Chapter 3
Shock and Compression Models and Transport Equations

In this work, the motion of cosmic ray particles along magnetic field lines in many situations of interest needs to be explained. Normally, Newton's laws (or their relativistic analogs) could be used to explain this if the number of particles is not too great. Nevertheless, in cases involving a large number of non-interacting particles such as in this work, the time evolution of the system can be described by using a transport equation giving information via a distribution function. In this chapter, descriptions of our models of shocks and compression regions are first presented. Afterwards, the general form of the pitch-angle transport equation and its specific forms for our situations are unveiled. Finally, the diffusion approximation is applied to the pitch-angle transport equation to yield the diffusion-convection equation, which is a simplified but approximate transport equation.

3.1 Shock and Compression Models

Before specializing to particle transport in our situations of interest, we have to model those situations first. In particular, the magnetic field lines influence cosmic ray particle motion and must be modelled.

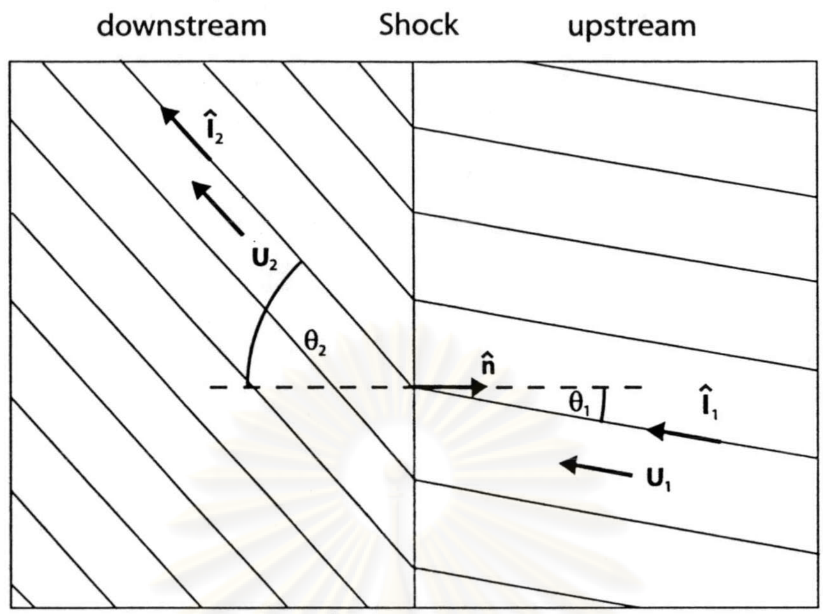

Figure 3.1: Our model of an astrophysical shock; $\hat{\mathbf{l}}$ is a unit vector in the direction of the magnetic field line, $\mathbf{U}$ is the plasma velocity, $\hat{\mathbf{n}}$ is the shock normal vector, and θ is the field angle. The subscripts "1" and "2" are labels for upstream and downstream, respectively.

3.1.1 Shock Modelling

In the cases of shocks, we model them as planar shocks. Plasma properties are constant on both the upstream and downstream sides, and they only have a sudden change at the shock. Figure 3.1 shows our model of a shock in a special reference frame used in this work, called the "de Hoffmann-Teller frame" (de Hoffman and Teller 1950), where $\mathbf{U}$ is the plasma velocity, $\mathbf{B}$ is the mean magnetic field, $\hat{\mathbf{n}}$ is the shock normal vector, θ is the "field angle," the angle between magnetic field lines and the shock normal vector. In Figure 3.1, the shock can be characterized by θ_1 and θ_2. In the de Hoffmann-Teller frame, the shock front is stationary, the electric field disappears on both sides of the shock, and the plasma flows along the magnetic field lines (Kirk et al. 1994).

3.1.2 Compression Region Modelling

Apart from shocks, there are continuous "compression regions," that can accelerate cosmic ray particles also (Klappong et al. 2001, K. Klappong M.Sc. thesis). In such a region, plasma properties continuously change; of course, magnetic field lines change gradually as well.

In this case, we need a gradually changing magnetic field line configuration, but the field line configuration should be compared with the shock magnetic field line configuration also. In particular, a field line should be a straight line when $z \rightarrow \pm\infty$, where z is the distance relative to a reference compression plane (at this plane z=0; analogous to the shock plane). Furthermore, the curvature of the field lines should be adjustable, and when we set the radius of curvature to be zero, the field line configuration should become the shock magnetic field line configuration. To satisfy these requirements, the hyperbolic magnetic field line configuration is chosen (Ruffolo and Chuychai 1999; P. Chuychai senior project 1999). In this compression region model, the compression width b is defined as half of the conjugate axis length of the hyperbolic function (see Figure 3.2). In particular, the compression width is expressed as the ratio of b to a characteristic spatial scale of the acceleration mechanism, the parallel mean free path, $\lambda_{\|}$.

At this point a mathematical description of the hyperbolic magnetic field configuration should be introduced. From knowledge of conic sections, a hyperbolic function (see Figure 3.3) that has a transverse axis, $2a$, along the y' axis and a conjugate axis, $2b$, along z' axis, and is centered at (h, k), can be described by this equation:

$$\frac{(y'-k)^2}{a^2} - \frac{(z'-h)^2}{b^2} = 1. \tag{3.1}$$

We will use only the upper branch of the hyperbola. We choose to place the

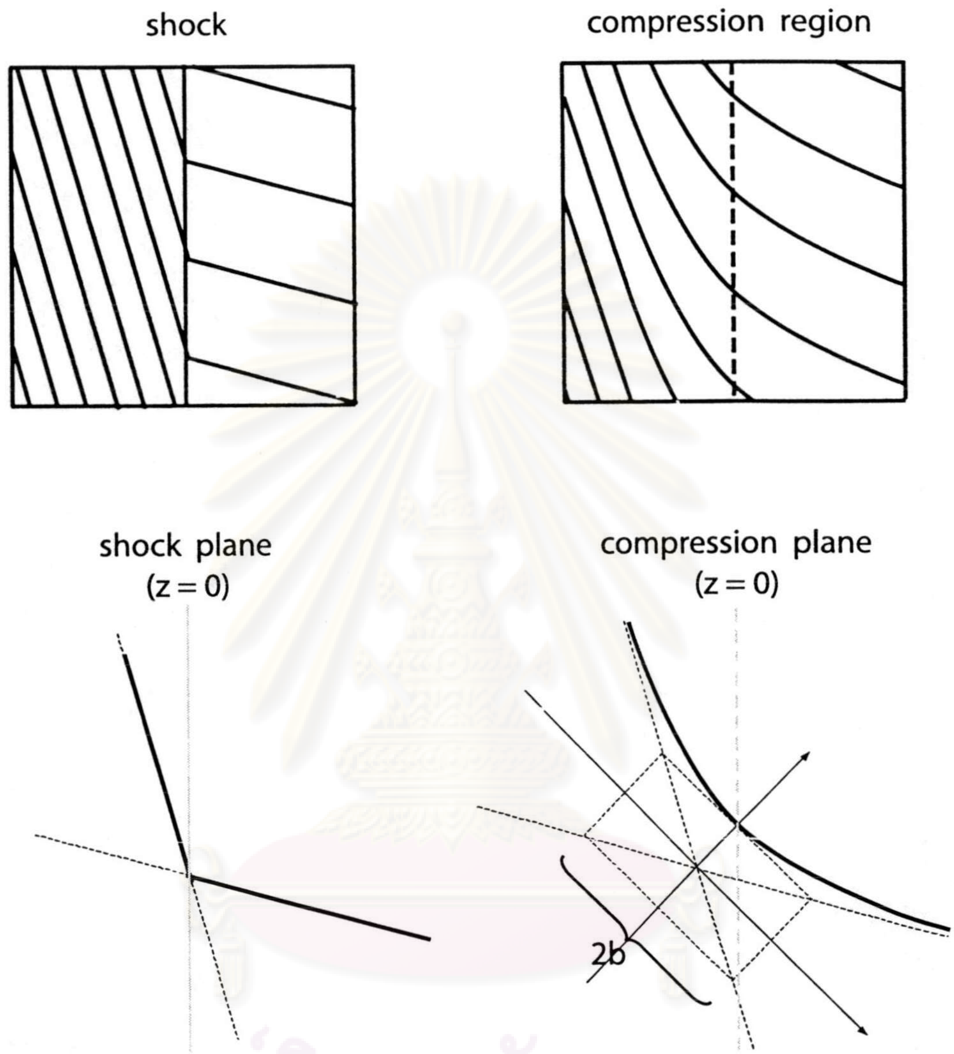

Figure 3.2: Magnetic field configuration model comparison between the "kink" configuration of a shock and hyperbolic configuration of a compression region. The half-conjugate axis, b, is used to parameterize the width of the compression.

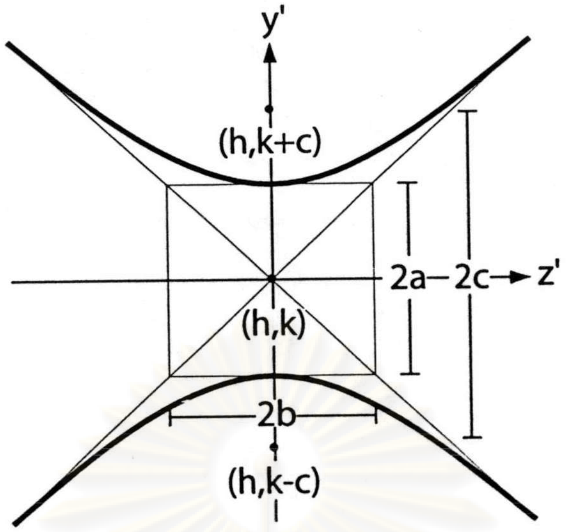

Figure 3.3: A hyperbolic function in the primed coordinates with center at (h, k); $2a$ is the length of the major axis, $2b$ is the length of the conjugate axis, $2c$ is the length between the two foci.

extremum point of the upper branch at the origin, i.e., $h = 0$ and $k = -a$, so the equation should be

$$\frac{(y' + a)^2}{a^2} - \frac{(z')^2}{b^2} = 1. \tag{3.2}$$

Over and above that, we need a rotated hyperbolic function. To describe this, the hyperbolic function in the primed coordinates needs to be transformed to the unprimed coordinates of interest (see Figure 3.4) by these relations

$$y' = y\cos\theta + z\sin\theta \tag{3.3}$$

and

$$z' = -y\sin\theta + z\cos\theta. \tag{3.4}$$

When equation (3.3) and (3.4) are substituted into equation (3.2), it yields

$$\frac{(y\cos\theta + z\sin\theta + a)^2}{a^2} - \frac{(-y\sin\theta + z\cos\theta)^2}{b^2} = 1. \tag{3.5}$$

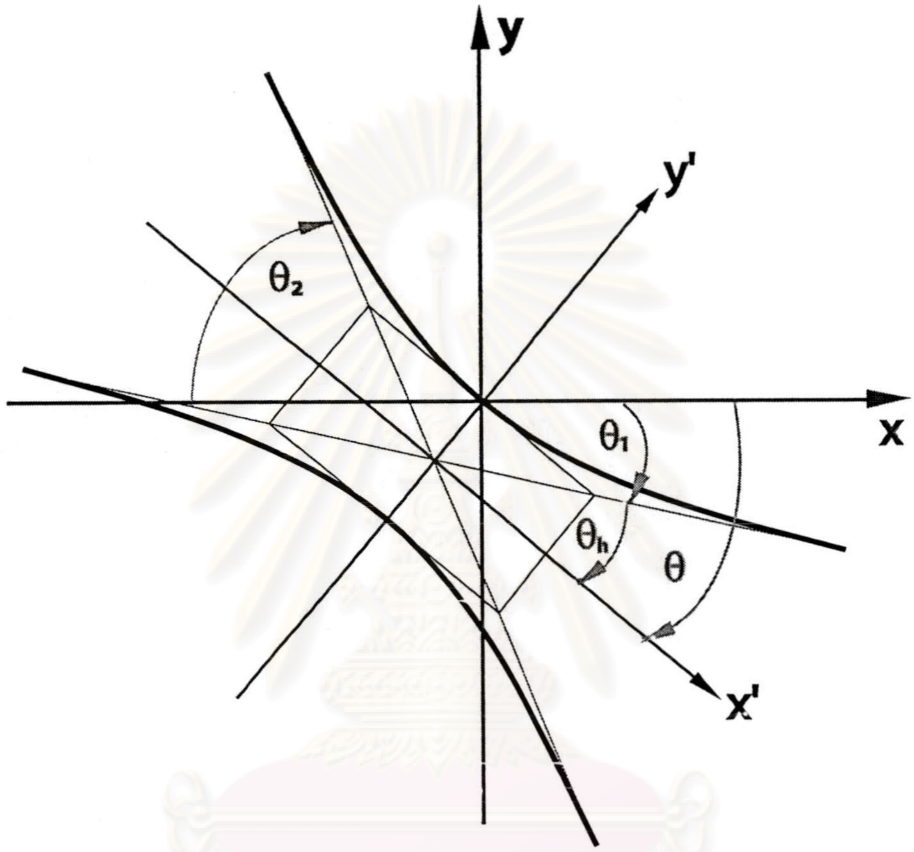

Figure 3.4: A shifted hyperbolic function in primed coordinates, following equation (3.2), in the view point of unprimed coordinates. $\theta = (\theta_2 + \theta_1)/2$ is the angle between the primed axes and unprimed axes (rotation angle) and $\theta_h = (\theta_2 - \theta_1)/2$ is the angle between the asymptotic lines and the z' axis, where θ_1 and θ_2 are angles between the upper asymptotic lines and the z axis upstream and downstream, respectively.

The above equation is a quadratic equation, and it has two solutions, the upper branch and the lower branch. In this work, only the upper branch is used. Moreover, the ratio of a to b is set to be constant, equal to $\tan\theta_h$, to retain the same asymptotic lines, implying that the width b can change independently without affecting the field angle. The solution function $y(z)$ is

$$y(z) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{3.6}$$

where A, B, and C are

$$\begin{aligned} A &= 1 - \sec^2\theta_h \sin^2\theta, \\ B &= z\sec^2\theta_h \sin 2\theta + 2b\tan\theta_h \cos\theta, \\ C &= z^2\sec^2\theta_h \sin^2\theta + 2zb\tan\theta_h \sin\theta - z^2\tan^2\theta_h, \end{aligned} \tag{3.7}$$

where $\theta = (\theta_2+\theta_1)/2$, $\theta_h = (\theta_2-\theta_1)/2$, and θ_1 and θ_2 mean the angles between the upper asymptotic lines and the z axis upstream and downstream, respectively.

In this work, we characterize the compression region magnetic field configuration by θ_1 and θ_2 (see Figure 3.4). They can be directly compared with the shock-field angles in the case of a shock. Furthermore, for the compression regions there is a variable b used to specify the compression width as well.

For a compression region, we work in the "normal incidence frame." In this frame, the compression plane is stationary and plasma flows into the compression plane normal to the direction of the plane (see Figure 3.5).

As a note, although plasma properties change gradually in the cases of compression regions, they change in only one direction, the z-direction, as in the shock model. An illustration of the constant magnetic field intensity and constant plasma speed in the y-direction is shown in Figure 3.5.

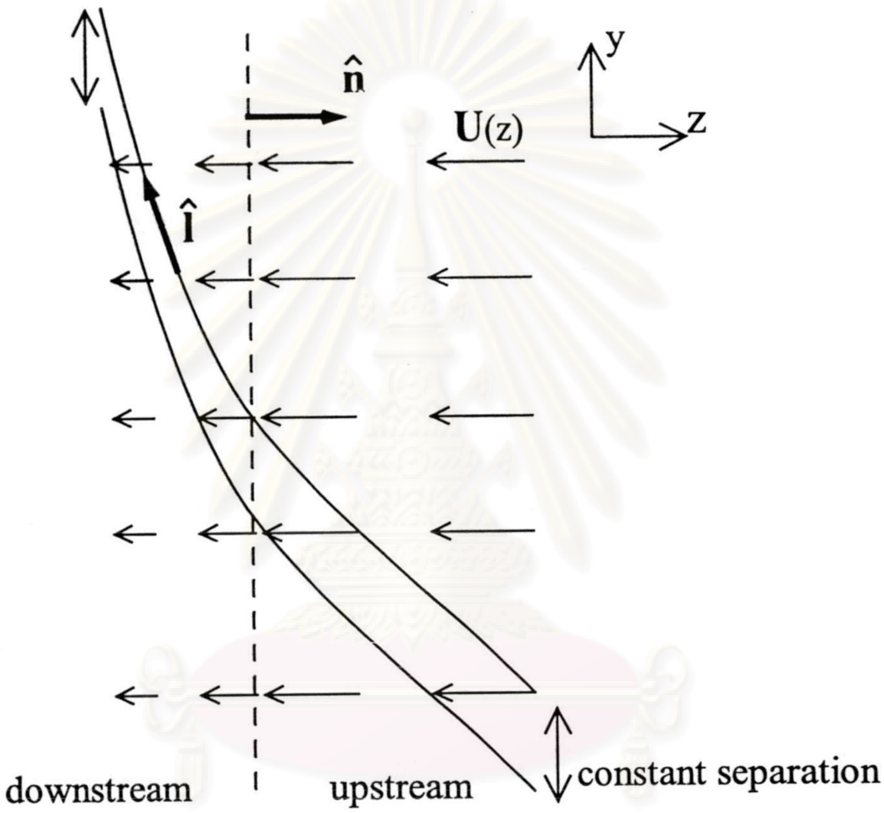

Figure 3.5: The structure of our compression region model in the normal incidence frame, where plasma flows to encounter the compression plane in the direction normal to the plane. Plasma speed and magnetic intensity are changed in only z direction. $\hat{\mathbf{l}}$ is the unit vector in the direction of magnetic field, $\mathbf{U}$ is the plasma velocity, $\hat{\mathbf{n}}$ is compression plane normal vector.

3.2 Pitch-Angle Transport Equation

At this stage, we would like to introduce the "pitch-angle transport equation," used in order to explain the dynamics of cosmic ray particles.

3.2.1 Pitch-Angle Distribution Function

For a system with a large number of particles, the concept of a distribution function can be used if the particles are assumed to be identical. Generally, the phase space distribution function, $f(\mathbf{x}, \mathbf{p}, t)$, tells us the number of particles in a small 6-dimensional volume of phase space at position $\mathbf{x}$ and momentum $\mathbf{p}$ at a time of interest, t:

$$f(\mathbf{x}, \mathbf{p}, t) = \frac{d^6N}{d^3\mathbf{x}d^3\mathbf{p}}, \tag{3.8}$$

where d^6N is the number of particles in the small 6-dimensional volume $d^3\mathbf{x}d^3\mathbf{p}$.

Notwithstanding, the distribution function in the pitch-angle transport equation used in this work is a bit different. Since all plasma properties are assumed to depend only on z, this single variable is enough to describe the particle position. In the pitch-angle transport equation, the momentum is treated in spherical coordinates: p describes the magnitude of the momentum, ϕ is the gyrophase of the momentum, and θ is the pitch angle. We can usually assume gyrotropy, which means that particles are uniformly distributed in gyrophase. Therefore, a description of the particle motion in ϕ is not needed. In addition, for mathematical convenience, the cosine of the pitch angle, μ, is used instead of θ. Therefore, the variables used for the momentum description of the distribution function are p and μ. Now the pitch-angle transport distribution function can be written as

$$F(z, p, \mu, t) = \frac{d^3N}{dzdpd\mu}, \tag{3.9}$$

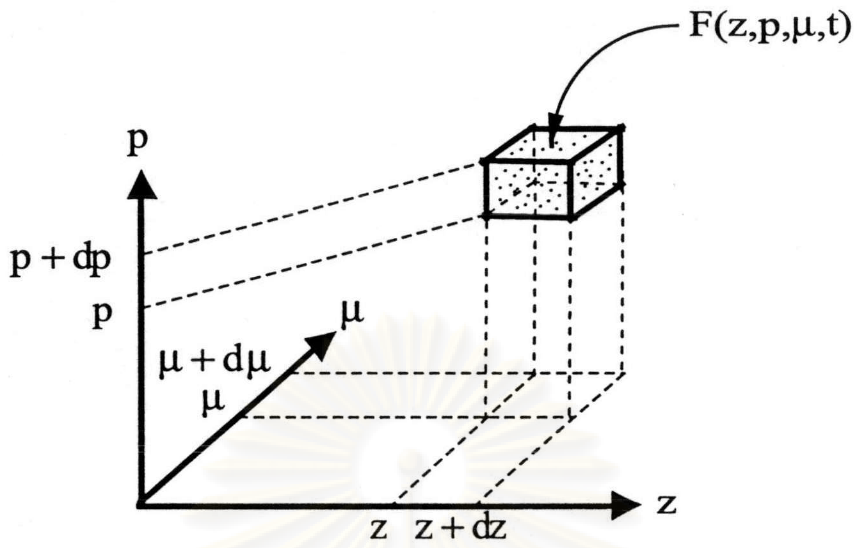

Figure 3.6: Schematic showing a distribution volume in our pitch-angle transport space

where d^3N is the number of particles in the small 3-dimensional volume $dzdpd\mu$, illustrated in Figure 3.6.

The relation between the pitch-angle distribution function and the phase space distribution function can be found by integrating $f(\mathbf{x}, \mathbf{p}, t)$ over dx, dy, and $d\phi$, where $f(\mathbf{x}, \mathbf{p}, t)$ is constant in x, y, and ϕ; after the integration, we get

$$F(z, p, \mu, t) = 2\pi A p^2 f(\mathbf{x}, \mathbf{p}, t), \tag{3.10}$$

where A is the flux-tube cross section perpendicular to the z-direction. Similarly, $F(z, p, \mu, t)$ also has a relation with the differential flux, $j(x, y, z, p, \mu, \phi, t) = d^6N/(dx\, dy\, dz\, dp\, d\mu\, d\phi)$, introduced in §2.1.2, by

$$F(z, p, \mu, t) = 2\pi A j(x, y, z, p, \mu, \phi, t). \tag{3.11}$$

In addition, in our work, p, μ, z and t are not treated in the same frame of reference. The z- and t-coordinates are treated in a fixed frame, a stationary frame such as the de Hoffmann-Teller frame or the normal incidence frame, while the momentum components, p and μ, are treated in the plasma frame. In other words, our distribution function, $F(z, p, \mu, t)$, is treated in a "mixed" frame.

3.2.2 General Form of the Pitch-Angle Transport Equation

To explain the dynamics of a large number of cosmic ray particles, transport equations can be used. In this subsection, we would like to introduce the pitch-angle transport equation used in this work.

For simplicity, we will start with particle transport in one dimension. Here, we define a one-dimensional distribution function by

$$f(x,t) = \frac{dN}{dx}, \tag{3.12}$$

where dN is the number of particles in the small one-dimensional cell dx. The number of particles in a cell can change due to inflow/outflow at the left or right boundary. Then the change in the number of particles in the cell, $f(x,t)\Delta x$, can be explained in terms of the flux, $S(x,t)$, flowing through the cell in x direction:

$$\frac{\partial}{\partial t}(f \cdot \Delta x) = S\left(x - \frac{\Delta x}{2}\right) - S\left(x + \frac{\Delta x}{2}\right), \tag{3.13}$$

or as $\Delta x \to 0$,

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial S(x,t)}{\partial x}. \tag{3.14}$$

Concentrating on fluxes, there are two basic kinds of processes leading to fluxes: systematic and random processes. A systematic or convection process is a process in which all particles in a cell move together with the same speed v (see Figure 3.7a). The flux due to the systematic process can be called the

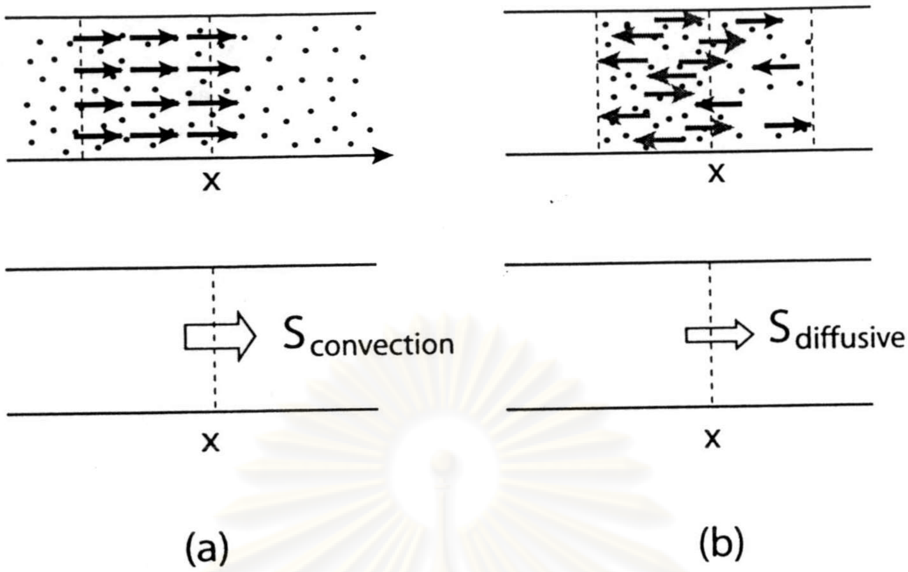

Figure 3.7: (a) Systematic process leading to convective flux. (b) Random process leading to diffusive flux.

"convective flux," described by

$$S(x,t)_{\text{convective}} = \left(\frac{\langle \Delta x \rangle}{\Delta t}\right) f(x,t), \tag{3.15}$$

where $\langle \Delta x \rangle / \Delta t$ refers to the rate of change in the x-coordinate of particles. In this case, $\langle \Delta x \rangle / \Delta t$ is simply v, so $S(x,t)_{\text{convective}} = vf(x,t)$.

On the other hand, a random process leads to a flux due to the random movement of the particles. The flux due to this random process depends on the gradient in f at that location. For example, as shown in Figure 3.7b, the particle density on the left-hand side is more than on the right-hand side. Due to random motion of particles, particles diffuse from the left-hand side to the right-hand side more than in the opposite way. For the net result, there is the a "diffusive flux" which is proportional to particle gradient (Fick's law),

$$S(x,t)_{\text{diffusive}} \propto -\left[\frac{\partial f(x,t)}{\partial x}\right] \tag{3.16}$$

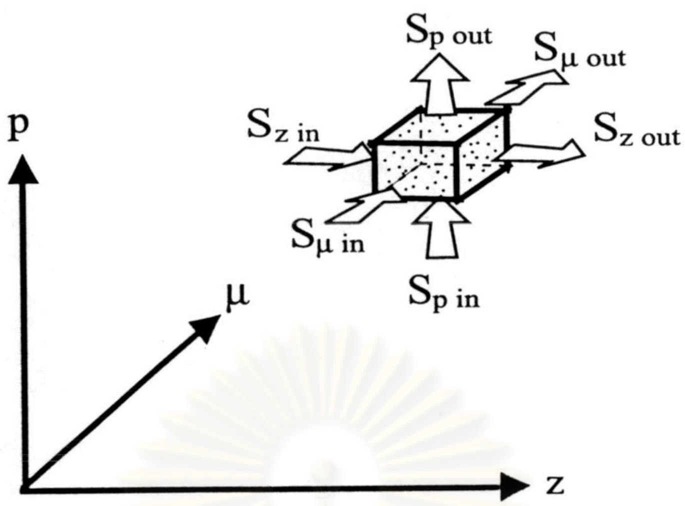

Figure 3.8: Fluxes move through a given volume in the z, p, and μ directions; if the incoming S does not balance the outgoing S, the particle number in the volume changes.

or

$$S(x,t)_{\text{diffusive}} = -D\left[\frac{\partial f(x,t)}{\partial x}\right], \tag{3.17}$$

where D is the diffusion coefficient and the minus sign means the diffusive flux has a flow direction opposite to the direction of the gradient.

In the case of our distribution function, $F(z,p,\mu,t)$, fluxes can flow through a given volume $(dz\,dp\,d\mu)$ in three dimensions, z, p, and μ, so the convective change in the density of particles in the volume should be described by a combination of changes in particle number due to fluxes in three dimensions (see Figure 3.8):

$$\begin{aligned}\left[\frac{\partial F(z,p,\mu,t)}{\partial t}\right]_{\text{convective}} &= \left(\frac{\partial F}{\partial t}\right)_z + \left(\frac{\partial F}{\partial t}\right)_p + \left(\frac{\partial F}{\partial t}\right)_\mu \\ &= -\frac{\partial S_z}{\partial z} - \frac{\partial S_p}{\partial p} - \frac{\partial S_\mu}{\partial \mu} \\ &= -\frac{\partial}{\partial z}\left(\frac{\langle \Delta z\rangle}{\Delta t}F\right) - \frac{\partial}{\partial p}\left(\frac{\langle \Delta p\rangle}{\Delta t}F\right) - \frac{\partial}{\partial \mu}\left(\frac{\langle \Delta \mu\rangle}{\Delta t}F\right)\end{aligned} \tag{3.18}$$

In the pitch-angle transport equation, for particle motion along a magnetic field, we consider convective fluxes in all dimensions, but only the μ-dimension has a diffusive flux (for reasons to be explained shortly). Then equation (3.18) can be re-written as

$$\begin{aligned}\frac{\partial F(z,p,\mu,t)}{\partial t} &= -\frac{\partial}{\partial z}\left(\frac{\langle\Delta z\rangle}{\Delta t}F\right)-\frac{\partial}{\partial p}\left(\frac{\langle\Delta p\rangle}{\Delta t}F\right)-\frac{\partial}{\partial \mu}\left(\frac{\langle\Delta \mu\rangle}{\Delta t}F\right)\\ &+\frac{\partial}{\partial\mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}\left(1-\frac{\mu v\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\right)F\right], \end{aligned} \tag{3.19}$$

where $\varphi(\mu)$ is the pitch-angle scattering coefficient,

$$\varphi(\mu) = a\,|\mu|^2\left(1-\mu^2\right), \tag{3.20}$$

where a is the scattering amplitude, q is the form of the scattering coefficient, v is the particle speed, $\mathbf{U}$ is the solar wind velocity, $\hat{\mathbf{l}}$ is the unit vector along the magnetic field line, and c is the speed of light (Skilling 1975, Ruffolo and Chuychai 1999). Note that the factor $1-(\mu v\mathbf{U}\cdot\hat{\mathbf{l}})/c^2$ is required to transform our mixed frame distribution function to be a pure plasma frame distribution function used in the "pitch-angle scattering term," i.e., the last term of equation (3.19) (Ruffolo 1995).

More specifically, the $\langle\Delta\mathrm{x}\rangle/\Delta t$, $\langle\Delta p\rangle/\Delta t$, and $\langle\Delta\mu\rangle/\Delta t$ terms can be expressed following the work of Ruffolo and Chuychai (1999) and P. Chuychai senior project (1999):

$$\frac{\langle\Delta\mathbf{x}\rangle}{\Delta t} = \mathbf{U}+\mu v\,\hat{\mathbf{l}}-\frac{\mu^2 v^2\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\hat{\mathbf{l}}, \tag{3.21}$$

$$\frac{\langle\Delta p\rangle}{\Delta t} = p\left[\frac{1-3\mu^2}{2}l_i\,l_j\frac{\partial U_j}{\partial x_i}-\frac{1-\mu^2}{2}\nabla\cdot\mathbf{U}-\frac{\mu}{v}\hat{\mathbf{l}}\cdot\frac{\partial\mathbf{U}}{\partial t}\right], \tag{3.22}$$

$$\frac{\langle\Delta\mu\rangle}{\Delta t} = \frac{1-\mu^2}{2}\left[v\,\nabla\cdot\hat{\mathbf{l}} + \mu\,\nabla\cdot\mathbf{U} - 3\,\mu\,l_i\,l_j\,\frac{\partial U_j}{\partial x_i} + \frac{v\,\mathbf{U}}{c^2}\cdot\frac{\partial\hat{\mathbf{l}}}{\partial t}\right.$$

$$\left. - \frac{2}{v}\hat{\mathbf{l}}\cdot\frac{\partial\mathbf{U}}{\partial t} - \frac{\mu\,v^2\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\nabla\cdot\hat{\mathbf{l}}\right]. \tag{3.23}$$

Note that the Einstein summation convention is used in the equations (3.22) and (3.23). Our shock and compression models are not time-dependent, so all time-dependent terms in equations (3.22) and (3.23) are zero. Moreover, in our work, $\langle\Delta\mathbf{x}\rangle/\Delta t$ reduces to $\langle\Delta z\rangle/\Delta t$ because we are interested in particle motion along the magnetic field line, and the position along the magnetic field can be described by z alone. Mathematically, F, $\hat{\mathbf{l}}$, and also $\mathbf{U}$ are functions of only one spatial coordinate, z. When the terms $\langle\Delta\mathbf{x}\rangle/\Delta t$, $\langle\Delta p\rangle/\Delta t$, and $\langle\Delta\mu\rangle/\Delta t$ in equations (3.21), (3.22), and (3.23) are substituted into equation (3.19), the pitch-angle transport equation for energetic particle motion along a general, static magnetic field configuration is obtained:

$$\begin{aligned}\frac{\partial F(t,z,\mu,p)}{\partial t} = & -\frac{\partial}{\partial z}\left[U_z + \mu\,v\,l_z - \frac{\mu^2 v^2\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\,l_z\right]F \\ & - \frac{\partial}{\partial p}p\left[\frac{1-3\mu^2}{2}\,l_i\,l_j\,\frac{\partial U_j}{\partial x_i} - \frac{1-\mu^2}{2}\nabla\cdot\mathbf{U}\right]F \\ & - \frac{\partial}{\partial\mu}\frac{1-\mu^2}{2}\left[v\nabla\cdot\hat{\mathbf{l}} + \mu\nabla\cdot\mathbf{U} - 3\mu\,l_i\,l_j\frac{\partial U_j}{\partial x_i} - \frac{\mu\,v^2\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\nabla\cdot\hat{\mathbf{l}}\right]F \\ & + \frac{\partial}{\partial\mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}\left(1 - \frac{\mu\,v\,\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\right)F\right].\end{aligned} \tag{3.24}$$

The first term on the right hand side of equation (3.24) is the "streaming and convection term." It describes the change of F due to the systematic particle motion in the z-direction with respect to our fixed frame. In particular, streaming explains the z-motion of particles with respect to the plasma, and convection explains the z-motion of plasma with respect to our fixed frame. Next, the second term can be called the "acceleration term." It explains the change of F due to systematic changes in particle momentum in the plasma frame, caused by the parallel and perpendicular divergence of the fluid flow. The third term is the "focusing and differential convection term." Adiabatic focusing, also known as magnetic mirroring, explains the changes in F caused by systematic changes in μ due to gradual changes in the mean field (see §2.2.1). Differential convection explains the change in F caused by systematic changes in μ due to the change in plasma speed, that is to say, the change in the our reference frame causes changes in μ values of the particles of interest. The last term describes the change of F due to random changes in μ of particles of interest caused by small scale irregularities of the magnetic field, known as pitch-angle scattering (see §2.2.1). This term is called the "scattering term."

3.2.3 Pitch-Angle Transport Equation for Cases of Interest

For special cases of interest, we need to specify the functional form of various quantities used in the pitch-angle transport equation for a general static magnetic field configuration (equation 3.24). In particular, we need expressions for U_z, l_z, $\mathbf{U} \cdot \hat{\mathbf{l}}$, $l_i l_j \partial U_j / \partial x_i$, $\nabla \cdot \mathbf{U}$, and $\nabla \cdot \hat{\mathbf{l}}$. In this work, these expressions are specified for two cases of interest, the cases of shocks and compression regions.

Shock Case

In the case of shocks (see Figure 3.1), our fixed frame is the de Hoffmann-Teller frame, in which plasma flows along magnetic field lines. The z-direction is normal to the shock plane. Moreover, plasma properties and the magnetic field are constant on a given side of the shock. Therefore, for each side of the shocks the specific terms should be

$$U_z = U\cos\theta, \tag{3.25}$$

$$l_z = \cos\theta, \tag{3.26}$$

$$\mathbf{U}\cdot\hat{\mathbf{l}} = U, \tag{3.27}$$

$$l_i l_j \partial U_j/\partial x_i = 0, \tag{3.28}$$

$$\nabla\cdot\mathbf{U} = 0, \tag{3.29}$$

$$\nabla\cdot\hat{\mathbf{l}} = 0, \tag{3.30}$$

where θ is θ_1 or θ_2 for the upstream or downstream side, respectively. After substitution of equations (3.25)-(3.30) into equation (3.24), we obtain the pitch-angle transport equation for the case of shocks:

$$\frac{\partial F(t,z,\mu,p)}{\partial t} = -\frac{\partial}{\partial z}\left[U\cos\theta + \mu v\, l\cos\theta - \frac{\mu^2 v^2 U}{c^2}\, l\cos\theta\right] F + \frac{\partial}{\partial\mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}\left(1-\frac{\mu v U}{c^2}\right)F\right]. \tag{3.31}$$

Note that although the equation (3.31) is the pitch-angle transport equation for a shock, it can be used only on one side of a shock with a set of appropriate parameters for either upstream or downstream. For the transport of particles across the shock, we use a special treatment that considers the actual orbits of particles (Sanguansak and Ruffolo 1999, Leerungnavarat et al. 2000).

Compression Region Case

Turn to the case of compression regions (see Figure 3.5), plasma properties are not constant as in the case of shocks. Additionally, compression regions are modelled in the normal incidence frame, with plasma flow in the direction normal to the compression plane. In this case, the transport equation is more complicated than in the case of shocks.

In order to find expression for specific terms involving $\hat{\mathbf{l}}$ and $\mathbf{U}$ in equation (3.24) for the case of compression regions, we would like to start by considering an infinitesimal distance along the magnetic field line direction, $d\mathbf{l}$, which satisfies

$$d\mathbf{l} \times \mathbf{B} = 0. \tag{3.32}$$

In Cartesian coordinates, $\mathbf{l}$ and $\mathbf{B}$ can be written

$$\mathbf{l} = l_x\hat{\mathbf{x}} + l_y\hat{\mathbf{y}} + l_z\hat{\mathbf{z}}, \tag{3.33}$$

$$\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}. \tag{3.34}$$

From equations (3.32)-(3.34), we get the relation,

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}. \tag{3.35}$$

Because in our models the magnetic field line only lies in the y-z plane, equation (3.35) can be re-written as

$$\frac{dy}{dz} = \frac{B_y}{B_z} \tag{3.36}$$

or

$$B_y = \frac{dy}{dz}B_z. \tag{3.37}$$

With substitution of the equations (3.36) and (3.37) and $B_x = 0$ into equation (3.34), we obtain

$$\mathbf{B} = B_z\left(\frac{dy}{dz}\hat{\mathbf{y}} + \hat{\mathbf{z}}\right), \tag{3.38}$$

where dy/dz can be found from the magnetic configuration of our models (§3.1.2).

Now we can find the unit vector along the magnetic field line, $\hat{\mathbf{l}}$. Start with the definition of $\hat{\mathbf{l}}$:

$$\hat{\mathbf{l}} = \frac{\mathbf{B}}{|\mathbf{B}|}, \tag{3.39}$$

where $|\mathbf{B}| = \sqrt{B_y^2 + B_z^2}$. Then, from the equations (3.38) and (3.39), the value of $\hat{\mathbf{l}}$ is unveiled:

$$\hat{\mathbf{l}} \;=\; \frac{dy/dz}{\sqrt{1+(dy/dz)^2}}\,\hat{\mathbf{y}} + \frac{1}{\sqrt{1+(dy/dz)^2}}\,\hat{\mathbf{z}}\;. \tag{3.40}$$

In the normal incidence frame, plasma flows in the z-direction only, so for the case of compression regions, the plasma velocity can be written

$$\mathbf{U} = U_z\hat{\mathbf{z}}. \tag{3.41}$$

Sometimes we will also refer to U_z as U_n, the normal component. Furthermore, as discussed in §2.2.2, the magnetic field can be dragged by the plasma. Therefore, we can describe the plasma velocity by considering changes in the magnetic field. From the magnetic configuration of our model shown in Figure 3.5, the relation between the plasma speed and the magnetic field line can be found:

$$U_z \propto \frac{dz}{dy} \quad \text{or} \quad U_z = \frac{c}{dy/dz}\;, \tag{3.42}$$

where c is a constant determined by specifying the plasma speed and dy/dz at a boundary. In this work, we specify the plasma speed to be U_{1n} and dy/dz to be $\tan\theta_1$ far upstream, so we get

$$c = -|U_{1n}| \cdot \tan\theta_1, \tag{3.43}$$

where the minus sign means that plasma flows in the $-z$-direction. Using equations (3.42) and (3.43), equation (3.41) yields

$$\mathbf{U} = \frac{-|U_{1n}| \cdot \tan\theta_1}{dy/dz}\,\hat{\mathbf{z}}. \tag{3.44}$$

Finally, since, we can express $\hat{\mathbf{l}}$ and $\mathbf{U}$ in terms of known parameters, the others specific terms in equation (3.24) can also be written in terms of known parameters, as shown here:

$$l_z = \frac{1}{\sqrt{1+(dy/dz)^2}}, \tag{3.45}$$

$$U_z = \frac{-|U_{1n}| \cdot \tan\theta_1}{dy/dz}, \tag{3.46}$$

$$\begin{aligned} \mathbf{U}\cdot\hat{\mathbf{l}} &= U_z l_z \\ &= \frac{-|U_{1n}|\tan\theta_1}{dy/dz} \cdot \frac{1}{\sqrt{1+(dy/dz)^2}}, \end{aligned} \tag{3.47}$$

$$\begin{aligned} \nabla\cdot\hat{\mathbf{l}} &= \frac{\partial l_z}{\partial z} \\ &= -\frac{dy}{dz} \cdot \frac{d^2y}{d\,z^2} \cdot {l_z}^3, \end{aligned} \tag{3.48}$$

$$\begin{aligned} \nabla\cdot\mathbf{U} &= \frac{\partial U_z}{\partial z} \\ &= \frac{|U_{1n}|\tan\theta_1}{(dy/dz)^2} \cdot \frac{d^2y}{d\,z^2}, \end{aligned} \tag{3.49}$$

$$\begin{aligned} l_i\, l_j \frac{\partial U_j}{\partial x_i} &= l_z\, l_z \frac{\partial U_z}{\partial z} \\ &= l_z^2 \left(\frac{|U_{1n}|\,\tan\theta_1}{(dy/dz)^2} \cdot \frac{d^2y}{d\,z^2} \right) . \end{aligned} \tag{3.50}$$

3.3 Diffusion-Convection Equation

Here, another transport equation is introduced. It is the "diffusion-convection equation," obtained by using the "diffusion approximation" on the pitch-angle transport equation. The diffusion approximation uses the concept that the effects of pitch-angle scattering and μ-dependent streaming together with the existence of anisotropy: a non-uniform distribution in the particles' direction of motion (indicated by μ). These can lead to particle diffusion in z-space; then, if the anisotropy is assumed to be nearly constant, the pitch-angle transport equation will be greatly simplified to be the diffusion-convection equation.

Consider the μ-z plane at a constant momentum. There is pitch-angle scattering making particles undergo a random walk in μ. Furthermore, there is the streaming process making particles systematically change their position in the z-direction with a μ-dependent rate of change, $v_z \approx \mu v$ or $v \cos\theta$ (see Figure 3.9). If there is a collection of particles crowded in a small range in μ and z, diffusion in the z-direction will result. These particles spread out to the other μ values due to pitch-angle scattering, while the streaming velocity depends on μ. The combination of the two processes yields diffusion in the z-direction, demonstrated in Figure 3.10.

More specifically, in the mathematical treatment, the distribution function, $F(z, p, \mu, t)$, is first separated into three components: the isotropic component that is constant in μ, $F_0(z, p, t)$; an anisotropic component that is an odd function of μ, $F_1(z, p, \mu, t)$; and an anisotropic component that is an even function of μ with integral zero, $F_2(z, p, \mu, t)$ (Earl 1974):

$$F(z, p, \mu, t) = F_0(z, p, t) + F_1(z, p, \mu, t) + F_2(z, p, \mu, t). \tag{3.51}$$

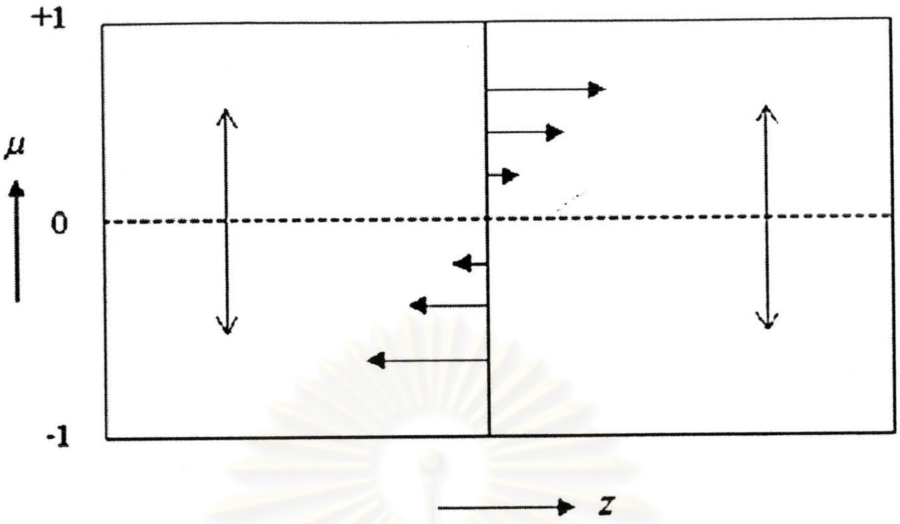

Figure 3.9: The pitch-angle scattering making particles undergo a random walk in μ, represented by up-down arrows, while the streaming process makes particles move along z with velocity $v_z \approx \mu v$, represented by horizontal arrows with μ-dependent magnitudes. (Picture credit: Klappong M.Sc. Thesis 2002)

Then, equation (3.48) is substituted into the pitch-angle transport equation for a general static magnetic field, equation (3.24). Consequently, the equation can be separated into two equations: μ-odd terms and μ-even terms:

ODD :

$$\begin{aligned}
\frac{\partial F_1}{\partial t} &= -\frac{\partial}{\partial z}\left[U_z - \frac{\mu^2 v^2 \mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\, l_z\right] F_1 - \frac{\partial}{\partial z}\left[\mu\, v\, l_z\right](F_0 + F_2) \\
&\quad - \frac{\partial}{\partial p}\, p\left[\frac{1-3\mu^2}{2}\, l_i\, l_j\, \frac{\partial U_j}{\partial x_i} - \frac{1-\mu^2}{2}\nabla\cdot\mathbf{U}\right] F_1 \\
&\quad - \frac{\partial}{\partial \mu}\frac{1-\mu^2}{2}\left[\mu\nabla\cdot\mathbf{U} - 3\mu\, l_i\, l_j \frac{\partial U_j}{\partial x_i} - \frac{\mu\, v^2 \mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\nabla\cdot\hat{\mathbf{l}}\right] F_1 \\
&\quad - \frac{\partial}{\partial \mu}\frac{1-\mu^2}{2}\left[v\nabla\cdot\hat{\mathbf{l}}\right](F_0 + F_2)
\end{aligned}$$

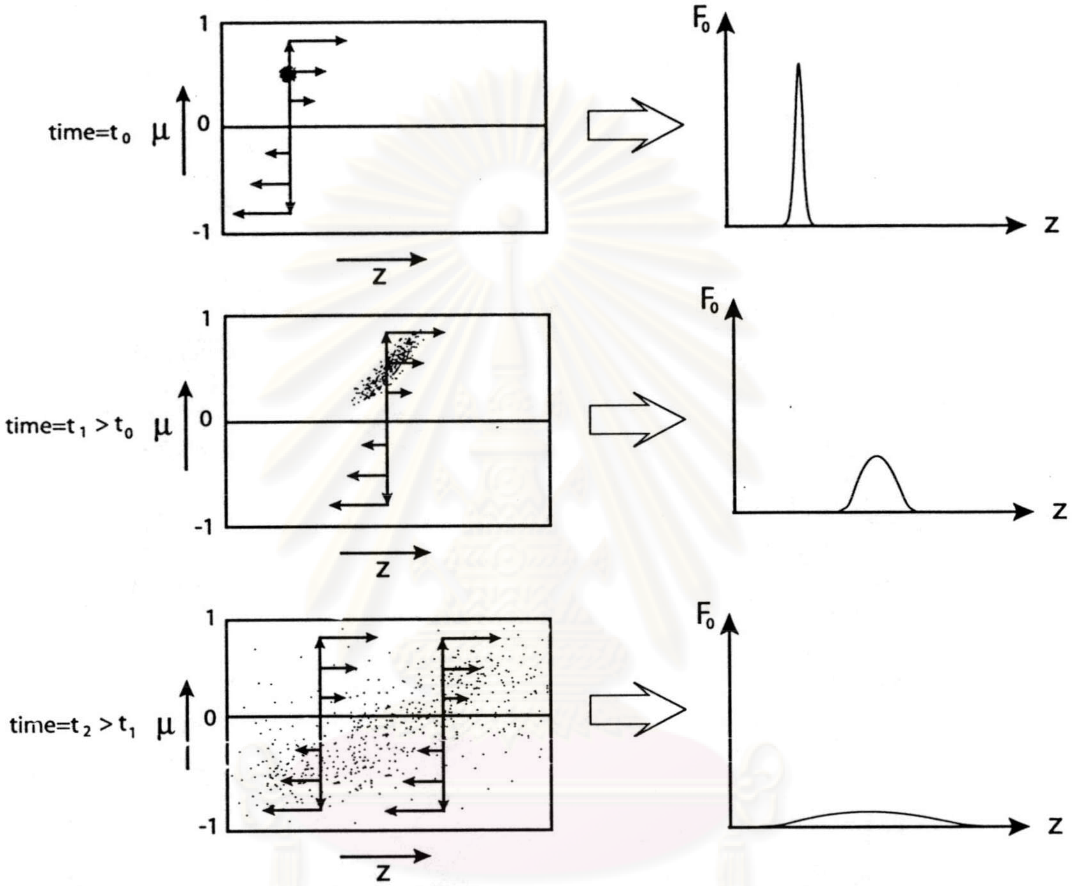

Figure 3.10: Demonstration of diffusion of anisotropic particles in the z-direction due to the effect of the pitch-angle scattering and the μ-dependent streaming. Left panels: density plots of particles in the μ-z plane. Right panels: particle density F vs. z.

$$+\frac{\partial}{\partial\mu}\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}F_1 - \frac{\partial}{\partial\mu}\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}\left(\frac{\mu v\,\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\right)(F_0+F_2), \quad (3.52)$$

EVEN :

$$\frac{\partial F_0}{\partial t}+\frac{\partial F_2}{\partial t} = -\frac{\partial}{\partial z}\left[U_z - \frac{\mu^2 v^2 \mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\,l_z\right](F_0+F_2) - \frac{\partial}{\partial z}\left[\mu v\, l_z\right]F_1$$

$$-\frac{\partial}{\partial p}p\left[\frac{1-3\mu^2}{2}\,l_i\,l_j\frac{\partial U_j}{\partial x_i} - \frac{1-\mu^2}{2}\nabla\cdot\mathbf{U}\right](F_0+F_2)$$

$$-\frac{\partial}{\partial\mu}\frac{1-\mu^2}{2}\left[\mu\nabla\cdot\mathbf{U} - 3\mu\, l_i\, l_j\frac{\partial U_j}{\partial x_i} - \frac{\mu v^2\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\nabla\cdot\hat{\mathbf{l}}\right](F_0+F_2)$$

$$-\frac{\partial}{\partial\mu}\frac{1-\mu^2}{2}\left[v\nabla\cdot\hat{\mathbf{l}}\right]F_1$$

$$+\frac{\partial}{\partial\mu}\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}(F_0+F_2) - \frac{\partial}{\partial\mu}\frac{\varphi(\mu)}{2}\frac{\partial}{\partial\mu}\left(\frac{\mu v\,\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\right)F_1. \quad (3.53)$$

Next, some simplifying approximations are made. First, the even anisotropic component of the distribution function, F_2, is neglected compared with F_0 and F_1 (Earl 1974). Then averaging over μ, the even equation (3.53) becomes

$$\frac{\partial F_0}{\partial t} = -\frac{\partial}{\partial z}\left[U_z - \frac{1}{3}\frac{v^2\mathbf{U}\cdot\hat{\mathbf{l}}}{c^2}\,l_z\right]F_0 - \frac{\partial}{\partial z}\left[v\, l_z\right]\frac{1}{2}\int_{-1}^{1}\mu F_1 d\mu$$

$$+\frac{\partial}{\partial p}p\left[\frac{1}{3}\nabla\cdot\mathbf{U}\right]F_0. \quad (3.54)$$

Evidently, this is much simpler than equation (3.50) (if we are not interested in the pitch-angle distribution). However, there is the odd anisotropic distribution function, F_1, embedded in the streaming term. Since at the end we would like to describe particle transport without μ, the term of $\int_{-1}^{1} \mu F_1 d\mu$ will have to be rewritten. To do this, we integrate the odd equation (3.52) from -1 to ν, along with the assumption that F_1 is small and nearly constant in space and time. Since we also consider U to be small (compared with v), we neglect terms of order UF_1, which are very small terms. Then integration over μ from -1 to ν makes equation (3.52) become

$$\begin{aligned} 0 &= -\left(\frac{1-\nu^2}{2}\right) v \frac{\partial}{\partial z} l_z F_0 - \left(\frac{1-\nu^2}{2}\right) \left[v \nabla \cdot \hat{\mathbf{1}} \right] F_0 \\ &\quad + \frac{\varphi(\nu)}{2} \frac{\partial F_1(\nu)}{\partial \nu} - \frac{v \mathbf{U} \cdot \hat{\mathbf{1}}}{c^2} \frac{\varphi(\nu)}{2} F_0. \end{aligned} \tag{3.55}$$

Now, we divide by $\varphi(\nu)/2$ and move the $\partial F_1(\nu)/\partial \nu$ term to the left-hand side, while leaving the other terms on the right-hand side; after that, we integrate over ν from 0 to μ. Then we get

$$\begin{aligned} F_1(\mu) &= -\left[\int_0^{\mu} \frac{1-\nu^2}{\varphi(\nu)} d\nu \right] v l_z \frac{\partial F_0}{\partial z} - \left[\int_0^{\mu} \frac{1-\nu^2}{\varphi(\nu)} d\nu \right] v \left(\nabla \cdot \hat{\mathbf{1}} - \frac{\partial l_z}{\partial z} \right) F_0 \\ &\quad + \frac{\mu v \mathbf{U} \cdot \hat{\mathbf{1}}}{c^2} F_0. \end{aligned} \tag{3.56}$$

Next, we substitute equation (3.56) into equation (3.54), and integrate and rearrange terms. Under the assumption of a constant flux-tube cross section along the z-direction, we get

$$\frac{\partial F_0}{\partial t} = \frac{\partial}{\partial z} D \frac{\partial F_0}{\partial z} - \frac{\partial}{\partial z} U_z F_0 + \frac{\partial}{\partial p} p \left[\frac{1}{3} \nabla \cdot \mathbf{U} \right] F_0, \tag{3.57}$$

where D represents a coefficient of diffusion in the z-direction, called the "spatial diffusion coefficient,"

$$D = \frac{v^2 l_z^2}{2} \int_{-1}^{1} \mu \left[\int_0^{\mu} \frac{1-\nu^2}{\varphi(\nu)} d\nu \right] d\mu$$

$$= \frac{v^2 l_z^2}{4} \int_{-1}^{1} \frac{(1-\mu^2)^2}{\varphi(\mu)} d\mu \tag{3.58}$$

(using integration by parts).

For a steady state, equation (3.57) becomes

$$\frac{\partial}{\partial z} D \frac{\partial F_0}{\partial z} - \frac{\partial}{\partial z} U_z F_0 + \frac{\partial}{\partial p} p \left[\frac{1}{3} \nabla \cdot \mathbf{U}\right] F_0 = 0, \tag{3.59}$$

called the "diffusion-convection-acceleration equation."

Finally, one last assumption can be used to make equation (3.59) simpler. The assumption is

$$F_0(z, p, t) \propto p^{-\gamma}, \tag{3.60}$$

where γ is a constant called the "power-law index" or "spectral index," as discussed in §2.1.2. With this assumption, equation (3.59) becomes an ordinary differential equation,

$$\frac{d}{dz} D \frac{dF_0}{dz} - \frac{d}{dz} U_z F_0 - \frac{\gamma - 1}{3} (\nabla \cdot \mathbf{U}) F_0 = 0. \tag{3.61}$$

Since the first term of this differential equation explains particle diffusion in z and the second term represents convection in z, this equation can be called the "diffusion-convection equation."

It can be seen that the diffusion approximation makes the equations much easier; therefore, much of the literature on shock acceleration is based on the diffusion approximation. However, this work will present evidence that using this approximation can give incorrect results in some cases.