References

[1] I. Daubechies, editor. *Ten Lectures on Wavelets.* Vermont: Capital City Press, 1992.

[2] G. Kaiser. *A Friendly Guide to Wavelets.* Birkuser, Boston, 1994.

[3] C. S. Burrus, R. A. Gopinath, and H. Gao, editors. *Introduction to Wavelets and Wavelet Transforms.* New Jersey: Prentice-Hall, Inc., 1998.

[4] C. K. Chui, editor. *An Introduction to Wavelets.* San Diego: Academic Press, Inc., 1992.

[5] J. C. van den Berg, editor. *Wavelets in Physics.* Cambridge: Cambridge University Press, 2004.

[6] M. Tegmark, A. de Oliveira-Costa, and A. J. S. Hamilton. *Phys. Rev.* **D68** (2003): 123523.

[7] C. L. Bennett et al. *Astrophys. J. Suppl.* **148** (2003): 1.

[8] W. Hu and S. Dodelson. *Annu. Rev. Astron. and Astrophys.* **40** (2002): 171.

[9] A. R. Liddle and D. H. Lyth. *Cosmological Inflation and Large-Scale Structure.* Cambridge: Cambridge University Press, 2000.

[10] L. Page et al. *Astrophys. J .Suppl.* **148** (2003): 233.

[11] R. K. Sachs and A. M. Wolfe. *Astrophys. J.* **147** (1967): 73.

[12] M. Zaldarriaga and U. Seljak. *Astrophys. J. Suppl.* **129** (2000): 431.

[13] J. Silk. *Astrophys. J.* **151** (1968): 459.

[14] G. Hinshaw et al. *Astrophys. J. Suppl.* **148** (2003): 135.

[15] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto. *Phys. Rep.* **402** (2004): 103.

[16] A. Guth. *Phys. Rev.* **D23** (1981): 347.

[17] R. J. Adler. *The Geometry of Random Fields.* John Wiley & Sons, New York, 1981.

[18] Z. Fan and J. M. Bardeen. *Phys. Rev.* **D51** (1995): 6714.

[19] L. Z. Fang and J. Pando. *Proc. International School of Astrophysics 'D. Chalonge'.* 5th Course: Current Topics in Astrofundamental Physics. Singapore: World Scientific, 1996.

[20] http://www.fyma.ucl.ac.be.

[21] J. Pando, D. Valls-Gabaud, and L-Z. Fang. *Phys. Rev. Lett.* **81** (1998): 4568.

[22] K. M. Gorski et al. astro-ph/9905275.

[23] R. B. Barreiro et al. *MNRAS* **318** (2000): 475.

[24] L. Cayon et al. *MNRAS* **326** (2001): 1243.

[25] E. Martinez-Gonzalez et al. *MNRAS* **336** (2002): 22.

[26] J-P. Antoine and P. Vanderghenst. *Appl. Comput. Harmon. Anal.* **7** (1999): 1.

[27] P. Vielva et al. *Astrophys. J.* **609** (2004): 22.

[28] P. Mukherjee and Y. Wang. *Astrophys. J.* **613** (2004): 51.

[29] E. Komatsu et al. *Astrophys. J. Suppl.* **148** (2003): 119.

[30] C. L. Bennett et al. *Astrophys. J. Suppl.* **148** (2003): 97.

[31] E. Komatsu et al. *Astrophys. J.* **566** (2002): 19.

[32] L. Cayon et al. *MNRAS* **339** (2003): 1189.

[33] M. Liguori, S. Matarrese, and L. Moscardini. *Astrophys. J.* **597** (2003): 57.

[34] R. A. Fisher. *Ann. Eugen.* **7** (1936): 179.

[35] G. Cowan. *Statistical Data Analysis.* Oxford University Press, Oxford, 1998.

[36] J. D. McEwen, M. P. Hobson, A. N. Lasenby, and D. J. Mortlock. astro-ph/0406604.

[37] J.-L. Starck, N. Aghanim, and O. Forni. astro-ph/0311577.

[38] E. Komatsu and D. N. Spergel. *Phys. Rev.* **D63** (2001): 63002.

APPENDICES

Appendix A

The Friedmann-Robertson-Walker Universe

Because gravity is the strongest force on cosmological scale, the universe can be well described by the Einstein field equation, a master equation of general relativity, that relates the Einstein tensor $G_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad , \tag{A.1}$$

where G is the Newtonian gravitational constant, and we have used the convention that the speed of light $c = 1$. In cosmology, the universe is assumed to be homogeneous and isotropic on large scale and described by the Friedmann-Robertson-Walker (FRW) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right] \quad , \tag{A.2}$$

where t is the cosmic time, r, θ, ϕ are the spherical polar coordinates in comoving frame of reference, $a(t)$ is the scale factor, and K is the curvature constant of the spatial space such that $K = 1$ for a closed universe, $K = 0$ for a flat universe, and $K = -1$ for an open universe. Note that, in the present, the homogeneity and isotropy of the observable universe have been established as the consequence of the existence of the inflationary period, any inhomogeneity had been redshifted away by the exponential expansion. When the universe is homogeneous and isotropic, the energy-momentum tensor $T_{\mu\nu}$ can be represented by that for a perfect fluid, in which an observer comoving with the fluid would see the universe around it as isotropic. It is

$$T_{\mu\nu} = \mathrm{diag}(\rho, -P, -P, -P) \quad . \tag{A.3}$$

Using Eq. (A.2) and Eq. (A.3), the Einstein field equation, that is in general a complicated tensor equation, is simplified and so provides the main equations of cosmology: the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \quad , \tag{A.4}$$

and the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad , \tag{A.5}$$

where $H = \dot{a}/a$ is the Hubble expansion rate and dots denote the cosmic time derivative. The Friedmann equation describes how the expansion rate of the universe is related to the energy density ρ and the curvature constant K, while the acceleration equation describes how the energy density and the pressure contribute to the acceleration, defined by $\ddot{a}/a$, of the universe.

Appendix B

Gaussian Statistics

The CMB anisotropy map can be referred to be Gaussian if the value of the anisotropies ($\Delta T/T$), as a random variable, at every point in the sky follows a Gaussian distribution. As a result, the histogram of anisotropy values collecting from all points should have the shape of Gaussian distribution. However, this is not sufficient to describe the statistical properties of a Gaussian map that is called in statistical mathematics "Gaussian random field". While a random variable is determined by a probability distribution, a random field is determined by a joint probability distribution , which tells us the probability of all points getting values at the same time. If we have a Gaussian random field $S(\mathbf{x})$, where $S \ = S(\mathbf{x} \ )$, $i = 1, \ldots, N$, is a random variable at the point $\mathbf{x}$ following Gaussian distribution law, its joint probability distribution is a multivariate Gaussian distribution:

$$P_N(S_1, \ldots, S_N) = \frac{\|\mathbf{C}^{-1}\|^{1\ 2}}{(2\pi)^{N\ 2}} \exp\left(-\frac{1}{2}\mathbf{S} \ \cdot \mathbf{C}^{-1} \cdot \mathbf{S}\right) \tag{B.1}$$

where $\mathbf{C}$ is the correlation matrix whose the element $C \ = \langle S\, S \ \rangle$ is the two-point correlation between the point $\mathbf{x}$ and $\mathbf{x}$, where the bracket denotes the ensemble average. This manifests us that the two-point correlation function specifies all statistical properties of the Gaussian map. In other words, every odd higher-order correlation $\langle S_1 S_2 \ldots S_{2\ +1}\rangle$, for $n \geq 1$, is zero and every even higher-order correlation is the sum of all possible combinations of correlation between different pair of points; for example, the four-point correlation is

$$\langle S_1 S_2 S_3 S_4\rangle = \langle S_1 S_2\rangle\langle S_3 S_4\rangle + \langle S_1 S_3\rangle\langle S_2 S_4\rangle + \langle S_1 S_4\rangle\langle S_2 S_3\rangle \quad . \tag{B.2}$$

For the CMB anisotropy map, which is assumed to be a homogeneous and isotropic random field, the angular two-point correlation function $C(\alpha)$ is the function only of the angular separation of two points α; it is independent of the specific pair of points (homogeneous) and the direction of the line connecting the two points (isotropic) [17]. From the so-called Wiener-Khintchine theorem, the power spectrum is the Fourier transform of the two-point correlation function. So, either the angular two-point correlation function or the angular power spectrum completely describe the statistical properties of the CMB sky if it is Gaussian. On the contrary, if it is non-Gaussian, obtaining the angular power spectrum has not finished yet the work of analysing CMB data.

Appendix C

The Quantization of the Inflaton Field

In quantum field theory, we take into account the quantum effect by implementing the second quantization. Here, we use the canonical quantization technique. However, it is more convenient in quantization to modify Eq. (2.14) conformal to the Klein-Gordon equation in Minkowskian space, which the method of quantization have been well-known, using the conformal time τ, where $\mathrm{d}\tau = \mathrm{d}t/a(t)$, instead of the cosmic time t and redefining the field from φ to $\widetilde{\varphi} \equiv a\,\varphi$. In the quantization, we mimic the treatment for the Klein-Gordon field; we write the field as a Hermitian operator as follows

$$\widetilde{\varphi}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left[u\ (\tau)\, a_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} + u^*(\tau)\, a^\dagger_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{C.1}$$

where $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ are the annihilation and creation operators which satisfy the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \quad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{C.2}$$

and impose the commutation relation between $\widetilde{\varphi}$ and its conjugate momentum Π, where $\Pi = \widetilde{\varphi}' \equiv \partial\,\widetilde{\varphi}/\partial\tau$:

$$\left[\,\widetilde{\varphi}(\tau, \mathbf{x})\,,\, \Pi(\tau, \mathbf{x}')\,\right] = i\,\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{C.3}$$

It follows that the eigen function $u\ (\tau)$ obey the equation of motion

$$u'' + \left[k^2 + M^2(\tau)\right] u\ = 0, \tag{C.4}$$

where the primes denote the differentiation with respect to the conformal time τ, and

$$M^2(\tau) = m^2\, a^2 - a''/a, \tag{C.5}$$

with the normalization condition

$$u_k\, u_k^{*\prime} - u_k^*\, u_k' = i, \tag{C.6}$$

that satisfy Eq. (C.3).

The solution of Eq. (C.4) depends on the model of inflation that give different forms, as a function of τ, of a''/a and a. In inflationary models that give a de Sitter stage, H is constant in time, the exact solution can be obtained. However, during inflation H is not exactly constant, it is therefore more general, but not really general, to consider the inflation with a quasi de Sitter stage when H changes slightly with time as $\dot{H} = -\epsilon H^2$. Recall that we mentioned it in order to introduce the slow-roll parameter ϵ, that have a small value, in the last section. From Eq. (2.9) that can be used to describe the quasi de Sitter expansion, we have

$$\frac{1}{\dot{a}}\frac{\mathrm{d}\dot{a}}{\mathrm{d}t} = (1-\epsilon)\frac{1}{a}. \tag{C.7}$$

Integrate both sides of the equation over cosmic time t, we obtain

$$\int \frac{1}{\dot{a}^2}\,\mathrm{d}\dot{a} = \int (1-\epsilon)\,\mathrm{d}\tau, \tag{C.8}$$

where we have used $\mathrm{d}\tau = \mathrm{d}t/a$ in the right hand side of the equation, then we arrive at

$$a(\tau) = -\frac{1}{H}\frac{1}{\tau(1-\epsilon)}. \tag{C.9}$$

Note that this equation implies that τ lies in $-\infty < \tau < 0$ for $-\infty < t < \infty$ $(0 < a < \infty)$. Using Eqs. (2.9) and (C.9), we can obtain the form of a''/a as

$$\begin{aligned} \frac{a''}{a} &= a^2\left(\frac{\ddot{a}}{a} + H^2\right) \\ &= a^2(2-\epsilon)H^2 \\ &= \frac{(2-\epsilon)}{\tau^2(1-\epsilon)^2} \\ &\simeq \frac{1}{\tau}(2+3\,\epsilon). \end{aligned} \tag{C.10}$$

Using Eqs. (C.9) and (C.10), we can write (C.5) as

$$\begin{aligned} M^2(\tau) &\simeq \frac{m_\phi^2}{H^2}\frac{1}{\tau^2(1-\epsilon)^2} - \frac{1}{\tau^2}(2+3\,\epsilon) \\ &= \frac{1}{\tau^2}\left[\frac{3\,\eta}{(1-\epsilon)^2} - (2+3\,\epsilon)\right], \end{aligned} \tag{C.11}$$

where

$$\eta \equiv \frac{m_\phi^2}{3H^2} = \frac{1}{3H^2}\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}$$

is another slow-roll parameter, with $\eta \ll 1$ in the quasi de Sitter stage, derived from the condition that we mentioned in the last section: $\ddot{\phi} \ll 3H\dot{\phi}$. Notice that the existence of η, i.e. $\eta > 0$, indicates that there is a small deviation of the potential from being perfectly flat, while $\epsilon > 0$ specifies that $\dot{H} \neq 0$. We will see later that both characteristics cause the deviation from scale-invariant power spectrum. We can approximate Eq. (C.11) in first order of the slow-roll parameters as

$$M^2(\tau) \simeq -\frac{1}{\tau^2}\left(2 + 3(\epsilon - \eta)\right). \tag{C.12}$$

Substituting Eq. (C.12) into Eq. (C.4), we obtain the equation that is conformal to the Bessel differential equation:

$$u_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right] = 0, \tag{C.13}$$

where

$$\nu^2 = \frac{9}{4} + 3(\epsilon - \eta). \tag{C.14}$$

Therefore, the exact solution, for ν real, is

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k)\, H_\nu^{(1)}(-k\tau) + c_2(k)\, H_\nu^{(2)}(-k\tau)\right], \tag{C.15}$$

where $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x) = [H_\nu^{(1)}(x)]^*$ are the Hankel's functions of the first and second kind, respectively, and $c_1(k)$ and $c_2(k)$ are integration constants.

In the ultraviolet limit $k \gg aH$ $(-k\tau \approx (k/aH) \gg 1)$, the solution u_k should be the plane-wave solution $e^{-ik}\,/\sqrt{2k}$ which matches that of the Klein-Gordon field in Minkowskian space, and we have the asymptotic form of the Hankel's function

$$H_\nu^{(1)}(x \gg 1) \approx \sqrt{\frac{2}{\pi x}}\, e^{i\left(\,-\frac{\pi}{2}\nu - \frac{\pi}{4}\right)}, \tag{C.16}$$

and

$$H_{\nu}^{(2)}(x \gg 1) \approx \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}, \tag{C.17}$$

where $x = -k\tau$, hence we can set the integration constants $c_1(k) = (\sqrt{\pi}/2)e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}}$ and $c_2(k) = 0$. We are interested in the perturbations with the modes well outside the horizon because the CMB anisotropies corresponding to them are not much affected from the photon-baryon fluid dynamics (see Fig. 2.1), so that their imprints provide the information about the early universe. On superhorizon scales, $x = -k\tau \ll 1$, we obtain

$$u_k(\tau) = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu}, \tag{C.18}$$

by using the asymptotic form of Hankel's function

$$H_{\nu}^{(1)}(x \ll 1) \approx \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} (x)^{-\nu}, \tag{C.19}$$

and the integration constants we have obtained above.

Vitae

Mr. Ekapong Hirunsirisawat was born on 23 November 1979 in Bangkok. He received his Bachelor of Science in Physics from Chulalongkorn University, in 2000.