CHAPTER I

COMPLEX ANALYTIC HOMOMORPHISMS

In this chapter, we shall prove that a complex analytic homomorphism of the multiplicative semigroup $M(n,\mathbb{C})$ to $\mathbb{C}$ which take 0 to 0 is of the form $(\det A)^m$ for some $m \in \mathbb{N}$, for all $A \in M(n,\mathbb{C})$ or is identically zero.

We first find all analytic homomorphisms $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{C}$ and $\phi(0) = 0$. Suppose that $\phi : \mathbb{C} \to \mathbb{C}$ is such an analytic homomorphism then ϕ can be written as

$$\phi(x) = c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n + \ldots$$

in some neighborhood of 0. It suffices to find c_i such that $\phi(x)\phi(y) = \phi(xy)$. We have that

$$(1.1) \qquad \phi(xy) = c_1xy + c_2x^2y^2 + c_3x^3y^3 + c_4x^4y^4 + \ldots$$

and

$$(1.2) \qquad \phi(x)\phi(y) = (c_1x+c_2x^2+c_3x^3+\ldots)(c_1y+c_2y^2+c_3y^3+\ldots)$$
$$= c_1^2xy+c_1c_2xy^2+c_2c_1x^2y+c_3c_1x^3y+c_2^2x^2y^2+c_1c_3xy^3+\ldots$$

If $c_i = 0 \ \forall i$, then $\phi \equiv 0$. Now assume that there exists k such that $c_k \neq 0$. Let n be the smallest natural number such that $c_n \neq 0$. We claim that $c_m = 0 \ \forall m \neq n$. We prove this by comparing the coefficient of the term x^my^n $(m \neq n)$ in (1.1) and (1.2); respectively.

Then,

$$0 = c_m c_n .$$

But $c_n \neq 0$ implying that $c_m = 0$. Now, we consider the coefficient of the term $x^n y^n$ in (1.1) and (1.2); respectively. Then we get that

$$c_n = c_n^2$$

which implies that $c_n = 1$ since $c_n \neq 0$.

Hence the analytic homomorphisms $\phi : \mathbb{C} \to \mathbb{C}$ taking 0 to 0 are the functions $\phi(x) = x^n$ for some $n \in \mathbb{N}$ and the 0 function.

Before proving this for arbitrary matrix semigroups we need two lemma

Let S and S' be semigroups and $S \times S' = \{(s,s') | s \in S, s' \in S'\}$. Define multiplication on $S \times S'$ by

$$(s,s')(s_1,s_1') = (ss_1, s's'_1)$$

for all s, s_1 in S and s', s_1' in S'. Then $S \times S'$ with this multiplication forms a semigroup.

Remark: If F is a field and $a \in F$ is such that $a^2 = a$, then a = 0 or 1.

Lemma 1.1 Let (S,0,1) and (S',0',1') be semigroups having zero and multiplicative identity and let (S*,0*,1*) be a field. If $\psi : S \times S' \to S^*$ is a homomorphism such that $\psi((0,0')) = 0^*$, then one of the following must be true :

(i) There exists a homomorphism $\alpha : S \to S^*$ such that $\alpha(s) = \psi((s,s'))$ for all s in S, s' in S' and $\alpha(0) = 0^*$.

(ii) There exists a homomorphism $\beta : S' \to S^*$ such that $\beta(s') = \psi((s,s'))$ for all s in S, s" in S' and $\beta(0') = 0^*$.

(iii) There exist homomorphisms $\alpha : S \to S^*$ and $\beta : S' \to S^*$ such that $\psi((s,s')) = \alpha(s)\beta(s')$ for all $s \in S$, $s' \in S'$ and $\alpha(0) = 0^*$, $\beta(0') = 0^*$.

Furthermore, case (i) occurs if and only if $\psi(0,1') = 0^*$ and $\psi(1,0') = 1^*$, case (ii) occurs if and only if $\psi(0,1') = 1^*$ and $\psi(1,0') = 0^*$ and case (iii) occurs if and only if $\psi(0,1') = 0^*$ and $\psi(1,0') = 0^*$.

<u>Proof</u>. Since ψ is a homomorphism, $\psi((0,1')) = \psi((0,1')(0,1'))$ $= \psi((0,1'))\psi((0,1')) = (\psi((0,1')))^2$, so $\psi((0,1')) = 0^*$ or 1^* by the above remark. Similarly, $\psi((1,0')) = 0^*$ or 1^*. Now, we have 4 cases to consider.

<u>Case 1</u>. $\psi((1,0')) = 0^*$ and $\psi((0,1')) = 0^*$. Claim that (iii) must occur.

Let $\alpha : S \to S^*$ be defined by $\alpha(s) = \psi((s,1'))$ for all s in S, and $\beta : S' \to S^*$ be defined by $\beta(s') = \psi((1,s'))$ for all s' in S'. Then α, β are homomorphisms since $\alpha(s_1s_2) = \psi((s_1s_2,1')) = \psi((s_1,1')(s_2,1'))$ $= \psi((s_1,1'))\psi((s_2,1')) = \alpha(s_1)\alpha(s_2)$ and $\beta(s_1's_2') = \psi((1,s_1's_2'))$ $= \psi((1,s_1')(1,s_2')) = \psi((1,s_1'))\psi((1,s_2')) = \beta(s_1')\beta(s_2')$. $\alpha(0) = \psi((0,1'))$ $= 0^*$ and $\beta(0') = \psi((1,0')) = 0^*$. For all $s \in S$, $s' \in S'$ we have that $\psi((s,s')) = \psi((s,1')(1,s')) = \psi((s,1'))\psi((1,s')) = \alpha(s)\beta(s')$; i.e., $\psi((s,s')) = \alpha(s)\beta(s')$ for all $s \in S$, $s' \in S'$.

<u>Case 2</u>. $\psi((0,1')) = 1^*$ and $\psi((1,0')) = 1^*$.

Since $0^* = \psi((0,0')) = \psi((0,1')(1,0')) = \psi((0,1'))\psi((1,0'))$ $= 1^* \cdot 1^* = 1^*$, this case is impossible.

Case 3. $\psi((0,1')) = 0^*$ and $\psi((1,0')) = 1^*$.

We have that $\psi((s,0')) = \psi((s,s')(1,0')) = \psi((s,s'))\psi((1,0'))$ $= \psi((s,s'))1^* = \psi((s,s'))$ for all $s \in S$, $s' \in S'$, so $\psi((s,0')) = \psi((s,s'))$ for all s' in S'. Define $\alpha(s) = \psi((s,0'))$ for all $s \in S$. Then $\psi((s,s'))$ $= \psi((s,0')) = \alpha(s)$ and $\alpha(s_1s_2) = \psi((s_1s_2,0')) = \psi((s_1,0')(s_2,0'))$ $= \psi((s_1,0'))\psi((s_2,0')) = \alpha(s_1)\alpha(s_2)$, $\alpha(0) = \psi((0,0')) = 0^*$. Therefore α satisfies (i).

Case 4. $\psi((0,1')) = 1^*$ and $\psi((1,0')) = 0^*$.

We also have that for each $s \in S$, $s' \in S'$, $\psi((0,s')) = \psi((s,s')(0,1'))$ $= \psi((s,s'))\psi((0,1')) = \psi((s,s'))$. Therefore $\psi((0,s')) = \psi((s,s'))$ for all s in S, s' in S'. For each $s' \in S'$, define $\beta(s') = \psi((0,s'))$. Then $\beta(s')$ $= \psi((0,s')) = \psi((s,s'))$, $\beta(s_1's_2') = \psi((0,s_1's_2')) = \psi((0,s_1')(0,s_2'))$ $= \psi((0,s_1'))\psi((0,s_2')) = \beta(s_1')\beta(s_2')$ and $\beta(0') = \psi((0,0')) = 0^*$. Hence β is a homomorphism and $\beta(0') = 0^*$ and $\beta(s') = \psi((s,s'))$ for all s in S, s' in S'. So we have (ii). #

Corollary. If S and S' are open subsets of $\mathbb{R}^n(\mathbb{C}^n)$ for some n and S^* is $\mathbb{R}(\mathbb{C})$ and if ψ is an analytic homomorphism, then α and β are also real (complex) analytic homomorphisms.

Proof. It follows immediately from the definitions of α and β. #

Now we shall begin to study real and complex analytic homomorphisms $\Phi : M(n,F) \to F$ taking 0 to 0 where F is either $\mathbb{R}$ or $\mathbb{C}$ and $n > 1$.

Since Φ is real or complex analytic, we have that

$$\Phi\left(\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}\right) = \sum_0^{\infty} \lambda_{m_{11}m_{12}\cdots m_{1n}\cdots m_{n1}\cdots m_{nn}} x_{11}^{m_{11}} x_{12}^{m_{12}} \cdots x_{1n}^{m_{1n}} \cdots x_{n1}^{m_{n1}} \cdots x_{nn}^{m_{nn}},$$

for all $x_{11},\ldots,x_{nn}$ in some neighborhood U of $\bar{0}$, where the constant term is 0.

Since

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix}$$

where $z_{ij} = \sum_{k=1}^{n} x_{ik}y_{kj}$ and ϕ is a multiplicative homomorphism, we get that

$$\left(\sum_0^{\infty} \lambda_{m_{11}m_{12}\cdots m_{1n}\cdots m_{n1}\cdots m_{nn}} x_{11}^{m_{11}} x_{12}^{m_{12}} \cdots x_{1n}^{m_{1n}} \cdots x_{n1}^{m_{n1}} \cdots x_{nn}^{m_{nn}}\right)\left(\sum_0^{\infty} \lambda_{m_{11}m_{12}\cdots m_{1n}\cdots m_{n1}\cdots m_{nn}} y_{11}^{m_{11}} y_{12}^{m_{12}} \cdots y_{1n}^{m_{1n}} \cdots y_{n1}^{m_{n1}} \cdots y_{nn}^{m_{nn}}\right)$$

$$= \sum_0^{\infty} \lambda_{m_{11}m_{12}\cdots m_{1n}\cdots m_{n1}\cdots\ \cdots m_{nn}} z_{11}^{m_{11}} z_{12}^{m_{12}} \cdots z_{1n}^{m_{1n}} \cdots z_{n1}^{m_{n1}} \cdots z_{nn}^{m_{nn}}.$$ So,

$$(1.3)\quad \sum_0^{\infty} \lambda_{m_{11}\cdots m_{22}\cdots m_{ii}\cdots m_{nn}} \cdot \lambda_{k_{11}\cdots k_{22}\cdots k_{ii}\cdots k_{nn}} x_{11}^{m_{11}} \cdots x_{22}^{m_{22}} \cdots x_{ii}^{m_{ii}} \cdots x_{nn}^{m_{nn}} y_{11}^{k_{11}} \cdots y_{22}^{k_{22}} \cdots y_{ii}^{k_{ii}} \cdots y_{nn}^{k_{nn}}$$

$$= \sum_0^{\infty} \lambda_{m_{11}\cdots m_{22}\cdots m_{ii}\cdots m_{nn}} z_{11}^{m_{11}} \cdots z_{22}^{m_{22}} \cdots z_{ii}^{m_{ii}} \cdots z_{nn}^{m_{nn}}$$

$$(1.3)' = \sum_{0}^{\infty} \lambda_{\ell_{11}\ldots\ell_{22}\ldots\ell_{ii}\ldots\ell_{nn}} (x_{11}y_{11}+x_{12}y_{21}+x_{13}y_{31}+\ldots+x_{1n}y_{n1})^{\ell_{11}} \ldots$$

$$\ldots\ldots\ldots\ldots (x_{21}y_{12}+x_{22}y_{22}+x_{23}y_{32}+\ldots+x_{2n}y_{n2})^{\ell_{22}} \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots (x_{n1}y_{1n}+x_{n2}y_{2n}+x_{n3}y_{3n}+\ldots+x_{nn}y_{nn})^{\ell_{nn}}.$$

The coefficient of $x_{12}^{m_{12}}x_{23}^{m_{23}}\ldots x_{n-1,n}^{m_{n-1,n}}y_{21}^{k_{21}}y_{32}^{k_{32}}\ldots y_{n,n-1}^{k_{n,n-1}}$ in (1.3) is $\lambda_{0m_{12}0\ldots0m_{23}0\ldots0m_{i,i+1}0\ldots0m_{n-1,n}0\ldots0}\cdot\lambda_{0\ldots0k_{21}0\ldots0k_{32}0\ldots0k_{i+1,i}0\ldots0k_{n,n-1}0}$. If $m_{i,i+1} = k_{i+1,i}$ $\forall\, 1 \leq i \leq n-1$, and $\ell_{ii} = m_{i,i+1}$ $\forall\, 1 \leq i \leq n-1$, then the coefficient of $x_{12}^{m_{12}}x_{23}^{m_{23}}\ldots x_{n-1,n}^{m_{n-1,n}}y_{21}^{k_{21}}y_{32}^{k_{32}}\ldots y_{n,n-1}^{k_{n,n-1}}$ in (1.3)$'$ is $\lambda_{\ell_{11}0\ldots0\ell_{22}0\ldots0\ell_{ii}0\ldots0\ell_{n-1,n-1}0\ldots0}$. Therefore,

$$(1.4)\quad \lambda_{\ell_{11}0\ldots0\ell_{22}0\ldots0\ell_{ii}0\ldots0\ell_{n-1,n-1}0\ldots0} = \lambda_{0m_{12}0\ldots0m_{23}0\ldots0m_{i,i+1}0\ldots0m_{n-1,n}0\ldots0}\cdot\lambda_{0\ldots0k_{21}0\ldots0k_{32}0\ldots0k_{i+1,i}0\ldots0k_{n,n-1}0}$$

whenever $m_{i,i+1} = k_{i+1,i} = \ell_{ii}$ $\forall\, 1 \leq i \leq n-1$. Therefore, if we can show that the RHS. of (1.4) equals zero for all $m_{i,i+1} \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq n-1$, then $\lambda_{\ell_{11}0\ldots0\ell_{22}0\ldots0\ell_{ii}0\ldots0\ell_{n-1,n-1}0\ldots0}$ must be zero for all $\ell_{ii} \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq n-1$.

Let $m_{i,i+1} \in \mathbb{N} \cup \{0\}$ for all i $(1 \leq i \leq n-1)$. The coefficient of the term $x_{12}^{m_{12}} x_{23}^{m_{23}} \ldots x_{n-1,n}^{m_{n-1,n}} y_{12}^{m_{12}} y_{23}^{m_{23}} \ldots y_{n-1,n}^{m_{n-1,n}}$ in (1.3) is

$\lambda^2_{0m_{12}0\ldots0m_{23}0\ldots0m_{i,i+1}0\ldots0m_{n-1,n}0\ldots0}$.

If $m_{12} \neq 0$, then the coefficient of the term $x_{12}^{m_{12}} x_{23}^{m_{23}} \ldots x_{n-1,n}^{m_{n-1,n}} y_{12}^{m_{12}} y_{23}^{m_{23}} \ldots$ $\ldots y_{n-1,n}^{m_{n-1,n}}$ in $(1.3)'$ is 0. Therefore,

$$(1.5) \qquad \lambda_{0m_{12}0\ldots0m_{23}0\ldots0m_{i,i+1}0\ldots0m_{n-1,n}0\ldots0} = 0$$

for all $m_{i,i+1} \in \mathbb{N} \cup \{0\}$, for all i $(2 \leq i \leq n-1)$, for all $m_{12} \in \mathbb{N}$.

Suppose that $m_{12} = 0$. Let i be the smallest natural number such that $m_{i,i+1} \neq 0$. The coefficient of the term $x_{i,i+1}^{m_{i,i+1}} x_{i+1,i+2}^{m_{i+1,i+2}} \ldots x_{n-1,n}^{m_{n-1,n}} y_{i,i+1}^{m_{i,i+1}}$ $y_{i+1,i+2}^{m_{i+1,i+2}} \ldots y_{n-1,n}^{m_{n-1,n}}$ in (1.3) is $\lambda^2_{0\ldots0m_{i,i+1}0\ldots0m_{i+1,i+2}0\ldots0m_{n-1,n}0\ldots0}$ and is 0 in $(1.3)'$. Therefore,

$$(1.6) \qquad \lambda_{0\ldots0m_{i,i+1}0\ldots0m_{i+1,i+2}0\ldots0m_{n-1,n}0\ldots0} = 0$$

for all $m_{i,i+1}, m_{i+1,i+2}, \ldots, m_{n-1,n} \in \mathbb{N} \cup \{0\}$, $m_{i,i+1} \neq 0$.

Since $\Phi(0) = 0$, $\lambda_{0\ldots0} = 0$. Hence by (1.5) and (1.6)

$$(1.7) \qquad \lambda_{0m_{12}0\ldots0m_{23}0\ldots0m_{i,i+1}0\ldots0m_{n-1,n}0\ldots0} = 0$$

$\forall m_{i,i+1} \in \mathbb{N} \cup \{0\}$, $\forall i$ $(1 \leq i \leq n-1)$. Consequently, the equation (1.4) equals zero. So

(1.8) $$\lambda_{\ell_{11}0\ldots0\ell_{22}0\ldots0\ell_{ii}0\ldots0\ell_{n-1,n-1}0\ldots0} = 0$$

for all $\ell_{ii} \in \mathbb{N} \cup \{0\}$, for all i $(1 \leq i \leq n-1)$.

The coefficient of $x_{21}^{m_{21}} x_{32}^{m_{32}} \ldots x_{n,n-1}^{m_{n,n-1}} y_{12}^{k_{12}} y_{23}^{k_{23}} \ldots y_{n-1,n}^{k_{n-1,n}}$ in (1.3) is

$$\lambda_{0\ldots0m_{21}0\ldots0m_{32}0\ldots0m_{i+1,i}0\ldots0m_{n,n-1}0} \cdot \lambda_{0k_{12}0\ldots0k_{23}0\ldots0k_{i,i+1}0\ldots0k_{n-1,n}0\ldots0} .$$

If $m_{i,i-1} = k_{i-1,i} = \ell_{ii}$ for all i $(2 \leq i \leq n)$, then the coefficient of $x_{21}^{m_{21}} x_{32}^{m_{32}} \ldots x_{n,n-1}^{m_{n,n-1}} y_{12}^{k_{12}} y_{23}^{k_{23}} \ldots y_{n-1,n}^{k_{n-1,n}}$ in (1.3') is $\lambda_{0\ldots0\ell_{22}0\ldots0\ell_{33}0\ldots0\ell_{ii}0\ldots0\ell_{nn}}$. Therefore,

(1.9) $$\lambda_{0\ldots0\ell_{22}0\ldots0\ell_{33}0\ldots0\ell_{ii}0\ldots0\ell_{nn}} = \lambda_{0\ldots0m_{21}0\ldots0m_{32}0\ldots0m_{i+1,i}0\ldots0m_{n,n-1}0} \cdot \lambda_{0k_{12}0\ldots0k_{23}0\ldots0k_{i,i+1}0\ldots0k_{n-1,n}0\ldots0}$$

whenever $m_{i,i-1} = k_{i-1,i} = \ell_{ii}$ for all i $(2 \leq i \leq n)$. By (1.7) we then have that (1.9) equals zero. Therefore, (1.10) $$\lambda_{0\ldots 0\ell_{22}0\ldots 0\ell_{33}0\ldots 0\ell_{ii}0\ldots 0\ell_{nn}} = 0$$

for all $\ell_{ii} \in \mathbb{N} \cup \{0\}$, for all i $(2 \leq i \leq n)$. Now we are ready to prove the following lemma. When we write F in the following lemma it will always stand for either $\mathbb{R}$ or $\mathbb{C}$.

Lemma 1.2 If $\Phi : M(n,F) \to F$ is a (real or complex) analytic multiplicative homomorphism and $\Phi(0) = 0$, then

$$\Phi\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right) = 0 \quad \text{and} \quad \Phi\left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) = 0.$$

Proof. Since

$$\Phi\left(\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}\right) = \sum_{0}^{\infty} \lambda_{m_{11}m_{12}\ldots m_{1n}\ldots m_{n1}\ldots m_{nn}} x_{11}^{m_{11}} x_{12}^{m_{12}} \ldots x_{1n}^{m_{1n}} \ldots x_{n1}^{m_{n1}} \ldots x_{nn}^{m_{nn}}$$

in some neighborhood U of $\bar{0}$,

(1.11) $$\Phi\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{bmatrix}\right) = \sum_{0}^{\infty} \lambda_{0\ldots 0m_{22}0\ldots 0m_{33}0\ldots 0m_{ii}0\ldots 0m_{nn}} x_{22}^{m_{22}} x_{33}^{m_{33}} \ldots x_{nn}^{m_{nn}}$$

and

$$(1.12)\quad \Phi\left(\begin{bmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) = \sum_{0}^{\infty} \lambda_{m_{11}0\ldots0}\, x_{11}^{m_{11}}$$

in some neighborhood U of $\bar{0}$. By (1.10) we then have that equation (1.11) equals zero and (1.8) also implies that equation (1.12) equals zero in U. Therefore, there exist $x_1,\ldots,x_n \neq 0$ sufficiently small so that

$$\Phi\left(\begin{bmatrix} x_{1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) = 0$$

and

$$\Phi\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{bmatrix}\right) = 0.$$

Therefore,

$$\Phi\left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} x_{1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{x_{1}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$= \Phi\left(\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \Phi\left(\begin{bmatrix} \frac{1}{x_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$= 0 \cdot \Phi\left(\begin{bmatrix} \frac{1}{x_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) = 0, \text{ and}$$

$$\Phi\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_n} \end{bmatrix}\right)$$

$$= \Phi\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}\right) \Phi\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_n} \end{bmatrix}\right)$$

$$= 0 \cdot \Phi\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{x_n} \end{bmatrix}\right) = 0.$$

Therefore, the lemma is completely proved. #

Again in what follows F shall stand for either ℝ or ℂ. Given n > 1, let

$$S = \left\{ \begin{bmatrix} x_{11} & 0 & \cdots & 0 \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & x_{n2} & \cdots & x_{nn} \end{bmatrix} ; x_{ij} \in F, \ \forall i,j = 1,2,\ldots,n \right\}.$$

Then, with respect to matrix multiplication, S is a semigroup. Notice that for all natural numbers n > 1, $S \cong F \times M(n-1,F)$. If Φ is a (real or complex) analytic homomorphism from M(n,F) to F and Φ(0)=0, then by lemma 1.2 $\Phi((1,\bar{0})) = 0$ and $\Phi((0,I)) = 0$ where I is the identity in M(n-1,F) and $\bar{0}$ is the zero matrix in M(n-1,F). Therefore, by case 1 of lemma 1.1 there are homomorphisms $\alpha : F \to F$ and $\beta : M(n-1,F) \to F$ such that $\Phi((s,s')) = \alpha(s)\beta(s')$ for all $s \in F$, $s' \in M(n-1,F)$ and $\alpha(0) = 0$, $\beta(0) = 0$. Also, α and β are (real or complex) analytic by the corollary to lemma 1.1. Now we are ready for our main theorem.

<u>Theorem 1.3</u> Let $\Phi : M(n,\mathbb{C}) \to \mathbb{C}$ be a complex analytic multiplicative homomorphism such that Φ(0) = 0. Then Φ ≡ 0 or there is a natural number m such that $\Phi(A) = (\det A)^m$ for all $A \in M(n,\mathbb{C})$.

<u>Proof.</u> We shall prove this theorem by induction on n. For n = 1, we have already proven that Φ ≡ 0 or there exists an $m \in N$ such that $\Phi(x) = x^m$ ($= (\det x)^m$) for all x in ℂ. Suppose that n > 1 and the assertion is true for n-1. We must show that it is true for n. Since Φ is a homomorphism,

$\Phi(I) = \Phi(I^2) = (\Phi(I))^2$ and so $\Phi(I) = 0$ or 1. If $\Phi(I) = 0$, then $\Phi(A) = \Phi(AI) = \Phi(A)\Phi(I) = \Phi(A)\cdot 0 = 0$ for all A in $M(n,\mathbb{C})$. Hence $\Phi \equiv 0$. Suppose that $\Phi(I) = 1$. Let $A \in M(n,\mathbb{C})$.

Case 1. A has an eigenvalue λ of order 1. From chapter 0 p. 6 there exists an invertible matrix B such that

$$B^{-1}AB = \begin{bmatrix} \lambda & 0 \cdots & 0 \\ 0 & & \\ \vdots & C_{(n-1)\times(n-1)} & \\ 0 & & \end{bmatrix}$$

$\Phi(B^{-1}AB) = \Phi(B^{-1})\Phi(A)\Phi(B) = \Phi(B^{-1})\Phi(B)\Phi(A) = \Phi(B^{-1}B)\Phi(A) = \Phi(I)\Phi(A) = \Phi(A)$, i.e., $\Phi(A) = \Phi(B^{-1}AB)$. By the previous remark we have that there are complex analytic homomorphisms $\alpha : \mathbb{C} \to \mathbb{C}$ and $\beta : M(n-1,\mathbb{C}) \to \mathbb{C}$ such that $\Phi((s,s')) = \alpha(s)\beta(s')$ for all $s \in \mathbb{C}$, $s' \in M(n-1,\mathbb{C})$ and $\alpha(0) = 0$, $\beta(0) = 0$. Then by the induction hypothesis either $\beta \equiv 0$ or $\beta(C) = (\det C)^t$ for some $t \in \mathbb{N}$, for all $C \in M(n-1,\mathbb{C})$ and, either $\alpha \equiv 0$ or $\alpha(\lambda) = \lambda^m$ for some $m \in \mathbb{N}$, for all $\lambda \in \mathbb{C}$. Since $1 = \Phi(I) = \Phi((1,I_{(n-1)})) = \alpha(1)\beta(I_{(n-1)})$, $\alpha \not\equiv 0$ and $\beta \not\equiv 0$. Therefore $\Phi(A) = \Phi(B^{-1}AB) = \lambda^m(\det C)^t$. Claim that $m = t$. Suppose that $m \neq t$. Choose $D \in M(n,\mathbb{C})$ such that no eigenvalues of D is zero and λ_1, λ_2 are eigenvalues of D of order 1 such that $\lambda_1^{m-t} \neq \lambda_2^{m-t}$. From chapter 0 p. 6 there is an invertible matrix $E \in M(n,\mathbb{C})$ such that

$$E^{-1}DE = \begin{bmatrix} \lambda_1 & 0 & 0 \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ 0 & 0 & & \\ \vdots & \vdots & F_{(n-2)\times(n-2)} & \\ 0 & 0 & & \end{bmatrix}.$$

Therefore $\Phi(D) = \Phi(E^{-1}DE) = \lambda_1^m \left(\det \begin{bmatrix} \lambda_2 & 0 \cdots & 0 \\ 0 & & \\ \vdots & F_{(n-2)\times(n-2)} & \\ 0 & & \end{bmatrix}\right)^t$

$$= \lambda_1^m \lambda_2^t \left(\det \left[F_{(n-2)\times(n-2)}\right]\right)^t .$$

Similarly, there exists an invertible matrix $G \in M(n,\mathbb{C})$ such that

$$G^{-1}DG = \begin{bmatrix} \lambda_2 & 0 & 0 \cdots & 0 \\ 0 & \lambda_1 & 0 \cdots & 0 \\ 0 & 0 & & \\ \vdots & \vdots & F_{(n-2)\times(n-2)} & \\ 0 & 0 & & \end{bmatrix} .$$

Therefore, $\Phi(D) = \Phi(G^{-1}DG) = \lambda_2^m \left(\det \begin{bmatrix} \lambda_1 & 0 \cdots & 0 \\ 0 & & \\ \vdots & F_{(n-2)\times(n-2)} & \\ 0 & & \end{bmatrix}\right)^t$

$$= \lambda_2^m \lambda_1^t \left(\det \left[F_{(n-2)\times(n-2)}\right]\right)^t .$$

So $\lambda_1^m \lambda_2^t \left(\det\left[F_{(n-2)\times(n-2)}\right]\right)^t = \lambda_2^m \lambda_1^t \left(\det\left[F_{(n-2)\times(n-2)}\right]\right)^t$, and hence $\lambda_1^{m-t} = \lambda_2^{m-t}$. This contradicts the fact that $\lambda_1^{m-t} \neq \lambda_2^{m-t}$. Therefore $m = t$. This shows that $\Phi(A) = \lambda^m\left(\det\left[C_{(n-1)\times(n-1)}\right]\right)^m = (\det(B^{-1}AB))^m = ((\det B^{-1})(\det A)(\det B))^m = ((\det B^{-1})(\det B)(\det A))^m = ((\det B^{-1}B)(\det A))^m = (\det (I)\cdot \det A)^m = (\det A)^m$, i.e. $\Phi(A) = (\det A)^m$.

<u>Case 2</u>. Suppose that all eigenvalues of A are of order > 1. In this case we claim that there is a sequence $(A_n)_{n\in\mathbb{N}}$ of matrices in $M(n;\mathbb{C})$ such that A_n has at least one eigenvalue of order 1 for each n, and $A = \lim\limits_{n\to\infty} A_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct eigenvalues of A of orders $\alpha_1, \alpha_2, \ldots, \alpha_m > 1$, respectively. From chapter 0 p.6, there is an invertible matrix $B_0 \in M(n,\mathbb{C})$ such that

$$B_0^{-1}AB_0 = \left[\begin{array}{cccc:c} \lambda_1 & 1 & \cdots & 0 & \\ \vdots & & \ddots & \vdots & \bar{0} \\ & & & 1 & \\ 0 & 0 & \cdots & \lambda_1 & \\ \hdashline & \bar{0} & & & P_{(n-k)\times(n-k)} \end{array}\right] = \Upsilon ,$$

where $\begin{bmatrix} \lambda_1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 0 & 0 & \cdots & \lambda_1 \end{bmatrix}$ is a $(k \times k)$ matrix and $k \leq \alpha_1$.

Let $$d = \min\{|\lambda_1-\lambda_2|, \ldots, |\lambda_1-\lambda_m|\}.$$

Therefore $d > 0$. Let $d' \in \mathbb{R}$ such that $0 < d' < d$, and let

$$\Upsilon_n = \left[\begin{array}{cccc:c} \lambda_1 + d'/n & 1 & \cdots & 0 & \\ 0 & \lambda_1 & \ddots & \vdots & \\ \vdots & \vdots & \ddots & 1 & \bar{0} \\ 0 & 0 & \cdots & \lambda_1 & \\ \hdashline & \bar{0} & & & P_{(n-k)\times(n-k)} \end{array}\right].$$

Then the distinct eigenvalues of γ_n are $\lambda_1 + d'/n, \lambda_2, \ldots, \lambda_m$ and $\lambda_1 + d'/n$ has order 1. As $n \to \infty$, $\gamma_n \to \gamma$. Since $\gamma = B^{-1}AB$, $A = B\gamma B^{-1}$. Let $A_n = B\gamma_n B^{-1}$ for all $n \in \mathbb{N}$. Therefore $\lambda_1 + d'/n$ is an eigenvalue of A_n of order 1 and $A_n \to B\gamma B^{-1} = A$ as $n \to \infty$, i.e., $A = \lim_{n\to\infty} A_n$. Thus we have the claim. Hence $\Phi(A) = \Phi(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \Phi(A_n) = \lim_{n\to\infty} (\det A_n)^m$ $= (\lim_{n\to\infty} (\det A_n))^m = (\det(\lim_{n\to\infty} A_n))^m = (\det A)^m$. #