $P_{\boldsymbol{u},\boldsymbol{v}}$ of degree less than α such that

$$|f(\boldsymbol{u} + \lambda\boldsymbol{v}) - P_{\boldsymbol{u},\boldsymbol{v}}(\lambda)| \leq C|\lambda|^{\alpha} \tag{I.2}$$

holds for all λ in a neighborhood of $0 \in \mathbb{R}$.

Let Ω_1 be a subset of $\mathbb{R}^d$ and Ω_2 an open neighborhood of Ω_1. f is said to be in $C^{\alpha}(\Omega_1, \Omega_2; \boldsymbol{v})$, if there exists a constant $C = C_{\boldsymbol{v}}$ so that for all $\boldsymbol{u} \in \Omega_1$ there is a polynomial $P_{\boldsymbol{u},\boldsymbol{v}}$ of degree less than α such that (I.2) holds for all $\lambda \in \mathbb{R}$ with $\boldsymbol{u} + \lambda\boldsymbol{v} \in \Omega_2$. If $\Omega_1 = \Omega_2$, then we denote $C^{\alpha}(\Omega_1, \Omega_2; \boldsymbol{v})$ simply by $C^{\alpha}(\Omega_1; \boldsymbol{v})$.

1.6 Continuous Wavelet Transform

Let us recall the definition of the wavelet transform. Let $\psi \in L^2(\mathbb{R}^d)$ be a locally integrable complex-valued function, which is in general well-localized and regular in the sense that there exist some constants $C > 0$ and $\epsilon > 0$ such that

$$|\psi(\boldsymbol{t})| + |\psi'(\boldsymbol{t})| \leq \frac{C}{1 + \|\boldsymbol{t}\|^{2+\epsilon}} \quad \text{for all } \boldsymbol{t} \in \mathbb{R}^d, \tag{I.3}$$

the first two moments of ψ vanish, i.e.

$$\int_{-\infty}^{\infty} \psi(t)\,dt = \int_{-\infty}^{\infty} t\psi(t)\,dt = 0, \tag{I.4}$$

and ψ satisfies admissible condition, i.e.

$$0 < c_\psi = \int_0^\infty |\hat{\psi}(a\boldsymbol{\xi})| \frac{da}{a^d} < \infty \text{ for a.e. } \boldsymbol{\xi} \in \mathbb{R}^d. \tag{I.5}$$

Such a function ψ is called *a wavelet function.*

These conditions are maximal in the sense that all theorems stated in this paper hold for those functions taken as wavelets. Dilating and translating the wavelet ψ, we obtain a two-parameter family of functions

$$\psi_{a,\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{\sqrt{a^d}} \psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right).$$

The parameter $\boldsymbol{b} \in \mathbb{R}^d$ is a position(translation) parameter, whereas $a > 0$ may be interpreted as a scale parameter. We can define the wavelet transform of an arbitrary function $f \in L^2(\mathbb{R}^d)$ with respect to a wavelet ψ as follows.

Definition 3. (Continuous Wavelet Transform). The *continuous wavelet transform* of an $L^2(\mathbb{R}^d)$ function f is defined by

$$W_f(a,\boldsymbol{b}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})\overline{\psi_{a,\boldsymbol{b}}(\boldsymbol{x})}\,d\boldsymbol{x} = \frac{1}{\sqrt{a^d}} \int_{\mathbb{R}^d} f(\boldsymbol{x}) \overline{\psi\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)}\,d\boldsymbol{x}$$

where this Lebesgue integral is well-defined for $a \in (0,\infty)$ and $\boldsymbol{b} \in \mathbb{R}^2$.

I. Daubechies([5]) showed that for $f, g \in L^2(\mathbb{R}^d)$

$$\int_0^\infty \int_{\mathbb{R}^d} W_f(a,\boldsymbol{b})\overline{W_g(a,\boldsymbol{b})} \frac{d\boldsymbol{b}\,da}{a^{d+1}} = c_\psi < f, g > .$$

We have Parseval's Formula that for $f \in L^2(\mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} |W_f(a,\boldsymbol{b})|^2 \frac{d\boldsymbol{b}\,da}{a^{d+1}} = c_\psi \, \|f\|_2^2 . \tag{I.6}$$

Definition (Admissibility Condition). A function ψ is said to be admissible if

$$0 < \int_0^\infty |\hat{\psi}(a)|^2 \frac{da}{a} < \infty.$$

Observe that for any $\xi \in \mathbb{R}$,

$$\int_0^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a} = \int_0^\infty |\hat{\psi}(a)|^2 \frac{da}{a} \equiv C_\psi.$$

The wavelet transform is invertible. An explicit inversion formula is given by the following theorem:

Theorem 1.6.1. *(Inversion Formula: Pointwise)*
Let $\psi \in L^2(\mathbb{R})$ be admissible. If $f \in (L^1 \cap L^2)(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then for each $x \in \mathbb{R}$,

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{\mathbb{R}} CW_f(a,b)\psi_{a,b}(x)\, db \frac{da}{a^2}$$

and the Plancherel Formula

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \frac{1}{C_\psi} \int_0^\infty \int_{\mathbb{R}} |CW_f(a,b)|^2\, db \frac{da}{a^2}$$

1.7 Characterization of Höder Regularities

Let $\alpha > 0$ and $k = [\alpha]$. Here and below we choose a wavelet ψ satisfying the following smoothness, decaying and oscillating properties

$$|\psi^{(i)}(x)| \leq C(1+|x|)^{-k-2} \quad \text{for } \ i = 0, ..., k+1$$

$$\int_{\mathbb{R}} x^j \psi(x)\, dx = 0 \quad \text{for } \ j = 0, ..., k$$

and

$$\int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi}\, d\xi = 1 \quad \text{with } \ \hat{\psi}(\xi) = 0 \ \text{ if } \ \xi < 0.$$

In this section we recall how to analyze uniform Hölder regularity thought wavelet transform. Gernerally speaking the amount of uniform regularity of function is reflected in its wavelet transform by the decrease of the wavelet coefficients at small scale as shown by the following well known theorem which give a necessary and sufficient condition.

Theorem 1.7.1. *A bounded function $f \in L^2(\mathbb{R})$ is Hölder continuous with regularity exponent α with $0 < \alpha < \infty$, if and only if its wavelet transform with respect to a wavelet ψ satisfies $|CW_f(a,b)| \leq Ca^{\alpha+\frac{1}{2}}$ for all $(a,b) \in (0,\infty) \times \mathbb{R}$.*

Above theorem give a characterization of the Hölder regularity over intervals but not at a point. The next theorem proved by Jaffard which found in [5, 15, 16, 17, 18] shoe that one can also estimate the hölder regularlity of function precisely at a point x_0. The theorems give a necessary condition and a sufficient condition, but not a necessary and sufficient condition. Now, also ψ satisfying the smoothness, decaying and oscillating properties.

Theorem 1.7.2. *If a bounded function f is Hölder continuous in x_0 with exponent $\alpha \in (0,1)$, then $|CW_f(a, b+x_0)| \leq Ca^{\frac{1}{2}}\,(a^\alpha + |b|^\alpha)$.*

We now turn our attention to the reciprocal proplem. The following theorem is similar to a theorem proved before by S.Jaffard.

Theorem 1.7.3. *Suppose that ψ is compactly supported, and $f \in L^2(\mathbb{R})$ is bounded and continuous. If, for some $\beta > 0$ and $\alpha \in (0,1)$, there exists a constant C such that*

$$|W_f(a,b)| \leq Ca^{\beta+\frac{1}{2}}$$

and

$$|W_f(a, b + x_0)| \leq C a^{\frac{1}{2}} \left(a^\alpha + \frac{|b|^\alpha}{|\log|b||} \right) \quad \textit{for all } a, b.$$

then f is Hölder continuous at x_0 with exponent α.

CHAPTER II

SHEARLET TRANSFORMS

In this chapter we will give a short introduction into the theory of shearlets. Section 2.1 introduces the definitions and theorems of the continuous shearlet transform. The definitions and theorems of the discrete shearlet transform, are then introduced in section 2.2.

2.1 Continuous Shearlet Transform

We will follow mainly the definitions and notations in G. Kutyniok and D. Labate [22]. Given ψ_1 and $\psi_2 \in L^2(\mathbb{R})$, let $\psi \in L^2(\mathbb{R}^2)$ be defined by

$$\hat{\psi}(\boldsymbol{\xi}) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right) \tag{II.1}$$

when $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ with $\xi_1 \neq 0$.

Definition 2.1.1. Let $\psi \in L^2(\mathbb{R}^2)$ be given by (II.1) where

1. $\psi_1 \in L^2(\mathbb{R})$ satisfies the admissibility condition, i.e.

$$\int_0^\infty \left|\hat{\psi}_1(a\xi)\right|^2 \frac{da}{a} = 1 \quad \text{for a.e. } \xi \in \mathbb{R},$$

   and $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$;

2. $\|\psi_2\| = 1$, and $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\hat{\psi}_2 \subset [-1, 1]$ and $\hat{\psi}_2 > 0$ on $(-1, 1)$.

Such a function ψ is called a *shearlet function*. A *continuous shearlet system* is the set of functions generated by ψ, namely,

$$\left\{\psi_{ast} = a^{-\frac{3}{4}}\psi\left(\mathrm{M}_{as}^{-1}(\cdot - \boldsymbol{t})\right) : a \in \mathrm{I} \subset \mathbb{R}^+,\, s \in \mathrm{S} \subset \mathbb{R},\, \boldsymbol{t} \in \mathbb{R}^2\right\},$$

where $\mathrm{M}_{as} = \mathrm{B}_s\mathrm{D}_a$, B_s is the *shear matrix* $\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$, and D_a is the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$. The *continuous shearlet transform* of f is then defined by

$$\mathcal{SH}_\psi f(a, s, \boldsymbol{t}) = \langle f, \psi_{ast}\rangle,\ a \in \mathrm{I} \subset \mathbb{R}^+,\ s \in \mathrm{S} \subset \mathbb{R},\ \boldsymbol{t} \in \mathbb{R}^2.$$

Here I is a set of parabolic scales and S is a set of shear parameters to be specified later.

Many properties of the continuous shearlets are more evident in the frequency domain. A direct computation shows that

$$\hat{\psi}_{ast}(\boldsymbol{\xi}) = a^{\frac{3}{4}}e^{-2\pi i\boldsymbol{\xi}\boldsymbol{t}}\hat{\psi}_1(a\xi_1)\hat{\psi}_2\left(\frac{1}{\sqrt{a}}\left(\frac{\xi_2}{\xi_1} - s\right)\right).$$

Thus, each function $\hat{\psi}_{ast}$ is supported on the set

$$\operatorname{supp}\hat{\psi}_{ast} \subseteq \left\{(\xi_1, \xi_2) \colon \xi_1 \in \left[-\frac{2}{a}, -\frac{1}{2a}\right] \cup \left[\frac{1}{2a}, \frac{2}{a}\right],\ \left|\frac{\xi_2}{\xi_1} - s\right| \leq \sqrt{a}\right\}.$$

Theorem 2.1.2. *Let* $\mathrm{I} = \mathbb{R}^+$, $\mathrm{S} = \mathbb{R}$, *and* $\psi \in L^2(\mathbb{R}^2)$ *be a shearlet function. Then, for all* $f \in L^2(\mathbb{R}^2)$,

$$f = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \langle \psi_{ast}, f\rangle\, \psi_{ast}\, \frac{da}{a^3}\, ds\, dt \quad in\ L^2(\mathbb{R}^2). \tag{II.2}$$

If the set S is not all of $\mathbb{R}$, then one needs some additional assumptions on ψ. Consider the subspace of $L^2(\mathbb{R}^2)$ given by $L^2(\mathrm{C})^{\check{}} = \{f \in L^2(\mathbb{R}^2) \colon \operatorname{supp}\hat{f} \subset C\}$ where

$$C = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\xi_1| \geq 2 \text{ and } \left|\frac{\xi_2}{\xi_1}\right| \leq 1\right\}.$$

Theorem 2.1.3. *Let* $\mathrm{I} = \{a\colon 0 < a < 1\}$, $\mathrm{S} = \{s\colon |s| \le 2\}$ *and* $\psi \in L^2(\mathbb{R}^2)$ *be a shearlet function. Then, for all* $f \in L^2(C)^\vee$,

$$f = \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 \langle \psi_{ast}, f\rangle \, \psi_{ast} \frac{da}{a^3} \, ds \, dt \quad \text{in } L^2(C)^\vee. \tag{II.3}$$

In this paper, we choose $\mathrm{S} = \{s\colon |s| \le 2\}$ and $\mathrm{I} = \{a\colon 0 < a < 1\}$. By Theorem 2.1.3, the continuous shearlet transform provides a reproducing formula only for functions in a proper subspace of $L^2(\mathbb{R}^2)$. However, even when S and I are bounded, it is possible to obtain a reproducing formula for all $f \in L^2(\mathbb{R}^2)$ as follows. Let

$$\hat{\psi}^{(v)}(\boldsymbol{\xi}) = \hat{\psi}^{(v)}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_2)\hat{\psi}_2\left(\frac{\xi_1}{\xi_2}\right) \tag{II.4}$$

where $\hat{\psi}_1, \hat{\psi}_2$ are defined as in Definition 2.1.1. The shearlets $\psi_{ast}^{(v)}$ are defined by $\psi_{ast}^{(v)} = a^{-\frac{3}{4}}\psi\left((\mathrm{M}_{as}^{(v)})^{-1}(\cdot - \boldsymbol{t})\right)$, where $\mathrm{M}_{as}^{(v)} = \mathrm{B}_s^{(v)}\mathrm{D}_a^{(v)}$ with $\mathrm{B}_s^{(v)} = \mathrm{B}_s^T$ and $\mathrm{D}_a^{(v)} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & a \end{pmatrix}$. Then $\left\{\psi_{ast}^{(v)}\right\}$ is a continuous shearlet system for $L^2(C^{(v)})^\vee$ where $C^{(v)}$ is the vertical cone

$$C^{(v)} = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2\colon |\xi_2| \ge 2 \text{ and } \left|\frac{\xi_2}{\xi_1}\right| > 1\right\}.$$

C and $C^{(v)}$ are illustrated in Figure 2.1.1 (a). Similar to "horizontal" shearlet in Definition 2.1.1, the associated "vertical" continuous shearlet transform is

$$\mathcal{SH}_\psi^{(v)} f(a, s, \boldsymbol{t}) = \left\langle f, \psi_{ast}^{(v)} \right\rangle,$$

and we have $\hat{\psi}_{ast}^{(v)}(\boldsymbol{\xi}) = a^{\frac{3}{4}} e^{-2\pi i \boldsymbol{\xi t}} \hat{\psi}_1(a\xi_2)\hat{\psi}_2\left(\frac{1}{\sqrt{a}}\left(\frac{\xi_1}{\xi_2} - s\right)\right)$, hence

$$\operatorname{supp} \hat{\psi}_{ast}^{(v)} \subseteq \left\{(\xi_1, \xi_2)\colon \xi_2 \in \left[-\frac{2}{a}, -\frac{1}{2a}\right] \cup \left[\frac{1}{2a}, \frac{2}{a}\right], \left|\frac{\xi_1}{\xi_2} - s\right| \le \sqrt{a}\right\}.$$

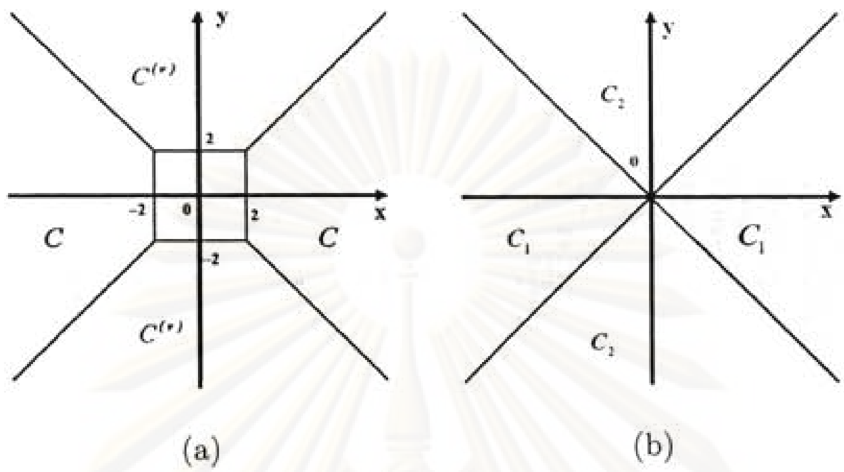


Figure 2.1.1: (a) An illustration of C and $C^{(v)}$. (b) An illustration of C_1 and C_2.

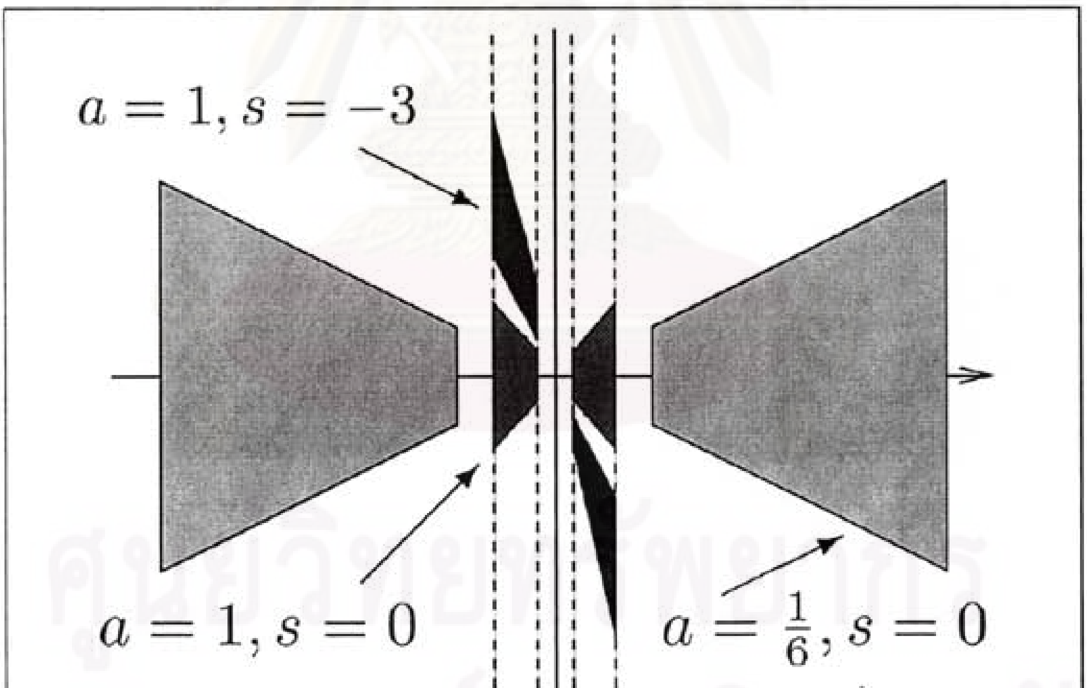


Figure 2.1.2: Support of the shearlets ψ_{ast} (in the frequency domain) for different values of a and s. (from [22])

Finally, let $W(x)$ be such that $\hat{W}(\xi) \in C^\infty(\mathbb{R}^2)$ and

$$|\hat{W}(\xi)|^2 + \chi_{C_1}(\boldsymbol{\xi}) \int_0^1 |\hat{\psi}_1(a\xi_1)|^2 \frac{da}{a} + \chi_{C_2}(\boldsymbol{\xi}) \int_0^1 |\hat{\psi}_1(a\xi_2)|^2 \frac{da}{a} = 1, \text{ for a.e. } \boldsymbol{\xi} \in \mathbb{R}^2,$$

where $C_1 = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\frac{\xi_2}{\xi_1}| \le 1\right\}$ and $C_2 = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\frac{\xi_2}{\xi_1}| > 1\right\}$, both of which are pictured in Figure 2.1.1(b).

Then it follows that W is a C^∞-window function in $\mathbb{R}^2$ with $\hat{W}(\boldsymbol{\xi}) = 1$ for $\boldsymbol{\xi} \in [-1/2, 1/2]^2$, $\hat{W} = 0$ outside the box $[-2, 2]^2$. Finally, let $(P_{C_1} f)^\frown = \hat{f}\chi_{C_1}$ and $(P_{C_2} f)^\frown = \hat{f}\chi_{C_2}$. Then, for each $f \in L^2(\mathbb{R}^2)$ we have

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \langle W(\cdot - \boldsymbol{t}), f\rangle\, W(\boldsymbol{x} - \boldsymbol{t})\, d\boldsymbol{t} + \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 \langle \psi_{ast}, P_{C_1} f\rangle\, \psi_{ast}(\boldsymbol{x})\, \frac{da}{a^3}\, ds\, dt$$
$$+ \int_{\mathbb{R}^2} \int_{-2}^{2} \int_0^1 \left\langle \psi_{ast}^{(v)}, P_{C_2} f\right\rangle \psi_{ast}^{(v)}(\boldsymbol{x})\, \frac{da}{a^3}\, ds\, d\boldsymbol{t}.$$

The functions ψ_{ast} and $\psi_{ast}^{(v)}$ are actually very similar. We will show in chapter III that ψ_{ast} and $\psi_{ast}^{(v)}$ have essentially the same decay properties (see Lemma 3.2.2).

2.2 Discrete Shearlet Transform

By modifying the construction of the previous section, K. Guo, G. Kutyniok and D. Labate [8] obtain a Parseval frame for functions whose Fourier transform is supported in the cone

$$\tilde{C} = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \ge \frac{1}{4}, \left|\frac{\xi_2}{\xi_1}\right| \le 1\right\} \tag{II.5}$$

They sample the scaling parameter of the continuous shearlet transform by choosing $a_j = 2^{-2j}, j \ge 0$, $s_{jk} = k\sqrt{a_j} = k2^{-j}, -2^j \le k \le 2^j$ and $\boldsymbol{t}_{jkm} = \mathrm{B}_{s_{jk}} \mathrm{D}_{a_j} \boldsymbol{m}$, $\boldsymbol{m} \in \mathbb{Z}^2$.

Note It is straightforward to verify that $T_{B_{s_{jk}} D_{a_j} \boldsymbol{m}} D_{B_{s_{jk}} D_{a_j}} = D_{D_{a_j} B_k} T_{\boldsymbol{m}}$. In fact,

$$T_{B_{s_{jk}} D_{a_j} \boldsymbol{m}} D_{B_{s_{jk}} D_{a_j}} \psi(\boldsymbol{x}) = a_j^{-\frac{3}{4}} \psi((B_{s_{jk}} D_{a_j})^{-1}(\boldsymbol{x} - B_{s_{jk}} D_{a_j} \boldsymbol{m})$$

$$
\begin{aligned}
&= a_j^{-\frac{3}{4}} \psi \left(\left(\begin{pmatrix} 1 & -k\sqrt{a_j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_j & 0 \\ 0 & \sqrt{a_j} \end{pmatrix} \right)^{-1} \boldsymbol{x} - D_{a_j}^{-1} B_{-s_{jk}} B_{s_{jk}} D_{a_j} \boldsymbol{m} \right) \\
&= a_j^{-\frac{3}{4}} \psi \left(\left(\begin{pmatrix} a_j & -ka_j \\ 0 & \sqrt{a_j} \end{pmatrix} \right)^{-1} \boldsymbol{x} - \boldsymbol{m} \right) \\
&= a_j^{-\frac{3}{4}} \psi \left(\left(\begin{pmatrix} a_j & 0 \\ 0 & \sqrt{a_j} \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \right)^{-1} \boldsymbol{x} - \boldsymbol{m} \right) \\
&= D_{D_{a_j} B_k} T_{\boldsymbol{m}} \psi(\boldsymbol{x}).
\end{aligned}
$$

Hence, using the calculation similar to the previous section, we obtain the discrete system

$$\{\psi_{jk\boldsymbol{m}}(\boldsymbol{x}) = 2^{\frac{3j}{2}} \psi(\mathrm{B}_{-k} \mathrm{D}_{2^{2j}} \boldsymbol{x} - \boldsymbol{m}) : j \geq 0, -2^j \leq k \leq 2^j, \boldsymbol{m} \in \mathbb{Z}^2\}, \tag{II.6}$$

when $\psi \in L^2(\mathbb{R}^2)$ is defined as

$$\widehat{\psi}(\boldsymbol{\xi}) = \widehat{\psi_1}(\xi_1) \widehat{\psi_2}(\frac{\xi_2}{\xi_1}) \tag{II.7}$$

with $\psi_1 \in L^2(\mathbb{R})$, $\widehat{\psi_1} \in C^\infty(\mathbb{R})$, $\operatorname{supp} \widehat{\psi_1} \subset [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ satisfying

$$\sum_{j \geq 0} |\widehat{\psi_1}(2^{-2j} \xi_1)|^2 = 1 \quad \text{for } |\xi_1| \geq \frac{1}{4}, \tag{II.8}$$

and $\psi_2 \in L^2(\mathbb{R})$, $\widehat{\psi_2} \in C^\infty(\mathbb{R})$, $\operatorname{supp} \widehat{\psi_2} \subset [-1, 1]$ satisfying

$$|\widehat{\psi_2}(\xi_2 - 1)|^2 + |\widehat{\psi_2}(\xi_2)|^2 + |\widehat{\psi_2}(\xi_2 + 1)|^2 = 1 \quad \text{for } |\xi_2| \leq 1. \tag{II.9}$$

It follows that, for any $j \geq 0$,

$$\sum_{k=2^j}^{2^j} |\widehat{\psi_2}(2^j \xi_2 + k)|^2 = 1 \quad \text{for } |\xi_2| \leq 1. \tag{II.10}$$

Let us quote a result proved by K. Guo, D. Labate, W. Lim, G. Weiss, and E. Wilson [14].

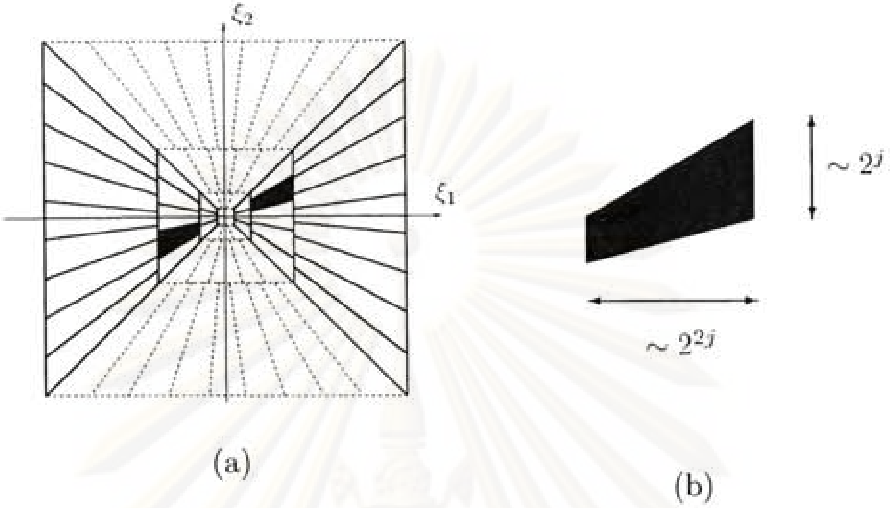


Figure 2.2.1: (a) The tiling of the frequency plane $\hat{\mathbb{R}}^2$ induced by the shearlets. The tiling of $\tilde{C}$ is illustrated in solid line, the tiling of $\tilde{C}^{(v)}$is in dashed line. (b) The frequency support of a shearlet ψ_{jkm} satisfies parabolic scaling. (from [10])

Lemma 2.2.1. *Let $\psi \in L^2(\mathbb{R}^2)$ be such that* $\operatorname{supp}\widehat{\psi} \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ *and*

$$\sum_{j,k\in\mathbb{Z}} \left|\widehat{\psi}(\boldsymbol{\xi} a^j b^k)\right|^2 = 1 \ a.e.\ \boldsymbol{\xi} \in \widehat{\mathbb{R}}^2$$

where $a, b \in GL_n(\mathbb{R}^2)$. Then the system $\{\psi_{jkm}(\boldsymbol{x}) = D_a^j D_b^k \psi(\boldsymbol{x} - \boldsymbol{m}) : j, k \in \mathbb{Z}, \boldsymbol{m} \in \mathbb{Z}^2\}$ is a Parseval frame for $L^2(\mathbb{R}^2)$.

Proof. Using the hypotheses on ψ, Plancherel theorem, the change of variable $\boldsymbol{\eta} = \boldsymbol{\xi} a^j b^k$, and Parseval theorem , then, for each $f \in L^2(\mathbb{R}^2)$, we have that

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} |< f, \psi_{jk\boldsymbol{m}} >|^2 \\
&\quad= \sum_{j,k\in\mathbb{Z}} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \left|< \hat{f}, \widehat{\psi_{jk\boldsymbol{m}}} >\right|^2 \\
&\quad= \sum_{j,k\in\mathbb{Z}} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \left|\int_{\widehat{\mathbb{R}^2}} \hat{f}(\boldsymbol{\xi}) \overline{\widehat{\psi}(\boldsymbol{\xi} a^j b^k)} e^{2\pi i \boldsymbol{\xi} a^j b^k \boldsymbol{m}} |\det a|^{\frac{j}{2}} |\det b|^{\frac{k}{2}} d\boldsymbol{\xi}\right|^2
\end{aligned}$$

$$= \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left| \int_Q \hat{f}(\boldsymbol{\eta} b^{-k} a^{-j}) \overline{\widehat{\psi}(\boldsymbol{\eta})} e^{2\pi i \eta m} |\det a|^{-\frac{j}{2}} |\det b|^{-\frac{k}{2}} d\boldsymbol{\eta} \right|^2$$

$$= \sum_{j,k\in\mathbb{Z}} \int_Q |\hat{f}(\boldsymbol{\eta} b^{-k} a^{-j})|^2 |\widehat{\psi}(\boldsymbol{\eta})|^2 |\det a|^{-j} |\det b|^{-k} d\boldsymbol{\eta}$$

$$= \sum_{j,k\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^2}} |\hat{f}(\boldsymbol{\xi})|^2 |\widehat{\psi}(\boldsymbol{\xi} a^j b^k)|^2 d\boldsymbol{\xi}$$

$$= \int_{\widehat{\mathbb{R}^2}} |\hat{f}(\boldsymbol{\xi})|^2 \left(\sum_{j,k\in\mathbb{Z}} |\widehat{\psi}(\boldsymbol{\xi} a^j b^k)|^2 \right) d\boldsymbol{\xi}$$

$$= \|f\|^2$$

when $Q = \widehat{\mathbb{R}^2} a^j b^k$. □

In 2006, Guo et al.[8] used this Lemma to show that (II.6) is a Parseval frame for $L^2(\tilde{C})^{\vee}$.

Theorem 2.2.2. *Let $\psi \in L^2(\mathbb{R}^2)$ be given in (II.7), where ψ_1 and ψ_2 satisfy (II.8) and (II.9), respectively. Then the system (II.6) is a Parseval frame for $L^2(\tilde{C})^{\vee}$.*

Proof. Using the hypotheses on ψ_1 and ψ_2, we obtain

$$\begin{aligned}
\sum_{j,k\in\mathbb{Z}} \left|\widehat{\psi}(\boldsymbol{\xi} A^j B^k)\right|^2 &= \sum_{j,k\in\mathbb{Z}} \left|\widehat{\psi}(\xi_1 2^j, k2^j\xi_1 + 2^{\frac{j}{2}}\xi_2)\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}} \left|\widehat{\psi_1}(\xi_1 2^j)\right|^2 \left|\widehat{\psi_2}(k + 2^{\frac{-j}{2}}\frac{\xi_2}{\xi_1})\right|^2 \\
&= \sum_{j\in\mathbb{Z}} \left|\widehat{\psi_1}(\xi_1 2^j)\right|^2 \sum_{k\in\mathbb{Z}} \left|\widehat{\psi_2}(k + 2^{\frac{-j}{2}}\frac{\xi_2}{\xi_1})\right|^2 \\
&= 1
\end{aligned}$$

By Lemma 2.2.1, the system (II.6) is Parseval frame for $L^2(\mathbb{R}^2)$. □

In order to obtain a reproducing system for the larger space $L^2(\mathbb{R}^2)$, we will choose $\psi^{(v)} \in L^2(\mathbb{R}^2)$ such that

$$\widehat{\psi^{(v)}}(\boldsymbol{\xi}) = \widehat{\psi_1}(\xi_2)\widehat{\psi_2}(\frac{\xi_1}{\xi_2}).$$

We can construct in a similar way a Parseval frame for $L^2(\tilde{C}^{(v)})^{\vee}$, where

$$\tilde{C}^{(v)} = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \geq \frac{1}{4}, \left|\frac{\xi_1}{\xi_2}\right| < 1\right\}.$$

Finally, a Parseval frame for $L^2(\tilde{T})^{\vee}$ where

$$\tilde{T} = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| < \frac{1}{4} \text{ and } |\xi_2| < \frac{1}{4}\right\}$$

can be constructed by taking translates of an appropriate smooth window function. In the next Chapter, we will show that ψ_{ast} and $\psi_{ast}^{(v)}$ have essentially the same decay properties (see Lemma 3.3.2).

CHAPTER III

VANISHING DIRECTIONAL MOMENTS AND DECAY PROPERTIES

In this section we shall investigate vanishing directional moments and decay properties of ψ_{ast} and $\psi_{ast}^{(v)}$. This properties will be needed in proving theorems in Chapter IV.

3.1 Directional Vanishing Moments and Properties of Shearlet Function

Let us defined definition of an L-order vanishing directional moments along a direction $\boldsymbol{v}$.

Definition 3.1.1. A function f of two variables is said to have an *L-order directional vanishing moments along a direction* $\boldsymbol{v} = (v_1, v_2)^T \neq \boldsymbol{0}$ if

$$\int_{\mathbb{R}} b^n f(b\boldsymbol{v} + \boldsymbol{w})db = 0, \quad \text{for all } \boldsymbol{w} \in \mathbb{R}^2 \text{ and } 0 \leq n < L.$$

Observe that this is a general definition which reduces to the definition in [27] when $v_1 \neq 0$. The above definition means essentially that any 1-D slices of the function have vanishing moments of order L. Notice from the definition that f has vanishing directional moment along direction $\boldsymbol{v}$ if and only if the same holds along direction $-\boldsymbol{v}$.

In the following proposition we show a relation between vanishing directional moment property of ψ_{ast} and ψ_{a0t}. Moreover, this property is also valid for vertical shearlet system.

Proposition 3.1.2. *Let* $a \in (0,1), \boldsymbol{t} \in \mathbb{R}^2$, $s \in [-2,2]$ *and* $\boldsymbol{v} = (v_1, v_2)^T$. *Then*

(i) ψ_{ast} *has vanishing directional moments along the direction* $\boldsymbol{v}$ *if and only if* ψ_{a0t} *has vanishing directional moments along the direction*

$$\mathrm{B}_{-s}\boldsymbol{v} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + sv_2 \\ v_2 \end{pmatrix},$$

(ii) $\psi_{ast}^{(v)}$ *has vanishing directional moments along the direction* $\boldsymbol{v}$ *if and only if* $\psi_{a0t}^{(v)}$ *has vanishing directional moments along the direction*

$$\mathrm{B}_{-s}^{(v)}\boldsymbol{v} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ sv_1 + v_2 \end{pmatrix}$$

Proof. By a straightforward computation, we have

$$\begin{aligned}
\int_{\mathbb{R}} b^n \psi_{a0t}\left(b\mathrm{B}_{-s}\boldsymbol{v} + \boldsymbol{w}\right) db &= \int_{\mathbb{R}} b^n \psi_{a0t}\left(b\begin{pmatrix} v_1 + sv_2 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) db \\
&= \int_{\mathbb{R}} b^n \psi_{as0}\left(\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} bv_1 \\ bv_2 \end{pmatrix} + \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \boldsymbol{t}\right)\right) db \\
&= \int_{\mathbb{R}} b^n \psi_{as0}\left(\begin{pmatrix} bv_1 \\ bv_2 \end{pmatrix} + \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 - t_1 \\ w_2 - t_2 \end{pmatrix}\right) db \\
&= \int_{\mathbb{R}} b^n \psi_{as0}\left(b\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 - t_1 - s(w_2 - t_2) \\ w_2 - t_2 \end{pmatrix}\right) db \\
&= \int_{\mathbb{R}} b^n \psi_{ast}\left(b\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 - s(w_2 - t_2) \\ w_2 \end{pmatrix}\right) db
\end{aligned}$$

$$= \int_{\mathbb{R}} b^n \psi_{ast} \left(b\boldsymbol{v} + \mathrm{B}_s \boldsymbol{w} - \begin{pmatrix} 0 & -s \\ 0 & 0 \end{pmatrix} \boldsymbol{t} \right) db$$

which implies that

$$\int_{\mathbb{R}} b^n \psi_{a0t} \left(b\mathrm{B}_{-s}\boldsymbol{v} + \boldsymbol{w} \right) db = \int_{\mathbb{R}} b^n \psi_{ast} \left(b\boldsymbol{v} + \boldsymbol{u} \right) db$$

where $\boldsymbol{u} = \mathrm{B}_s \boldsymbol{w} - \begin{pmatrix} 0 & -s \\ 0 & 0 \end{pmatrix} \boldsymbol{t}$.

Similarly, we have

$$\int_{\mathbb{R}} b^n \psi^{(v)}_{a0t} \left(b\mathrm{B}^{(v)}_{-s}\boldsymbol{v} + \boldsymbol{w} \right) db = \int_{\mathbb{R}} b^n \psi^{(v)}_{ast} \left(b\boldsymbol{v} + \boldsymbol{u}^{(v)} \right) db$$

where $\boldsymbol{u}^{(v)} = \mathrm{B}^{(v)}_s \boldsymbol{w} - \begin{pmatrix} 0 & 0 \\ -s & 0 \end{pmatrix} \boldsymbol{t}$.

(i) Assume that ψ_{ast} has vanishing directional moments along the direction $\boldsymbol{v}$. Then, for each $\boldsymbol{w} \in \mathbb{R}^2$ and $0 \leq n < L$,

$$\int_{\mathbb{R}} b^n \psi_{a0t} \left(b\mathrm{B}_{-s}\boldsymbol{v} + \boldsymbol{w} \right) db = \int_{\mathbb{R}} b^n \psi_{ast} \left(b\boldsymbol{v} + \boldsymbol{u} \right) db = 0.$$

Conversely, assume that ψ_{a0t} has vanishing directional moments along the direction $\mathrm{B}_{-s}\boldsymbol{v}$. Let $\boldsymbol{u} \in \mathbb{R}^2$ and $\boldsymbol{w} = \mathrm{B}_{-s} \left(\boldsymbol{u} + \begin{pmatrix} 0 & -s \\ 0 & 0 \end{pmatrix} \boldsymbol{t} \right)$.

Then $\boldsymbol{u} = \mathrm{B}_s \boldsymbol{w} - \begin{pmatrix} 0 & -s \\ 0 & 0 \end{pmatrix} \boldsymbol{t}$. And, for $0 \leq n < L$,

$$\int_{\mathbb{R}} b^n \psi_{ast} \left(b\boldsymbol{v} + \boldsymbol{u} \right) db = \int_{\mathbb{R}} b^n \psi_{a0t} \left(b\mathrm{B}_{-s}\boldsymbol{v} + \boldsymbol{w} \right) db = 0.$$

(ii) Assume that $\psi^{(v)}_{ast}$ has vanishing directional moments along the direction $\boldsymbol{v}$. Then, for each $\boldsymbol{w} \in \mathbb{R}^2$ and $0 \leq n < L$,

$$\int_{\mathbb{R}} b^n \psi^{(v)}_{a0t} \left(b\mathrm{B}_{-s}\boldsymbol{v} + \boldsymbol{w} \right) db = \int_{\mathbb{R}} b^n \psi^{(v)}_{ast} \left(b\boldsymbol{v} + \boldsymbol{u}^{(v)} \right) db = 0.$$

Conversely, assume that $\psi_{a0t}^{(v)}$ has vanishing directional moments along the direction $\mathrm{B}_{-s}^{(v)}\boldsymbol{v}$. Let $\boldsymbol{u}^{(v)} \in \mathbb{R}^2$ and $\boldsymbol{w} = \mathrm{B}_{-s}^{(v)}\left(\boldsymbol{u}^{(v)} + \begin{pmatrix} 0 & 0 \\ -s & 0 \end{pmatrix}\boldsymbol{t}\right)$.

Then $\boldsymbol{u} = \mathrm{B}_{s}^{(v)}\boldsymbol{w} - \begin{pmatrix} 0 & 0 \\ -s & 0 \end{pmatrix}\boldsymbol{t}$. And, for $0 \leq n < L$,

$$\int_{\mathbb{R}} b^n \psi_{ast}^{(v)}\left(b\boldsymbol{v} + \boldsymbol{u}^{(v)}\right)\, db = \int_{\mathbb{R}} b^n \psi_{a0t}^{(v)}\left(b\mathrm{B}_{-s}\boldsymbol{v} + \boldsymbol{w}\right)\, db = 0.$$

□

3.2 Continuous Shearlet Transform

Lemmas 3.2.1 and 3.2.2 are useful for analysis of the Hölder regularity and directional regularity of function by continuous shearlet transforms.

In Lemma 3.2.1 we found what is condition that ψ_{ast} and $\psi_{ast}^{(v)}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$.

Lemma 3.2.1. *Let $a \in (0,1), \boldsymbol{t} \in \mathbb{R}^2$, $s \in [-2,2]$ and $\boldsymbol{v} = (v_1, v_2)^T$ be a unit vector.*

(i) If $\left|s + \frac{v_1}{v_2}\right| > \sqrt{a}$ then shearlet functions ψ_{ast} have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$.

(ii) If $\left|s + \frac{v_2}{v_1}\right| > \sqrt{a}$ then shearlet functions $\psi_{ast}^{(v)}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$.

Here, if v_2 or v_1 is 0 then $\frac{v_1}{v_2}$ or $\frac{v_2}{v_1}$, respectively, are treated as ∞ so that the assumed inequality holds for all $a \in (0,1)$ and $s \in [-2,2]$. For instance, if $v_2 = 0$, then for any $a \in (0,1)$, $s \in [-2,2]$ and $\boldsymbol{t} \in \mathbb{R}^2$ ψ_{ast} have vanishing directional *moments of any order $L < \infty$ along the direction $\boldsymbol{v}$ and if $v_1 = 0$, then for any $a \in (0,1)$, $s \in [-2,2]$ and $\boldsymbol{t} \in \mathbb{R}^2$ $\psi_{ast}^{(v)}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$.*

Proof. (i) Because $\text{supp}(\widehat{\psi_{a0t}}) \subseteq \left\{(\xi_1,\xi_2) : \frac{1}{2a} \le |\xi_1| \le \frac{2}{a} \text{ and } |\xi_2| \le \sqrt{a}|\xi_1|\right\}$ and $\left|s + \frac{v_1}{v_2}\right| > \sqrt{a}$, it follows that $\text{supp}(\widehat{\psi_{a0t}}) \cap \{(\xi_1,\xi_2) \colon \xi_2 = (s+\frac{v_1}{v_2})\xi_1\} = \varnothing$. Note that $\text{B}_s^{(v)} \text{supp}(\widehat{\psi_{a0t}}) = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} \text{supp}(\widehat{\psi_{a0t}}) = \text{supp}(\widehat{\psi_{ast}})$ and so

$$\begin{aligned}
&\text{supp}(\widehat{\psi_{a0t}}) \cap \{(\xi_1,\xi_2) \colon \xi_2 = (s+\frac{v_1}{v_2})\xi_1\} = \varnothing \\
\iff &\text{B}^{(v)}_{(s+\frac{v_1}{v_2})} \text{supp}(\widehat{\psi_{a0t}}) \cap \text{B}^{(v)}_{(s+\frac{v_1}{v_2})} \{(\xi_1,\xi_2) \colon \xi_2 = (s+\frac{v_1}{v_2})\xi_1\} = \varnothing \\
\iff &\text{supp}(\widehat{\psi_{a(s+\frac{v_1}{v_2})t}}) \cap \{(\xi_1,\xi_2) \colon \xi_2 = 0\} = \varnothing.
\end{aligned}$$

Consequently, we have that all partial derivatives of $\widehat{\psi_{a(s+\frac{v_1}{v_2})t}}$ vanish on the ξ_1-axis. Next, we show that $\psi_{a(s+\frac{v_1}{v_2})t}$ has vanishing directional moments along the direction of x_2-axis of any order L. Let $g(x_1) := \int x_2^n \psi_{a(s+\frac{v_1}{v_2})t}(x_1,x_2)dx_2$. For each $\xi_1 \in \hat{\mathbb{R}}$,

$$\begin{aligned}
\hat{g}(\xi_1) &= \int g(x_1) e^{-2\pi i x_1 \xi_1}\, dx_1 \\
&= \int\int x_2^n \psi_{a(s+\frac{v_1}{v_2})t}(x_1,x_2) e^{-2\pi i x_1 \xi_1}\, dx_1\, dx_2 \\
&= \left(x_2^n \psi_{a(s+\frac{v_1}{v_2})t}\right)^{\wedge}(\xi_1, 0) \\
&= (-2\pi i)^{-n} \partial_2^n \widehat{\psi_{a(s+\frac{v_1}{v_2})t}}(\xi_1, 0) \\
&= 0,
\end{aligned}$$

which implies that $g(x_1) \equiv 0$. Therefore, $\psi_{a(s+\frac{v_1}{v_2})t}$ has vanishing moments along the direction $(0,1)^T$. Hence ψ_{ast} has vanishing moments along the direction $\text{B}_{-\frac{v_1}{v_2}}(0,1)^T = \begin{pmatrix} 1 & \frac{v_1}{v_2} \\ 0 & 1 \end{pmatrix} (0,1)^T = (\frac{v_1}{v_2}, 1)^T$, i.e. ψ_{ast} has vanishing moments along the direction $\boldsymbol{v}$.

Finally, If $v_2 = 0$, then we use the fact that, for all $a > 0$, $\boldsymbol{t} \in \mathbb{R}^2$ and $s \in [-2, 2]$, $\text{supp}(\widehat{\psi_{ast}}) \cap \{(\xi_1, \xi_2) \colon \xi_1 = 0\} = \varnothing$ and so, by the same line of proof, we have ψ_{ast} has vanishing moments along the direction $(1,0)^T$.

(ii) Because $\text{supp}(\widehat{\psi^{(v)}_{a0t}}) \subseteq \{(\xi_1, \xi_2) : \frac{1}{2a} \le |\xi_2| \le \frac{2}{a} \text{ and } |\xi_1| \le \sqrt{a}|\xi_2|\}$ and $\left|s + \frac{v_2}{v_1}\right| > \sqrt{a}$, it follows that $\text{supp}(\widehat{\psi^{(v)}_{a0t}}) \cap \{(\xi_1, \xi_2) \colon \xi_1 = (s + \frac{v_2}{v_1})\xi_2\} = \varnothing$. Note that $\mathrm{B}_s \,\text{supp}(\widehat{\psi^{(v)}_{a0t}}) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \text{supp}(\widehat{\psi^{(v)}_{a0t}}) = \text{supp}(\widehat{\psi^{(v)}_{ast}})$ and so

$$
\begin{aligned}
&\text{supp}(\widehat{\psi^{(v)}_{a0t}}) \cap \{(\xi_1, \xi_2) \colon \xi_1 = (s + \frac{v_2}{v_1})\xi_2\} = \varnothing \\
\iff &\mathrm{B}_{(s+\frac{v_2}{v_1})} \,\text{supp}(\widehat{\psi^{(v)}_{a0t}}) \cap \mathrm{B}_{(s+\frac{v_2}{v_1})}\{(\xi_1, \xi_2) \colon \xi_2 = (s + \frac{v_1}{v_2})\xi_1\} = \varnothing \\
\iff &\text{supp}(\widehat{\psi_{a(s+\frac{v_1}{v_2})t}}) \cap \{(\xi_1, \xi_2) \colon \xi_1 = 0\} = \varnothing.
\end{aligned}
$$

Consequently, we have that all partial derivatives of $\widehat{\psi^{(v)}_{a(s+\frac{v_1}{v_2})t}}$ vanish on the ξ_2-axis. Next, we show that $\psi^{(v)}_{a(s+\frac{v_1}{v_2})t}$ has vanishing directional moments along the direction of x_1-axis of any order L. Let $g(x_2) := \int x_1^n \psi^{(v)}_{a(s+\frac{v_2}{v_1})t}(x_1, x_2) dx_1$. For each $\xi_2 \in \hat{\mathbb{R}}$,

$$
\begin{aligned}
\hat{g}(\xi_2) &= \int g(x_2) e^{-2\pi i x_2 \xi_2}\, dx_2 \\
&= \int\int x_1^n \psi^{(v)}_{a(s+\frac{v_2}{v_1})t}(x_1, x_2) e^{-2\pi i x_2 \xi_2}\, dx_1\, dx_2 \\
&= \left(x_1^n \psi_{a(s+\frac{v_2}{v_1})t}\right)^{\widehat{}}(0, \xi_2) \\
&= (-2\pi i)^{-n} \partial_1^n \widehat{\psi_{a(s+\frac{v_2}{v_1})t}}(0, \xi_2) \\
&= 0,
\end{aligned}
$$

which implies that $g(x_2) \equiv 0$. Therefore, $\psi^{(v)}_{a(s+\frac{v_2}{v_1})t}$ has vanishing moments along the direction $(1,0)^T$. Hence $\psi^{(v)}_{ast}$ has vanishing moments along the direction $\mathrm{B}^{(v)}_{-\frac{v_2}{v_1}}(1,0)^T = \begin{pmatrix} 1 & 0 \\ \frac{v_1}{v_2} & 1 \end{pmatrix} (1,0)^T = (\frac{v_2}{v_1}, 1)^T$, i.e. $\psi^{(v)}_{ast}$ has vanishing moments along the direction $\boldsymbol{v}$.

Finally, If $v_1 = 0$, then we use the fact that, for all $a > 0$, $\boldsymbol{t} \in \mathbb{R}^2$ and $s \in [-2,2]$, $\operatorname{supp}(\widehat{\psi^{(v)}_{ast}}) \cap \{(\xi_1, \xi_2) \colon \xi_2 = 0\} = \varnothing$ and so, by the same line of proof, we have $\psi^{(v)}_{ast}$ has vanishing moments along the direction $(0,1)^T$. □

In the following lemma we obtain a decay property of all partial derivatives of shearlet functions.

Lemma 3.2.2. *For each $N = 1, 2, \ldots$ there is a constant C_N such that for all $a \in (0,1)$, $s \in [-2,2]$, $\boldsymbol{t} \in \mathbb{R}^2$, $\boldsymbol{x} \in \mathbb{R}^2$ and $\nu \in \mathbb{N}_0^2$*

$$|\partial^\nu \psi_{ast}(\boldsymbol{x})| \leq \frac{C_N a^{-3/4-|\nu|}(\sqrt{a}+|s|)^{\nu_2}}{1 + \left\|\mathrm{D}_{1/a}\mathrm{B}_{-s}(\boldsymbol{x}-\boldsymbol{t})\right\|^{2N}} \tag{III.1}$$

and

$$\left|\partial^\nu \psi^{(v)}_{ast}(\boldsymbol{x})\right| \leq \frac{C_N a^{-3/4-|\nu|}(\sqrt{a}+|s|)^{\nu_1}}{1 + \left\|\mathrm{D}^{(v)}_{1/a}\mathrm{B}^{(v)}_{-s}(\boldsymbol{x}-\boldsymbol{t})\right\|^{2N}}. \tag{III.2}$$

Proof. Firstly, we will prove the equation (III.1). We restrict first to the case $s = 0$ and $\boldsymbol{t} = \boldsymbol{0}$. Fix an index vector $\nu := (\nu_1, \nu_2)$ and define

$$h_a(\boldsymbol{x}) := \psi_{a\mathbf{0}0}(\mathrm{D}_a\boldsymbol{x}) \quad \text{and} \quad g_a(\boldsymbol{x}) := \partial^\nu h_a(\boldsymbol{x}) = a^{\nu_1+\frac{\nu_2}{2}}(\partial^\nu \psi_{a\mathbf{0}0})(\mathrm{D}_a\boldsymbol{x}).$$

By a straightforward computation we have

$$\begin{aligned}
\hat{g}_a(\boldsymbol{\xi}) &= (2\pi i\boldsymbol{\xi})^\nu \hat{h}_a(\boldsymbol{\xi}) \\
&= (2\pi i\boldsymbol{\xi})^\nu a^{-3/2}\widehat{\psi}_{a\mathbf{0}0}(\mathrm{D}_{1/a}\boldsymbol{\xi}) \\
&= (2\pi i\boldsymbol{\xi})^\nu a^{-3/2}a^{-3/4}\hat{\psi}(\boldsymbol{\xi})a^{3/2} \\
&= (2\pi i\boldsymbol{\xi})^\nu a^{-3/4}\hat{\psi}(\boldsymbol{\xi}).
\end{aligned}$$

Now, replacing $\boldsymbol{x}$ by $\mathrm{D}_{1/a}\boldsymbol{x}$ in the equation $(-4\pi^2\|\boldsymbol{x}\|^2)^k g_a(\boldsymbol{x}) = \int_{\mathbb{R}^2} \Delta^k \hat{g}_a(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$, where Δ is the Laplacian, yields

$$\left|(-4\pi^2\left\|\mathrm{D}_{1/a}\boldsymbol{x}\right\|^2)^k(\partial^\nu \psi_{a\mathbf{0}0})(\boldsymbol{x})\right| = \left|(-4\pi^2\left\|\mathrm{D}_{1/a}\boldsymbol{x}\right\|^2)^k a^{-(\nu_1+\nu_2/2)} g_a(\mathrm{D}_{1/a}\boldsymbol{x})\right|$$

$$= a^{-(\nu_1+\nu_2/2)} \left| \int_{\mathbb{R}^2} (\Delta^k \hat{g}_a)(\boldsymbol{\xi}) e^{2\pi i \mathrm{D}_{1/a}\boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi} \right|$$
$$\leq a^{-(\nu_1+\nu_2/2)} \int_{\mathbb{R}^2} |(\Delta^k \hat{g}_a)(\boldsymbol{\xi})| \, d\boldsymbol{\xi}$$
$$= a^{-(3/4+\nu_1+\nu_2/2)} \int_{\mathbb{R}^2} \left| \Delta^k((2\pi i\boldsymbol{\xi})^\nu \hat{\psi}(\boldsymbol{\xi})) \right| d\boldsymbol{\xi}$$
$$\leq C_k a^{-(3/4+\nu_1+\nu_2/2)}.$$

In the last step we used the notification that $\int_{\mathbb{R}^2} \left| \Delta^k((2\pi i\boldsymbol{\xi})^\nu \hat{\psi}(\boldsymbol{\xi})) \right| d\boldsymbol{\xi} \leq C_k$ where C_k is in fact independent of k. Consequently, if $k = 0$ then we have inequality

$$|(\partial^\nu \psi_{a\mathbf{0}0})(\boldsymbol{x})| \leq C a^{-(3/4+\nu_1+\nu_2/2)}.$$

Since

$$(1 + (2\pi)^{2k} \left\| \mathrm{D}_{1/a}\boldsymbol{x} \right\|^{2k}) \left| \partial^\nu (\psi_{a\mathbf{0}0})(\boldsymbol{x}) \right|$$
$$= |(\partial^\nu \psi_{a\mathbf{0}0})(\boldsymbol{x})| + \left| (4\pi^2 \left\| \mathrm{D}_{1/a}\boldsymbol{x} \right\|^2)^k (\partial^\nu \psi_{a\mathbf{0}0})(\boldsymbol{x}) \right| \leq C a^{-(3/4+\nu_1+\nu_2/2)},$$

we have the inequality

$$|(\partial^\nu \psi_{a\mathbf{0}0})(\boldsymbol{x})| \leq \frac{C a^{-3/4-\nu_1-\nu_2/2}}{1 + (2\pi)^{2k} \left\| \mathrm{D}_{1/a}\boldsymbol{x} \right\|^{2k}} \leq \frac{C a^{-3/4-\nu_1-\nu_2/2}}{1 + \left\| \mathrm{D}_{1/a}\boldsymbol{x} \right\|^{2k}}.$$

Next, we show how to estimate for general $s \in \mathbb{R}$:

$$\partial_2^{\nu_2}\partial_1^{\nu_1}\psi_{as\mathbf{0}}(\boldsymbol{x}) = \partial_2^{\nu_2}\partial_1^{\nu_1-1}(\partial_1\psi_{as\mathbf{0}}(\boldsymbol{x}))$$
$$= \partial_2^{\nu_2}\partial_1^{\nu_1-1}(\partial_1\psi_{a\mathbf{0}0}(x_1 + sx_2, x_2))$$
$$= \partial_2^{\nu_2}(\partial_1^{\nu_1}\psi_{a\mathbf{0}0}(x_1 + sx_2, x_2))$$
$$= \sum_{l=0}^{\nu_2} \binom{\nu_2}{l} s^{\nu_2-l} \partial_1^{\nu_1+\nu_2-l} \partial_2^l \psi_{a\mathbf{0}0}(x_1 + sx_2, x_2).$$

Thus, we have

$$|\partial_2^{\nu_2}\partial_1^{\nu_1}\psi_{as\mathbf{0}}(\boldsymbol{x})| \leq \sum_{l=0}^{\nu_2} \binom{\nu_2}{l} |s|^{\nu_2-l} \left| \partial_1^{\nu_1+\nu_2-l} \partial_2^l \psi_{a\mathbf{0}0}(x_1 + sx_2, x_2) \right|$$

$$\leq \sum_{l=0}^{\nu_2} \begin{pmatrix} \nu_2 \\ l \end{pmatrix} |s|^{\nu_2 - l} \frac{Ca^{-3/4-\nu_1-\nu_2+l-\frac{l}{2}}}{1 + \left\| \mathrm{D}_{1/a}(x_1 + sx_2, x_2)^T \right\|^{2k}}$$

$$= \frac{Ca^{-3/4-\nu_1-\nu_2}}{1 + \left\| \mathrm{D}_{1/a}\mathrm{B}_{-s}(\boldsymbol{x}) \right\|^{2k}} \sum_{l=0}^{\nu_2} \begin{pmatrix} \nu_2 \\ l \end{pmatrix} |s|^{\nu_2 - l} a^{\frac{l}{2}}$$

$$= \frac{Ca^{-3/4-\nu_1-\nu_2}(\sqrt{a} + |s|)^{\nu_2}}{1 + \left\| \mathrm{D}_{1/a}\mathrm{B}_{-s}(\boldsymbol{x}) \right\|^{2k}}$$

$$= \frac{Ca^{-3/4-|\nu_1|}(\sqrt{a} + |s|)^{\nu_2}}{1 + \left\| \mathrm{D}_{1/a}\mathrm{B}_{-s}(\boldsymbol{x}) \right\|^{2k}}.$$

It is clear that all above hold also for a general $\boldsymbol{t}$ because translation does not change regularity properties. Then, the proof of equation(III.1) is complete.

Next, we will prove the equation (III.2). We restrict first to the case $s = 0$ and $\boldsymbol{t} = \mathbf{0}$. Fix an index vector $\nu := (\nu_1, \nu_2)$ and define

$$h_a^{(v)}(\boldsymbol{x}) := \psi_{a00}^{(v)}(\mathrm{D}_a^{(v)}\boldsymbol{x}) \quad \text{and} \quad g_a^{(v)}(\boldsymbol{x}) := \partial^\nu h_a^{(v)}(\boldsymbol{x}) = a^{\nu_2 + \frac{\nu_1}{2}} (\partial^\nu \psi_{a00}^{(v)})(\mathrm{D}_a^{(v)}\boldsymbol{x}).$$

By a straightforward computation we have

$$\hat{g}_a^{(v)}(\boldsymbol{\xi}) = (2\pi i \boldsymbol{\xi})^\nu \hat{h}_a^{(v)}(\boldsymbol{\xi})$$
$$= (2\pi i \boldsymbol{\xi})^\nu a^{-3/2} \widehat{\psi}_{a00}^{(v)}(\mathrm{D}_{1/a}^{(v)}\boldsymbol{\xi})$$
$$= (2\pi i \boldsymbol{\xi})^\nu a^{-3/2} a^{-3/4} \hat{\psi}^{(v)}(\boldsymbol{\xi}) a^{3/2}$$
$$= (2\pi i \boldsymbol{\xi})^\nu a^{-3/4} \hat{\psi}^{(v)}(\boldsymbol{\xi}).$$

Now, replacing $\boldsymbol{x}$ by $\mathrm{D}_{1/a}^{(v)}x$ in the equation $(-4\pi^2 \|\boldsymbol{x}\|^2)^k g_a^{(v)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \Delta^k \hat{g}_a^{(v)}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$, where Δ is the Laplacian, yields

$$\left| (-4\pi^2 \left\| \mathrm{D}_{1/a}^{(v)}\boldsymbol{x} \right\|^2)^k (\partial^\nu \psi_{a00}^{(v)})(\boldsymbol{x}) \right| = \left| (-4\pi^2 \left\| \mathrm{D}_{1/a}^{(v)}\boldsymbol{x} \right\|^2)^k a^{-(\nu_2+\nu_1/2)} g_a(\mathrm{D}_{1/a}^{(v)}\boldsymbol{x}) \right|$$
$$= a^{-(\nu_2+\nu_1/2)} \left| \int_{\mathbb{R}^2} (\Delta^k \hat{g}_a^{(v)})(\boldsymbol{\xi}) e^{2\pi i \mathrm{D}_{1/a}^{(v)}\boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi} \right|$$
$$\leq a^{-(\nu_2+\nu_1/2)} \int_{\mathbb{R}^2} \left| (\Delta^k \hat{g}_a^{(v)})(\boldsymbol{\xi}) \right| d\boldsymbol{\xi}$$

$$= a^{-(3/4+\nu_2+\nu_1/2)} \int_{\mathbb{R}^2} \left| \Delta^k((2\pi i \boldsymbol{\xi})^\nu \hat{\psi}(\boldsymbol{\xi})) \right| d\boldsymbol{\xi}$$
$$\leq C_k a^{-(3/4+\nu_2+\nu_1/2)}.$$

In the last step we used the notification that $\int_{\mathbb{R}^2} \left| \Delta^k((2\pi i \boldsymbol{\xi})^\nu \hat{\psi}^{(v)}(\boldsymbol{\xi})) \right| d\boldsymbol{\xi} \leq C_k$ where C_k is in fact independent of k. Consequently, if $k = 0$ then we have inequality

$$\left| (\partial^\nu \psi^{(v)}_{a00})(\boldsymbol{x}) \right| \leq C a^{-(3/4+\nu_2+\nu_1/2)}.$$

Since

$$\begin{aligned} &(1 + (2\pi)^{2k} \left\| \mathrm{D}^{(v)}_{1/a} \boldsymbol{x} \right\|^{2k}) \left| \partial^\nu (\psi^{(v)}_{a00})(\boldsymbol{x}) \right| \\ &= \left| (\partial^\nu \psi^{(v)}_{a00})(\boldsymbol{x}) \right| + \left| (4\pi^2 \left\| \mathrm{D}^{(v)}_{1/a} \boldsymbol{x} \right\|^2)^k (\partial^\nu \psi^{(v)}_{a00})(\boldsymbol{x}) \right| \leq C a^{-(3/4+\nu_2+\nu_1/2)}, \end{aligned}$$

we have the inequality

$$\left| (\partial^\nu \psi^{(v)}_{a00})(\boldsymbol{x}) \right| \leq \frac{C a^{-3/4-\nu_2-\nu_1/2}}{1 + (2\pi)^{2k} \left\| \mathrm{D}^{(v)}_{1/a} \boldsymbol{x} \right\|^{2k}} \leq \frac{C a^{-3/4-\nu_2-\nu_1/2}}{1 + \left\| \mathrm{D}^{(v)}_{1/a} \boldsymbol{x} \right\|^{2k}}.$$

Next, we show how to estimate for general $s \in \mathbb{R}$:

$$\begin{aligned} \partial_1^{\nu_1} \partial_2^{\nu_2} \psi^{(v)}_{as0}(\boldsymbol{x}) &= \partial_1^{\nu_1} \partial_2^{\nu_2 - 1} (\partial_2 \psi^{(v)}_{as0}(\boldsymbol{x})) \\ &= \partial_1^{\nu_1} \partial_2^{\nu_2 - 1} (\partial_2 \psi^{(v)}_{a00}(x_1, s x_1 + x_2)) \\ &= \partial_1^{\nu_1} (\partial_2^{\nu_2} \psi^{(v)}_{a00}(x_1, s x_1 + x_2)) \\ &= \sum_{l=0}^{\nu_1} \binom{\nu_1}{l} s^{\nu_1 - l} \partial_1^{\nu_2 + \nu_1 - l} \partial_1^l \psi^{(v)}_{a00}(x_1, s x_1 + x_2). \end{aligned}$$

Thus, we have

$$\begin{aligned} \left| \partial_1^{\nu_1} \partial_2^{\nu_2} \psi^{(v)}_{as0}(\boldsymbol{x}) \right| &\leq \sum_{l=0}^{\nu_1} \binom{\nu_1}{l} |s|^{\nu_1 - l} \left| \partial_2^{\nu_2 + \nu_1 - l} \partial_1^l \psi^{(v)}_{a00}(x_1, s x_1 + x_2) \right| \\ &\leq \sum_{l=0}^{\nu_1} \binom{\nu_1}{l} |s|^{\nu_1 - l} \frac{C a^{-3/4 - \nu_2 - \nu_1 + l - \frac{l}{2}}}{1 + \left\| \mathrm{D}^{(v)}_{1/a} (x_1, s x_1 + x_2)^T \right\|^{2k}} \end{aligned}$$

$$= \frac{Ca^{-3/4-\nu_1-\nu_2}}{1+\left\|\mathrm{D}_{1/a}^{(v)}\mathrm{B}_{-s}^{(v)}(\boldsymbol{x})\right\|^{2k}} \sum_{l=0}^{\nu_1} \binom{\nu_1}{l} |s|^{\nu_1-l} a^{\frac{l}{2}}$$
$$= \frac{Ca^{-3/4-\nu_1-\nu_2}(\sqrt{a}+|s|)^{\nu_1}}{1+\left\|\mathrm{D}_{1/a}^{(v)}\mathrm{B}_{-s}^{(v)}(\boldsymbol{x})\right\|^{2k}}$$
$$= \frac{Ca^{-3/4-|\nu|}(\sqrt{a}+|s|)^{\nu_1}}{1+\left\|\mathrm{D}_{1/a}^{(v)}\mathrm{B}_{-s}^{(v)}(\boldsymbol{x})\right\|^{2k}}.$$

It is clear that all above hold also for a general $\boldsymbol{t}$ because translation does not change regularity properties. Then, the proof of equation(III.2) is complete. □

3.3 Discrete Shearlet Transform

Lemma 3.3.1 and 3.3.2 are importent tools in the analysis of Hölder regularity and directional regularity of function by discrete shearlet transforms.

Because we sample the scaling parameter of the continuous shearlet transform by choosing $a_j = 2^{-2j}, j \geq 0$, $s_{jk} = k\sqrt{a_j} = k2^{-j}, -2^j \leq k \leq 2^j$ and $\boldsymbol{t}_{jkm} = B_{s_{jk}} D_{a_j} \boldsymbol{m}$, $\boldsymbol{m} \in \mathbb{Z}^2$, we have vanishing moments and decay property of ψ_{jkm} and $\psi^{(v)}_{jkm}$ by the same proof as those of Lemmas 3.2.1 and 3.2.2, respectively.

Lemma 3.3.1. *Let $j \geq 0, \boldsymbol{m} \in \mathbb{Z}^2$, $-2^j \leq k \leq 2^j$ and $\boldsymbol{v} = (v_1, v_2)^T$ be a unit vector.*

(i) if $\left|k2^{-j} + \frac{v_1}{v_2}\right| > 2^{-j}$ then shearlet functions ψ_{jkm} have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$,

(ii) if $\left|k2^{-j} + \frac{v_2}{v_1}\right| > 2^{-j}$ then shearlet functions $\psi^{(v)}_{jkm}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$.

Here, if v_2 or v_1 is 0 then $\frac{v_1}{v_2}$ or $\frac{v_2}{v_1}$, respectively, are treated as ∞ so that the assumed inequality holds for all $j \geq 0, \boldsymbol{m} \in \mathbb{Z}^2$, and $-2^j \leq k \leq 2^j$. For instance, *if $v_2 = 0$, then for any $j \geq 0, \boldsymbol{m} \in \mathbb{Z}^2$, and $-2^j \leq k \leq 2^j$ $\psi_{jk\boldsymbol{m}}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$ and if $v_1 = 0$, then for any $j \geq 0, \boldsymbol{m} \in \mathbb{Z}^2$, and $-2^j \leq k \leq 2^j$ $\psi^{(v)}_{jk\boldsymbol{m}}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$..*

Lemma 3.3.2. *For each $N = 1, 2, \ldots$ there is a constant C_N such that for all $\boldsymbol{x} \in \mathbb{R}^2$ and $\nu \in \mathbb{N}_0^2$,*

$$|\partial^\nu \psi_{j,k,\boldsymbol{m}}(\boldsymbol{x})| \leq \frac{C_N 2^{(3/2+2\nu_1+\nu_2)j}(1+|k|)^{\nu_2}}{1 + \|B_{-k}D_{2^{2j}}\boldsymbol{x} - \boldsymbol{m}\|^{2N}} \tag{III.3}$$

and

$$\left|\partial^\nu \psi^{(v)}_{j,k,\boldsymbol{m}}(\boldsymbol{x})\right| \leq \frac{C_N 2^{(3/2+\nu_1+2\nu_2)j}(1+|k|)^{\nu_1}}{1 + \|B_{-k}D_{2^{2j}}\boldsymbol{x} - \boldsymbol{m}\|^{2N}}. \tag{III.4}$$

Proof. Firstly, we will prove the equation (III.3). We restrict first to the case $k = 0$ and $\boldsymbol{m} = \boldsymbol{0}$. Fix an index vector $\nu := (\nu_1, \nu_2)$ and define

$$g_j(\boldsymbol{x}) := \psi_{j00}(\mathrm{D}_{2^{-2j}}\boldsymbol{x}) \quad \text{and} \quad h_j(\boldsymbol{x}) := \partial^\nu g_j(\boldsymbol{x}) = 2^{-(2\nu_1+\nu_2)j}(\partial^\nu \psi_{j00})(\mathrm{D}_{2^{-2j}}\boldsymbol{x}).$$

A straightforward computation yields

$$\begin{aligned} \hat{h}_j(\boldsymbol{\xi}) &= (i2\pi\boldsymbol{\xi})^\nu \hat{g}_j(\boldsymbol{\xi}) \\ &= (2\pi i\boldsymbol{\xi})^\nu 2^{3j}\widehat{\psi}_{j00}(\mathrm{D}_{2^{2j}}\boldsymbol{\xi}) \\ &= (2\pi i\boldsymbol{\xi})^\nu 2^{3j}2^{3j/2}\hat{\psi}(\boldsymbol{\xi})2^{-3j} \\ &= (2\pi i\boldsymbol{\xi})^\nu 2^{3j/2}\hat{\psi}(\boldsymbol{\xi}). \end{aligned} \tag{III.5}$$

Now, replacing $\boldsymbol{x}$ by $\mathrm{D}_{2^{2j}}\boldsymbol{x}$ in the equation $(-4\pi^2\|\boldsymbol{x}\|^2)^k g_j(\boldsymbol{x}) = \int_{\mathbb{R}^2} \Delta^k \hat{g}_j(\boldsymbol{\xi})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}$, where Δ is the Laplacian, yields

$$\begin{aligned} \left|(-4\pi^2\|\mathrm{D}_{2^{2j}}\boldsymbol{x}\|^2)^l 2^{-(2\nu_1+\nu_2)j}(\partial^\nu\psi_{j00})(\boldsymbol{x})\right| &= \left|(-4\pi^2\|\mathrm{D}_{2^{2j}}\boldsymbol{x}\|^2)^l h_j(\mathrm{D}_{2^{2j}}\boldsymbol{x})\right| \\ &= \left|\int_{\mathbb{R}^2}(\Delta^l\hat{h}_j)(\boldsymbol{\xi})e^{2\pi i\mathrm{D}_{2^{2j}}\boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}\right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^2} \left|(\Delta^l \hat{h}_j)(\boldsymbol{\xi})\right| \left|e^{2\pi i \mathrm{D}_{2^{2j}} \boldsymbol{x}\cdot\boldsymbol{\xi}}\right| d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^2} \left|(\Delta^l \hat{h}_j)(\boldsymbol{\xi})\right| d\boldsymbol{\xi} \\
&= 2^{3j/2} \int_{\mathbb{R}^2} \left|\Delta^l((2\pi i\boldsymbol{\xi})^\nu \hat{\psi}(\boldsymbol{\xi}))\right| d\boldsymbol{\xi} \\
&\leq C2^{3j/2}.
\end{aligned}$$

In the last step we used the notification that $\left|\Delta^l((2\pi i\boldsymbol{\xi})^\nu \hat{\psi}(\boldsymbol{\xi}))\right| \leq C$ where C is independent of a. When $k = 0$ this reduces to

$$\left|2^{-(2\nu_1+\nu_2)j}(\partial^\nu \psi_{j0\mathbf{0}})(\boldsymbol{x})\right| \leq C2^{3j/2}.$$

By combining the above estimates, we get

$$\begin{aligned}
&2^{-(2\nu_1+\nu_2)j}(1 + 4^l\pi^{2l} \left\|\mathrm{D}_{2^{2j}}\boldsymbol{x}\right\|^{2l}) \left|\partial^\nu \psi_{j0\mathbf{0}})(\boldsymbol{x})\right| \\
&= \left|2^{-(2\nu_1+\nu_2)j}(\partial^\nu \psi_{j0\mathbf{0}})(\boldsymbol{x})\right| + \left|(-4\pi^2 \left\|\mathrm{D}_{2^{2j}}\boldsymbol{x}\right\|^2)^l 2^{-(2\nu_1+\nu_2)j}(\partial^\nu \psi_{j0\mathbf{0}})(\boldsymbol{x})\right| \\
&\leq C2^{3j/2},
\end{aligned}$$

which finally gives the inequality

$$\left|\partial^\nu \psi_{j0\mathbf{0}})(\boldsymbol{x})\right| \leq \frac{C2^{(3/2+2\nu_1+\nu_2)j}}{1 + (2\pi)^{2l} \left\|\mathrm{D}_{2^{2j}}\boldsymbol{x}\right\|^{2l}} \leq \frac{C2^{(3/2+2\nu_1+\nu_2)j}}{1 + \left\|\mathrm{D}_{2^{2j}}\boldsymbol{x}\right\|^{2k}}.$$

Now we give brief arguments how we get estimate for general k. For $-2^j \leq k \leq 2^j$,

$$\begin{aligned}
\partial_2^{\nu_2}\partial_1^{\nu_1}\psi_{jk\mathbf{0}}(\boldsymbol{x}) &= \partial_2^{\nu_2}\partial_1^{\nu_1-1}(\partial_1\psi_{(2^{-2j})(k2^{-j})\mathbf{0}}(\boldsymbol{x})) \\
&= \partial_2^{\nu_2}\partial_1^{\nu_1-1}(\partial_1\psi_{(2^{-2j})0\mathbf{0}}(x_1 + k2^{-j}x_2, x_2)) \\
&= \partial_2^{\nu_2}(\partial_1^{\nu_1}\psi_{(2^{-2j})0\mathbf{0}}(x_1 + k2^{-j}x_2, x_2)) \\
&= \sum_{l=0}^{\nu_2} \begin{pmatrix} \nu_2 \\ l \end{pmatrix} (k2^{-j})^l \partial_1^{\nu_1+\nu_2-l}\partial_2^l \psi_{(2^{-2j})0\mathbf{0}}(x_1 + k2^{-j}x_2, x_2)).
\end{aligned}$$

Since $\mathrm{B}_{k\sqrt{a_j}}\mathrm{D}_{a_j} = \mathrm{D}_{a_j}\mathrm{B}_k$, we have $\mathrm{B}_{k2^{-j}}\mathrm{D}_{2^{-2j}} = \mathrm{D}_{2^{-2j}}\mathrm{B}_k$ and

$$\begin{aligned}
|\partial_2^{\nu_2}\partial_1^{\nu_1}\psi_{jk\mathbf{0}}(\boldsymbol{x})| &\le \sum_{m=0}^{\nu_2}\binom{\nu_2}{m}|k2^{-j}|^{\nu_2-m}\left|\partial_1^{\nu_1+\nu_2-m}\partial_2^{m}\psi_{(2^{-2j})\mathbf{0}\mathbf{0}}(x_1+k2^{-j}x_2,x_2))\right| \\
&\le \sum_{m=0}^{\nu_2}\binom{\nu_2}{m}|k2^{-j}|^{\nu_2-m}\frac{C2^{(3/2+2\nu_1+2\nu_2-2m+m)j}}{1+\|\mathrm{D}_{2^{2j}}(x_1+k2^{-j}x_2,x_2)^T\|^{2l}} \\
&= \frac{C2^{(3/2+2\nu_1+2\nu_2)j}}{1+\|\mathrm{D}_{2^{2j}}\mathrm{B}_{-k2^{-j}}(\boldsymbol{x})\|^{2k}}\sum_{m=0}^{\nu_2}\binom{\nu_2}{m}|k2^{-j}|^{\nu_2-m}2^{-mj} \\
&= \frac{C2^{(3/2+2\nu_1+2\nu_2)j}(2^{-j}+|k2^{-j}|)^{\nu_2}}{1+\|\mathrm{D}_{2^{2j}}\mathrm{B}_{-k2^{-j}}(\boldsymbol{x})\|^{2l}} \\
&= \frac{C2^{(3/2+2\nu_1+\nu_2)j}(1+|k|)^{\nu_2}}{1+\|\mathrm{B}_{-k}\mathrm{D}_{2^{2j}}(\boldsymbol{x})\|^{2l}}
\end{aligned}$$

Because translation does not change regularity properties, it is clear that all above would hold also for a general $\boldsymbol{m}$.

Next, we will prove the equation (III.4). We restrict first to the case $k = 0$ and $\boldsymbol{m} = \mathbf{0}$. Fix an index vector $\nu := (\nu_1, \nu_2)$ and define

$$g_j^{(v)}(\boldsymbol{x}) := \psi_{j00}^{(v)}(\mathrm{D}_{2^{-2j}}^{(v)}\boldsymbol{x}) \quad \text{and} \quad h_j^{(v)}(\boldsymbol{x}) := \partial^{\nu} g_j^{(v)}(\boldsymbol{x}) = 2^{-(2\nu_2+\nu_1)j}(\partial^{\nu}\psi_{j00}^{(v)})(\mathrm{D}_{2^{-2j}}^{(v)}\boldsymbol{x}).$$

A straightforward computation yields

$$\begin{aligned}
\hat{h}_j^{(v)}(\boldsymbol{\xi}) &= (i2\pi\boldsymbol{\xi})^{\nu}\hat{g}_j^{(v)}(\boldsymbol{\xi}) \\
&= (2\pi i\boldsymbol{\xi})^{\nu}2^{3j}\widehat{\psi}_{j00}^{(v)}(\mathrm{D}_{2^{2j}}^{(v)}\boldsymbol{\xi}) \\
&= (2\pi i\boldsymbol{\xi})^{\nu}2^{3j}2^{3j/2}\hat{\psi}^{(v)}(\boldsymbol{\xi})2^{-3j} \\
&= (2\pi i\boldsymbol{\xi})^{\nu}2^{3j/2}\hat{\psi}^{(v)}(\boldsymbol{\xi}).
\end{aligned} \tag{III.6}$$

Now, replacing $\boldsymbol{x}$ by $\mathrm{D}_{2^{2j}}^{(v)}\boldsymbol{x}$ in the equation $(-4\pi^2\|\boldsymbol{x}\|^2)^k g_j^{(v)}(\boldsymbol{x}) = \int_{\mathbb{R}^2}\Delta^k\hat{g}_j^{(v)}(\boldsymbol{\xi})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}$, where Δ is the Laplacian, yields

$$\begin{aligned}
\left|(-4\pi^2 \left\|D^{(v)}_{2^{2j}}\boldsymbol{x}\right\|^2)^l 2^{-(2\nu_2+\nu_1)j}(\partial^\nu \psi^{(v)}_{j00})(\boldsymbol{x})\right| &= \left|(-4\pi^2 \left\|D^{(v)}_{2^{2j}}\boldsymbol{x}\right\|^2)^l h_j(D^{(v)}_{2^{2j}}\boldsymbol{x})\right| \\
&= \left|\int_{\mathbb{R}^2}(\Delta^l \hat{h}^{(v)}_j)(\boldsymbol{\xi})e^{2\pi i D^{(v)}_{2^{2j}}\boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}\right| \\
&\le \int_{\mathbb{R}^2}\left|(\Delta^l \hat{h}^{(v)}_j)(\boldsymbol{\xi})\right|\left|e^{2\pi i D^{(v)}_{2^{2j}}\boldsymbol{x}\cdot\boldsymbol{\xi}}\right|d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^2}\left|(\Delta^l \hat{h}^{(v)}_j)(\boldsymbol{\xi})\right|d\boldsymbol{\xi} \\
&= 2^{3j/2}\int_{\mathbb{R}^2}\left|\Delta^l((2\pi i\boldsymbol{\xi})^\nu \hat{\psi}^{(v)}(\boldsymbol{\xi}))\right|d\boldsymbol{\xi} \\
&\le C2^{3j/2}.
\end{aligned}$$

In the last step we used the notification that $\left|\Delta^l((2\pi i\boldsymbol{\xi})^\nu \hat{\psi}^{(v)}(\boldsymbol{\xi}))\right| \le C$ where C is independent of a. When $k = 0$ this reduces to

$$\left|2^{-(2\nu_2+\nu_1)j}(\partial^\nu \psi^{(v)}_{j00})(\boldsymbol{x})\right| \le C2^{3j/2}.$$

By combining the above estimates, we get

$$\begin{aligned}
&2^{-(2\nu_2+\nu_1)j}(1 + 4^l\pi^{2l}\left\|D^{(v)}_{2^{2j}}\boldsymbol{x}\right\|^{2l})\left|\partial^\nu \psi^{(v)}_{j00})(\boldsymbol{x})\right| \\
&= \left|2^{-(2\nu_2+\nu_1)j}(\partial^\nu \psi^{(v)}_{j00})(\boldsymbol{x})\right| + \left|(-4\pi^2 \left\|D^{(v)}_{2^{2j}}\boldsymbol{x}\right\|^2)^l 2^{-(2\nu_2+\nu_1)j}(\partial^\nu \psi^{(v)}_{j00})(\boldsymbol{x})\right| \\
&\le C2^{3j/2},
\end{aligned}$$

which finally gives the inequality

$$\left|\partial^\nu \psi^{(v)}_{j00})(\boldsymbol{x})\right| \le \frac{C2^{(3/2+2\nu_2+\nu_1)j}}{1 + (2\pi)^{2l}\left\|D^{(v)}_{2^{2j}}\boldsymbol{x}\right\|^{2l}} \le \frac{C2^{(3/2+\nu_1+2\nu_2)j}}{1 + \left\|D^{(v)}_{2^{2j}}\boldsymbol{x}\right\|^{2k}}.$$

Now we give brief arguments how we get estimate for general k. For $-2^j \le k \le 2^j$,

$$\begin{aligned}
\partial_1^{\nu_1}\partial_2^{\nu_2}\psi^{(v)}_{jk\mathbf{0}}(\boldsymbol{x}) &= \partial_1^{\nu_1}\partial_2^{\nu_2-1}(\partial_2\psi^{(v)}_{(2^{-2j})(k2^{-j})\mathbf{0}}(\boldsymbol{x})) \\
&= \partial_1^{\nu_1}\partial_2^{\nu_2-1}(\partial_2\psi_{(2^{-2j})0\mathbf{0}}(x_1, x_2 + k2^{-j}x_1))
\end{aligned}$$

$$= \partial_1^{\nu_1}(\partial_2^{\nu_2}\psi^{(v)}_{(2^{-2j})\mathbf{0}\mathbf{0}}(x_1, x_2 + k2^{-j}x_1))$$

$$= \sum_{l=0}^{\nu_1} \begin{pmatrix} \nu_1 \\ l \end{pmatrix} (k2^{-j})^l \partial_2^{\nu_2+\nu_1-l}\partial_1^l \psi^{(v)}_{(2^{-2j})\mathbf{0}\mathbf{0}}(x_1, x_2 + k2^{-j}x_1)).$$

Since $\mathrm{B}^{(v)}_{k\sqrt{a_j}}\mathrm{D}^{(v)}_{a_j} = \mathrm{D}^{(v)}_{a_j}\mathrm{B}^{(v)}_k$, we have $\mathrm{B}^{(v)}_{k2^{-j}}\mathrm{D}^{(v)}_{2^{-2j}} = \mathrm{D}^{(v)}_{2^{-2j}}\mathrm{B}^{(v)}_k$ and

$$\begin{aligned}
|\partial_1^{\nu_1}\partial_2^{\nu_2}\psi_{jk\mathbf{0}}(\boldsymbol{x})| &\leq \sum_{m=0}^{\nu_1} \begin{pmatrix} \nu_1 \\ m \end{pmatrix} |k2^{-j}|^{\nu_1-m} \left|\partial_2^{\nu_2+\nu_1-m}\partial_1^m \psi_{(2^{-2j})\mathbf{0}\mathbf{0}}(x_1, x_2 + k2^{-j}\boldsymbol{x_1}))\right| \\
&\leq \sum_{m=0}^{\nu_1} \begin{pmatrix} \nu_1 \\ m \end{pmatrix} |k2^{-j}|^{\nu_1-m} \frac{C2^{(3/2+2\nu_2+2\nu_1-2m+m)j}}{1 + \left\|\mathrm{D}^{(v)}_{2^{2j}}(x_1, x_2 + k2^{-j}x_1)^T\right\|^{2l}} \\
&= \frac{C2^{(3/2+2\nu_2+2\nu_1)j}}{1 + \left\|\mathrm{D}^{(v)}_{2^{2j}}\mathrm{B}^{(v)}_{-k2^{-j}}(\boldsymbol{x})\right\|^{2k}} \sum_{m=0}^{\nu_1} \begin{pmatrix} \nu_1 \\ m \end{pmatrix} |k2^{-j}|^{\nu_1-m}2^{-mj} \\
&= \frac{C2^{(3/2+2\nu_2+2\nu_1)j}(2^{-j} + |k2^{-j}|)^{\nu_1}}{1 + \left\|\mathrm{D}^{(v)}_{2^{2j}}\mathrm{B}^{(v)}_{-k2^{-j}}(\boldsymbol{x})\right\|^{2l}} \\
&= \frac{C2^{(3/2+\nu_1+2\nu_2)j}(1 + |k|)^{\nu_1}}{1 + \left\|\mathrm{B}_{-k}\mathrm{D}^{(v)}_{2^{2j}}(\boldsymbol{x})\right\|^{2l}}
\end{aligned}$$

Because translation does not change regularity properties, it is clear that all above would hold also for a general $\boldsymbol{m}$. □

CHAPTER IV

REGULARITY AND DECAY OF TRANSFORMS

In this chapter, necessary decay rates and sufficient decay rates of continuous shearlet transform across scales for a function to have certain uniform or pointwise regularities are obtained.

4.1 Continuous Shearlet Transform

4.1.1 Uniform Regularity

The following theorem gives a necessary condition for Hölder regularity in terms of decay of shearlet transforms.

Theorem 4.1.1. *If a bounded function $f \in C^\alpha(\mathbb{R}^2)$, then there exists a constant C such that*

$$|\langle \psi_{ast}, f \rangle| \le C a^{\alpha+\frac{3}{4}} \text{ and } \left|\left\langle \psi^{(v)}_{ast}, f \right\rangle\right| \le C a^{\alpha+\frac{3}{4}}$$

for all $0 < a < 1$, $s \in [-2, 2]$ and $\boldsymbol{t} \in \mathbb{R}^2$.

Proof. Since uniform regularity is translation invariant, without loss of generality, we can assume that $\boldsymbol{t} = \boldsymbol{0}$. By assumption there exists a uniform constant C such that, for each $x_2 \in \mathbb{R}$, there exists a polynomial $P_{\mathrm{B}_s(0,x_2)^T}$ such that, for all $x_1 \in \mathbb{R}$,

$$|f(\mathrm{B}_s\boldsymbol{x}) - P_{\mathrm{B}_s(0,x_2)^T}(\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(0,x_2)^T)| \le C\|\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(0,x_2)^T\|^\alpha$$

$$\leq C\|\mathrm{B}_s((x_1,0)^T)\|^\alpha$$
$$\leq Cc(s)^\alpha\|(x_1,0)\|^\alpha = C|x_1|^\alpha, \quad \text{(IV.1)}$$

when $c(s) = \|\mathrm{B}_s\|_{op} = \left(1+\frac{s^2}{2}+\left(s^2+\frac{s^4}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \leq \sqrt{3+\sqrt{8}} = 1+\sqrt{2}$ (See G. Kutyniok and D. Labate [22]). By the rapid decay property of $\psi_{a\mathbf{00}}$ the integral

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|P_{\mathrm{B}_s(0,x_2)^T}(\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(x_1,0)^T)\psi_{a00}(x_1,x_2)\right| dx_1dx_2 < \infty$$

This justifies our use of Fubini's theorem. By the assumption that $\psi_{a\mathbf{00}}$ has vanishing directional moments of any order along the x_1-axis for $a \in (0,1)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} P_{\mathrm{B}_s(0,x_2)^T}(\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(x_1,0)^T)\psi_{a00}(x_1,x_2)dx_1dx_2 \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} P_{\mathrm{B}_s(0,x_2)^T}(\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(x_1,0)^T)\psi_{a00}(x_1,x_2)dx_1\right)dx_2 \\
&= \int_{\mathbb{R}} 0\,dx_2 = 0. \qquad \text{(IV.2)}
\end{aligned}$$

So

$$\begin{aligned}
|\langle\psi_{as\mathbf{0}}, f\rangle| &= \left|\int_{\mathbb{R}^2}\left(f(\mathrm{B}_s\boldsymbol{x}) - P_{\mathrm{B}_s(0,x_2)^T}(\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(0,x_2)^T)\right)\psi_{a\mathbf{00}}(\boldsymbol{x})d\boldsymbol{x}\right| \\
&\leq \int_{\mathbb{R}^2}\left|f(\mathrm{B}_s\boldsymbol{x}) - P_{\mathrm{B}_s(0,x_2)^T}(\mathrm{B}_s\boldsymbol{x} - \mathrm{B}_s(0,x_2)^T)\right||\psi_{a\mathbf{00}}(\boldsymbol{x})|\,d\boldsymbol{x} \\
&\leq C\int_{\mathbb{R}^2}|x_1|^\alpha\left|\frac{a^{-3/4}}{1+\|\mathrm{D}_{1/a}\boldsymbol{x}\|^{2N}}\cdot\right|d\boldsymbol{x} \\
&\leq C\int_{\mathbb{R}^2}|ay_1|^\alpha\left|\frac{a^{3/4}}{1+\|\boldsymbol{y}\|^{2N}}\cdot\right|d\boldsymbol{y} \\
&\leq Ca^{\alpha+3/4},
\end{aligned}$$

where we have used (IV.1), (IV.2) and (III.1). Similarly, using vanishing directional moments and decay property of $\psi^{(v)}_{a\mathbf{00}}$, we can show that $\left|\left\langle\psi^{(v)}_{as\boldsymbol{t}}, f\right\rangle\right| \leq Ca^{\alpha+\frac{3}{4}}$. □

In the next theorem, we obtained a sufficient condition for a function f to be uniformly C^α. For $f \in L(\mathbb{R}^2)$, let us recall that $(P_E f)\hat{} = \hat{f}\chi_E$ where E is C_1, C_2, defined in Section 2.1, or $T = \{(\xi_1, \xi_2) : |\xi_1| < 2 \text{ and } |\xi_2| < 2\}$.

Theorem 4.1.2. *Let $f \in L^2(\mathbb{R}^2)$ and $\alpha > 0$ a non-integer. If there is a constant $C < \infty$ such that, for each $0 < a < 1$, $s \in [-2, 2]$ and $\boldsymbol{t} \in \mathbb{R}^2$,*

$$|\langle \psi_{ast}, P_{C_1} f\rangle| \leq C a^{\alpha + \frac{5}{4}} \text{ and } \left|\left\langle \psi_{ast}^{(v)}, P_{C_2} f\right\rangle\right| \leq C a^{\alpha+\frac{5}{4}},$$

then $f \in C^\alpha(\mathbb{R}^2)$.

Proof. First, we recall that for each $f \in L^2(\mathbb{R}^2)$ we have:

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \langle W(\cdot - \boldsymbol{t}), P_C f\rangle\, W(\boldsymbol{x} - \boldsymbol{t})\, d\boldsymbol{t} + \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \langle \psi_{ast}, P_{C_1} f\rangle\, \psi_{ast}(\boldsymbol{x})\, \frac{da}{a^3}\, ds\, d\boldsymbol{t}$$
$$+ \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \left\langle \psi_{ast}^{(v)}, P_{C_2} f\right\rangle \psi_{ast}^{(v)}(\boldsymbol{x})\, \frac{da}{a^3}\, ds\, d\boldsymbol{t}.$$

Let

$$f^{(h)}(\boldsymbol{x}) = \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \langle \psi_{ast}, P_{C_1} f\rangle\, \psi_{ast}(\boldsymbol{x})\, \frac{da}{a^3}\, ds\, d\boldsymbol{t}$$

and

$$f^{(v)}(\boldsymbol{x}) = \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \left\langle \psi_{ast}^{(v)}, P_{C_2} f\right\rangle \psi_{ast}^{(v)}(\boldsymbol{x})\, \frac{da}{a^3}\, ds\, d\boldsymbol{t}.$$

We use the notation $\boldsymbol{x} = \boldsymbol{u} + \lambda \boldsymbol{v}$, where $\|\boldsymbol{v}\| = 1$. Here we can set $\boldsymbol{u} = \boldsymbol{0}$, the general case follows by a simple translation. Since $W \in C^\infty$, regularity of the function f depends only on regularity of $f^{(h)}$ and $f^{(v)}$. Next, we will show regularity of the function $f^{(h)}$ and, by reconstruction formula, we have

$$f^{(h)}(\boldsymbol{x}) = \sum_{j=-\infty}^{-1} \Delta_j(\boldsymbol{x}), \tag{IV.3}$$

where

$$\Delta_j(\boldsymbol{x}) := \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \langle \psi_{ast}, P_{C_1} f \rangle \, \psi_{ast}(\boldsymbol{x}) \, dt \, ds \, \frac{da}{a^3}.$$

We approximate $f^{(h)}$ by the polynomial

$$P_{\mathbf{0},\boldsymbol{v}}(\lambda) := \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} \sum_{j=-\infty}^{-1} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\mathbf{0}), |\lambda| \leq \frac{1}{2}.$$

Choose J such that $2^{J-1} \leq |\lambda| \leq 2^J$. In all calculations, our generic constant C will remain independent of J. By the triangle inequality,

$$\begin{aligned} |f^{(h)}(\lambda \boldsymbol{v}) - P_{\mathbf{0},\boldsymbol{v}}(\lambda)| &= \left| \sum_{j=-\infty}^{-1} \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} \sum_{j=-\infty}^{-1} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\mathbf{0}) \right| \\ &\leq \sum_{j=-\infty}^{-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\mathbf{0}) \right|. \end{aligned} \tag{IV.4}$$

We then give an estimate of $(\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{v})$. By the assumption and the decay estimate (III.1),

$$\begin{aligned} \left| \partial_1^{k-l} \partial_2^l \Delta_j(\boldsymbol{x}) \right| &= \left| \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \langle \psi_{ast}, P_{C_1} f \rangle \, \partial_1^{k-l} \partial_2^l \psi_{ast}(\boldsymbol{x}) \, dt \, ds \, \frac{da}{a^3} \right| \\ &\leq C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} a^{\alpha+5/4} \frac{a^{-3/4-k}(\sqrt{a} + |s|)^l}{1 + \left\| \mathrm{D}_{1/a} \mathrm{B}_{-s}(\boldsymbol{x} - \boldsymbol{t}) \right\|^{2N}} \, dt \, ds \, \frac{da}{a^3} \\ &= C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \frac{a^{\alpha+5/4-k+3/2-3/4}}{1 + \|\boldsymbol{y}\|^{2N}} \, d\boldsymbol{y} \, ds \, \frac{da}{a^3} \\ &\leq C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \frac{a^{\alpha-k+2}}{1 + \|\boldsymbol{y}\|^{2N}} \, d\boldsymbol{y} \, ds \, \frac{da}{a^3} \\ &\leq C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \frac{a^{\alpha-k+2}}{1 + \|\boldsymbol{y}\|^{2N}} \, d\boldsymbol{y} \, ds \, \frac{da}{a^3} \\ &\leq C \int_{2^j}^{2^{j+1}} a^{\alpha-k-1} \, da \\ &\leq C 2^{j(\alpha-k)}. \end{aligned} \tag{IV.5}$$

Therefore

$$\left| (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\lambda \boldsymbol{v}) \right| = \left| (v_1 \partial_1 + v_2 \partial_2)^k \, \Delta_j(\lambda \boldsymbol{v}) \right|$$

$$= \left| \sum_{l=0}^{k} \binom{k}{l} v_1^{k-l} v_2^l \partial_1^{k-l} \partial_2^l \Delta_j(\lambda \boldsymbol{v}) \right|$$

$$\leq C \sum_{l=0}^{k} \binom{k}{l} \left| v_1^{k-l} v_2^l \right| 2^{j(\alpha-k)}$$

$$\leq C 2^{j(\alpha-k)}. \tag{IV.6}$$

Since $|v_1| \leq 1$ and $|v_2| \leq 1$, C can be chosen to be independent from $\boldsymbol{v}$. Equation (IV.6) clearly also holds for $k = 0$. Since each summand in (IV.4) is the absolute error of the approximation of $\Delta_j(\lambda \boldsymbol{v})$ by its Taylor polynomial of degree $\lfloor \alpha \rfloor$, we get

$$\left| f^{(h)}(\lambda \boldsymbol{v}) - P_{\boldsymbol{0},\boldsymbol{v}}(\lambda) \right|$$

$$\leq \sum_{j=-\infty}^{J-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}) \right| + \sum_{j=J}^{-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}) \right|$$

$$\leq C \sum_{j=-\infty}^{J-1} \left[2^{j\alpha} + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|\lambda|^k}{k!} 2^{j(\alpha-k)} \right] + \sum_{j=J}^{-1} \frac{|\lambda|^{\lfloor \alpha \rfloor + 1}}{(\lfloor \alpha \rfloor + 1)!} \sup_{h \in [0,\lambda]} \left| (\boldsymbol{v} \cdot \nabla)^{\lfloor \alpha \rfloor + 1} \Delta_j(h \boldsymbol{v}) \right|$$

$$\leq C |\lambda|^\alpha + \sum_{j=J}^{-1} C |\lambda|^{\lfloor \alpha \rfloor + 1} 2^{j(\alpha - \lfloor \alpha \rfloor - 1)}$$

$$\leq C |\lambda|^\alpha + C |\lambda|^{\lfloor \alpha \rfloor + 1} |\lambda|^{(\alpha - \lfloor \alpha \rfloor - 1)}$$

$$\leq C |\lambda|^\alpha$$

The assumption $\alpha' < 2\alpha$ was needed to make infinite summations to converge. Because $\|\boldsymbol{x}\| = |\lambda|$, the theorem is proved. The regularity of $f^{(v)}$ can be proved similarly starting by the reconstruction formula

$$f^{(v)}(\boldsymbol{x}) = \sum_{j=-\infty}^{-1} \Delta_j^{(v)}(\boldsymbol{x}), \tag{IV.7}$$

where

$$\Delta_j^{(v)}(\boldsymbol{x}) := \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \left\langle \psi_{ast}^{(v)}, P_{C_2} f \right\rangle \psi_{ast}^{(v)}(\boldsymbol{x}) \, dt \, ds \, \frac{da}{a^3}$$

Approximating $f^{(v)}$ by the polynomial

$$P^{(v)}_{\mathbf{0},\boldsymbol{v}}(\lambda) := \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^k}{k!} \sum_{j=-\infty}^{-1} (\boldsymbol{v}\cdot\nabla)^k \Delta_j^{(v)}(\mathbf{0})$$

gives

$$|f^{(v)}(\lambda\boldsymbol{v}) - P^{(v)}_{\mathbf{0},\boldsymbol{v}}(\lambda)| \leq C\,|\lambda|^{\alpha}.$$

□

4.1.2 Pointwise Regularity

In this section we shall study pointwise regularity estimates and find estimates on the shearlet transform which are necessary conditions and sufficient conditions.

Theorem 4.1.3. *If a bounded function $f \in C^{\alpha}(\boldsymbol{u})$ then there exists $C < \infty$ such that*

$$\begin{aligned} |\langle\psi_{ast}, f\rangle| &\leq Ca^{\frac{\alpha}{2}+\frac{3}{4}}\left(1 + \left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha}\right) \text{ and} \\ \left|\left\langle\psi^{(v)}_{ast}, f\right\rangle\right| &\leq Ca^{\frac{\alpha}{2}+\frac{3}{4}}\left(1 + \left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha}\right) \end{aligned} \tag{IV.8}$$

for all $0 < a < 1$, $s \in [-2, 2]$ and $\boldsymbol{t} \in \mathbb{R}^2$.

Proof. Since the proofs are similar, we will prove only the first inequality. By definition, the polynomial approximation property (I.1) holds only in some neighborhood of point $\boldsymbol{u}$ but f is bounded and so this property holds in all $\mathbb{R}^2$. Since

$$\int_{\mathbb{R}^2} \psi_{ast}(\boldsymbol{x})P_{\boldsymbol{u}}(\boldsymbol{x}-\boldsymbol{u})d\boldsymbol{x} = \int_{\mathbb{R}^2} \psi_{a00}(\boldsymbol{x})P_{\boldsymbol{u}}(\mathrm{B}_s(\boldsymbol{x}-\boldsymbol{u})+\boldsymbol{t})d\boldsymbol{x} = 0.$$

we have

$$|\langle\psi_{ast}, f\rangle| \leq \int_{\mathbb{R}^2} |\psi_{ast}(\boldsymbol{x})|\,|f(\boldsymbol{x}) - P_{\boldsymbol{u}}(\boldsymbol{x}-\boldsymbol{u})|\;d\boldsymbol{x}$$

$$
\begin{aligned}
&\leq Ca^{-\frac{3}{4}} \int_{\mathbb{R}^2} \frac{\|\boldsymbol{x}-\boldsymbol{u}\|^{\alpha}}{1+\|\mathrm{D}_{1/a}\mathrm{B}_{-s}(\boldsymbol{x}-\boldsymbol{t})\|^{2N}}\, d\boldsymbol{x} \\
&= Ca^{-\frac{3}{4}+\frac{3}{2}} \int_{\mathbb{R}^2} \frac{\|\mathrm{B}_s\mathrm{D}_a\boldsymbol{y}+\boldsymbol{t}-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\, d\boldsymbol{y} \\
&\leq Ca^{-\frac{3}{4}+\frac{3}{2}} \int_{\mathbb{R}^2} \frac{\|\mathrm{B}_s\mathrm{D}_a\boldsymbol{y}\|^{\alpha}+\|\boldsymbol{t}-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\, d\boldsymbol{y} \\
&= Ca^{-\frac{3}{4}+\frac{3}{2}} \int_{\mathbb{R}^2} \frac{c(s)^{\alpha}a^{\frac{\alpha}{2}}\|\boldsymbol{y}\|^{\alpha}+\|\boldsymbol{t}-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\, d\boldsymbol{y} \\
&\leq Ca^{\frac{3}{4}+\frac{\alpha}{2}} \left(1+\left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha}\right),
\end{aligned}
$$

since we can choose N large enough so that the last integral is finite. We have also used the fact that $\mathrm{B}_s\mathrm{D}_a$ is a bounded linear operator with norm $\|\mathrm{B}_s\mathrm{D}_a\| = c(s)a^{1/2}$ and $c(s) \leq 1+\sqrt{2}$. □

Theorem 4.1.4. *Let $f \in L^2(\mathbb{R}^2)$, $\boldsymbol{u} \in \mathbb{R}^2$, and α be a non-integer positive number. If there exist $C < \infty$ and $\alpha' < 2\alpha$ such that, for each $0 < a < 1$, $s \in [-2,2]$ and $\boldsymbol{t} \in \mathbb{R}^2$,*

$$
|\langle \psi_{ast}, P_{C_1} f\rangle| \leq Ca^{\alpha+\frac{5}{4}} \left(1+\left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right) \tag{IV.9}
$$

and

$$
\left|\left\langle \psi^{(v)}_{ast}, P_{C_2} f\right\rangle\right| \leq Ca^{\alpha+\frac{5}{4}} \left(1+\left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), \tag{IV.10}
$$

then $f \in C^{\alpha}(\boldsymbol{u})$.

Proof. First, we recall that for each $f \in L^2(\mathbb{R}^2)$ we have:

$$
f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \langle W(\cdot-\boldsymbol{t}), P_C f\rangle\, W(\boldsymbol{x}-\boldsymbol{t})\, d\boldsymbol{t} + f^{(h)}(\boldsymbol{x}) + f^{(v)}(\boldsymbol{x}).
$$

When $f^{(h)}$ and $f^{(v)}$ are defined in Theorem 4.1.2. We use the notation $\boldsymbol{x} = \boldsymbol{u}+\lambda\boldsymbol{v}$, where $\|\boldsymbol{v}\| = 1$. Here we can set $\boldsymbol{u} = \boldsymbol{0}$, the general case follows by a simple translation. Since $W \in C^{\infty}$, regularity of the function f depends only on regularity of $f^{(h)}$ and $f^{(v)}$. Next, we will show regularity of the function $f^{(h)}$ and, by reconstruction formula, we have

$$f^{(h)}(\boldsymbol{x}) = \sum_{j=-\infty}^{-1} \Delta_j(\boldsymbol{x}), \tag{IV.11}$$

where

$$\Delta_j(\boldsymbol{x}) := \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \langle \psi_{ast}, P_{C_1} f \rangle \, \psi_{ast}(\boldsymbol{x}) \, dt \, ds \, \frac{da}{a^3}.$$

We approximate $f^{(h)}$ by the polynomial

$$P_{\boldsymbol{0},\boldsymbol{v}}(\lambda) := \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^k}{k!} \sum_{j=-\infty}^{-1} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}), |\lambda| \le \frac{1}{2}.$$

Choose J such that $2^{J-1} \le |\lambda| \le 2^J$. In all calculations, our generic constant C will remain independent of J. By the triangle inequality,

$$\begin{aligned}
|f^{(h)}(\lambda \boldsymbol{v}) - P_{\boldsymbol{0},\boldsymbol{v}}(\lambda)| &= \left| \sum_{j=-\infty}^{-1} \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^k}{k!} \sum_{j=-\infty}^{-1} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}) \right| \\
&\le \sum_{j=-\infty}^{-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}) \right| \\
&\le \sum_{j=-\infty}^{J-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}) \right| + \sum_{j=J}^{-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{0}) \right|.
\end{aligned} \tag{IV.12}$$

We then give an estimate of $(\boldsymbol{v} \cdot \nabla)^k \Delta_j(\boldsymbol{v})$. By the assumption and the decay estimate (III.1),

$$\begin{aligned}
|\partial_1^{k-l} \partial_2^l \Delta_j(\boldsymbol{x})| &= \left| \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \langle \psi_{ast}, P_{C_1} f \rangle \, \partial_1^{k-l} \partial_2^l \psi_{ast}(\boldsymbol{x}) \, dt \, ds \, \frac{da}{a^3} \right| \\
&\le C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} a^{\alpha+5/4} \left(1 + \left\| \frac{\boldsymbol{t}}{a^{1/2}} \right\|^{\alpha'} \right) \frac{a^{-3/4-k} (\sqrt{a} + |s|)^l}{1 + \left\| \mathrm{D}_{1/a} \mathrm{B}_{-s} (\boldsymbol{x} - \boldsymbol{t}) \right\|^{2N}} \, dt \, ds \, \frac{da}{a^3} \\
&= C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \left(1 + \left\| \frac{\boldsymbol{x} - \mathrm{B}_s \mathrm{D}_a \boldsymbol{y}}{a^{1/2}} \right\|^{\alpha'} \right) \frac{a^{\alpha+5/4-k+3/2-3/4}}{1 + \|\boldsymbol{y}\|^{2N}} \, d\boldsymbol{y} \, ds \, \frac{da}{a^3}
\end{aligned}$$

$$\leq C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \frac{a^{\alpha-k+2}\left(1 + a^{-\alpha'/2}\left(\|\boldsymbol{x}\|^{\alpha'} + \|\mathrm{B}_s\mathrm{D}_a\boldsymbol{y}\|^{\alpha'}\right)\right)}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y}\,ds\,\frac{da}{a^3}$$
$$\leq C \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \frac{a^{\alpha-k+2}\left(1 + c(s)^{\alpha'}\|\boldsymbol{y}\|^{\alpha'} + a^{-\alpha'/2}\|\boldsymbol{x}\|^{\alpha'}\right)}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y}\,ds\,\frac{da}{a^3}$$
$$\leq C \int_{2^j}^{2^{j+1}} a^{\alpha-k-1}\left(1 + a^{-\alpha'/2}\|\boldsymbol{x}\|^{\alpha'}\right) da$$
$$\leq C\left(2^{j(\alpha-k)} + 2^{j(\alpha-k-\alpha'/2)}\|\boldsymbol{x}\|^{\alpha'}\right). \tag{IV.13}$$

Therefore

$$\begin{aligned}\left|(\boldsymbol{v}\cdot\nabla)^k\Delta_j(\lambda\boldsymbol{v})\right| &= \left|(v_1\partial_1 + v_2\partial_2)^k\,\Delta_j(\lambda\boldsymbol{v})\right| \\ &= \left|\sum_{l=0}^{k}\binom{k}{l} v_1^{k-l}v_2^l\partial_1^{k-l}\partial_2^l\Delta_j(\lambda\boldsymbol{v})\right| \\ &\leq C\sum_{l=0}^{k}\binom{k}{l}\left|v_1^{k-l}v_2^l\right|\left(2^{j(\alpha-k)} + 2^{j(\alpha-k-\alpha'/2)}|\lambda|^{\alpha'}\right) \\ &\leq C\left(2^{j(\alpha-k)} + 2^{j(\alpha-k-\alpha'/2)}|\lambda|^{\alpha'}\right). \end{aligned} \tag{IV.14}$$

Since $|v_1| \leq 1$ and $|v_2| \leq 1$, C can be chosen to be independent from $\boldsymbol{v}$. Equation (IV.14) clearly also holds for $k = 0$. Since each summand in (IV.12) is the absolute error of the approximation of $\Delta_j(\lambda\boldsymbol{v})$ by its Taylor polynomial of degree $\lfloor\alpha\rfloor$, we get

$$
\begin{aligned}
&\left|f^{(h)}(\lambda \boldsymbol{v}) - P_{\mathbf{0},\boldsymbol{v}}(\lambda)\right| \\
&\leq \sum_{j=-\infty}^{J-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(0) \right| + \sum_{j=J}^{-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} (\boldsymbol{v} \cdot \nabla)^k \Delta_j(\mathbf{0}) \right| \\
&\leq C \sum_{j=-\infty}^{J-1} \left[\left(2^{j\alpha} + 2^{j(\alpha - \alpha'/2)} |\lambda|^{\alpha'} \right) + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|\lambda|^k}{k!} 2^{j(\alpha - k)} \right] \\
&\quad + \sum_{j=J}^{-1} \frac{|\lambda|^{\lfloor \alpha \rfloor + 1}}{(\lfloor \alpha \rfloor + 1)!} \sup_{h \in [0,\lambda]} \left| (\boldsymbol{v} \cdot \nabla)^{\lfloor \alpha \rfloor + 1} \Delta_j(h \boldsymbol{v}) \right| \\
&\leq C\, |\lambda|^{\alpha} + \sum_{j=J}^{-1} C\, |\lambda|^{\lfloor \alpha \rfloor + 1} \left(2^{j(\alpha - \lfloor \alpha \rfloor - 1)} + 2^{j(\alpha - \lfloor \alpha \rfloor - 1 - \alpha'/2)} |\lambda|^{\alpha'} \right) \\
&\leq C\, |\lambda|^{\alpha} + C\, |\lambda|^{\lfloor \alpha \rfloor + 1} \left(|\lambda|^{(\alpha - \lfloor \alpha \rfloor - 1)} + |\lambda|^{(\alpha - \lfloor \alpha \rfloor - 1 - \alpha'/2)} |\lambda|^{\alpha'} \right) \\
&\leq C\, |\lambda|^{\alpha}
\end{aligned}
$$

The assumption $\alpha' < 2\alpha$ was needed to make infinite summations to converge. Because $\|\boldsymbol{x}\| = |\lambda|$, the theorem is proved. The regularity of $f^{(v)}$ can be proved similarly starting by the reconstruction formula

$$
f^{(v)}(\boldsymbol{x}) = \sum_{j=-\infty}^{-1} \Delta_j^{(v)}(\boldsymbol{x}), \tag{IV.15}
$$

where

$$
\Delta_j^{(v)}(\boldsymbol{x}) := \int_{2^j}^{2^{j+1}} \int_{-2}^{2} \int_{\mathbb{R}^2} \left\langle \psi_{ast}^{(v)}, P_{C_2} f \right\rangle \psi_{ast}^{(v)}(\boldsymbol{x})\, dt\, ds\, \frac{da}{a^3}
$$

Approximating $f^{(v)}$ by the polynomial

$$
P_{\mathbf{0},\boldsymbol{v}}^{(v)}(\lambda) := \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} \sum_{j=-\infty}^{-1} (\boldsymbol{v} \cdot \nabla)^k \Delta_j^{(v)}(\mathbf{0})
$$

gives

$$
|f^{(v)}(\lambda \boldsymbol{v}) - P_{\mathbf{0},\boldsymbol{v}}^{(v)}(\lambda)| \leq C\, |\lambda|^{\alpha}.
$$

□

4.2 Discrete Shearlet Transform

Because we sample the scaling parameter of the continuous shearlet transform by choosing $a_j = 2^{-2j}, j \geq 0$, $s_{jk} = k\sqrt{a_j} = k2^{-j}, -2^j \leq k \leq 2^j$ and $\boldsymbol{t}_{jkm} = B_{s_{jk}} D_{a_j} \boldsymbol{m}$, $\boldsymbol{m} \in \mathbb{Z}^2$, we can analyse Hölder regularity of $\psi_{jk\boldsymbol{m}}$ and $\psi^{(v)}_{jk\boldsymbol{m}}$ in Theorem 4.2.1 and 4.2.2 by the same proof of Theorem 4.1.1 and 4.1.2, respectively.

Theorem 4.2.1. *If a bounded function $f \in C^\alpha(\mathbb{R}^2)$, then there exists a constant C such that*

$$|\langle \psi_{j,k,\boldsymbol{m}}, f \rangle| \leq C 2^{-2j(\frac{\alpha}{2}+\frac{3}{4})} \text{ and } \left| \left\langle \psi^{(v)}_{j,k,\boldsymbol{m}}, f \right\rangle \right| \leq C 2^{-2j(\frac{\alpha}{2}+\frac{3}{4})}$$

for all $j \geq 0$, $-2^j \leq k \leq 2^j$ and $\boldsymbol{m} \in \mathbb{Z}^2$.

Proof. We first recall that uniform regularity of f means that there exist a constant C independent of $\boldsymbol{u} \in \mathbb{R}^2$ and, for each $\boldsymbol{u}$, a polynomial $P_{\boldsymbol{u}}$ of degree less than α such that

$$|f(\boldsymbol{x}) - P_{\boldsymbol{u}}(\boldsymbol{x} - \boldsymbol{u})| \leq C\|\boldsymbol{x} - \boldsymbol{u}\|^\alpha$$

for all $\boldsymbol{x} \in \mathbb{R}^2$. Therefore, for each $x_2 \in \mathbb{R}$, there exists a polynomial $P_{\mathrm{D}_{2^{-2j}}\mathrm{B}_k(0,x_2)^T}$ such that, for all $x_1 \in \mathbb{R}$,

$$\begin{aligned}
&|f(\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{x}) - P_{\mathrm{D}_{2^{-2j}}\mathrm{B}_k(0,x_2)^T}(\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{x} - \mathrm{D}_{2^{-2j}}\mathrm{B}_k(0,x_2)^T)| \\
&\leq C\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{x} - \mathrm{D}_{2^{-2j}}\mathrm{B}_k(0,x_2)^T\|^\alpha \\
&\leq C\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k(\boldsymbol{x} - (0,x_2)^T)\|^\alpha \\
&= C\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k(x_1,0)^T)\|^\alpha \\
&\leq C2^{-j\alpha}|x_1|^\alpha. \qquad \text{(IV.16)}
\end{aligned}$$

By using (IV.16) and Lemma 3.2.1 we get

$$|\langle \psi_{j,k,\boldsymbol{m}}, f \rangle| = \left| \int_{\mathbb{R}^2} f(\boldsymbol{x})\psi_{j,k,\boldsymbol{m}}(\boldsymbol{x})d\boldsymbol{x} \right|$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}^2} f(\boldsymbol{x})\psi(\mathrm{B}_{-k}\mathrm{D}_{2^{2j}}\boldsymbol{x} - \boldsymbol{m}) 2^{\frac{3j}{2}}\, d\boldsymbol{x} \right| \\
&= \left| \int_{\mathbb{R}^2} f(\mathrm{D}_{2^{-2j}}\mathrm{B}_k(\boldsymbol{x} - \boldsymbol{m}))\psi(\boldsymbol{x}) 2^{-\frac{3j}{2}}\, d\boldsymbol{x} \right| \\
&\leq \int_{\mathbb{R}^2} |f(\mathrm{D}_{2^{-2j}}\mathrm{B}_k(\boldsymbol{x} - \boldsymbol{m})) - \\
&\qquad P_{\mathrm{D}_{2^{-2j}}\mathrm{B}_k((0,x_2)^T-(0,m_2)^T)}(\mathrm{D}_{2^{-2j}}\mathrm{B}_k(\boldsymbol{x} - \boldsymbol{m}) - \\
&\qquad\qquad \mathrm{D}_{2^{-2j}}\mathrm{B}_k((0,x_2)^T - (0,m_2)^T))||\psi(\boldsymbol{x})| 2^{-\frac{3j}{2}}\, d\boldsymbol{x} \\
&\leq C \int_{\mathbb{R}^2} 2^{-j\alpha} |x_1 - m_1|^\alpha |\psi(\boldsymbol{x})|\, 2^{-\frac{3j}{2}}\, d\boldsymbol{x} \\
&\leq C 2^{-j\alpha - \frac{3j}{2}} \int_{\mathbb{R}^2} |x_1 - m_1|^\alpha \frac{1}{1 + \|\boldsymbol{x}\|^{2N}}\, d\boldsymbol{x} \\
&\leq C 2^{-j(\alpha + \frac{3}{4})} \int_{\mathbb{R}^2} |x_1 - m_1|^\alpha \frac{1}{1 + \|\boldsymbol{x}\|^{2N}}\, d\boldsymbol{x} \\
&\leq C 2^{-2j(\frac{\alpha}{2} + \frac{3}{4})}.
\end{aligned}
$$

Similarly, we can show that $\left| \left\langle \psi^{(v)}_{j,k,\boldsymbol{m}}, f \right\rangle \right| \leq C 2^{-2j(\frac{\alpha}{2} + \frac{3}{4})}$. □

A sufficient condition for a function f to be C^α is given in the next theorem. Unfortunately, the condition here is not closed to the necessary condition presented above due to the effect of parabolic scaling. Let us denote $\tilde{C}_1 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\frac{\xi_1}{\xi_2}| \leq 1\}$, $\tilde{C}_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\frac{\xi_1}{\xi_2}| > 1\}$, $(P_{\tilde{C}_1} f)^\frown = \hat{f}\chi_{\tilde{C}_1}$, and $(P_{\tilde{C}_2} f)^\frown = \hat{f}\chi_{\tilde{C}_2}$.

Theorem 4.2.2. *Let $f \in L^2(\mathbb{R}^2)$ and $\alpha > 0$ a non-integer. If there is a constant $C < \infty$ such that, for each $j \geq 0$, $-2^j \leq k \leq 2^j$ and $\boldsymbol{m} \in \mathbb{Z}^2$,*

$$
\left|\left\langle \psi_{j,k,\boldsymbol{m}}, P_{\tilde{C}_1} f \right\rangle\right| \leq C 2^{-2j(\alpha + \frac{5}{4})} \text{ and } \left| \left\langle \psi^{(v)}_{j,k,\boldsymbol{m}}, P_{\tilde{C}_2} f \right\rangle \right| \leq C 2^{-2j(\alpha + \frac{5}{4})}, \qquad \text{(IV.17)}
$$

then $f \in C^\alpha(\mathbb{R}^2)$.

Proof. First, we recalled that, for each $f \in L^2(\mathbb{R}^2)$ we have:

$$
f(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^2} \langle W(\cdot - \boldsymbol{m}), P_{\tilde{C}} f \rangle\, W(\boldsymbol{x} - \boldsymbol{m})
$$

$$+\sum_{j=0}^{\infty}\sum_{k=-2^j}^{2^j}\sum_{m\in\mathbb{Z}^2}\left\langle \psi_{ast}, P_{\tilde{C}_1}f\right\rangle \psi_{ast}(\boldsymbol{x})\,\frac{da}{a^3}\,ds\,dt$$
$$+\sum_{j=0}^{\infty}\sum_{k=-2^j}^{2^j}\sum_{m\in\mathbb{Z}^2}\left\langle \psi^{(v)}_{ast}, P_{\tilde{C}_2}f\right\rangle \psi^{(v)}_{ast}(\boldsymbol{x})\,\frac{da}{a^3}\,ds\,dt$$

where $\tilde{C} = \{(\xi_1,\xi_2)\in\mathbb{R}^2\colon |\xi_1|,|\xi_2|\le \frac{1}{4}\}$ and $(P_{\tilde{C}}f)\hat{} = \hat{f}\chi_{\tilde{C}}$. Let

$$\tilde{f}^{(h)}(\boldsymbol{x}) = \sum_{j=0}^{\infty}\sum_{k=-2^j}^{2^j}\sum_{m\in\mathbb{Z}^2}\left\langle \psi_{ast}, P_{\tilde{C}_1}f\right\rangle \psi_{ast}(\boldsymbol{x})\,\frac{da}{a^3}\,ds\,dt$$

and

$$\tilde{f}^{(v)}(\boldsymbol{x}) = \sum_{j=0}^{\infty}\sum_{k=-2^j}^{2^j}\sum_{m\in\mathbb{Z}^2}\left\langle \psi^{(v)}_{ast}, P_{\tilde{C}_2}f\right\rangle \psi^{(v)}_{ast}(\boldsymbol{x})\,\frac{da}{a^3}\,ds\,dt.$$

We use the notation $\boldsymbol{x} = \boldsymbol{u}+\lambda\boldsymbol{v}$, where $\|\boldsymbol{v}\| = 1$. Here we can set $\boldsymbol{u} = \boldsymbol{0}$, the general case follows by simple translation. Since regularity of the function f dependent only on regularity of $\tilde{f}^{(h)}$ and $\tilde{f}^{(v)}$, we will show regularity of the function $P_{\tilde{C}_1}f$ and, by reconstruction formulas, we have

$$\tilde{f}^{(h)}(\boldsymbol{x}) = \sum_{j=0}^{\infty}\Delta_j(\boldsymbol{x}), \tag{IV.18}$$

where

$$\Delta_j(\boldsymbol{x}) := \sum_{k=-2^j}^{2^j}\sum_{m\in\mathbb{Z}^2}\left\langle \psi_{j,k,\boldsymbol{m}}, P_{\tilde{C}_1}f\right\rangle \psi_{j,k,\boldsymbol{m}}(\boldsymbol{x}).$$

We try to approximate $\tilde{f}^{(h)}$ by polynomial

$$P_{\boldsymbol{0}}(\lambda\boldsymbol{v}) := \sum_{h=0}^{\lfloor\alpha\rfloor}\frac{\lambda^h}{h!}\sum_{j=0}^{\infty}(\boldsymbol{v}\cdot\nabla)^h\Delta_j(\boldsymbol{0}).$$

Next we investigate how fast the terms $(\lambda\boldsymbol{v}\cdot\nabla)^h\Delta_j(\lambda\boldsymbol{v})$ go to zero when $j\to-\infty$. First, by using assumptions of the theorem, implication (IV.17) and the decay property in Lemma 3.2.2, we can derive

$$\left|\partial_1^{h-l}\partial_2^{l}\Delta_j(\boldsymbol{x})\right|$$

$$
\begin{aligned}
&= \left| \sum_{k=-2^j}^{2^j} \sum_{m \in \mathbb{Z}^2} \left\langle \psi_{j,k,m}, P_{\tilde{C}_1} f \right\rangle \partial_1^{h-l} \partial_2^l \psi_{j,k,m}(x) \right| \\
&\leq C \sum_{k=-2^j}^{2^j} \sum_{m \in \mathbb{Z}^2} 2^{-2j(\alpha+\frac{5}{4})} \frac{2^{(3/2+2(h-l)+l)j}(1+|k|)^l}{1+\|\mathrm{B}_{-k}\mathrm{D}_{2^{2j}} x - m\|^{2N}} \\
&= C 2^{-2j(\alpha+5/4)+(3/2+2(h-l)+l)j} \sum_{k=-2^j}^{2^j} \sum_{m \in \mathbb{Z}^2} \frac{(1+|k|)^l}{1+\|\mathrm{B}_{-k}\mathrm{D}_{2^{2j}} x - m\|^{2N}} \\
&\leq C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2})} \sum_{k=-2^j}^{2^j} (1+|k|)^l \\
&\leq C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2})} 2^{j(l+1)} \\
&= C 2^{-2j(\alpha-h)}. \qquad \text{(IV.19)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left|(v \cdot \nabla)^h \Delta_j(\lambda v)\right| &= \left|(v_1\partial_1 + v_2\partial_2)^h \Delta_j(\lambda v)\right| \\
&= \left| \sum_{l=0}^{h} \binom{h}{l} v_1^{h-l} v_2^l \partial_1^{h-l} \partial_2^l \Delta_j(\lambda v) \right| \\
&\leq C \sum_{l=0}^{h} \binom{h}{l} \left|v_1^{h-l} v_2^l\right| (2^{-2j(\alpha-h)} + 2^{-2j(\alpha-h-\frac{\alpha'}{2})} \|\lambda\|^{\alpha'}) \\
&\leq C(2^{-2j(\alpha-h)} + 2^{-2j(\alpha-h-\frac{\alpha'}{2})} |\lambda|^{\alpha'}). \qquad \text{(IV.20)}
\end{aligned}
$$

Notice that C can be really independent from v since $|v_1| \leq 1$ and $|v_2| \leq 1$. Equation (IV.20) clearly also holds for $h = 0$. Then, by the triangle inequality,

$$
\begin{aligned}
|\tilde{f}^{(h)}(\lambda v) - P_0(\lambda v)| &= \left| \sum_{j=0}^{\infty} \Delta_j(\lambda v) - \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{\lambda^h}{h!} \sum_{j=0}^{\infty} (v \cdot \nabla)^h \Delta_j(\mathbf{0}) \right| \\
&\leq \sum_{j=0}^{\infty} \left| \Delta_j(\lambda v) - \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{\lambda^h}{h!} (v \cdot \nabla)^h \Delta_j(\mathbf{0}) \right|.
\end{aligned}
$$

We investigate now coarse and fine scales separately and therefore we choose J such that $2^{(-2J-1)} \leq |\lambda| \leq 2^{-2J}$. It's essential to notice that our generic constant

C can remain independent of J in all calculations. By noticing that each summand is the absolute error of the approximation of $\Delta_j(\lambda \boldsymbol{v})$ by its Taylor polynomial of degree $\lfloor\alpha\rfloor$, we get, for coarse scales,

$$
\begin{aligned}
\sum_{j=0}^{J-1} & \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\lambda^h}{h!} (\boldsymbol{v} \cdot \nabla)^h \Delta_j(\mathbf{0}) \right| \\
& \leq \sum_{j=0}^{J-1} \frac{|\lambda|^{\lfloor\alpha\rfloor+1}}{(\lfloor\alpha\rfloor+1)!} \sup_{\epsilon\in[0,\lambda]} \left|(\boldsymbol{v}\cdot\nabla)^{\lfloor\alpha\rfloor+1} \Delta_j(\epsilon \boldsymbol{v})\right| \\
& \leq \sum_{j=0}^{J-1} C\,|\lambda|^{\lfloor\alpha\rfloor+1} \left(2^{-2j(\alpha-\lfloor\alpha\rfloor-1)} + 2^{-2j(\alpha-\lfloor\alpha\rfloor-1-\frac{\alpha'}{2})}|\lambda|^{\alpha'}\right) \\
& \leq C\,|\lambda|^{\lfloor\alpha\rfloor+1} \left[(2^J)^{-2(\alpha-\lfloor\alpha\rfloor-1)} + (2^J)^{-2(\alpha-\lfloor\alpha\rfloor-1-\frac{\alpha'}{2})}\,|\lambda|^{\alpha'} \right] \\
& = C\,|\lambda|^{\lfloor\alpha\rfloor+1}\, 2^{-2J(\alpha-\lfloor\alpha\rfloor-1)} \\
& \leq C\,|\lambda|^{\lfloor\alpha\rfloor+1}\, |\lambda|^{(\alpha-\lfloor\alpha\rfloor-1)} \\
& \leq C\,|\lambda|^{\alpha},
\end{aligned}
$$

where the estimate (IV.20) was used in the second inequality. At fine scales we use directly the estimate (IV.20) and get

$$
\begin{aligned}
\sum_{j=J}^{\infty} & \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{h=0}^{\lfloor\alpha\rfloor} \frac{\lambda^h}{h!} (\boldsymbol{v} \cdot \nabla)^h \Delta_j(\mathbf{0}) \right| \\
& \leq C \sum_{j=J}^{\infty} \left[\left(2^{-2j\alpha} + 2^{-2j(\alpha-\frac{\alpha'}{2})}|\lambda|^{\alpha'}\right) + \sum_{h=0}^{\lfloor\alpha\rfloor} \frac{|\lambda|^h}{h!} 2^{-2j(\alpha-h)} \right] \\
& = C \sum_{j=0}^{\infty} \left[\left(2^{-2(j+J)\alpha} + 2^{-2(j+J)(\alpha-\frac{\alpha'}{2})}|\lambda|^{\alpha'}\right) + \sum_{h=0}^{\lfloor\alpha\rfloor} \frac{|\lambda|^h}{h!} 2^{-2(j+J)(\alpha-h)} \right] \\
& = C 2^{-2J\alpha} \sum_{j=0}^{\infty} \left[\left(2^{-2j\alpha} + 2^{(-2j\alpha+j\alpha'+J\alpha')}|\lambda|^{\alpha'}\right) + \sum_{h=0}^{\lfloor\alpha\rfloor} \frac{|\lambda|^h\, 2^{2Jh}}{h!} 2^{-2j(\alpha-h)} \right] \\
& \leq C\,|\lambda|^{\alpha} \sum_{j=0}^{\infty} \left[\left(2^{-2j\alpha} + 2^{(-2j\alpha+j\alpha')}|\lambda|^{\alpha'-\frac{\alpha'}{2}}\right) + \sum_{h=0}^{\lfloor\alpha\rfloor} \frac{|\lambda|^h\,|\lambda|^{-h}}{h!} 2^{-2j(\alpha-h)} \right]
\end{aligned}
$$

$$= C\,|\lambda|^{\alpha} \sum_{j=0}^{\infty} \left[(2^{-2j\alpha} + 2^{(-2j\alpha + j\alpha')} |\lambda|^{\frac{\alpha'}{2}}) + \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{1}{h!} 2^{-2j(\alpha - h)} \right]$$

$$\leq C\,|\lambda|^{\alpha}.$$

The assumption $\alpha' < 2\alpha$ was needed to make infinite summations to converge. Because $\|\boldsymbol{x}\| = |\lambda|$, the theorem is proved. Finally, we will show regularity of the function $\tilde{f}^{(v)}$ and, by reconstruction formula, we have

$$\tilde{f}^{(v)}(\boldsymbol{x}) = \sum_{j=0}^{\infty} \Delta_j^{(v)}(\boldsymbol{x}), \qquad \text{(IV.21)}$$

where

$$\Delta_j^{(v)}(\boldsymbol{x}) := \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m} \in \mathbb{Z}^2} \left\langle \psi_{j,k,\boldsymbol{m}}^{(v)}, P_{C_2} f \right\rangle \psi_{j,k,\boldsymbol{m}}^{(v)}(\boldsymbol{x})\, dt\, ds\, \frac{da}{a^3}.$$

We try to approximate $\tilde{f}^{(v)}$ by polynomial

$$P_{\boldsymbol{0}}^{(v)}(\lambda \boldsymbol{v}) := \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} \sum_{j=0}^{\infty} (\boldsymbol{v} \cdot \nabla)^k \Delta_j^{(v)}(\boldsymbol{0}).$$

Similarly, by the triangle inequality,

$$|\tilde{f}^{(v)}(\lambda \boldsymbol{v}) - P_{\boldsymbol{0}}^{(v)}(\lambda \boldsymbol{v})| \leq C\,|\lambda|^{\alpha}.$$

□

Theorem 4.2.3. *If a bounded function $f \in C^{\alpha}(\boldsymbol{u})$ then there exists $C < \infty$ such that*

$$|\langle \psi_{j,k,\boldsymbol{m}}, f \rangle| \leq C 2^{-2j(\frac{\alpha}{2} + \frac{3}{4})} \left(1 + \left\| \frac{\mathrm{D}_{2^{-2j}} \mathrm{B}_k \boldsymbol{m} - \boldsymbol{u}}{2^{-j}} \right\|^{\alpha} \right) \text{ and}$$

$$\left| \left\langle \psi_{j,k,\boldsymbol{m}}^{(v)}, f \right\rangle \right| \leq C 2^{-2j(\frac{\alpha}{2} + \frac{3}{4})} \left(1 + \left\| \frac{\mathrm{D}_{2^{-2j}} \mathrm{B}_k \boldsymbol{m} - \boldsymbol{u}}{2^{-j}} \right\|^{\alpha} \right) \qquad \text{(IV.22)}$$

for all $j \geq 0$, $-2^j \leq k \leq 2^j$ and $\boldsymbol{m} \in \mathbb{Z}^2$.

Proof. We would like to remark that because f is bounded, the polynomial approximation property (I.1) holds in all $\mathbb{R}^2$ although by definition it holds only in some neighborhood of point $\boldsymbol{u}$. Therefore the proof is similar to that for uniform regularity, but we obviously do not have varying polynomials for different x_2. We do have

$$\begin{aligned}
|\langle \psi_{j,k,\boldsymbol{m}}, f\rangle| &\leq \int_{\mathbb{R}^2} |\psi_{j,k,\boldsymbol{m}}(\boldsymbol{x})|\,|f(\boldsymbol{x}) - P_{\boldsymbol{u}}(\boldsymbol{x}-\boldsymbol{u})|\,d\boldsymbol{x} \\
&\leq C2^{\frac{3j}{2}} \int_{\mathbb{R}^2} \frac{\|\boldsymbol{x}-\boldsymbol{u}\|^{\alpha}}{1+\|\mathrm{B}_{-k}\mathrm{D}_{2^{2j}}\boldsymbol{x}-\boldsymbol{m}\|^{2N}}\,d\boldsymbol{x} \\
&= C2^{\frac{3j}{2}-3j} \int_{\mathbb{R}^2} \frac{\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k(\boldsymbol{y}+\boldsymbol{m})-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y} \\
&\leq C2^{-\frac{3j}{2}} \int_{\mathbb{R}^2} \frac{\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{y}\|^{\alpha}+\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{m}-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y} \\
&= C2^{-\frac{3j}{2}} \int_{\mathbb{R}^2} \frac{\|\mathrm{B}_{k2^{-j}}\mathrm{D}_{2^{-2j}}\boldsymbol{y}\|^{\alpha}+\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{m}-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y} \\
&= C2^{-\frac{3j}{2}} \int_{\mathbb{R}^2} \frac{c(k2^{-j})^{\alpha}2^{-j\alpha}\,\|\boldsymbol{y}\|^{\alpha}+\|\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{m}-\boldsymbol{u}\|^{\alpha}}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y} \\
&\leq Cc(k2^{-j})^{\alpha}2^{-\frac{3j}{2}-j\alpha}\left(1+\left\|\frac{\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{m}-\boldsymbol{u}}{2^{-j}}\right\|^{\alpha}\right) \\
&\leq C2^{-j(\alpha+\frac{3}{2})}\left(1+\left\|\frac{\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{m}-\boldsymbol{u}}{2^{-j}}\right\|^{\alpha}\right),
\end{aligned}$$

since we can choose N large enough so that the last integral is finite. We have also used the fact that $\mathrm{B}_{k2^{-j}}\mathrm{D}_{2^{-2j}}$ is a bounded linear operator with norm $\|\mathrm{B}_{k2^{-j}}\mathrm{D}_{2^{-2j}}\| = c(k2^{-j})2^{-j}$, since $-2^j \leq k \leq 2^j$, $-1 \leq k2^{-j} \leq 1$ and $c(k2^{-j}) = \left(1+\frac{(k2^{-j})^2}{2}+\left((k2^{-j})^2+\frac{(k2^{-j})^4}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \leq \sqrt{\frac{3}{2}+\frac{\sqrt{5}}{2}}$. Similarly,

$$\left|\left\langle \psi^{(v)}_{j,k\boldsymbol{m}}, f\right\rangle\right| \leq Ca^{2^{-j(\alpha+\frac{3}{2})}}\left(1+\left\|\frac{\mathrm{D}_{2^{-2j}}\mathrm{B}_k\boldsymbol{m}-\boldsymbol{u}}{2^{-j}}\right\|^{\alpha}\right).$$

□

Theorem 4.2.4. *Let $f \in L^2(\mathbb{R}^2)$ and α be a non-integer positive number. If there* *exist $C < \infty$ and $\alpha' < 2\alpha$ such that, for all $j \geq 0$, $-2^j \leq k \leq 2^j$ and $\boldsymbol{m} \in \mathbb{Z}^2$,*

$$\left|\langle \psi_{j,k,\boldsymbol{m}}, P_{\tilde{C}_1} f\rangle\right| \leq C 2^{-2j(\alpha+\frac{5}{4})} \left(1 + \left\|\frac{\mathrm{D}_{2^{-2j}}\mathrm{B}_k \boldsymbol{m} - \boldsymbol{u}}{2^{-j}}\right\|^{\alpha'}\right) \text{ and}$$

$$\left|\left\langle \psi^{(v)}_{j,k,\boldsymbol{m}}, P_{\tilde{C}_2} f\right\rangle\right| \leq C 2^{-2j(\alpha+\frac{5}{4})} \left(1 + \left\|\frac{\mathrm{D}_{2^{-2j}}\mathrm{B}_k \boldsymbol{m} - \boldsymbol{u}}{2^{-j}}\right\|^{\alpha'}\right), \qquad \text{(IV.23)}$$

then $f \in C^\alpha(\boldsymbol{u})$.

Proof. First, we recalled that, for each $f \in L^2(\mathbb{R}^2)$ we have:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \langle W(\cdot - \boldsymbol{m}), P_{\tilde{C}} f\rangle\, W(\boldsymbol{x} - \boldsymbol{m}) + \tilde{f}^{(h)}(\boldsymbol{x}) + \tilde{f}^{(v)}(\boldsymbol{x}).$$

When $\tilde{f}^{(h)}$ and $\tilde{f}^{(v)}$ are defined in Theorem 4.2.2. We use the notation $\boldsymbol{x} = \boldsymbol{u} + \lambda\boldsymbol{v}$, where $\|\boldsymbol{v}\| = 1$. Here we can set $\boldsymbol{u} = \boldsymbol{0}$, the general case follows by simple translation. Since the regularity of the function f depends only on regularity of $\tilde{f}^{(h)}$ and $\tilde{f}^{(v)}$, we will show regularity of the function $P_{\tilde{C}_1} f$ and, by reconstruction formulas, we have

$$\tilde{f}^{(h)}(\boldsymbol{x}) = \sum_{j=0}^{\infty} \Delta_j(\boldsymbol{x}), \qquad \text{(IV.24)}$$

where

$$\Delta_j(\boldsymbol{x}) := \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \langle \psi_{j,k,\boldsymbol{m}}, P_{\tilde{C}_1} f\rangle\, \psi_{j,k,\boldsymbol{m}}(\boldsymbol{x}).$$

We try to approximate $\tilde{f}^{(h)}$ by polynomial

$$P_{\boldsymbol{0}}(\lambda\boldsymbol{v}) := \sum_{h=0}^{\lfloor\alpha\rfloor} \frac{\lambda^h}{h!} \sum_{j=0}^{\infty} (\boldsymbol{v}\cdot\nabla)^h \Delta_j(\boldsymbol{0}).$$

Next we investigate how fast the terms $(\lambda\boldsymbol{v}\cdot\nabla)^h\Delta_j(\lambda\boldsymbol{v})$ go to zero when $j \to -\infty$. First, by using assumptions of the theorem, implication (IV.23) and the decay property in Lemma 3.2.2, we can derive

$$\left|\partial_1^{h-l}\partial_2^l \Delta_j(\boldsymbol{x})\right|$$

$$
\begin{aligned}
&= \left| \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \left\langle \psi_{j,k,\boldsymbol{m}}, P_{\tilde{C}_1} f \right\rangle \partial_1^{h-l} \partial_2^{l} \psi_{j,k,\boldsymbol{m}}(\boldsymbol{x}) \right| \\
&\leq C \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} 2^{-2j(\alpha+\frac{5}{4})} \left(1 + \left\| \frac{\mathrm{D}_{2^{-2j}} \mathrm{B}_k \boldsymbol{m}}{2^{-j}} \right\|^{\alpha'} \right) \frac{2^{(3/2+2(h-l)+l)j} (1+|k|)^l}{1 + \left\| \mathrm{B}_{-k} \mathrm{D}_{2^{2j}} \boldsymbol{x} - \boldsymbol{m} \right\|^{2N}} \\
&= C 2^{-2j(\alpha+5/4)+(3/2+2(h-l)+l)j} \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \frac{(1+|k|)^l}{1 + \left\| \mathrm{B}_{-k} \mathrm{D}_{2^{2j}} \boldsymbol{x} - \boldsymbol{m} \right\|^{2N}} \\
&\quad + C 2^{-2j(\alpha+5/4)+(3/2+2(h-l)+l)j+\alpha' j} \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \frac{\left\| \mathrm{D}_{2^{-2j}} \mathrm{B}_k \boldsymbol{m} \right\|^{\alpha'} (1+|k|)^l}{1 + \left\| \mathrm{B}_{-k} \mathrm{D}_{2^{2j}} \boldsymbol{x} - \boldsymbol{m} \right\|^{2N}} \\
&\leq C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2})} \sum_{k=-2^j}^{2^j} (1+|k|)^l \\
&\quad + C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2}-\frac{\alpha'}{2})} \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \frac{\left\| \mathrm{D}_{2^{-2j}} \mathrm{B}_k (\boldsymbol{m} - \mathrm{B}_{-k} \mathrm{D}_{2^{2j}} \boldsymbol{x}) \right\|^{\alpha'} (1+|k|)^l}{1 + \|\boldsymbol{m}\|^{2N}} \\
&\leq C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2})} 2^{j(l+1)} + C 2^{-2j(\alpha-h+\frac{l}{2}-\frac{\alpha'}{2})} \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \frac{\left\| \mathrm{D}_{2^{-2j}} \mathrm{B}_k \boldsymbol{m} - \boldsymbol{x} \right\|^{\alpha'} (1+|k|)^l}{1 + \|\boldsymbol{m}\|^{2N}} \\
&= C 2^{-2j(\alpha-h)} + C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2}-\frac{\alpha'}{2})} \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \frac{(\left\| \mathrm{D}_{2^{-2j}} \mathrm{B}_k \boldsymbol{m} \right\|^{\alpha'} + \|\boldsymbol{x}\|^{\alpha'})(1+|k|)^l}{1 + \|\boldsymbol{m}\|^{2N}} \\
&\leq C 2^{-2j(\alpha-h)} + C 2^{-2j(\alpha-h+\frac{l}{2}+\frac{1}{2}-\frac{\alpha'}{2})} 2^{j(l+1)} \|\boldsymbol{x}\|^{\alpha'} \\
&\leq C 2^{-2j(\alpha-h)} + C 2^{-2j(\alpha-h-\frac{\alpha'}{2})} \|\boldsymbol{x}\|^{\alpha'} \qquad \text{(IV.25)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left| (\boldsymbol{v}\cdot\nabla)^h \Delta_j(\lambda \boldsymbol{v}) \right| &= \left| (v_1\partial_1 + v_2\partial_2)^h \, \Delta_j(\lambda \boldsymbol{v}) \right| \\
&= \left| \sum_{l=0}^{h} \binom{h}{l} v_1^{h-l} v_2^{l} \partial_1^{h-l} \partial_2^{l} \Delta_j(\lambda \boldsymbol{v}) \right| \\
&\leq C \sum_{l=0}^{h} \binom{h}{l} \left| v_1^{h-l} v_2^{l} \right| (2^{-2j(\alpha-h)} + 2^{-2j(\alpha-h-\frac{\alpha'}{2})} \|\lambda\|^{\alpha'}) \\
&\leq C(2^{-2j(\alpha-h)} + 2^{-2j(\alpha-h-\frac{\alpha'}{2})} |\lambda|^{\alpha'}). \qquad \text{(IV.26)}
\end{aligned}
$$

Notice that C can be really independent from $\boldsymbol{v}$ since $|v_1| \leq 1$ and $|v_2| \leq 1$. Equation (IV.26) clearly also holds for $h = 0$. Then, by the triangle inequality,

$$|\tilde{f}^{(h)}(\lambda \boldsymbol{v}) - P_0(\lambda \boldsymbol{v})| = \left| \sum_{j=0}^{\infty} \Delta_j(\lambda \boldsymbol{v}) - \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{\lambda^h}{h!} \sum_{j=0}^{\infty} (\boldsymbol{v} \cdot \nabla)^h \Delta_j(\mathbf{0}) \right|$$
$$\leq \sum_{j=0}^{\infty} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{\lambda^h}{h!} (\boldsymbol{v} \cdot \nabla)^h \Delta_j(\mathbf{0}) \right|.$$

We investigate now coarse and fine scales separately and therefore we choose J such that $2^{(-2J-1)} \leq |\lambda| \leq 2^{-2J}$. It's essential to notice that our generic constant C can remain independent of J in all calculations. By noticing that each summand is the absolute error of the approximation of $\Delta_j(\lambda \boldsymbol{v})$ by its Taylor polynomial of degree $\lfloor \alpha \rfloor$, we get, for coarse scales,

$$\sum_{j=0}^{J-1} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^h}{h!} (\boldsymbol{v} \cdot \nabla)^h \Delta_j(\mathbf{0}) \right|$$
$$\leq \sum_{j=0}^{J-1} \frac{|\lambda|^{\lfloor \alpha \rfloor + 1}}{(\lfloor \alpha \rfloor + 1)!} \sup_{\epsilon \in [0,\lambda]} \left| (\boldsymbol{v} \cdot \nabla)^{\lfloor \alpha \rfloor + 1} \Delta_j(\epsilon \boldsymbol{v}) \right|$$
$$\leq \sum_{j=0}^{J-1} C\, |\lambda|^{\lfloor \alpha \rfloor + 1} \left(2^{-2j(\alpha - \lfloor \alpha \rfloor - 1)} + 2^{-2j(\alpha - \lfloor \alpha \rfloor - 1 - \frac{\alpha'}{2})} |\lambda|^{\alpha'}\right)$$
$$\leq C\, |\lambda|^{\lfloor \alpha \rfloor + 1} \left[\left(2^J\right)^{-2(\alpha - \lfloor \alpha \rfloor - 1)} + \left(2^J\right)^{-2(\alpha - \lfloor \alpha \rfloor - 1 - \frac{\alpha'}{2})} |\lambda|^{\alpha'} \right]$$
$$= C\, |\lambda|^{\lfloor \alpha \rfloor + 1}\, 2^{-2J(\alpha - \lfloor \alpha \rfloor - 1)}$$
$$\leq C\, |\lambda|^{\lfloor \alpha \rfloor + 1}\, |\lambda|^{(\alpha - \lfloor \alpha \rfloor - 1)}$$
$$\leq C\, |\lambda|^{\alpha},$$

where the estimate (IV.26) was used in the second inequality. At fine scales we use directly the estimate (IV.26) and get

$$\sum_{j=J}^{\infty} \left| \Delta_j(\lambda \boldsymbol{v}) - \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{\lambda^h}{h!} (\boldsymbol{v} \cdot \nabla)^h \Delta_j(\mathbf{0}) \right|$$

$$
\begin{aligned}
&\leq C \sum_{j=J}^{\infty} \left[(2^{-2j\alpha} + 2^{-2j(\alpha - \frac{\alpha'}{2})} |\lambda|^{\alpha'}) + \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{|\lambda|^h}{h!} 2^{-2j(\alpha - h)} \right] \\
&= C \sum_{j=0}^{\infty} \left[\left(2^{-2(j+J)\alpha} + 2^{-2(j+J)(\alpha - \frac{\alpha'}{2})} |\lambda|^{\alpha'} \right) + \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{|\lambda|^h}{h!} 2^{-2(j+J)(\alpha - h)} \right] \\
&= C 2^{-2J\alpha} \sum_{j=0}^{\infty} \left[(2^{-2j\alpha} + 2^{(-2j\alpha + j\alpha' + J\alpha')} |\lambda|^{\alpha'}) + \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{|\lambda|^h \, 2^{2Jh}}{h!} 2^{-2j(\alpha - h)} \right] \\
&\leq C \, |\lambda|^{\alpha} \sum_{j=0}^{\infty} \left[(2^{-2j\alpha} + 2^{(-2j\alpha + j\alpha')} |\lambda|^{\alpha' - \frac{\alpha'}{2}}) + \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{|\lambda|^h \, |\lambda|^{-h}}{h!} 2^{-2j(\alpha - h)} \right] \\
&= C \, |\lambda|^{\alpha} \sum_{j=0}^{\infty} \left[(2^{-2j\alpha} + 2^{(-2j\alpha + j\alpha')} |\lambda|^{\frac{\alpha'}{2}}) + \sum_{h=0}^{\lfloor \alpha \rfloor} \frac{1}{h!} 2^{-2j(\alpha - h)} \right] \\
&\leq C \, |\lambda|^{\alpha} .
\end{aligned}
$$

The assumption $\alpha' < 2\alpha$ was needed to make infinite summations to converge. Because $\|\boldsymbol{x}\| = |\lambda|$, the theorem is proved. Finally, we will show regularity of the function $\tilde{f}^{(v)}$ and, by reconstruction formula, we have

$$
\tilde{f}^{(v)}(\boldsymbol{x}) = \sum_{j=0}^{\infty} \Delta_j^{(v)}(\boldsymbol{x}), \tag{IV.27}
$$

where

$$
\Delta_j^{(v)}(\boldsymbol{x}) := \sum_{k=-2^j}^{2^j} \sum_{\boldsymbol{m} \in \mathbb{Z}^2} \left\langle \psi_{j,k,\boldsymbol{m}}^{(v)}, P_{C_2} f \right\rangle \psi_{j,k,\boldsymbol{m}}^{(v)}(\boldsymbol{x}) \, d\boldsymbol{t} \, ds \, \frac{da}{a^3}.
$$

We try to approximate $\tilde{f}^{(v)}$ by polynomial

$$
P_{\boldsymbol{0}}^{(v)}(\lambda \boldsymbol{v}) := \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\lambda^k}{k!} \sum_{j=0}^{\infty} (\boldsymbol{v} \cdot \nabla)^k \Delta_j^{(v)}(\boldsymbol{0}).
$$

Similarly, by the triangle inequality,

$$
|\tilde{f}^{(v)}(\lambda \boldsymbol{v}) - P_{\boldsymbol{0}}^{(v)}(\lambda \boldsymbol{v})| \leq C \, |\lambda|^{\alpha} .
$$

□

CHAPTER V

DIRECTION OF SINGULARITY

5.1 Continuous Shearlet Transform

Theorem 5.1.1. *Let* $f \in L^2(\mathbb{R}^2)$, $\boldsymbol{u} \in \mathbb{R}^2$, *and assume that* $\alpha > 0$ *is not an integer. If there exist* $\alpha' < 2\alpha$, $-2 \leq s_0 \leq 2$, *and* $C, C' < \infty$ *such that, for each* $0 < a < 1$, $-2 \leq s \leq 2$, *and* $\boldsymbol{t} \in \mathbb{R}^2$,

$$|\langle \psi_{ast}, P_{C_1} f\rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}\left(1 + \left\|\dfrac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } |s - s_0| > C'\sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(1 + \left\|\dfrac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } |s - s_0| \leq C'\sqrt{a}, \end{cases} \tag{V.1}$$

and

$$\left|\left\langle \psi^{(v)}_{ast}, P_{C_2} f\right\rangle\right| \leq Ca^{\alpha+\frac{5}{4}}\left(1 + \left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), \tag{V.2}$$

then $f \in C^{\alpha}(\boldsymbol{u})$. *Similar statement holds if the inequality (V.1) holds for* $\left\langle \psi^{(v)}_{ast}, P_{C_2} f\right\rangle$ *and the inequality (V.2) holds for* $\langle \psi_{ast}, P_{C_1} f\rangle$.

Proof. Let us denote $I_{s_0,a} := s_0 + Ca^{1/2}[-1, 1]$. The only difference from the proof of Theorem 4.1.4 is that, in the derivation of (IV.13), we split the integral with respect to the shear parameter s into

$$\int_{-2}^{2} ds = \int_{[-2,2]\setminus I_{s_0,a}} ds + \int_{I_{s_0,a}} ds.$$

This integral obviously gives the same estimate as in (IV.13). □

In the following theorems, for $L > 0$, $s_0 \in [-2, 2]$ and $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$, let $\Gamma_{\boldsymbol{u}}$ denote the vertical line passing though $\boldsymbol{u}$ and $\Gamma_{\boldsymbol{u},s_0}$ denote the line passing through (u_1, u_2) with slope $-\frac{1}{s_0}$. Observe that we may write $\Gamma_{\boldsymbol{u}} = \Gamma_{\boldsymbol{u},0}$ so that $(x_1, x_2) \in \Gamma_{\boldsymbol{u},s_0}$ if and only if $x_1 = -s_0(x_2 - u_2) + u_1$. Recall that if $N \subseteq \mathbb{R}^2$, then $N(L)$ is the L-neighborhood of N, i.e. the set of all points whose distance to N is less than L.

Theorem 5.1.2. *Let f be bounded with $f \in C^{\alpha}(\Gamma_{(u_1,0)}, \mathbb{R}^2; (1,0))$ when $\alpha \in (0,1]$ and $f \in C^{2\alpha+1+\varepsilon}(\mathbb{R}^2; (0,1))$ for any fixed $\varepsilon > 0$ and $u_1 \in \mathbb{R}$. Then there exist $C < \infty$ such that for $0 < a < 1$, $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2$, and $s \in [-2, 2]$,*

$$|\langle \psi_{ast}, f\rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(1 + \left|\frac{t_1 - u_1}{a}\right|^{\alpha}\right), & \text{if } |s| \leq \sqrt{a}. \end{cases}$$

Proof. For $u_1 \in \mathbb{R}$ and $|s| \leq \sqrt{a}$, we have that

$$\begin{aligned} |\langle \psi_{ast}, f\rangle| &= \left|\int_{\mathbb{R}^2} (f(\boldsymbol{x}) - f(u_1, x_2))\, \psi_{ast}(\boldsymbol{x}) d\boldsymbol{x}\right| \\ &\leq C \int_{\mathbb{R}^2} |x_1 - u_1|^{\alpha} \left(\frac{a^{-3/4}}{1 + \left\|\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}(\boldsymbol{x} - \boldsymbol{t})\right\|^{2N}}\right) d\boldsymbol{x} \\ &= C \int_{\mathbb{R}^2} \left|ay_1 - s\sqrt{a}y_2 + t_1 - u_1\right|^{\alpha} \left(\frac{a^{3/4}}{1 + \|\boldsymbol{y}\|^{2N}}\right) d\boldsymbol{y} \\ &\leq C \int_{\mathbb{R}^2} (|ay_1|^{\alpha} + (|a|\,|y_2|)^{\alpha} + |t_1 - u_1|^{\alpha}) \left(\frac{a^{3/4}}{1 + \|\boldsymbol{y}\|^{2N}}\right) d\boldsymbol{y} \\ &= Ca^{\alpha+3/4}\left(1 + \left|\frac{t_1 - u_1}{a}\right|^{\alpha}\right). \end{aligned}$$

Next, let $|s| > \sqrt{a}$. We start by taking the rectangle $R_a := [-a^{-c}, a^{-c}]^2$ for some $0 < c < 1/2$, to be determined later. We notice that $\mathrm{B}_s\mathrm{D}_a R_a$ is sheared similarly to the essential support of ψ_{ast} and $R_a \to \mathbb{R}^2$ whiile $\mathrm{B}_s\mathrm{D}_a R_a \to \{\mathbf{0}\}$ when $a \to 0$. We will also use here the notation $\boldsymbol{v}(\boldsymbol{x}) := (x_1, \frac{x_1}{|s|})$. Since the line

$x_2 = \frac{x_1}{|s|}$ is parallel to major axis of $\mathrm{B}_s\mathrm{D}_a R_a$, $\boldsymbol{v}(\boldsymbol{x})$ lies on major axis of $\mathrm{B}_s\mathrm{D}_a R_a$ and $\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{x}$ is always parallel to x_2-axis. Let $h_t(\boldsymbol{y}) = f(\boldsymbol{y} + \boldsymbol{t})$. By assumption of f we have $h_t \in C^{2\alpha+1+\varepsilon}(\mathbb{R}^2, (0,1))$.

$$
\begin{aligned}
|\langle \psi_{ast}, f\rangle| &= |\langle \psi_{as\mathbf{0}}, h_t\rangle| \\
&= \left| \int_{\mathbb{R}^2} \left(h_t(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right) \psi_{as\mathbf{0}}(\boldsymbol{x}) d\boldsymbol{x} \right| \\
&\leq \left| \int_{\mathbb{R}^2 \setminus \mathrm{B}_s\mathrm{D}_a R_a} \left(h_t(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right) \psi_{as\mathbf{0}}(\boldsymbol{x}) d\boldsymbol{x} \right| \\
&\quad + \left| \int_{\mathrm{B}_s\mathrm{D}_a R_a} \left(h_t(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right) \psi_{as\mathbf{0}}(\boldsymbol{x}) d\boldsymbol{x} \right|.
\end{aligned}
$$

Suppose h_t is bounded by M. So the first integral can be bounded by

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2 \setminus \mathrm{B}_s\mathrm{D}_a R_a} \left(h_t(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x})\right) \psi_{as\mathbf{0}}(\boldsymbol{x}) d\boldsymbol{x} \right| \\
&\leq C a^{-\frac{3}{4}} \int_{\mathbb{R}^2 \setminus \mathrm{B}_s\mathrm{D}_a R_a} \frac{\left|h_t(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right|}{1 + \|\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}\boldsymbol{x}\|^{2N}} d\boldsymbol{x} \\
&\leq C a^{\frac{3}{4}} \int_{\mathbb{R}^2 \setminus R_a} \frac{M + P_{\boldsymbol{y}'}(C' \|\boldsymbol{y}\|)}{1 + \|\boldsymbol{y}\|^{2N}} d\boldsymbol{y} (\text{ for some } C' > 0) \\
&= C a^{\frac{3}{4} + c(2N - 1 - \text{degree } P_{\boldsymbol{y}'})}
\end{aligned}
$$

where $\boldsymbol{y}' = \boldsymbol{v}(\mathrm{B}_s\mathrm{D}_a\boldsymbol{y})$. Since c is fixed, we just choose N large enough to make $\frac{3}{4} + c(2N - 1 - \text{degree } P_{\boldsymbol{y}'})$ as large as necessary.

Assume that $f \in C^{2\alpha+1+\varepsilon}(\mathbb{R}^2, (0,1))$. Thus, for every $\boldsymbol{y} \in \mathbb{R}^2$, there exists a polynomial $P_{\boldsymbol{y}}$ such that for $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{y}$ such that $(\boldsymbol{x} - \boldsymbol{y}) \parallel (0,1)$

$$
|f(\boldsymbol{x}) - P_{\boldsymbol{y}}(\boldsymbol{x} - \boldsymbol{y})| \leq C \|\boldsymbol{x} - \boldsymbol{y}\|^{2\alpha+1+\varepsilon}.
$$

Since a is small enough, we can choose $r > 0$ with $\mathrm{B}_s\mathrm{D}_a R_a \subset B(\mathbf{0}, r)$. Therefore, for $x \in \mathrm{B}_s\mathrm{D}_a R_a$, $\|x - v(x)\|$ is less than the length l of the part of the line parallel to the x_2-axis lying inside the rectangle $\mathrm{B}_s\mathrm{D}_a R_a$. Observe that $|l| \leq 2a^{1/2-c}$ and so

$$\left|f(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right| \leq C \left\|\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x})\right\|^{2\alpha+1+\varepsilon} \leq C \left|l\right|^{2\alpha+1+\varepsilon} \leq C a^{(1/2-c)2\alpha+1+\varepsilon}.$$

Recall that we can choose any $c \in (0, 1/2)$ and hence for any small ε we can choose $c = \dfrac{\varepsilon}{4\alpha + 2 + 2\varepsilon}$. With these, we obtain an estimate for the second integral

$$\begin{aligned}
&\left|\int_{\mathrm{B}_s \mathrm{D}_a R_a} \left(f(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right) \psi_{as\boldsymbol{0}}(\boldsymbol{x}) d\boldsymbol{x}\right| \\
&\quad \leq \int_{\mathrm{B}_s \mathrm{D}_a R_a} \left|f(\boldsymbol{x}) - P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{x}))\right| \left|\psi_{as\boldsymbol{0}}(\boldsymbol{x})\right| d\boldsymbol{x} \\
&\quad \leq C \int_{\mathrm{B}_s \mathrm{D}_a R_a} \frac{a^{\left(\frac{1}{2}-c\right)2\alpha+1+\varepsilon} a^{-\frac{3}{4}}}{1 + \|\mathrm{D}_{\frac{1}{a}} \mathrm{B}_{-s} \boldsymbol{x}\|^{2N}} d\boldsymbol{x} \\
&\quad \leq C \int_{R_a} \frac{a^{\left(\frac{1}{2}-c\right)2\alpha+1+\varepsilon} a^{\frac{3}{4}}}{1 + \|\boldsymbol{y}\|^{2N}} d\boldsymbol{y} \\
&\quad \leq C a^{(2\alpha+1+\varepsilon)\left(\frac{1}{2}-c\right)+\frac{3}{4}} \quad \leq C a^{\left(\alpha+\frac{5}{4}\right)}.
\end{aligned}$$

□

Theorem 5.1.3. *Let f be bounded with $f \in C^{\alpha}(\Gamma_{(u_1,0)}(L); (1,0))$ when $\alpha \in (0,1]$, $L > 1$ and $f \in C^{2\alpha+1+\varepsilon}(\Gamma_{(u_1,0)}, \Gamma_{(u_1,0)}(L); (0,1))$ for any fixed $\varepsilon > 0$ and $u_1 \in \mathbb{R}$. Then there exists $C < \infty$ is that, for $0 < a < 1$ and $a_0 < 1$, if $0 < a < a_0$ and $\boldsymbol{t} = (t_1, t_2) \in \Gamma_{(u_1,0)}(r)$ with $r < L/2$ and $s \in [-2, 2]$, we have*

$$|\langle \psi_{ast}, f \rangle| \leq \begin{cases} C a^{\alpha + \frac{5}{4}}, & \text{if } |s| > \sqrt{a}, \\ C a^{\alpha + \frac{3}{4}} \left(1 + \left|\dfrac{t_1 - u_1}{a}\right|^{\alpha}\right), & \text{if } |s| \leq \sqrt{a}. \end{cases}$$

Proof. For simplicity we assume again that $u_1 = 0$, the general case follows by simple translation. Let $|s| \leq \sqrt{a}$. Since

$$|\langle \psi_{ast}, f \rangle| = \left|\int_{\mathbb{R}^2} \left(f(\boldsymbol{x}) - f(0, x_2)\right) \psi_{ast}(\boldsymbol{x}) d\boldsymbol{x}\right|$$

$$
\begin{aligned}
&\leq C\int_{\mathbb{R}^2}|x_1|^\alpha\left(\frac{a^{-3/4}}{1+\left\|\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}(\boldsymbol{x}-\boldsymbol{t})\right\|^{2N}}\right)d\boldsymbol{x}\\
&= C\int_{\mathbb{R}^2}\left|ay_1-s\sqrt{a}y_2+t_1\right|^\alpha\left(\frac{a^{-3/4}a^{3/2}}{1+\|\boldsymbol{y}\|^{2N}}\right)d\boldsymbol{y}\\
&\leq C\int_{\mathbb{R}^2}\left(|ay_1|^\alpha+\left|s\sqrt{a}y_2\right|^\alpha+|t_1|^\alpha\right)\left(\frac{a^{3/4}}{1+\|\boldsymbol{y}\|^{2N}}\right)d\boldsymbol{y}\\
&\leq C\int_{\mathbb{R}^2}\left(|ay_1|^\alpha+|ay_2|^\alpha+|t_1|^\alpha\right)\left(\frac{a^{3/4}}{1+\|\boldsymbol{y}\|^{2N}}\right)d\boldsymbol{y}\\
&\leq C\int_{\mathbb{R}^2}\left((2a\|\boldsymbol{y}\|)^\alpha+|t_1|^\alpha\right)\left(\frac{a^{3/4}}{1+\|\boldsymbol{y}\|^{2N}}\right)d\boldsymbol{y}\\
&= Ca^{\alpha+3/4}\left(1+\left|\frac{t_1}{a}\right|^\alpha\right).
\end{aligned}
$$

Next, let $|s|>\sqrt{a}$. We start by taking the rectangle $R_a:=[-a^{-c},a^{-c}]^2$ for some $0<c<1/2$, to be determined later. We notice that $\mathrm{B}_s\mathrm{D}_aR_a$ is sheared similarly to the essential support of $\psi_{as\boldsymbol{t}}$ and $R_a\to\mathbb{R}^2$ while $\mathrm{B}_s\mathrm{D}_aR_a\to\{\mathbf{0}\}$ when $a\to 0$. We will also use here the notation $\boldsymbol{v}(\boldsymbol{x}):=(x_1,\frac{x_1}{|s|})$. Since the line $x_2=\frac{x_1}{|s|}$ is parallel to major axis of $\mathrm{B}_s\mathrm{D}_aR_a$, $\boldsymbol{v}(\boldsymbol{x})$ lies on major axis of $\mathrm{B}_s\mathrm{D}_aR_a$ and $\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{x}$ is always parallel to x_2-axis. Let $\boldsymbol{t}\in\Gamma_{(u_1,0)}(r)$ and $h_{\boldsymbol{t}}(\boldsymbol{y})=f(\boldsymbol{y}+\boldsymbol{t})$. By assumption of f we have $h_{\boldsymbol{t}}\in C^{2\alpha+1+\varepsilon}(\Gamma_{(u_1,0)}(r);(0,1))$ and $h_{\boldsymbol{t}}$ is bounded.

$$
\begin{aligned}
|\langle\psi_{as\boldsymbol{t}},f\rangle| &= \left|\left\langle a^{\frac{-3}{4}}\psi(\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}(\cdot-\boldsymbol{t})),f(\cdot)\right\rangle\right|\\
&= \left|\left\langle a^{-\frac{3}{4}}\psi(\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}(\cdot)),f(\cdot+\boldsymbol{t})\right\rangle\right|\\
&= \left|\left\langle a^{-\frac{3}{4}}\psi(\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}(\cdot)),h_{\boldsymbol{t}}(\cdot)\right\rangle\right|\\
&= |\langle\psi_{as\mathbf{0}},h_{\boldsymbol{t}}\rangle|\\
&= \left|\int_{\mathbb{R}^2}\left(h(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right)\psi_{as\mathbf{0}}(\boldsymbol{x})d\boldsymbol{x}\right|\\
&\leq \left|\int_{\mathbb{R}^2\setminus\mathrm{B}_s\mathrm{D}_aR_a}\left(h_{\boldsymbol{t}}(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right)\psi_{as\mathbf{0}}(\boldsymbol{x})d\boldsymbol{x}\right|
\end{aligned}
$$

$$+\left|\int_{\mathrm{B}_s\mathrm{D}_aR_a}\left(h_t(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right)\psi_{as\mathbf{0}}(\boldsymbol{x})d\boldsymbol{x}\right|.$$

By the proof of Theorem 5.1.2, we have that the first integral can writing

$$\left|\int_{\mathbb{R}^2\smallsetminus\mathrm{B}_s\mathrm{D}_aR_a}\left(h_t(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x})\right)\psi_{as\mathbf{0}}(\boldsymbol{x})d\boldsymbol{x}\right|\leq Ca^K$$

where $\boldsymbol{y}'=\boldsymbol{v}(\mathrm{B}_s\mathrm{D}_a\boldsymbol{y})$ and K can be chosen arbitrarily large.

Let $a_0<1$ be such that, for all $0<a<a_0$ and $s\in[-2,2]$, $\mathrm{B}_s\mathrm{D}_aR_a\subseteq\Gamma_{(u_1,0)}(r)\subseteq\Gamma_{(u_1,0)}(L)$. Let us now assume that $h_t\in C^{2\alpha+1+\varepsilon}(\Gamma_{(u_1,0)}(r);(0,1))$ i.e. for every $\boldsymbol{y}\in\mathbb{R}^2$ there exists a polynomial $P_{\boldsymbol{y}}$ such that

$$|h_t(\boldsymbol{x})-P_{\boldsymbol{y}}(\boldsymbol{x}-\boldsymbol{y})|\leq C\left\|\boldsymbol{x}-\boldsymbol{y}\right\|^{2\alpha+1+\varepsilon},\quad\text{when }(\boldsymbol{x}-\boldsymbol{y})\parallel(0,1)$$

for all $\boldsymbol{x}$ in some neighborhood of $\boldsymbol{y}$. Therefore, for $x\in\mathrm{B}_s\mathrm{D}_aR_a$, $\|(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\|$ less than the length l of the part of the line parallel to the x_2-axis lying inside the rectangle $\mathrm{B}_s\mathrm{D}_aR_a$ is at most $|l|\leq Ca^{1/2-c}$ and so

$$\left|h_t(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right|\leq C\left\|(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right\|\leq C\left|l\right|^{2\alpha+1+\varepsilon}\leq Ca^{(1/2-c)2\alpha+1+\varepsilon}.$$

Now we remember that we can choose any $c\in(0,1/2)$ and, therefore, for any small ε we can choose $c=\dfrac{\varepsilon}{4\alpha+2+2\varepsilon}$. With these, we obtain an estimate for the second integral

$$\begin{aligned}
&\left|\int_{\mathrm{B}_s\mathrm{D}_aR_a}\left(h_t(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right)\psi_{as\mathbf{0}}(\boldsymbol{x})d\boldsymbol{x}\right|\\
&\leq\int_{\mathrm{B}_s\mathrm{D}_aR_a}\left|h_t(\boldsymbol{x})-P_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{x}))\right|\left|\psi_{as\mathbf{0}}(\boldsymbol{x})\right|d\boldsymbol{x}\\
&\leq C\int_{\mathrm{B}_s\mathrm{D}_aR_a}\frac{a^{\left(\frac{1}{2}-c\right)2\alpha+1+\varepsilon}a^{-\frac{3}{4}}}{1+\|\mathrm{D}_{\frac{1}{a}}\mathrm{B}_{-s}\boldsymbol{x}\|^{2N}}\,d\boldsymbol{x}\\
&\leq C\int_{R_a}\frac{a^{\left(\frac{1}{2}-c\right)2\alpha+1+\varepsilon}a^{\frac{3}{4}}}{1+\|\boldsymbol{y}\|^{2N}}\,d\boldsymbol{y}\\
&\leq Ca^{2\alpha+1+\varepsilon\left(\frac{1}{2}-c\right)+\frac{3}{4}}\quad\leq Ca^{\left(\alpha+\frac{5}{4}\right)}.
\end{aligned}$$

□

Lemma 5.1.4. *Let f be bounded with $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}, \mathbb{R}^2; (1,0))$ for some $s_0 \in [-2,2]$ and $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$. Then $f \circ \mathrm{B}_{s_0} \in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)}, \mathbb{R}^2; (1,0))$.*

Moreover, if $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}, \Gamma_{(u_1,u_2),s_0}(L); (1,0))$ for some $s_0 \in [-2,2]$ and $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ then $f \circ \mathrm{B}_{s_0} \in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)}, \Gamma_{(u_1+s_0u_2,0)}(L); (1,0))$.

Proof. Assume that $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}, \mathbb{R}^2; (1,0))$. Then, for each $\boldsymbol{x} \in \Gamma_{(u_1,u_2),s_0}$, there exists a polynomial $P_{\boldsymbol{x}}$ and a constant $C > 0$ such that

$$|f(\boldsymbol{y}) - P_{\boldsymbol{x}}(\boldsymbol{y} - \boldsymbol{x})| \le C \|\boldsymbol{y} - \boldsymbol{x}\|^{\alpha}, \quad \text{when } (\boldsymbol{y} - \boldsymbol{x}) \parallel (1,0).$$

We have that $\mathrm{B}_{s_0}\Gamma_{(u_1+s_0u_2,0)} = \Gamma_{(u_1,u_2),s_0}$ and $\mathrm{B}_{s_0}(1,0) = (1,0)$. Then, for $\boldsymbol{x}' \in \Gamma_{(u_1+s_0u_2,0)}$ and $\boldsymbol{y}' \in \mathbb{R}^2$ with $(\boldsymbol{y}' - \boldsymbol{x}') \parallel (1,0)$, $\mathrm{B}_{s_0}\boldsymbol{x}' \in \Gamma_{(u_1,u_2),s_0}$ and $\mathrm{B}_{s_0}(\boldsymbol{y}' - \boldsymbol{x}') \parallel (1,0)$. Thus

$$|f \circ \mathrm{B}_{s_0}(\boldsymbol{y}') - P_{\boldsymbol{x}'} \circ \mathrm{B}_{s_0}(\boldsymbol{y}' - \boldsymbol{x}')| \le C \|B_{s_0}(\boldsymbol{y}' - \boldsymbol{x}')\|^{\alpha} \le C \|\boldsymbol{y}' - \boldsymbol{x}'\|^{\alpha}.$$

So we have $f \circ \mathrm{B}_{s_0} \in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)}, (1,0))$. The latter part of the Lemma can be proved in a similar way. □

Lemma 5.1.5. *Let f be bounded with $f \in C^{\alpha}(\mathbb{R}^2; \mathrm{B}_{s_0}(0,1))$ for some $s_0 \in [-2,2]$. Then $f \circ \mathrm{B}_{s_0} \in C^{\alpha}(\mathbb{R}^2; (0,1))$.*

Moreover, if $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}(L); \mathrm{B}_{s_0}(0,1))$ for some $s_0 \in [-2,2]$ and $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ then $f \circ \mathrm{B}_{s_0} \in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)}(L); (0,1))$.

Proof. Assume that $f \in C^{\alpha}(\mathbb{R}^2; \mathrm{B}_{s_0}(0,1))$. Then, for each $\boldsymbol{y} \in \mathbb{R}^2$, there exist a polynomial $P_{\boldsymbol{y}}$ and constant $C > 0$ such that

$$|f(\boldsymbol{x}) - P_{\boldsymbol{y}}(\boldsymbol{x} - \boldsymbol{y})| \le C \|\boldsymbol{x} - \boldsymbol{y}\|^{\alpha}, \quad \text{when } (\boldsymbol{x} - \boldsymbol{y}) \parallel \mathrm{B}_{s_0}(0,1).$$

Let $(\boldsymbol{x}-\boldsymbol{y}) \parallel (0,1)$. For $-2 \le s_0 \le 2$, we have $\mathrm{B}_{s_0}(\boldsymbol{x}-\boldsymbol{y}) \parallel \mathrm{B}_{s_0}(0,1)$. So

$$|f \circ B_{s_0}(\boldsymbol{x}) - P_{\boldsymbol{y}} \circ B_{s_0}(\boldsymbol{x}-\boldsymbol{y})| \le C \left\| B_{s_0}(\boldsymbol{x}-\boldsymbol{y}) \right\|^{\alpha} \le C \left\| \boldsymbol{x}-\boldsymbol{y} \right\|^{\alpha}.$$

From this inequality we have $f \circ B_{s_0} \in C^{\alpha}(\mathbb{R}^2; (0,1))$.

The latter part of the Lemma can be proved in a similar way. □

Theorem 5.1.6. *Let f be bounded with $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}, \mathbb{R}^2; (1,0))$ when $\alpha \in (0,1]$ and $f \in C^{2\alpha+1+\varepsilon}(\mathbb{R}^2; \mathrm{B}_{s_0}(0,1))$ for some $s_0 \in [-2,2]$ with any fixed $\varepsilon > 0$ and $\boldsymbol{u} = (u_1,u_2) \in \mathbb{R}^2$. Then there exists $C < \infty$ such that for $0 < a < 1$, $\boldsymbol{t} = (t_1,t_2) \in \mathbb{R}^2$, and $s \in [-2,2]$,*

$$|\langle \psi_{ast}, f \rangle| \le \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s-s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}} \left(1 + \left| \dfrac{t_1 + s_0 t_2 - u_1 - s_0 u_2}{a} \right|^{\alpha} \right), & \text{if } |s-s_0| \le \sqrt{a}. \end{cases}$$

Proof. Consider

$$\begin{aligned} \langle \psi_{ast}, f \rangle &= a^{-\frac{3}{4}} \left\langle \psi(\mathrm{D}_{\frac{1}{a}} \mathrm{B}_{-s}(\cdot - \boldsymbol{t})), f(\cdot) \right\rangle \\ &= a^{-\frac{3}{4}} \left\langle \psi(\mathrm{D}_{\frac{1}{a}} \mathrm{B}_{-s} \mathrm{B}_{s_0} \mathrm{B}_{-s_0}(\cdot - \boldsymbol{t})), f(\mathrm{B}_{s_0} \mathrm{B}_{-s_0} \cdot) \right\rangle \\ &= a^{-\frac{3}{4}} \left\langle \psi(\mathrm{D}_{\frac{1}{a}} \mathrm{B}_{-s} \mathrm{B}_{s_0}(\mathrm{B}_{-s_0} \cdot - \mathrm{B}_{-s_0} \boldsymbol{t})), f(\mathrm{B}_{s_0} \mathrm{B}_{-s_0} \cdot) \right\rangle \\ &= a^{-\frac{3}{4}} \left\langle \psi(\mathrm{D}_{\frac{1}{a}} \mathrm{B}_{-s} \mathrm{B}_{s_0}(\cdot - \mathrm{B}_{-s_0} \boldsymbol{t})), f(\mathrm{B}_{s_0} \cdot) \right\rangle \\ &= a^{-\frac{3}{4}} \left\langle \psi(\mathrm{D}_{\frac{1}{a}} \mathrm{B}_{-(s-s_0)}(\cdot - \mathrm{B}_{-s_0} \boldsymbol{t})), f(\mathrm{B}_{s_0} \cdot) \right\rangle \\ &= \left\langle \psi_{a(s-s_0)\mathrm{B}_{-s_0}\boldsymbol{t}}, f \circ \mathrm{B}_{s_0} \right\rangle. \end{aligned}$$

By Lemma 5.1.4 and Lemma 5.1.5, $f \circ \mathrm{B}_{s_0} \in C^{\alpha}(\Gamma_{(u_1+s_0u_2,0)}; (1,0))$ and $f \circ B_{s_0} \in C^{2\alpha+1+\epsilon}(\mathbb{R}^2; (0,1))$. Using Theorem 5.1.2 with above equation we have

$$|\langle \psi_{ast}, f \rangle| = \left| \left\langle \psi_{a(s-s_0)\mathrm{B}_{-s_0}\boldsymbol{t}}, f \circ \mathrm{B}_{s_0} \right\rangle \right|$$

$$\leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s-s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(1+\left|\dfrac{t_1+s_0t_2-u_1-s_0u_2}{a}\right|^{\alpha}\right), & \text{if } |s-s_0| \leq \sqrt{a}. \end{cases}$$

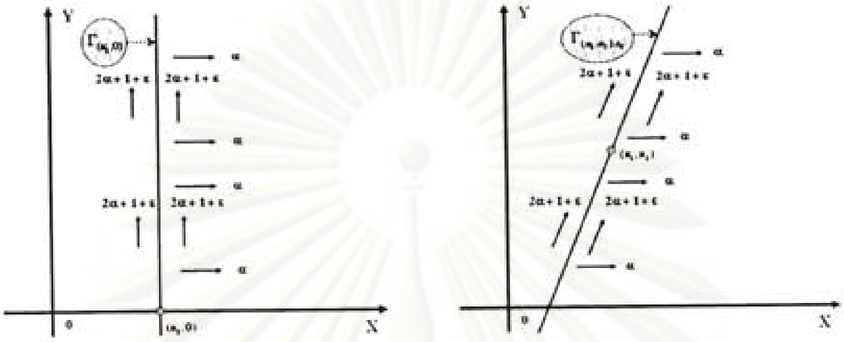
□

Theorem 5.1.7. *Let f be bounded with $f \in C^{\alpha}(\Gamma_{(u_1,u_2),s_0}, \Gamma_{(u_1,u_2),s_0}(L); (1,0))$ when $\alpha \in (0,1]$, $L > 1$ and $f \in C^{2\alpha+1+\varepsilon}(\Gamma_{(u_1,u_2),s_0}(L); \mathrm{B}_{s_0}(0,1))$ for some $s_0 \in [-2,2]$ with any fixed $\varepsilon > 0$ and $\boldsymbol{u} = (u_1,u_2) \in \mathbb{R}^2$. Then there exists $C < \infty$ is that, for $0 < a < 1$ and $a_0 < 1$, if $0 < a < a_0$ and $\boldsymbol{t} = (t_1,t_2) \in \Gamma_{(u_1,u_2),s_0}(r)$ with $r < L/2$ and $s \in [-2,2]$, we have*

$$|\langle \psi_{ast}, f\rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s-s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(1+\left|\dfrac{t_1+s_0t_2-u_1-s_0u_2}{a}\right|^{\alpha}\right), & \text{if } |s-s_0| \leq \sqrt{a}. \end{cases}$$

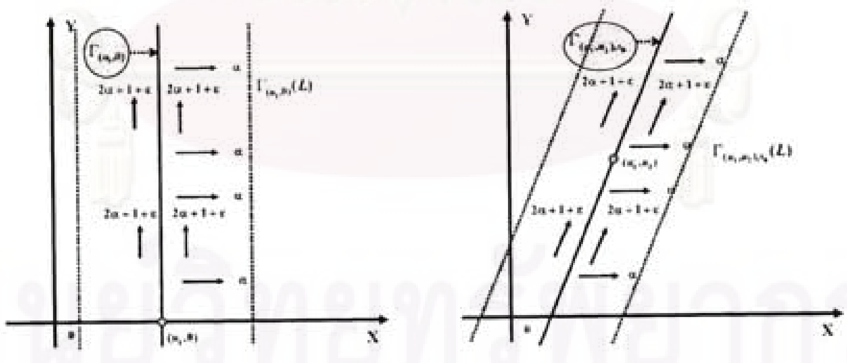
Proof. By Lemma 5.1.4, Lemma 5.1.5 and the same way of the proof of theorem 5.1.6, the proof is complete. □

Theorem 5.1.6 says essentially that, given a line L in the direction of $\mathrm{B}_{s_0}(0,1)$, a set of necessary conditions for a bounded function f to be smooth enough globally in the direction of L ($f \in C^{\alpha}(\mathbb{R}^2; \mathrm{B}_{s_0}(0,1))$) and to have low regularity on L in the horizontal direction ($f \in C^{\alpha}(\Gamma_{\boldsymbol{u},s_0}, \mathbb{R}^2; (1,0))$) is that the continuous shearlet transform $\langle \psi_{ast}, f\rangle$ decays like $a^{\alpha+\frac{5}{4}}$ in directions "away" from the direction of L and that the needed decay rate in directions "near" the line is half an order lower and depends also on the horizontal distance from the line to the parallel line containing the $\boldsymbol{t}$. See Figure 5.1.2 for an illustration. Theorem 5.1.7 can be considered as the same result with weakened conditions where only regularity information on a neighborhood of the singularity line is assumed.



(a) (b)

Figure 5.1.1: (a) An illustration of regularity in Theorem 5.1.2 . (b) An illustration of regularity in Theorem 5.1.6.



(a) (b)

Figure 5.1.2: (a) An illustration of regularity in Theorem 5.1.3 . (b) An illustration of regularity in Theorem 5.1.7.

5.2 Discrete Shearlet Transform

We recall $\tilde{C}_1 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\frac{\xi_1}{\xi_2}| \leq 1\}$, $\tilde{C}_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon |\frac{\xi_1}{\xi_2}| > 1\}$, $(P_{\tilde{C}_1} f)\hat{} = \hat{f}\chi_{\tilde{C}_1}$, and $(P_{\tilde{C}_2} f)\hat{} = \hat{f}\chi_{\tilde{C}_2}$.

Theorem 5.2.1. *Let $f \in L^2(\mathbb{R}^2)$, $\boldsymbol{u} \in \mathbb{R}^2$, and assume that $\alpha > 0$ is not an integer. If there exist $\alpha' < 2\alpha$, $-2^j \leq k_0 \leq 2^j$, and $C, C' < \infty$ such that, for each $j \geq 0$, $-2^j \leq k \leq 2^j$, and $\boldsymbol{m} \in \mathbb{Z}^2$,*

$$|\langle \psi_{jk\boldsymbol{m}}, P_{C_1} f \rangle| \leq \begin{cases} C2^{-j(2\alpha + \frac{5}{2})} \left(1 + \left\| \dfrac{\boldsymbol{m} - \boldsymbol{u}}{2^{-j}} \right\|^{\alpha'}\right), & \text{if } \left|\frac{k}{2^j} - \frac{k_0}{2^{j_0}}\right| > C'2^{-j} \\ C2^{-j(2\alpha + \frac{3}{2})} \left(1 + \left\| \dfrac{\boldsymbol{m} - \boldsymbol{u}}{2^{-j}} \right\|^{\alpha'}\right), & \text{if } \left|\frac{k}{2^j} - \frac{k_0}{2^{j_0}}\right| \leq C'2^{-j} \end{cases} \quad \text{(V.3)}$$

and

$$\left|\left\langle \psi^{(v)}_{jk\boldsymbol{m}}, P_{C_2} f \right\rangle\right| \leq C2^{-j(2\alpha + \frac{5}{2})} \left(1 + \left\| \frac{\boldsymbol{m} - \boldsymbol{u}}{2^{-j}} \right\|^{\alpha'}\right), \quad \text{(V.4)}$$

then $f \in C^\alpha(\boldsymbol{u})$. Similar statement holds if the inequality (V.3) is valid for $\left\langle \psi^{(v)}_{ast}, P_{C_2} f \right\rangle$ and the inequality (V.4) is valid for $\langle \psi_{ast}, P_{C_1} f \rangle$.

CHAPTER VI

DIRECTIONAL CALDÉRON-ZYGMUND REGULARITY

6.1 Orthonormal Wavelet Bases and Multiresolution Analysis

Let H be a Hilbert space (complete inner product space) and a sequence $\{x_n\}$ in H. Then

1. $\{x_n\}$ is a *basis* for H if for each $x \in H$ there is unique decomposition

$$x = \sum_{n \in \mathbb{Z}} c_n x_n \text{ when } c_n \in \mathbb{C}.$$

2. $\{x_n\}$ is an *orthonormal basis* for H if $\{x_n\}$ is a basis for H and

$$\langle x_n, x_m \rangle = \delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \quad \text{for all } n, m \in \mathbb{Z}. \tag{VI.1}$$

Let $\psi^{(i)} \in L^2(\mathbb{R}^d)$ for $i \in \{0, 1, ..., 2^d - 1\}$. We define the system

$$\left\{ \psi_{j,\boldsymbol{k}}^{(i)}(\cdot) = 2^{\frac{jd}{2}} \psi^{(i)}(2^j \cdot - \boldsymbol{k}),\ i \in \{0, 1, ..., 2^d - 1\},\ j \in \mathbb{Z}, \text{ and } \boldsymbol{k} \in \mathbb{Z}^d \right\} \tag{VI.2}$$

If the system (VI.2) is a basis for $L^2(\mathbb{R}^d)$, $\psi^{(i)}$ is called a *wavelet function*. Moreover, if the system (VI.2) is an orthonormal basis for $L^2(\mathbb{R}^d)$, $\psi^{(i)}$ is called an *orthonormal wavelet* function.

Orthonormal wavelet bases are constructed through the help of a *multiresolution analysis*. It is a sequence of closed subspaces of $L^2(\mathbb{R}^d)$ denoted by V_j ($j \in \mathbb{Z}$) and satisfying:

1. $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}$,
2. $\forall j \in \mathbb{Z},\ f(\cdot) \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$,
3. $\exists \varphi \in V_0$ such that the functions $\varphi(\cdot - k)(k \in \mathbb{Z}^d)$ form an orthonormal basis of V_0.
4. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

The subspace W_j is defined as the orthogonal complement of V_j in V_{j+1}. Consequently, W_j are orthonormal, and so we can use two possible decompositions of $L^2(\mathbb{R}^d)$ as a direct orthogonal sum

$$L^2(\mathbb{R}^d) = \oplus_{j=-\infty}^{+\infty} W_j \text{ or } L^2(\mathbb{R}^d) = V_0 \oplus \oplus_{j=0}^{+\infty} W_j. \tag{VI.3}$$

The multiresoltion analysis is *r-smooth* if $\varphi \in C^r$ and if the $\partial^\alpha \varphi$, for $|\alpha| \leq r$, have fast decay, that is there exists $C > 0$ such that

$$|\partial^\alpha \varphi| \leq \frac{C}{(1 + \|\boldsymbol{x}\|)^{d+1}} \text{ for all } |\alpha| \leq r$$

Under these assumptions, there exist $2^d - 1$ function $\psi^{(i)}$ satisfying the same regularity and decay properties as φ and such that the $\psi^{(i)}(\boldsymbol{x} - \boldsymbol{k})$ for $i = 1, ..., 2^d - 1,\ \boldsymbol{k} \in \mathbb{Z}^d$ form an orthonormal basis of W_0. Using property of V_j number 2, we have that the $\psi^{(i)}(2^j \boldsymbol{x} - \boldsymbol{k})$ for $i = 1, ..., 2^d - 1,\ \boldsymbol{k} \in \mathbb{Z}^d$ form an orthonormal basis of W_j.

Hence a function in $L^2(\mathbb{R}^d)$ can be written as

$$f(\boldsymbol{x}) = \sum_{j\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \sum_{i} c^i_{j,\boldsymbol{k}} \psi^{(i)}(2^j\boldsymbol{x} - \boldsymbol{k}),$$

or

$$f(x) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} C_k \varphi(\boldsymbol{x} - \boldsymbol{k}) + \sum_{j=0}^{\infty} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \sum_{i} c^i_{j,\boldsymbol{k}} \psi^{(i)}(2^j\boldsymbol{x} - \boldsymbol{k});$$

where the $c^i_{j,\boldsymbol{k}}$ are the wavelet coefficients of f given by

$$c^i_{j,k} = 2^{dj} \int_{\mathbb{R}^d} f(\boldsymbol{x}) \psi^{(i)}(2^j\boldsymbol{x} - \boldsymbol{k})\, d\boldsymbol{x}, \tag{VI.4}$$

and

$$C_k = \int_{\mathbb{R}^d} f(\boldsymbol{x}) \varphi(2^j\boldsymbol{x} - \boldsymbol{k})\, d\boldsymbol{x}. \tag{VI.5}$$

We will often assume that the multiresolution analysis is of *tensor product type* i.e., if $\boldsymbol{x} = (x_1, ..., x_d)$, then

$$\varphi(\boldsymbol{x}) = \Phi(x_1)\Phi(x_2)\cdots\Phi(x_d),$$

where $\Phi(x)$ is associated to a 1-dimensional multiresolution analysis.

If $\Psi(x)$ denotes the corresponding 1-dimensional wavelet, we can take for d-dimensional wavelets function

$$\psi^{(i)}(\boldsymbol{x}) = \Psi_1(x_1)\Psi_2(x_2)\cdots\Psi_d(x_d),$$

where Ψ_l denotes either Φ or Ψ, and where the choice $\Phi(x_1)\cdots\Phi(x_d)$ is the only one excluded (thus there are indeed $2^d - 1$ wavelets).

6.2 Discrete Wavelet Transform and Caldéron and Zygmund

The following definition without restriction on r was introduced by Caldéron and Zygmund ([1]) in 1961.

Definition 6.2.1. Let $p \in [1, \infty)$, $f \in L^p_{\text{loc}}(\mathbb{R}^d)$, and $u \geq -\frac{d}{p}$. Then $f \in T^p_u(\boldsymbol{x}_0)$ if there exist constants $R > 0$, $C > 0$ and a polynomial $P_{\boldsymbol{x}_0}$ of degree less than u such that

$$\forall r \leq R, \left(\frac{1}{r^d}\int_{B(\boldsymbol{x}_0,r)} |f(\boldsymbol{x}) - P_{\boldsymbol{x}_0}(\boldsymbol{x}-\boldsymbol{x}_0)|^p dx\right)^{\frac{1}{p}} \leq Cr^u. \tag{VI.6}$$

Note that, for each $0 \leq u$ and $1 \leq p$, we have

1. $\left\|(f - P(\cdot - \boldsymbol{x}_0))1_{B(\boldsymbol{x}_0,r)}\right\|_p \leq Cr^{u+\frac{d}{p}}$.
2. $C^u(\boldsymbol{x}_0) \subseteq T^p_u(\boldsymbol{x}_0)$
3. If $1 \leq q \leq p < +\infty$, then $T^p_u(\boldsymbol{x}_0) \hookrightarrow T^q_u(\boldsymbol{x}_0)$.
4. If $f \in T^p_u(\boldsymbol{x}_0) \cap T^q_v(\boldsymbol{x}_0)$ and $0 < \alpha < 1$ then $f \in T^r_w(\boldsymbol{x}_0)$ where $w = \alpha u + (1-\alpha)v$ and $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.
5. If $0 \leq u \leq v$, $1 \leq p < \infty$, then $T^p_v(\boldsymbol{x}_0) \subset T^p_u(\boldsymbol{x}_0)$.
6. Let $1 \leq p \leq \infty$, $0 \leq u \leq v$, $f \in T^p_u(\boldsymbol{x}_0)$ and $g \in C^v(\boldsymbol{x}_0)$ Then $fg \in T^p_u(\boldsymbol{x}_0)$.
7. Let $1 \leq p < \infty$, $\frac{-d}{p} < u$, $0 < v$, $u \leq v$. If $f \in T^p_u(\boldsymbol{x})$ $g \in C^v(\boldsymbol{x}_0)$ and $g(\boldsymbol{x}_0) = 0$. Then $fg \in T^p_w(\boldsymbol{x}_0)$ where
   (a) $w = \min(u+v, v)$ if $v \leq 1$,
   (b) $w = \min(u+1, v)$ if $v \geq 1$, and
   (c) $w = u+1$ if $u > 0$, $v \geq 1$ and $f(\boldsymbol{x}_0) = 0$.

All of this section, we work only on $\mathbb{R}^2$ and so we start by introducing some definitions and notations on $\mathbb{R}^2$. Let $\psi^{(i)}$, $i = 1, 2, 3$ be compactly supported C^2 functions generating a wavelet basis, i.e. the functions $\psi_{i,j,k}(\boldsymbol{x}) = 2^j\psi^{(i)}(2^j\boldsymbol{x} -$

$\boldsymbol{k})(i = 1,2,3,\ j \in \mathbb{Z},\ \boldsymbol{k} \in \mathbb{Z}^2)$ form an orthonormal basis for $L^2(\mathbb{R}^2)$. For the purpose of detecting regularity, the wavelets $\psi^{(i)}$, $i = 1,2,3$, are also assumed to satisfy

$$\int \boldsymbol{x}^{\boldsymbol{m}} \psi^{(i)}(\boldsymbol{x})\, d\boldsymbol{x} = 0 \text{ for all } |\boldsymbol{m}| \leq 2,$$

$$\text{and } \exists C > 0, \text{ s.t. } |\partial^{\beta} \psi^{(i)}(\boldsymbol{x})| \leq \frac{C}{(1 + \|\boldsymbol{x}\|)^2} \text{ for all } |\beta| \leq 2.$$

For each scale $j \in \mathbb{Z}$ and $\boldsymbol{k} \in \mathbb{Z}^2$, the order pair $(j, \boldsymbol{k})$ can be represented in a unique fashion (1-1) by the dyadic square of width 2^{-j} whose lower left conner is $2^{-j}\boldsymbol{k}$. This representation is natural in the sense that if $\operatorname{supp} \psi \subseteq [0,1]^2$ Then $\operatorname{supp} \psi(2^j \boldsymbol{x} - \boldsymbol{k}) \subseteq 2^{-j}(\boldsymbol{k} + [0,1]^2)$.

Since there are 3 possible values for i, each i may be associated with a dyadic subcube λ_i of $[0,1]^2$ of width $\frac{1}{2}$ except for $[0, \frac{1}{2})^2$. With this choice of association λ_i, one can index multiwavelets $\psi_{j,\boldsymbol{k}}^{(i)}$ by relabelling each $(i, j, \boldsymbol{k})$ by a dyadic subcube $\lambda = 2^{-j}(\boldsymbol{k} + \lambda_i)$, i.e. $\lambda = \{\boldsymbol{x} : 2^j \boldsymbol{x} - \boldsymbol{k} \in \lambda_i\}$. We then denote $\psi_\lambda(\boldsymbol{x}) = \psi^{(i)}(2^j \boldsymbol{x} - \boldsymbol{k})$ and $c_\lambda = 2^{2j} \int \psi_\lambda(\boldsymbol{x}) f(\boldsymbol{x})\, d\boldsymbol{x}$. Let $\boldsymbol{x}_0 \in \mathbb{R}^2$ and $j_0 \in \mathbb{Z}$. If $\boldsymbol{x}_0$ is contained in a dyadic cube of width 2^{-j_0}, then $\lambda_{j_0}(\boldsymbol{x}_0)$ denotes this unique dyadic cube. Jaffard [21] then defined the *local square function*

$$S_f(j_0, \boldsymbol{x}_0)(\boldsymbol{x}) = \left(\sum_{\lambda \subset 3\lambda_{j_0}(\boldsymbol{x}_0)} |c_\lambda|^2 1_\lambda(\boldsymbol{x}) \right)^{\frac{1}{2}}$$

and gave an L^p-norm estimate of $S_f(j_0, \boldsymbol{x}_0)(\boldsymbol{x})$ for $f \in T_u^p(\boldsymbol{x}_0)$ as well as the converse.

Theorem 6.2.2. *Let $p \in [1, \infty)$ and $u > 0$; if $f \in T_u^p(\boldsymbol{x}_0)$, then $\exists C > 0$ such that*

$$\|S_f(j, x_0)\|_p \leq C 2^{-j(u + \frac{d}{p})} \quad \textit{for all } j \geq 0. \tag{VI.7}$$

Conversely, if (VI.7) holds and $u \notin \mathbb{N}$, then $f \in T_u^p(\boldsymbol{x}_0)$.

6.3 Directional $T_u^p(\boldsymbol{x}_0;\theta)$

We will define directional regularity based on parabolic scaling at $\boldsymbol{x}_0$ in $\mathbb{R}^2$. Fix $p \geq 1$. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to belong to $T_u^p(\boldsymbol{x}_0;\theta)$ when $u > 0$ and $\theta \in [0, 2\pi)$ if there exist a polynomial $P_{\boldsymbol{x}_0}$ of degree less than u, constants $C > 0$, and $R > 0$ such that

$$\forall r \leq R, \left(\frac{1}{r^{\frac{3}{2}}}\int_{E_r(\boldsymbol{x}_0;\theta)} |f(\boldsymbol{x}) - P_{\boldsymbol{x}_0}(\boldsymbol{x}-\boldsymbol{x}_0)|^p dx\right)^{\frac{1}{p}} \leq Cr^{\frac{u}{2}}$$

where $E_r(\boldsymbol{x}_0;\theta) = \boldsymbol{x}_0 + R_\theta E_r$, and E_r is the ellipse $\frac{x^2}{r} + \frac{y^2}{r^2} = 1$.

Note that, For any $\theta \in [0, 2\pi)$, $0 \geq u$ and $0 \geq v$, we have that

(i) $C^u(\boldsymbol{x}_0) \subseteq T_u^p(\boldsymbol{x}_0;\theta)$.

(ii) If $1 \leq q \leq p \leq +\infty$, then $T_u^p(\boldsymbol{x}_0;\theta) \hookrightarrow T_u^q(\boldsymbol{x}_0;\theta)$.

(iii) If $f \in T_u^p(\boldsymbol{x}_0;\theta) \cap T_v^q(\boldsymbol{x}_0;\theta)$ and if r is such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ with $0 < \alpha < 1$, then $f \in T_w^r(\boldsymbol{x}_0;\theta)$ when $w = \alpha u + (1-\alpha)v$.

In the next example, we have to show that $T_u^p(\boldsymbol{x}_0) \neq T_u^p(\boldsymbol{x}_0;\theta)$.

Example For $\theta_0 \in (0, \frac{\pi}{2})$ and $\beta + \frac{1}{p} \leq \gamma$, let function f be defined in polar coordinate such that

$$f(\rho,\theta) = \begin{cases} \rho^\beta, & \text{if } 0 \leq \theta \leq \theta_0 \\ \rho^\gamma, & \text{if } \theta_0 \leq \theta \leq \frac{\pi}{2}, \end{cases}$$

and f is symmetry with respect to $x, y-$axes, i.e. $f(\rho,\theta) = f(\rho,-\theta)$ and $f(\rho,\theta) = f(\rho, \pi-\theta)$.

We will show that $f \in T^p_{\beta+\frac{1}{p}}(\boldsymbol{0};0) - T^p_{\beta+\frac{1}{p}}(\boldsymbol{0})$.

$$\left[\frac{1}{r^2}\int\int_{B(\boldsymbol{0},r)} \left|\rho^{\alpha(\theta)}\right|^p \rho d\rho d\theta\right]^{\frac{1}{p}} = \left[\frac{4}{r^2}\int_0^{\frac{\pi}{2}}\int_0^r \rho^{p\alpha(\theta)+1} d\rho d\theta\right]^{\frac{1}{p}}$$

$$= \left[\frac{4}{r^2}\int_0^{\theta_0}\int_0^r \rho^{p\beta+1}d\rho d\theta + \frac{4}{r^2}\int_{\theta_0}^{\frac{\pi}{2}}\int_0^r \rho^{p\gamma+1}d\rho d\theta\right]^{\frac{1}{p}}$$
$$= \left[4\theta_0\frac{r^{p\beta}}{p\beta+2} + 4(\frac{\pi}{2}-\theta_0)\frac{r^{p\gamma}}{p\gamma+2}\right]^{\frac{1}{p}}$$
$$\leq Cr^{\beta}.(\because \beta < \gamma)$$

Then $f \in T^p_\beta(\mathbf{0})$. We can see that $f \notin T^p_{\beta+\epsilon}(\mathbf{0})$ any $\epsilon > 0$ because

$$\frac{1}{r^{\beta+\epsilon}}\left[\frac{1}{r^2}\int\int_{B(\mathbf{0},r)}\left|\rho^{\alpha(\theta)}\right|^p \rho d\rho d\theta\right]^{\frac{1}{p}} = \left[4\theta_0\frac{r^{-p\epsilon}}{p\beta+2} + 4(\frac{\pi}{2}-\theta_0)\frac{r^{p\gamma-p\beta-p\epsilon}}{p\gamma+2}\right]^{\frac{1}{p}} \to \infty$$

as $r \to 0$. Next to show that $f \in T^p_{\beta+\frac{1}{p}}(\mathbf{0};0)$.

$$\left[\frac{1}{r^{\frac{3}{2}}}\int\int_{E_r(\mathbf{0};0)}\left|\rho^{\alpha(\theta)}\right|^p \rho d\rho d\theta\right]^{\frac{1}{p}} = \left[\frac{1}{r^{\frac{3}{2}}}\int\int_{E_r(\mathbf{0};0)}(x^2+y^2)^{\frac{p\alpha(\theta)}{2}}dxdy\right]^{\frac{1}{p}}$$
$$= \left[\int\int_{B(\mathbf{0},1)}(r(u^2+v^2)+r(r-1)v^2)^{\frac{p\alpha(\theta)}{2}}dudv\right]^{\frac{1}{p}}$$

when $u = \frac{x}{\sqrt{r}}$ and $v = \frac{y}{r}$.

Thus $\cot\theta_0 = \frac{x}{y} = \frac{\sqrt{r}u}{rv} = \frac{1}{\sqrt{r}}\cot\sigma_0$ and so $\sigma_0 = \cot^{-1}(\sqrt{r}\cot\theta_0)$. Then

$$\frac{1}{r^{\frac{3}{2}}}\int\int_{E_r(\mathbf{0};0)}\left|\rho^{\alpha(\sigma)}\right|^p \rho d\rho d = \int_0^{2\pi}\int_0^1 (r\rho^2 + r(r-1)\rho^2\sin^2\sigma)^{\frac{p\alpha(\sigma)}{2}}\rho d\rho d\sigma$$
$$= \frac{4r^{\frac{p\beta}{2}}}{p\beta+2}\int_0^{\sigma_0}(1+(r-1)\sin^2\sigma)^{\frac{p\beta}{2}}d\sigma$$
$$+ \frac{4r^{\frac{p\gamma}{2}}}{p\gamma+2}\int_{\sigma_0}^{\frac{\pi}{2}}(1+(r-1)\sin^2\sigma)^{\frac{p\gamma}{2}}d\sigma$$

Since, for $0 < r < 1$, $-1 < (r-1) < 0$ and $0 < \sin^2\sigma < 1$ with $\sigma \in [0, \frac{\pi}{2}]$, $-1 < (r-1)\sin^2\sigma < 0$ and so $0 < 1+(r-1)\sin^2\sigma < 1$. Consequently, we have

$$\int_{\sigma_0}^{\frac{\pi}{2}}(1+(r-1)\sin^2\sigma)^{\frac{p\gamma}{2}}d\sigma \leq \int_0^{\frac{\pi}{2}}(1)^{\frac{p\gamma}{2}}d\sigma$$

and

$$\int_0^{\sigma_0}(1+(r-1)\sin^2\sigma)^{\frac{p\gamma}{2}}d\sigma \leq \sigma_0 = \cot^{-1}(\sqrt{r}\cot\theta_0) \leq \sqrt{r}\cot\theta_0$$

So

$$\left[\frac{1}{r^{\frac{3}{2}}}\int\int_{E_r(\mathbf{0};0)}\left|\rho^{\alpha(\theta)}\right|^p \rho d\rho d\theta\right]^{\frac{1}{p}} \leq \left[\frac{4r^{\frac{p\beta}{2}+\frac{1}{2}}}{p\beta+2}\cot\theta_0+\frac{4r^{\frac{p\gamma}{2}}}{p\gamma+2}\frac{\pi}{2}\right]^{\frac{1}{p}}$$

$$\leq C\left[\frac{4}{p\beta+2}+\frac{4}{p\gamma+2}\right]^{\frac{1}{p}} r^{\frac{\beta}{2}+\frac{1}{2p}}(\text{since } \beta+\frac{1}{p}\leq\gamma)$$

$$= Cr^{\frac{1}{2}(\beta+\frac{1}{p})}.$$

Therefore, $f \in T^p_{\beta+\frac{1}{p}}(\mathbf{0};0)$ We can see that $f \notin T^p_{\beta+\frac{1}{p}+\epsilon}(\mathbf{0};0)$ any $\epsilon > 0$ because

$$\frac{1}{r^{\frac{\beta+\epsilon}{2}}}\left[\frac{1}{r^{\frac{3}{2}}}\int\int_{E_r(\mathbf{0};0)}\left|\rho^{\alpha(\theta)}\right|^p \rho d\rho d\theta\right]^{\frac{1}{p}}$$

$$= \left[\frac{4r^{\frac{-p\epsilon}{2}}}{p\beta+2}\int_0^{\sigma_0}(1+(r-1)\sin^2\sigma)^{\frac{p\beta}{2}}d\sigma\right]^{\frac{1}{p}} + \left[\frac{4r^{\frac{p(\gamma-\beta-\epsilon)}{2}}}{p\gamma+2}\int_{\sigma_0}^{\frac{\pi}{2}}(1+(r-1)\sin^2\sigma)^{\frac{p\gamma}{2}}d\sigma\right]^{\frac{1}{p}}$$

$\rightarrow \infty$ as $r \rightarrow 0$.

Let us define a new local square function.

$$S_f(j_0,\boldsymbol{x}_0)(\boldsymbol{x}) = \left(\sum_{\lambda\subset\lambda_{[j_0]}(\boldsymbol{x}_0)}|c_\lambda|^2 1_\lambda(\boldsymbol{x})\right)^{(1/2)}$$

when M the smallest even integer greater then $\sqrt{3\cdot 2^{j_0+1}}$ and $\lambda_{[j_0]}(\boldsymbol{x}_0)$ denote the cube of same center as $\lambda_{j_0}(\boldsymbol{x}_0)$ with 3 times wider in the minor axis and M times wider in the major axis.

Next, we will be introduce Lemma 6.3.1 (see [25]) this is important tool for the proof of main Theorem.

Lemma 6.3.1. *For* $1 < p < \infty$, f, $(\sum_\lambda |c_\lambda|^2 1_\lambda(x))^{\frac{1}{2}}$ *and* $(\sum_\lambda |c_\lambda|^2|\psi_\lambda(x)|^2)^{\frac{1}{2}}$ *are equivalent norms in* $L^p(\mathbb{R}^2)$.

Theorem 6.3.2. *Let* $p \in (1,\infty)$ *and* $u < 1$; *if* $f \in T^p_u(\boldsymbol{x}_0;0)$, *then* $\exists C > 0$ *such that*

$$\|S_f(j,\boldsymbol{x}_0)\|_p \leq C2^{-j(\frac{u}{2}+\frac{3}{2p})} \quad \textit{for all } j \geq 0. \tag{VI.8}$$

Proof. Assume that $f \in T_u^p(\boldsymbol{x}_0;0)$.

Then, by definition of $T_u^p(\boldsymbol{x}_0;0)$, $\exists R, C > 0$ such that

$$\left\|(f - f(\boldsymbol{x}_0))1_{E(\boldsymbol{x}_0,\sqrt{r},r)}\right\|_p \leq Cr^{(\frac{u}{2}+\frac{3}{2p})}, \quad \forall r \leq R$$

where $E(\boldsymbol{x}_0, \sqrt{r}, r)$ is the ellipse $\frac{x^2}{r} + \frac{y^2}{r^2} = 1$.

Let $\lambda_{j_0}(\boldsymbol{x}_0)$ be the unique dyadic cube of width 2^{-j_0} which contain $\boldsymbol{x}_0$.

Let $g(\boldsymbol{x}) = (f - P)1_{E(\boldsymbol{x}_0,\sqrt{D2^{-j_0}},D2^{-j_0})}$ where D is large enough such that if $\lambda \subset \lambda_{[j_0]}(\boldsymbol{x}_0)$ then $\text{supp}(\lambda) \subset E(\boldsymbol{x}_0, \sqrt{D2^{-j_0}}, D2^{-j_0})$. Thus, for $\lambda \subset \lambda_{[j_0]}(\boldsymbol{x}_0)$,

$$\begin{aligned}
c_\lambda &= 2^{2j} \int \psi_\lambda(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} \\
&= 2^{2j} \int \psi_\lambda(\boldsymbol{x})(f - f(\boldsymbol{x}_0))1_{E(\boldsymbol{x}_0,\sqrt{D2^{-j_0}},D2^{-j_0})}(\boldsymbol{x}) d\boldsymbol{x} \\
&\quad (\text{since } \text{supp}(\lambda) \subset E(\boldsymbol{x}_0, \sqrt{D2^{-j_0}}, D2^{-j_0})) \\
&= 2^{2j} \int \psi_\lambda(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x} \\
&= c'_\lambda. \quad (\text{when } c'_\lambda \text{ is a wavelet coefficient of } g)
\end{aligned}$$

Then

$$\begin{aligned}
\left\|\Big(\sum_{\lambda \subset \lambda_{[j_0]}(\boldsymbol{x}_0)} |c_\lambda|^2 1_\lambda(\boldsymbol{x})\Big)^{\frac{1}{2}}\right\|_p &= \left\|\Big(\sum_{\lambda \subset \lambda_{[j_0]}(\boldsymbol{x}_0)} |c'_\lambda|^2 1_\lambda(\boldsymbol{x})\Big)^{\frac{1}{2}}\right\|_p \\
&\leq \left\|\Big(\sum_{\lambda} |c'_\lambda|^2 1_\lambda(\boldsymbol{x})\Big)^{\frac{1}{2}}\right\|_p \\
&\leq C \left\|g\right\|_p \quad \text{for some } C > 0 \\
&= \left\|(f - f(\boldsymbol{x}_0))1_{E(\boldsymbol{x}_0,\sqrt{D2^{-j_0}},D2^{-j_0})}\right\|_p \\
&\leq C(D2)^{-j_0(\frac{u}{2}+\frac{3}{2p})} \\
&= C(D)^{-j_0(\frac{u}{2}+\frac{3}{2p})} 2^{-j_0(\frac{u}{2}+\frac{3}{2p})} \\
&= C' 2^{-j_0(\frac{u}{2}+\frac{3}{2p})}.
\end{aligned}$$

□

Theorem 6.3.3. *Let $p \in (1,\infty)$ and $0 < u < 1$ $\exists C > 0$ such that*

$$\|S_f(j,\boldsymbol{x}_0)\|_p \le C2^{-j(\frac{u}{2}+\frac{2}{p})} \quad \text{for all } j \ge 0. \tag{VI.9}$$

Then $f \in T^p_u(\boldsymbol{x}_0;0)$, then

Proof. We can forget the low frequency component of f corresponding to $j < 0$.

Assume that $\exists C > 0$ s.t.

$$\forall j \ge 0,\ \ \|S_f(j,\boldsymbol{x}_0)\|_p \le C2^{-j(\frac{u}{2}+\frac{2}{p})}. \tag{VI.10}$$

Let Λ_j denote the set of dyadic cubes of width 2^{-j}, $\Delta_j f = \sum_{\lambda\in\Lambda_j} c_\lambda\psi_\lambda$. Since $f(\boldsymbol{x}) = \sum_\lambda c_\lambda\psi_\lambda(\boldsymbol{x}) = \sum_{j\in\mathbb{Z}} \Delta_j f(\boldsymbol{x})$ for all $\boldsymbol{x}\in\mathbb{R}^2$ (the convergence of the right-hand side to f takes place in $L^p(\mathbb{R}^2)$ norm), $f(\boldsymbol{x}_0) = \sum_\lambda c_\lambda\psi_\lambda(\boldsymbol{x}_0)$.

We can forget the low frequency component of f corresponding to $j < 0$.

Let $0 < \rho < 1$ be given nd let J be defined by $2^{-J} \le \rho < 2\cdot 2^{-J}$ and L be a constant which if $\lambda \in \Lambda_j$ for each $j \ge J+L$ and $\operatorname{supp}(\psi_\lambda)\cap E(\boldsymbol{x}_0,\sqrt{\rho},\rho) \ne \phi$ then $\operatorname{supp}(\psi_\lambda) \subset E(\boldsymbol{x}_0,\sqrt{2\rho},2\rho))$.

To show that $\left\|(f_H - f(\boldsymbol{x}_0))1_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)}\right\|_p \le C\rho^{(\frac{u}{2}+\frac{3}{2p})}$ for some $C > 0$.

$$\left\|(f_H - f(\boldsymbol{x}_0))1_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)}\right\|_p = \left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |f(\boldsymbol{x}) - f(\boldsymbol{x}_0)|^p d\boldsymbol{x}\right)^{\frac{1}{p}}$$

$$= \left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |\sum_0^\infty(\Delta_j f(\boldsymbol{x}_0) - \Delta_j f(\boldsymbol{x}_0)|^p d\boldsymbol{x}\right)^{\frac{1}{p}}$$

$$= \left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |\sum_{j=0}^{J+L}(\Delta_j f(\boldsymbol{x}) - \Delta_j f(\boldsymbol{x}_0)) + \sum_{j=J+L+1}^{\infty}(\Delta_j f(\boldsymbol{x}) - \Delta_j f(\boldsymbol{x}_0)|^p d\boldsymbol{x}\right)^{\frac{1}{p}}$$

$$\le \left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |\sum_{j=0}^{J+L}(\Delta_j f(\boldsymbol{x}) - \Delta_j f(\boldsymbol{x}_0)|^p d\boldsymbol{x}\right)^{\frac{1}{p}} +$$

$$\left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |\sum_{j=J+L+1}^{\infty} \Delta_j f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{\frac{1}{p}} + \left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |\sum_{j=J+L+1}^{\infty} \Delta_j f(\boldsymbol{x}_0)|^p d\boldsymbol{x}\right)^{\frac{1}{p}}$$

(By Minkowski's inequality).

Let T_1, T_2, and T_3 denote the three terms on the right-hand side.

Let $D = \max\limits_{0\le j\le J+L} \left\{ D_j : \forall\lambda \in \Lambda_j \text{ if } \operatorname{supp}(\psi_\lambda) \cap E(\boldsymbol{x}_0,\sqrt{\rho},\rho) \neq \phi \text{ then } \operatorname{dist}(\lambda,\boldsymbol{x}_0) \le D_j 2^{-\frac{j}{2}} \right\}$

Let $j \le J+L$. Thus at most N of the $\psi_\lambda(\lambda \in \Lambda_j)$ have a support intersecting $E = E(\boldsymbol{x}_0,\sqrt{\rho},\rho)$ and if $\lambda \in \Lambda_j$ s.t. $\operatorname{dist}(\lambda,\boldsymbol{x}_0) \le D2^{-\frac{j}{2}}$, then $|c_\lambda| \le C2^{-j(\frac{u}{2})}$.

$$\begin{aligned}
|c_\lambda|2^{-j\frac{2}{p}} = (\int |c_\lambda|^p 1_\lambda(x) d\boldsymbol{x})^{\frac{1}{p}} &\le \left(\int (\sum_{\lambda\subset\lambda_{[j-M]}(\boldsymbol{x}_0)} |c_\lambda|^2 1_\lambda)^{\frac{p}{2}} d\boldsymbol{x}\right)^{\frac{1}{p}} \\
&= ||S_f(j-M,\boldsymbol{x}_0)||_p \\
&\le C2^{-(j-M)(\frac{u}{2}+\frac{2}{p})} \qquad \text{(By assumption)} \\
&= C2^{M(\frac{u}{2}+\frac{2}{p})}2^{-j(\frac{u}{2}+\frac{2}{p})} \\
&= C2^{-j(\frac{u}{2}+\frac{2}{p})} \qquad (C = C2^{M(\frac{u}{2}+\frac{2}{p})}).
\end{aligned}$$

Let $\boldsymbol{x} \in E$, for $|\beta| = 1$,

$$\begin{aligned}
|\partial^\beta \Delta_j f| &= \left| \sum_{\lambda\in\Lambda_j,\operatorname{supp}(\psi_\lambda)\cap E(\boldsymbol{x}_0,\sqrt{\rho},\rho)\neq\phi} c_\lambda \partial^\beta \psi_\lambda(\boldsymbol{x}) \right| \\
&= 2^j \left| \sum_{\boldsymbol{k},i} c_\lambda \partial^\beta \psi^i(2^j\boldsymbol{x}-\boldsymbol{k}) \right| \\
&\le 2^j \left| \sum_{\boldsymbol{k}} \frac{2^{-j(\frac{u}{2})}}{(1+|2^j\boldsymbol{x}-\boldsymbol{k}|)^3} \right| \\
&\le C2^{-j(\frac{u}{2})}2^j = C2^{(1-\frac{u}{2})j}
\end{aligned}$$

when $C = \max\limits_{0\le j\le J+L}\{\frac{1}{(1+|2^j\boldsymbol{x}-\boldsymbol{k}|)^3} : \lambda \in \Lambda_j \text{ and } \operatorname{supp}(\psi_\lambda) \cap B \neq \phi\}$. Then

$$|\Delta_j f(\boldsymbol{x}) - \Delta_j f(\boldsymbol{x}_0)| \le \sup_{[\boldsymbol{x},\boldsymbol{x}_0]} \sum_{|\beta|=\lfloor u\rfloor+1} \frac{|\partial^\beta \Delta_j f|}{\beta!} |\boldsymbol{x}-\boldsymbol{x}_0|^\beta$$

$$= C \sup_{[\boldsymbol{x},\boldsymbol{x}_0]} \sum_{|\beta|=1} |\partial^\beta \Delta_j f| \prod_{i=1}^{d} |x_i - x_{i,0}|^{\beta_i}$$

$$\leq C \sup_{[\boldsymbol{x},\boldsymbol{x}_0]} \sum_{|\beta|=1} |\partial^\beta \Delta_j f| \prod_{i=1}^{d} \rho^{\frac{\beta_i}{2}}$$

$$= C \sup_{[\boldsymbol{x},\boldsymbol{x}_0]} \sum_{|\beta|=1} |\partial^\beta \Delta_j f| \rho^{\frac{|\beta|}{2}}$$

$$\leq C\rho^{\frac{1}{2}} 2^{(1-\frac{u}{2})j} (\sum_{|\beta|=1} 1)$$

$$= C\rho^{\frac{1}{2}} 2^{(1-\frac{u}{2})j}. \quad (C = C(\sum_{|\beta|=1} 1))$$

So,

$$\sum_{j=0}^{J+L} |\Delta_j f(\boldsymbol{x}) - \Delta_j f(\boldsymbol{x}_0)| \leq C \sum_{j=0}^{J+L} \rho^{\frac{1}{2}} 2^{j(1-\frac{u}{2})}$$

$$= C\rho^{\frac{1}{2}} \sum_{j=0}^{J+L} 2^{j(1-\frac{u}{2})}$$

$$\leq C_1 \rho^{\frac{1}{2}}$$

$$\leq C_1 \rho^{\frac{u}{2}}$$

Hence

$$T_1 \leq \left(\int_{E(\boldsymbol{x}_0, \sqrt{\rho}, \rho)} |C\rho^{\frac{u}{2}}|^p dx \right)^{\frac{1}{p}} = C\rho^{(\frac{u}{2} + \frac{3}{2p})}$$

And

$$\sum_{j=J+L}^{\infty} |\Delta_j f(\boldsymbol{x}_0)| \leq C \sum_{j=J+L}^{\infty} 2^{(-\frac{u}{2})j}$$

$$\leq C 2^{(-\frac{u}{2})(J+L)} \sum_{j=0}^{\infty} 2^{j(-\frac{u}{2})}$$

$$= C\rho^{(\frac{u}{2})} \sum_{j=0}^{\infty} 2^{j(-\frac{u}{2})} \qquad (2^{-(J+L)} < 2^{-J} < \rho)$$

$$\leq C\rho^{(\frac{u}{2})}.$$

Hence

$$T_2 \le \left(\int_{E(\boldsymbol{x}_0,\sqrt{\rho},\rho)} |C\rho^{(\frac{u}{2})}|^p d\boldsymbol{x} \right)^{\frac{1}{p}} = C\rho^{(\frac{u}{2}+\frac{3}{2p})}.$$

Next,we try to bound T_3. Let now $g_J(\boldsymbol{x}) = \sum_{j=J+L}^{\infty} \Delta_j f(\boldsymbol{x})$. We have that g_J and $\sum_{j=J+L}^{\infty} \sum_{\forall\lambda\in\Lambda_j,\lambda\subset E(\boldsymbol{x}_0,\sqrt{2\rho},2\rho)} c_\lambda\psi_\lambda$ coincide on E since if $\lambda \in \Lambda_j$, for each $j \ge J+L$, and $\operatorname{supp}(\psi_\lambda) \cap B(\boldsymbol{x}_0,\rho) \neq \phi$ then $\operatorname{supp}(\psi_\lambda) \subset B(\boldsymbol{x}_0, 2\rho)$. Let

$$A = \bigcup_{j\ge J+L,\lambda(\in\Lambda_j)\subset E(\boldsymbol{x}_0,\sqrt{2\rho},2\rho)} \operatorname{supp}(\psi_\lambda).$$

Thus

$$\begin{aligned}
\|g_J 1_E\|_p &= \left(\int_E | \sum_{j=J+L}^{\infty} \sum_{\lambda\in\Lambda_j} c_\lambda\psi_\lambda(\boldsymbol{x})|^p d\boldsymbol{x} \right)^{\frac{1}{p}} \\
&\le \left(\int_A | \sum_{j=J+L}^{\infty} \sum_{\lambda\in\Lambda_j} c_\lambda\psi_\lambda(\boldsymbol{x})|^p d\boldsymbol{x} \right)^{\frac{1}{p}} \\
&= \left\| \sum_{j=J+L}^{\infty} \sum_{\lambda(\lambda\in\Lambda_j)\subset E(\boldsymbol{x}_0,\sqrt{2\rho},2\rho)} c_\lambda\psi_\lambda \right\|_p \\
&\le C \left\| \left(\sum_{j=J+L}^{\infty} \sum_{\lambda(\lambda\in\Lambda_j)\subset E(\boldsymbol{x}_0,\sqrt{2\rho},2\rho)} |c_\lambda|^2 1_\lambda \right)^{\frac{1}{2}} \right\|_p \quad \text{(By equivalent norms)} \\
&\le C \left\| \left(\sum_{\lambda\subset[M,3]\lambda_{J-L}(\boldsymbol{x}_0)} |c_\lambda|^2 1_\lambda \right)^{\frac{1}{2}} \right\|_p \\
&= C \left\| S_f(J-L,\boldsymbol{x}_0) \right\|_p \\
&\le C2^{-J(\frac{u}{2}+\frac{2}{p})} \quad \text{(By assumption)} \\
&\le C2\rho^{(\frac{u}{2}+\frac{2}{p})} \quad (2^{-J} \ge \rho) \\
&\le C\rho^{(\frac{u}{2}+\frac{3}{2p})}.
\end{aligned}$$

□

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