IDEAL THEORY IN SOME TERNARY SEMIRINGS

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ทฤษฎีไอดีลในบางกึ่งริงไตรภาค

นางสาวภัทราวรรณ เพชรแก้ว

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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ในวิทยานิพนธ์ฉบับนี้ เราศึกษาทฤษฎีไอดีลในกึ่งริงไตรภาคของเซตของจำนวนเต็มที่ไม่เป็นลบ $\mathbb{Z}_0^+$ และทฤษฎีไอดีลในกึ่งริงไตรภาคของเซตของคู่อันดับของจำนวนเต็มที่ไม่เป็นบวก $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ จากการศึกษาเราได้ผลลัพธ์ที่เกี่ยวข้องกับไอดีลในกึ่งริงไตรภาคดังกล่าว และเราได้ศึกษาเพิ่มเติมว่ากึ่งริงไตรภาค $\mathbb{Z}_0^+$ และกึ่งริงไตรภาค $\mathbb{Z}_0^-$ สมสัณฐานกัน ในทำนองเดียวกันเราได้แสดงว่ากึ่งริงไตรภาค $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ และกึ่งริงไตรภาค $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ สมสัณฐานกัน นอกจากนี้เรายังได้เปรียบเทียบกึ่งริงไตรภาค $\mathbb{Z}$ กับริง $\mathbb{Z}$

ภาควิชา คณิตศาสตร์ และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..

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In this thesis, we study the ideal theory in the ternary semiring $\mathbb{Z}_0^+$ of non-negative integers and the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ of ordered pairs of non-positive integers and obtain some results regarding the ideals of these ternary semirings. Moreover, we show that the ternary semirings $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ are isomorphic as well as the ternary semirings $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. We also compare the ternary semiring $\mathbb{Z}$ to the ring $\mathbb{Z}$.

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CONTENTS

CHAPTER I

INTRODUCTION

In 1971, W. G. Lister have investigated ternary rings and their structures. In fact, W. G. Lister charaterized those additive subgroups of rings which are closed under the triple product. T. K. Dutta and S. Kar introduced the notion of a ternary semiring as a generalization of a ternary ring in 2003. In [2], T. K. Dutta and S. Kar introduced the notions of left/right/lateral ideals of ternary semirings and also characterized regular ternary semirings. In 2005, S. Kar introduced the notion of quasi-ideals and bi-ideals in the ternary semiring. The ideal theory in the ternary semiring $\mathbb{Z}_0^-$ has been introduced and studied by S. Kar in [1]. In 2011, S. Kar studied the ideal theory in the ternary semiring $\mathbb{Z}_0^-$ of non-positive integers and obtained some results regarding the ideals of the ternary semiring $\mathbb{Z}_0^-$. In his research, he studied the T_n-ideal in the ternary semirings $\mathbb{Z}_0^-$ where $T_n = \{t \in \mathbb{Z}_0^- \mid t \leq n\} \cup \{0\}$ for $n \in \mathbb{Z}_0^-$ and he had the following results concerning T_n.

Theorem 1.1. ([1] S. Kar, 2011) *T_n is an ideal in $\mathbb{Z}_0^-$ such that*

(i) $T_0 = T_{-1} = \mathbb{Z}_0^-$,

(ii) *For $n \leq -1, m \leq n$ if and only if $T_m \subseteq T_n$,*

(iii) $T_m \cup T_n = T_p$, *where* $p = \max\{m, n\}$,

(iv) $T_m \cap T_n = T_q$, *where* $q = \min\{m, n\}$,

(v) $\bigcap\{T_i : i \in \mathbb{Z}_0^-\} = \{0\}$.

Theorem 1.2. ([1] S. Kar, 2011) *$\mathbb{Z}_0^-$ satisfies the ascending chain condition on T_n- ideal.*

Let I be an ideal of a ternary semiring S. A subset B of I is called a *basis* for I if every element of I can be written in the form $\sum_{i=1}^{n} r_i s_i b_i$, where $r_i, s_i \in S$ and $b_i \in B$. If the set B is finite, then B is called a *finite basis* for I.

S. Kar denoted the set $S(a,b) = \{t \in \mathbb{Z}_0^- \mid a \leq t \leq b\}$ for $a, b \in \mathbb{Z}_0^-$ and $a \leq b$. He derived a theorem and some lemmas which were essential for the characterization of all ideals in the ternary semiring $\mathbb{Z}_0^-$. From those lemmas he had some methods by which he can determine if an ideal in $\mathbb{Z}_0^-$ contains a T_n-ideal.

Theorem 1.3. ([1] S. Kar, 2011) *If $n < -1$, then $S(2n, n)$ is a finite basis for T_n.*

Let I be an ideal in the ternary semiring $\mathbb{Z}_0^-$. If $a \in I, m \in \mathbb{Z}_0^-$, where $m \neq 0$, and $S(-(m-1)a, -ma) \subseteq I$, then there exists an $n \in \mathbb{Z}_0^-$ such that $T_n \subseteq I$. If there exists $a \in I$ such that $a + (-1) \in I$, then there exists an n such that $T_n \subseteq I$. If $a, b \in I$ and a, b are relatively prime, then there exists an n such that $T_n \subseteq I$.

Our main purpose of this thesis is to study the ideal theory in the ternary semiring $\mathbb{Z}_0^+$ of non-negative integers and the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ of ordered pairs of non-positive integers. We are going to give some ideals as examples and to prove some analogous results on such ideals. In Chapter 2, we give some basic definitions and examples. In Chapter 3, we study the ideal theory in the ternary semiring $\mathbb{Z}_0^+$. In Chapter 4, we study the ideal theory in the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Moreover, we show that the ternary semirings $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ are isomorphic as well as the ternary semirings $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. We also compare the ternary semiring $\mathbb{Z}$ to the ring $\mathbb{Z}$.

CHAPTER II

PRELIMINARIES

In this chapter, we present a number of elementary concepts, notations and definitions on semigroups, semirings and ternary semirings which will be used for this thesis.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Q}$ denote the set of rational numbers, $\mathbb{N}$ denote the set of natural numbers (positive integers), $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}^+$ denote the set of all positive integers, $\mathbb{Z}^-$ denote the set of all negative integers, $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$.

Definition 2.1. ([16] M. Petrich, 1973) A *semigroup* is a system $(S, \cdot)$ consisting of a nonempty set S together with an associative binary operation

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ for all } a, b, c \in S.$$

In this thesis, we denote $a \cdot b$ by ab .

Definition 2.2. ([16] M. Petrich, 1973) Let S be a semigroup. An element $e \in S$ is a *left* (*right*) *identity* of S if $ex = x$ $(xe = x)$ for all $x \in S$. $e \in S$ is an *identity* of S if $ex = x = xe$ for all $x \in S$.

Definition 2.3. ([16] M. Petrich, 1973) An element z of a semigroup S is called a *left (right) zero* of S if $zx = z$ $(xz = z)$ for all $x \in S$, and z is a *zero* of S if $zx = z = xz$ for all $x \in S$.

If a semigroup S has an identity, then we denote it by 1. If a semigroup S has a zero, then we denote it by 0.

Definition 2.4. ([16] M. Petrich, 1973) A semigroup S is called a *commutative semigroup* if $ab = ba$ for all $a, b \in S$.

Definition 2.5. ([16] M. Petrich, 1973) A nonempty subset T of a semigroup S is a *subsemigroup* of S if T is itself a semigroup with respect to the operation of S. (Notice that a subset T of a semigroup S is a subsemigroup of S if and only if $T \neq \varnothing$ and $ab \in T$ for all $a, b \in T$.)

Example. $(\mathbb{N} \cup \{0\}, \cdot)$ *is a commutative semigroup with identity 1 and zero 0.*
$(\mathbb{N} \cup \{0\}, +)$ *is a commutative semigroup with identity 0 but has no zero.*

Example. *([16] M. Petrich, 1973) Define operations $*$ and $\cdot$ on the closed interval $[0, 1]$ by $x * y = min\{x, y\}$ and $x \cdot y = max\{x, y\}$ for all $x, y \in [0, 1]$. Then $([0, 1], *)$ is a commutative semigroup with identity 1 and zero 0, and $([0, 1], \cdot)$ is a commutative semigroup with identity 0 and zero 1.*

Example. *Let $M_n(\mathbb{R})$ be the set of all real $n \times n$ matrices where $n \in \mathbb{N}$. Then $M_n(\mathbb{R})$ is a semigroup under usual matrix multiplication. If $n > 1$, then $M_n(\mathbb{R})$ is a noncommutative semigroup with identity I_n and zero $\underline{0}$.*

Definition 2.6. ([14] J. S. Golan, 1999) $(S, +, \cdot)$ is a *semiring* if the following conditions are satisfied:

(i) $(S, +)$ is a commutative semigroup,

(ii) $(S, \cdot)$ is a semigroup,

(iii) $\forall a, b, c \in S, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

A *proper semiring* is a semiring which is not a ring.

Let $(S, +, \cdot)$ be a semiring.

(i) If $(S, \cdot)$ has an identity e, we call e the *identity* of the semiring S.

(ii) If $(S, +)$ has an identity, we call this element the *zero* of the semiring S.

(iii) If the semiring S has a zero and $x + y = 0$, we denote y by $-x$ and call it the *opposite element* of x.

Note. If S is a semiring and $a \in S$ we defines an element $na \in S, n \in \mathbb{N}$ by $na = a_1 + a_2 + a_3 + \cdots + a_n$ where $a_1 = a_2 = a_3 = \cdots = a_n = a$.

Then for all $a, b \in S$ and $m, n \in \mathbb{N}$,

(i) $na + ma = (n + m)a$,

(ii) $m(na) = (mn)a$,

(iii) $m(a + b) = ma + mb$.

If S has a zero and a, b have opposite elements, then

(i) $-(a + b) = (-a) + (-b)$,

(ii) $-(-a) = a$,

(iii) $-(na) = n(-a)$, for all $n \in \mathbb{N}$.

Example. 1. *Each ring is a semiring.*

2. $(\mathbb{N}, +, \cdot)$ *and* $(\mathbb{N}_0, +, \cdot)$ *are proper semirings.*

3. *If* $m\mathbb{N} = \{mn \mid n \in \mathbb{N}\}, m \in \mathbb{N}$, *then* $(m\mathbb{N}, +, \cdot)$ *is a proper semiring.*

4. *For* $c \in \mathbb{N}$, $(\{c, c+1, c+2, \cdots\}, +, \cdot)$ *is a proper semiring.*

5. $(\mathbb{Q}^+, +, \cdot)$, $(\mathbb{Q}_0^+, +, \cdot)$, $(\mathbb{R}^+, +, \cdot)$, $(\mathbb{R}_0^+, +, \cdot)$ *are proper semirings.*

Definition 2.7. Let S be a semiring and A, B nonempty subsets of S. We define the following subsets of S:

$$A + B = \{a + b \mid a \in A, b \in B\},$$
$$A \cdot B = \{a \cdot b \mid a \in A, b \in B\}.$$

Note. If $A = \{a\}$, one simply writes $a + B$ and aB instead of $\{a\} + B$ and $\{a\} \cdot B$, respectively.

Example. *Let* $(S, \cdot)$ *be a semigroup and* $P(S)$ *the power set of* S. *For all* $A, B \in P(S) \setminus \{\varnothing\}$, $A \cdot B = \{a \cdot b \mid a \in A, b \in B\} \neq \varnothing$. *Then* $(P(S) \setminus \{\varnothing\}, \cup, \cdot)$ *is a semiring.*

Definition 2.8. ([1] S. Kar, 2011) A non-empty set S together with a binary operation, called addition, and a ternary multiplication, denoted by juxtaposition, is said to be a *ternary semiring* if S is an additive commutative semigroup satisfying the following conditions: for all $a, b, c, d, e \in S$

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a + b)cd = acd + bcd$,

(iii) $a(b + c)d = abd + acd$,

(iv) $ab(c + d) = abc + abd$.

We see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not nessesarily reduce to a semiring by this example. We consider $\mathbb{Z}_0^-$ under usual addition and multiplication, we see that $\mathbb{Z}_0^-$ is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, $\mathbb{Z}_0^-$ is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 2.9. Let S be a ternary semiring. If there exists an element $e \in S$ such that

$$eea = aee = eae = a$$

for all $a \in S$, then e is called an *identity* of the ternary semiring S.

Example. *With the usual binary addition and ternary multiplication*, 1 *is the identity of the ternary semiring* $\mathbb{Z}_0^+$ *and* -1 *is the identity of the ternary semiring* $\mathbb{Z}_0^-$.

Definition 2.10. ([1] S. Kar, 2011) Let S be a ternary semiring. If there exists an element $0 \in S$ such that

$$0 + x = x \text{ and } 0xy = x0y = xy0 = 0 \text{ for all } x, y \in S$$

then 0 is called the *zero element* or simply the *zero* of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Example. *With the usual binary addition and ternary multiplication*, $\mathbb{Z}_0^+$ *and* $\mathbb{Z}_0^-$ *form the ternary semirings with zero.*

Definition 2.11. ([1] S. Kar, 2011) An additive subsemigroup T of a ternary semiring S is called a *ternary subsemiring* of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.12. ([1] S. Kar, 2011) An additive subsemigroup I of a ternary semiring S is called :

A *left ideal* of S if $s_1s_2i \in I$ for all $s_1, s_2 \in S$ and $i \in I$.

A *right ideal* of S if $is_1s_2 \in I$ for all $s_1, s_2 \in S$ and $i \in I$.

A *lateral ideal* of S if $s_1 i s_2 \in I$ for all $s_1, s_2 \in S$ and $i \in I$.
If I is a left, a right and a lateral ideal of S, then I is called an *ideal* of S.

Note. ([11] R. Chinram, 2010) It is obvious that every ideal of a ternary semiring with zero contains a zero element.

Definition 2.13. ([1] S. Kar, 2011) An ideal I of a ternary semiring S is called a *k-ideal* if

$$x + y \in I; x \in S, y \in I \text{ imply that } x \in I.$$

Note. Since an ideal I of a ternary semiring S is an additive commutative semigroup, we get that $y + x \in I; x \in S, y \in I$ imply that $x \in I$.

Example. ([11] R. Chinram, 2010) *Consider the ternary semiring* $\mathbb{Z}_0^-$ *under usual binary addition and ternary multiplication*, *let* $I = \{0, -3\} \cup \{-5, -6, -7, \cdots\}$. *It is easy to prove that* I *is an ideal of* $\mathbb{Z}_0^-$ *but not a k-ideal of* $\mathbb{Z}_0^-$ *because* $(-2) + (-3) = (-5) \in I$ but $-2 \notin I$.

Example. ([11] R. Chinram, 2010) *Consider the ternary semiring* $\mathbb{Z}_0^-$ *under usual binary addition and ternary multiplication*, *let* $I = \{-3k \mid k \in \mathbb{N} \cup \{0\}\}$. *It is easy to show that* I *is a k-deal of* $\mathbb{Z}_0^-$.

Definition 2.14. ([1] S. Kar, 2011) Let I be an ideal of a ternary semiring S. A subset B of I is called a *basis* for I if every element of I can be written in the form $\sum_{i=1}^{n} r_i s_i b_i$, where $r_i, s_i \in S$, $b_i \in B$ and $n \in \mathbb{N}$.

If the set B is finite, then B is called a *finite basis* for I.

Definition 2.15. ([11] R. Chinram, 2010) Let S, T be ternary semirings. A map $\varphi : S \to T$ is a *homomorphism* if

$$\varphi(a + b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$$

for all $a, b, c \in S$.

A *monomorphism* is a homomorphism which is one-to-one.

An *epimorphism* is a homomorphism which is onto.

An *isomorphism* is a homomorphism which is one-to-one and onto.

We said that a ternary semiring S and a ternary semiring T are *isomorphic*, if there exists an isomorphism $\phi : S \to T$ or there exists an isomorphism $\psi : T \to S$. In this case, we will use the notation $S \cong T$.

Note. ([11] R. Chinram, 2010) Let $\varphi : S \to R$ be an epimorphism of ternary semiring. If I is an ideal of S, then $\varphi(I)$ is an ideal of R. If S and R be ternary semirings with zero 0, then $\varphi(0) = 0$.

Definition 2.16. Given two partially ordered sets A and B, the *lexicographical order* on the Cartesian product $A \times B$ is defined as

$$(a, b) \leq (c, d) \quad \text{if and only if} \quad a < c \quad \text{or} \quad (a = c \text{ and } b \leq d),$$

$$(a, b) < (c, d) \quad \text{if and only if} \quad a < c \quad \text{or} \quad (a = c \text{ and } b < d).$$

In this thesis, we define the additive and the ternary multiplicative operator on the Cartesian product $A \times B$ as
follows

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b)(c, d)(e, f) = (ace, bdf).$$

Example. *With the binary addition and ternary multiplication*, $(1, 1)$ *is the identity of the ternary semiring* $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ *and* $(-1, -1)$ *is the identity of the ternary semiring* $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

Example. *With the binary addition and ternary multiplication*, $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ *and* $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ *form the ternary semirings with zero.*

Definition 2.17. A partially ordered set P is said to satisfy the *ascending chain condition* (*ACC*) if every ascending chain of elements eventually terminates. Equivalently, given any sequence of elements of P

$$a_1 \leq a_2 \leq a_3 \leq \cdots$$

there exists a positive integer n such that

$$a_n = a_{n+1} = a_{n+2} = \cdots$$

Similarly, P is said to satisfy the descending chain condition (DCC) if every descending chain of elements eventually terminates, or equivalently if any descending sequence

$$\cdots \geq a_3 \geq a_2 \geq a_1$$

of elements of P eventually stabilizes.

CHAPTER III

IDEAL THEORY IN THE TERNARY SEMIRING $\mathbb{Z}_0^+$

In this chapter we study the ideal theory in the ternary semiring of non-negative integers $\mathbb{Z}_0^+$ and classify them.

Let $n \in \mathbb{Z}_0^+$ and $T_n = \{t \in \mathbb{Z}_0^+ \mid t \geq n\} \cup \{0\}$. Then we have the following results concerning T_n.

Theorem 3.1. *Let $n \in \mathbb{Z}_0^+$. Then T_n is an ideal in the ternary semiring $\mathbb{Z}_0^+$ such that*

(i) $T_0 = T_1 = \mathbb{Z}_0^+$,

(ii) *For $n \geq 1, n \leq m$ if and only if $T_m \subseteq T_n$,*

(iii) $T_m \cup T_n = T_p$, *where* $p = \min\{m, n\}$,

(iv) $T_m \cap T_n = T_q$, *where* $q = \max\{m, n\}$,

(v) $\bigcap\{T_n \mid n \in \mathbb{Z}_0^+\} = \{0\}$.

Proof. First, we want to prove that T_n is an ideal of the ternary semiring $\mathbb{Z}_0^+$. Let $a, b \in T_n$. We divide the proof of an additive subsemigroup into four cases.

Case 1: If $a = 0$ and $b = 0$, then $a + b = 0 + 0 = 0 \in T_n$.

Case 2: If $a = 0$ and $b \geq n$, then $a + b \geq 0 + n = n \in T_n$.

Case 3: If $a \geq n$ and $b = 0$, then it is similar to the Case 2.

Case 4: If $a \geq n$ and $b \geq n$, then $a + b \geq 2n \geq n \in T_n$.

From any cases, we have that T_n is an additive subsemigroup of $\mathbb{Z}_0^+$.

Next, we let $x \in T_n$ and $r, s \in \mathbb{Z}_0^+$ where $r \neq 0$ and $s \neq 0$ (In case of $r = 0$ or $s = 0$, we have that $rsx = 0 \in T_n$). If $x = 0$ then $rsx = rs(0) = 0 \in T_n$. If $x \geq n$ then $rsx \geq rsn \geq n \in T_n$. Similarly, we have that $rxs, xrs \in T_n$. Consequently, T_n is an ideal of $\mathbb{Z}_0^+$.

(i) By the definition of T_n, we have that

$$T_0 = \{t \in \mathbb{Z}_0^+ \mid t \geq 0\} \cup \{0\} = \mathbb{Z}_0^+ \cup \{0\} = \mathbb{Z}_0^+,$$
$$T_1 = \{t \in \mathbb{Z}_0^+ \mid t \geq 1\} \cup \{0\} = \mathbb{Z}^+ \cup \{0\} = \mathbb{Z}_0^+.$$

Thus $T_0 = T_1 = \mathbb{Z}_0^+$.

(ii) Let $m, n \in \mathbb{Z}_0^+$ be such that $1 \leq n \leq m$. We know that

$$T_n = \{0, n, n+1, n+2, n+3, \cdots\} \text{ and } T_m = \{0, m, m+1, m+2, m+3, \cdots\}.$$

Since $n \leq m$, $T_m \subseteq T_n$. Conversely, suppose $T_m \subseteq T_n$, it is easy to see that $n \leq m$.

(iii) Let $m, n, p \in \mathbb{Z}_0^+$ such that $p = \min\{m, n\}$. If $m = n$, then $T_m \cup T_n = T_p$. If $m > n$, from (ii) we get that $T_m \subseteq T_n$. Since $p = \min\{m, n\} = n$, $T_m \cup T_n = T_n = T_p$. In case of $n > m$ we can prove in the same way.

(iv) Let $m, n, q \in \mathbb{Z}_0^+$ such that $q = \max\{m, n\}$. If $m = n$, then $T_m \cap T_n = T_q$. If $m > n$, from (ii) we obtain $T_m \subseteq T_n$. Since $q = \max\{m, n\} = m$, $T_m \cap T_n = T_m = T_q$. In case of $n > m$ we can prove in the same way.

(v) By definition of T_n, we have that $\{0\} \subseteq \bigcap\{T_i \mid i \in \mathbb{Z}_0^+\}$. Next, suppose that there exists $x \in \bigcap\{T_i \mid i \in \mathbb{Z}_0^+\}$ be such that $x \neq 0$. By definition of T_n, it is clear that $x \in T_x$. Since $x \in \bigcap\{T_i \mid i \in \mathbb{Z}_0^+\}$, $x + 1 \in \mathbb{Z}_0^+$. We know that $T_{x+1} = \{t \in \mathbb{Z}_0^+ \mid t \geq x+1\} \cup \{0\}$. Since $x < x+1$, $x \notin T_{x+1}$. It is a contradiction with $x \in \bigcap\{T_i \mid i \in \mathbb{Z}_0^+\}$. Consequently, we obtain that $\bigcap\{T_i \mid i \in \mathbb{Z}_0^+\} = \{0\}$. $\square$

For $n \in \mathbb{Z}_0^+$, the notation T_n-ideal will be used to denote the ideal T_n.

Remark. *Let $n \in \mathbb{Z}_0^+$. If $n \neq 0, 1$, then T_n is not a k-ideal of the ternary semiring $\mathbb{Z}_0^+$.*

Theorem 3.2. *Let $n \in \mathbb{Z}_0^+$. Then $\mathbb{Z}_0^+$ satisfies the ascending chain condition on T_n-ideals.*

Proof. Let $\{T_{n_i} \mid n_i \in \mathbb{Z}_0^+ \text{ and } i \in \mathbb{N}\}$ be an ascending chain of T_n-ideals in $\mathbb{Z}_0^+$. Then it is finite since by the Theorem 3.1 (ii), the decreasing sequence $\{n_i\}$ of positive integers is finite. Thus there exists $j \in \mathbb{N}$ such that $T_{n_i} = T_{n_j}$ for each $i \geq j$ and hence $\mathbb{Z}_0^+$ satisfies the ascending chain condition on T_n-ideals. $\square$

For $a, b \in \mathbb{Z}_0^+$ and $a \leq b$, the notation $S(a, b)$ will be used to denote the set $\{t \in \mathbb{Z}_0^+ \mid a \leq t \leq b\}$.

Note For $a, b \in \mathbb{Z}_0^+$ and $a \leq b$, we have that $S(a, b) \subseteq T_a$.

Theorem 3.3. *Let $n \in \mathbb{Z}_0^+$. If $n > 1$, then $S(n, 2n)$ is a finite basis for T_n.*

Proof. Let $x \in T_n$. In case of $x \in S(n, 2n)$ or $x = cdn$ for some $c, d \in \mathbb{Z}_0^+$, it is easy to prove that the mentioned case is a basis for T_n. Suppose that $x > 2n$ and $x \neq cdn$ for any $c, d \in \mathbb{Z}_0^+$. Let k =max$\{ l \in \mathbb{Z}_0^+ \mid ln < x\}$. Then we have that $kn < x < (k+1)n$. However, this guarantees the existence of a positive integer $m < n$ such that $kn + m = x$ and it follow that $n + m \in S(n, 2n)$. Hence we have that $x = kn + m = (k - 1 + 1)n + m = ((k - 1) + 1)n + m = (k - 1)n + n + m$ where $n \in S(n, 2n)$ and $n + m \in S(n, 2n)$. Therefore $S(n, 2n)$ is a basis for T_n. Since the set $S(n, 2n)$ is finite, $S(n, 2n)$ is a finite basis for T_n. $\square$

Now we study some lemmas and propositions which will be essential for characterization of all ideals in the ternary semiring $\mathbb{Z}_0^+$. From these lemmas and proposition we have some methods by which we can determine if an ideal of $\mathbb{Z}_0^+$ contains a T_n-ideal.

Lemma 3.4. *Let I be an ideal of the ternary semiring $\mathbb{Z}_0^+$. If $a \in I, m \in \mathbb{Z}_0^+$, where $a \neq 0$ and $S(ma, (m+1)a) \subseteq I$, then there exists an $n \in \mathbb{Z}_0^+$ such that $T_n \subseteq I$.*

Proof. If $a = 1$, then $1 \in I$. Hence $I = \mathbb{Z}_0^+$. Therefore $T_n \subseteq I$ for all $n \in \mathbb{Z}_0^+$. If $a > 1$, we claim that $T_{(m+1)a} \subseteq I$. Let $x \in T_{(m+1)a}$. If $x = cda$ for some $c, d \in \mathbb{Z}_0^+$, then clearly $x \in I$. Next, suppose that $x > (m+1)a$ and $x \neq cda$ for $c, d \in \mathbb{Z}_0^+$. Let k =max$\{ l \in \mathbb{Z}_0^+ \mid ln < x\}$. Then we have that $ka < x < (k+1)a$. Thus there exists a positive integer $b < a$ such that $ka + b = x$ We have that $ma + b \in S(ma, (m+1)a) \subseteq I$. Hence $x = ka + b = ka - ma + ma + b = (ka - ma) + (ma + b) = (k - m)a + (ma + b) \in I$. Consequently, $T_{(m+1)a} \subseteq I$ and therefore the proof of the lemma follows. $\square$

Proposition 3.5. *Let I be an ideal of the ternary semiring $\mathbb{Z}_0^+$. If there exists $a \in I$ such that $a+1 \in I$, then there exists an $n \in \mathbb{Z}_0^+$ such that $T_n \subseteq I$.*

Proof. If I is a T_n-ideal, then the lemma is obvious. Suppose that I is not a T_n-ideal and x is the smallest element in I such that $x+1 \in I$. Since I is an ideal, a series of simple calculations shows that the following elements belong to I:

(i) $x+1, x$

(ii) $2x+2, 2x+1, 2x$;by (1)

(iii) $3x+3, 3x+2, 3x+1, 3x$;by $(1), (2)$

$\vdots$

(x) $(x)x+x, \cdots, (x)x+3, (x)x+2, (x)x+1, (x)x.$;by (i), (ii), (iii), $\cdots$, (x-1)

The last row of elements is $S((x)x, (x+1)x) \subseteq I$. Thus there exists an $n \in \mathbb{Z}_0^+$ such that $T_n \subseteq I$, by using Lemma 3.4. $\square$

Lemma 3.6. *Let $a, b \in \mathbb{Z}_0^+$ where $a, b \neq 0$. If d is the greatest common divisor of a and b, then there exist $s, t \in \mathbb{Z}_0^+$ such that $sa = tb + d$ or $tb = sa + d$.*

Proof. From elementary number theory, it is well known that $d = s'a + t'b$ for some integers s' and t'. Since $1 \leq d \leq a$ and $1 \leq d \leq b$, it follows that ($s' \geq 0$ and $t' \leq 0$) or ($s' \leq 0$ and $t' \geq 0$). If $s' \geq 0$ and $t' \leq 0$, then

$$d = s'a + t'b$$
$$s'a = -t'b + d$$
$$sa = tb + d$$

where $s = s' \geq 0$ and $t = -t' \geq 0$. On the other hand, if $s' \leq 0$ and $t' \geq 0$ then

$$d = s'a + t'b$$
$$t'b = -s'a + d$$
$$tb = sa + d$$

where $t = t' \geq 0$ and $s = -s' \geq 0$. Hence the proof of the lemma follows. $\square$

Proposition 3.7. *Let I be an ideal of the ternary semiring $\mathbb{Z}_0^+$, $a \in I$ and $b \in I$. If a and b are relatively prime, then there exists an $n \in \mathbb{Z}_0^+$ such that $T_n \subseteq I$.*

Proof. Since a and b are relatively prime, 1 is the greatest common divisor of a and b. From Lemma 3.6 guarantees the existence of $s \in \mathbb{Z}_0^+$ and $t \in \mathbb{Z}_0^+$ such that $sa = tb + 1$ or $tb = sa + 1$. Since I is an ideal, it is clearly that $sa \in I$ and $tb \in I$. Consequently, $sa + 1 \in I$ or $tb + 1 \in I$ and the lemma follows from Lemma 3.5. $\square$

CHAPTER IV

IDEAL THEORY IN THE TERNARY SEMIRING $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$

In this chapter, we study the ideal theory in the ternary semiring of ordered pairs of non-positive integers $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. We study about the ideal $T_{(m,n)} = \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) \leq (m, 0)\} \cup \{(0, t) \mid t \in \mathbb{Z}_0^-\}$ and the ideal $T^*_{(m,n)} = \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) = ((-1)k_1 m, (-1)k_2 n),\ k_1, k_2 \in \mathbb{Z}_0^-\}$ and classify them. Moreover, we show that the ternary semirings $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ are isomorphic as well as the ternary semirings $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. We will also compare the ternary semiring $\mathbb{Z}$ to the ring $\mathbb{Z}$.

Given two partially ordered set A and B, the lexicographical order on the Cartesian product $A \times B$ is defined as

$$(a, b) \leq (c, d) \quad \text{if and only if} \quad a < c \quad \text{or} \quad (a = c \text{ and } b \leq d),$$

$$(a, b) < (c, d) \quad \text{if and only if} \quad a < c \quad \text{or} \quad (a = c \text{ and } b < d).$$

In this research, we define the additive and the ternary multiplicative operator as follow:

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b)(c, d)(e, f) = (ace, bdf).$$

Proposition 4.1. *Let $(a, b), (c, d), (a', b'), (c', d') \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. If $(a, b) \leq (c, d)$ and $(a', b') \leq (c', d')$, then $(a, b) + (a', b') \leq (c, d) + (c', d')$.*

Proof. Suppose that $(a, b) \leq (c, d)$ and $(a', b') \leq (c', d')$. Since $(a, b) \leq (c, d)$, $a < c$ or $(a = c$ and $b \leq d)$. Since $(a', b') \leq (c', d')$, $a' < c'$ or $(a' = c'$ and $b' \leq d')$. We want to show that $(a, b) + (a', b') \leq (c, d) + (c', d')$. Since we known that

$(a,b)+(a',b')=(a+a',b+b')$ and $(c,d)+(c',d')=(c+c',d+d')$, we can show $(a+a',b+b') \leq (c+c',d+d')$ instead of $(a,b)+(a',b') \leq (c,d)+(c',d')$. We divide its proof into four cases.

Case 1: If $a < c$ and $a' < c'$, then $a+a' < c+c'$. Thus $(a+a',b+b') \leq (c+c',d+d')$.

Case 2: If $a < c$ and ($a' = c'$ and $b' \leq d'$), then $a+a' < c+c'$. Thus $(a+a',b+b') \leq (c+c',d+d')$.

Case 3: If ($a = c$ and $b \leq d$) and $a' < c'$, then it is similar to the Case 2.

Case 4: If ($a = c$ and $b \leq d$) and ($a' = c'$ and $b' \leq d'$), then $a+a' = c+c'$ and $b+b' \leq d+d'$. Thus $(a+a',b+b') \leq (c+c',d+d')$.

From any cases, we can conclude that $(a,b)+(a',b') \leq (c,d)+(c',d')$, as desired. $\square$

In this chapter, we first study the ideal theory in the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Let $(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $T_{(m,n)} = \{(t_1,t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1,t_2) \leq (m,0)\} \cup \{(0,t) \mid t \in \mathbb{Z}_0^-\}$. Then we have the following results concerning $T_{(m,n)}$.

Note. For $(a,b),(c,d) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$, if $a = c$ then $T_{(a,b)} = T_{(c,d)}$.

Theorem 4.2. *Let $(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then $T_{(m,n)}$ is an ideal in the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ such that*

(i) $T_{(0,m)} = T_{(-1,n)} = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ for all $m,n \in \mathbb{Z}_0^-$,

(ii) *If* $(a,b) \leq (m,n)$, *then* $T_{(a,b)} \subseteq T_{(m,n)}$,

(iii) *If* $(p,q) =$max$\{(a,b),(m,n)\}$, *then* $T_{(a,b)} \cup T_{(m,n)} = T_{(p,q)}$,

(iv) *If* $(p,q) =$min$\{(a,b),(m,n)\}$, *then* $T_{(a,b)} \cap T_{(m,n)} = T_{(p,q)}$,

(v) $\bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\} = \{(0,t) \mid t \in \mathbb{Z}_0^-\}$.

Proof. We first prove that $T_{(m,n)}$ is an ideal of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Let $(a_1,a_2),(b_1,b_2) \in T_{(m,n)}$. Then we have that

$$(a_1,a_2) \leq (m,0) \quad \text{or} \quad (a_1,a_2) = (0,t) \text{ for some } t \in \mathbb{Z}_0^-$$

and

$$(b_1,b_2) \leq (m,0) \quad \text{or} \quad (b_1,b_2) = (0,t') \text{ for some } t \in \mathbb{Z}_0^-.$$

We divide the proof of an additive subsemigroup into four cases.

Case 1: If $(a_1, a_2) \leq (m, 0)$ and $(b_1, b_2) \leq (m, 0)$, then

$$(a_1, a_2) + (b_1, b_2) \leq (m, 0) + (m, 0) = (2m, 0) \leq (m, 0).$$

Therefore $(a_1, a_2) + (b_1, b_2) \in T_{(m,n)}$.

Case 2: If $(a_1, a_2) \leq (m, 0)$ and $(b_1, b_2) = (0, t')$ for some $t' \in \mathbb{Z}_0^-$, then

$$(a_1, a_2) + (b_1, b_2) \leq (m, 0) + (0, t') = (m, t') \leq (m, 0).$$

Hence $(a_1, a_2) + (b_1, b_2) \in T_{(m,n)}$.

Case 3: If $(a_1, a_2) = (0, t)$ for some $t \in \mathbb{Z}_0^-$ and $(b_1, b_2) \leq (m, 0)$, then

$$(a_1, a_2) + (b_1, b_2) \leq (0, t) + (m, 0) = (m, t) \leq (m, 0).$$

Thus $(a_1, a_2) + (b_1, b_2) \in T_{(m,n)}$.

Case 4: If $(a_1, a_2) = (0, t)$ for some $t \in \mathbb{Z}_0^-$ and $(b_1, b_2) = (0, t')$ for some $t' \in \mathbb{Z}_0^-$, then

$$(a_1, a_2) + (b_1, b_2) = (0, t) + (0, t') = (0, t + t').$$

Since $t, t' \in \mathbb{Z}_0^-$, so $t + t' \in \mathbb{Z}_0^-$. Therefore $(a_1, a_2) + (b_1, b_2) \in T_{(m,n)}$.

By any cases, we obtain that $T_{(m,n)}$ is an additive subsemigroup of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

Next, let $(r_1, r_2), (s_1, s_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. We want to show that $(r_1, r_2)(s_1, s_2)(a_1, a_2) \in T_{(m,n)}$. We divide its proof into three cases.

Case 1: If $r_1 = 0$ or $s_1 = 0$, then

$$(r_1, r_2)(s_1, s_2)(a_1, a_2) = (r_1 s_1 a_1, r_2 s_2 a_2) = (0, r_2 s_2 a_2).$$

Since $r_2, s_2, a_2 \in \mathbb{Z}_0^-$, $r_2 s_2 a_2 \in \mathbb{Z}_0^-$. By definition of $T_{(m,n)}$, we have that $(r_1, r_2)(s_1, s_2)(a_1, a_2) \in T_{(m,n)}$.

Case 2: If $r_2 = 0$ or $s_2 = 0$, then

$$(r_1, r_2)(s_1, s_2)(a_1, a_2) = (r_1 s_1 a_1, r_2 s_2 a_2) = (r_1 s_1 a_1, 0).$$

If $r_1 s_1 a_1 = 0$, then $(r_1 s_1 a_1, 0) = (0, 0)$. If $r_1 s_1 a_1 \neq 0$, then

$$(r_1 s_1 a_1, 0) \leq (a_1, 0) \leq (m, 0).$$

Thus $(r_1, r_2)(s_1, s_2)(a_1, a_2) \in T_{(m,n)}$.
Case 3: $r_1, r_2, s_1, s_2 \neq 0$. If $(a_1, a_2) \leq (m, 0)$, then

$$(r_1, r_2)(s_1, s_2)(a_1, a_2) \leq (r_1, r_2)(s_1, s_2)(m, 0) = (r_1 s_1 m, 0) \leq (m, 0).$$

If $(a_1, a_2) = (0, t)$ for some $t \in \mathbb{Z}_0^-$, then

$$(r_1, r_2)(s_1, s_2)(a_1, a_2) = (r_1, r_2)(s_1, s_2)(0, t) = (0, r_2 s_2 t).$$

Since $r_2, s_2, t \in \mathbb{Z}_0^-$, $r_2 s_2 t \in \mathbb{Z}_0^-$. Therefore $(r_1, r_2)(s_1, s_2)(a_1, a_2) \in T_{(m,n)}$.
From Case 1, Case 2 and Case 3, we obtain $(r_1, r_2)(s_1, s_2)(a_1, a_2) \in T_{(m,n)}$ for all $(r_1, r_2), (s_1, s_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $(a_1, a_2) \in T_{(m,n)}$. Similarly, $(a_1, a_2)(r_1, r_2)(s_1, s_2) \in T_{(m,n)}$ and $(r_1, r_2)(a_1, a_2)(s_1, s_2) \in T_{(m,n)}$.

Therefore we can conclude that $T_{(m,n)}$ is an ideal in $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

(i) We want to show that $T_{(0,m)} = T_{(-1,n)} = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ for all $m, n \in \mathbb{Z}_0^-$.
Let $m, n \in \mathbb{Z}_0^-$. We have that

$$\begin{aligned} T_{(0,m)} &= \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) \leq (0, 0)\} \cup \{(0, t) \mid t \in \mathbb{Z}_0^-\} \\ &= \mathbb{Z}_0^- \times \mathbb{Z}_0^-. \\ T_{(-1,n)} &= \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) \leq (-1, 0)\} \cup \{(0, t) \mid t \in \mathbb{Z}_0^-\} \\ &= \mathbb{Z}_0^- \times \mathbb{Z}_0^-. \end{aligned}$$

Therefore $T_{(0,m)} = T_{(-1,n)} = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ for all $m, n \in \mathbb{Z}_0^-$, as desired.

(ii) Let $(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ be such that $(a,b) \leq (m,n)$. Suppose that $(t_1,t_2) \in T_{(a,b)}$, then

$$(t_1,t_2) \leq (a,0) \quad \text{or} \quad (t_1,t_2) = (0,t) \text{ for some } t \in \mathbb{Z}_0^-.$$

Case 1: $(t_1,t_2) \leq (a,0)$. Since $(a,b) \leq (m,n)$, $a < m$ or ($a = m$ and $b \leq n$).
If $a < m$, then

$$(t_1,t_2) \leq (a,0) < (m,0).$$

If $a = m$ and $b \leq n$, then

$$(t_1,t_2) \leq (a,0) = (m,0).$$

Therefore $(t_1,t_2) \in T_{(m,n)}$.
Case 2: $(t_1,t_2) = (0,t)$ for some $t \in \mathbb{Z}_0^-$. By definition of $T_{(m,n)}$, it is clear that $(t_1,t_2) \in T_{(m,n)}$.

From Case 1 and Case 2, we get that $T_{(a,b)} \subseteq T_{(m,n)}$.

(iii) Let $(p,q),(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ be such that $(p,q) =$max$\{(a,b),(m,n)\}$. If $(a,b) \geq (m,n)$, then $(p,q) = (a,b)$. Since $(a,b) \geq (m,n)$, by (ii) we get that $T_{(m,n)} \subseteq T_{(a,b)}$. Therefore

$$T_{(a,b)} \cup T_{(m,n)} = T_{(a,b)} = T_{(p,q)}.$$

If $(a,b) \leq (m,n)$, then $(p,q) = (m,n)$. Since $(a,b) \leq (m,n)$, by (ii) we have $T_{(a,b)} \subseteq T_{(m,n)}$. Hence

$$T_{(a,b)} \cup T_{(m,n)} = T_{(m,n)} = T_{(p,q)}.$$

Consequently $T_{(a,b)} \cup T_{(m,n)} = T_{(p,q)}$ in the event of $(p,q) =$max$\{(a,b),(m,n)\}$.

(iv) Let $(p,q),(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ be such that $(p,q) =$min$\{(a,b),(m,n)\}$. If $(a,b) \geq (m,n)$, then $(p,q) = (m,n)$. Since $(a,b) \geq (m,n)$, by (ii) we have

$T_{(m,n)} \subseteq T_{(a,b)}$. Therefore

$$T_{(a,b)} \cap T_{(m,n)} = T_{(m,n)} = T_{(p,q)}.$$

If $(a,b) \leq (m,n)$, then $(p,q) = (a,b)$. Since $(a,b) \leq (m,n)$, by (ii) we obtain that $T_{(a,b)} \subseteq T_{(m,n)}$. Hence

$$T_{(a,b)} \cap T_{(m,n)} = T_{(a,b)} = T_{(p,q)}.$$

Consequently $T_{(a,b)} \cap T_{(m,n)} = T_{(p,q)}$ in the event of (p,q) =min$\{(a,b),(m,n)\}$.

(v) First, we want to show that $\bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\} \subseteq \{(0,t) \mid t \in \mathbb{Z}_0^-\}$. Suppose that there exists $(x,y) \in \bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\} \setminus \{(0,t) \mid t \in \mathbb{Z}_0^-\}$. We have

$$T_{(x-1,y)} = \{(t_1,t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1,t_2) \leq (x-1,0)\} \cup \{(0,t) \mid t \in \mathbb{Z}_0^-\}.$$

Since $x > x-1$, it is clear that $(x,y) > (x-1,0)$. Now, we have

$$(x,y) > (x-1,0) \quad \text{and} \quad (x,y) \notin \{(0,t) \mid t \in \mathbb{Z}_0^-\}.$$

Hence $(x,y) \notin T_{(x-1,y)}$, it is a contradiction with $(x,y) \in \bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\}$. Therefore $\bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\} \subseteq \{(0,t) \mid t \in \mathbb{Z}_0^-\}$, as desired.

Conversely, by definition of $T_{(m,n)}$, it is clearly that

$$\{(0,t) \mid t \in \mathbb{Z}_0^-\} \subseteq \bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\}.$$

Consequently, $\bigcap\{T_{(i,j)} \mid (i,j) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-\} = \{(0,t) \mid t \in \mathbb{Z}_0^-\}$ as desired. $\square$

Remark. *If $(m,n) \neq (0,t),(-1,t')$ for all $t,t' \in \mathbb{Z}_0^-$ then $T_{(m,n)}$ is not a k-ideal of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.*

For $(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $(a,b) \leq (m,n)$, the notation $S((a,b),(m,n))$ will be used to denote the set $\{(t_1,t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (a,b) \leq (t_1,t_2) \leq (m,n)\}$.

Note. For $(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $(a,b) \leq (m,n)$, we have that $S((a,b),(m,n)) \subseteq T_{(m,n)}$.

Theorem 4.3. *Let $(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $(m,n) \leq (-2,0)$, then $S((2m,2n),(m,n))$ is a basis for $T_{(m,n)}$.*

Proof. Let $(t_1,t_2) \in T_{(m,n)}$. If $(t_1,t_2) \in S((2m,2n),(m,n))$ or $(t_1,t_2) = (c_1,c_2)(d_1,d_2)(m,n)$ for some $(c_1,c_2),(d_1,d_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$, it is easy to prove that the mentioned case is a basis for $T_{(m,n)}$. Suppose that $(t_1,t_2) \notin S((2m,2n),(m,n))$ and $(t_1,t_2) \neq (c_1,c_2)(d_1,d_2)(m,n)$ for any $(c_1,c_2),(d_1,d_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. If $(m,n) < (t_1,t_2) \leq (m,0)$, then $(t_1,t_2) = (m,x)$ for some $x \in \mathbb{Z}_0^-$. Hence, we can write

$$(t_1,t_2) = (m,x) = (-1,0)(-1,0)(m,n) + (0,x)(0,-1)(m-1,-1).$$

If $(t_1,t_2) = (0,t)$ for some $t \in \mathbb{Z}_0^-$, then we can write

$$(t_1,t_2) = (0,t) = (0,t)(0,-1)(m-1,-1).$$

Consider $(m-1,-1)$, Since $(m,n) \leq (-2,0)$, $m \leq -2$. Therefore

$$2m < m-1 < m,$$

so we get $(2m,2n) < (m-1,-1) < (m,n)$. Hence $(m-1,-1) \in S((2m,2n),(m,n))$. Next, suppose that $(t_1,t_2) < (2m,2n)$. Let $k = \min\{\, l \in \mathbb{Z}_0^- \mid -l(m,n) > (t_1,t_2)\}$. Then we have that

$$-(k-1)(m,n) < (t_1,t_2) < -(k)(m,n).$$

However, this guarantees the existence of an $(x,y) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$, $(x,y) > (m,n)$ such that

$$-(k)(m,n) + (x,y) = (t_1,t_2).$$

Since $(x, y) > (m, n)$,

$$(m, n) + (x, y) > (m, n) + (m, n).$$

Hence

$$(2m, 2n) = 2(m, n) = (m, n) + (m, n) < (m, n) + (x, y) < (m, n).$$

Therefore, we get $(m, n) + (x, y) \in S((2m, 2n), (m, n))$. Consider,

$$\begin{aligned}(t_1, t_2) &= -(k)(m, n) + (x, y) \\ &= -(k + 1)(m, n) + (m, n) + (x, y).\end{aligned}$$

Since $(m, n) \in S((2m, 2n), (m, n))$ and $(m, n) + (x + y) \in S((2m, 2n), (m, n))$, it follows that $S((2m, 2n), (m, n))$ is a basis for $T_{(m,n)}$. $\square$

Let $(m, n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $T^*_{(m,n)} = \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) = ((-1)k_1 m, (-1)k_2 n),$ $k_1, k_2 \in \mathbb{Z}_0^-\}$. Then we have the following results concerning $T^*_{(m,n)}$.

Theorem 4.4. *Let $(m, n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then $T^*_{(m,n)}$ is an ideal of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ such that*

(i) $T^*_{(0,0)} = \{(0, 0)\}$,

(ii) $T^*_{(-1,-1)} = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

Proof. First, we want to prove that $T^*_{(m,n)}$ is an ideal of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.
Let $(a_1, a_2), (b_1, b_2) \in T^*_{(m,n)}$. Then $(a_1, a_2) = ((-1)k_1 m, (-1)k_2 n)$ for some $k_1, k_2 \in \mathbb{Z}_0^-$ and $(b_1, b_2) = ((-1)k_3 m, (-1)k_4 n)$ for some $k_3, k_4 \in \mathbb{Z}_0^-$. Hence

$$\begin{aligned}(a_1, a_2) + (b_1, b_2) &= ((-1)k_1 m, (-1)k_2 n) + ((-1)k_3 m, (-1)k_4 n) \\ &= ((-1)k_1 m + (-1)k_3 m, (-1)k_2 n + (-1)k_4 n) \\ &= ((-1)(k_1 + k_3)m, (-1)(k_2 + k_4)n).\end{aligned}$$

Since $k_1 + k_3, k_2 + k_4 \in \mathbb{Z}_0^-$, $(a_1, a_2) + (b_1, b_2) \in T^*_{(m,n)}$.

Therefore $T^*_{(m,n)}$ is an additive subsemigroup of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

Next, let $(r_1, r_2), (s_1, s_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Consider

$$\begin{aligned}(r_1, r_2)(s_1, s_2)(a_1, a_2) &= (r_1 s_1, r_2 s_2)(a_1, a_2)\\ &= (r_1 s_1, r_2 s_2)((-1)k_1 m, (-1)k_2 n)\\ &= (r_1 s_1 (-1)k_1 m, r_2 s_2 (-1) k_2 n)\\ &= ((-1)(r_1 s_1 k_1)m, (-1)(r_2 s_2 k_2)n).\end{aligned}$$

Since $r_1 s_1 k_1, r_2 s_2 k_2 \in \mathbb{Z}_0^-$, $(r_1, r_2)(s_1, s_2)(a_1, a_2) \in T^*_{(m,n)}$. Similarly, we have that

$$(r_1, r_2)(a_1, a_2)(s_1, s_2) \in T^*_{(m,n)} \quad \text{and} \quad (a_1, a_2)(r_1, r_2)(s_1, s_2) \in T^*_{(m,n)}.$$

Consequently, $T^*_{(m,n)}$ is an ideal of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

(i) $T^*_{(0,0)} = \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) = ((-1)k_1(0), (-1)k_2(0)), k_1, k_2 \in \mathbb{Z}_0^-\} = \{(0,0)\}$.

(ii) $T^*_{(-1,-1)} = \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) = ((-1)k_1(-1), (-1)k_2(-1)), k_1, k_2 \in \mathbb{Z}_0^-\} = \{(t_1, t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (t_1, t_2) = (k_1, k_2), k_1, k_2 \in \mathbb{Z}_0^-\} = \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. □

Theorem 4.5. *Let $a, b, m, n \in \mathbb{Z}_0^-$ where $m, n \neq 0$. Then $m|a$ and $n|b$ if and only if $T^*_{(a,b)} \subseteq T^*_{(m,n)}$.*

Proof. Suppose that $m|a$ and $n|b$. Then $a = (-1)xm$ and $b = (-1)yn$ for some $x, y \in \mathbb{Z}_0^-$. We want to show that $T^*_{(a,b)} \subseteq T^*_{(m,n)}$. Let $(t_1, t_2) \in T^*_{(a,b)}$. Then $(t_1, t_2) = ((-1)k_1 a, (-1)k_2 b)$ for some $k_1, k_2 \in \mathbb{Z}_0^-$. Consider

$$\begin{aligned}(t_1, t_2) &= ((-1)k_1 a, (-1)k_2 b)\\ &= ((-1)k_1(-1)xm, (-1)k_2(-1)yn)\\ &= ((-1)(-k_1 x)m, (-1)(-k_2 y)n).\end{aligned}$$

Since $-k_1 x, -k_2 y \in \mathbb{Z}_0^-$, $(t_1, t_2) \in T^*_{(m,n)}$. Therefore $T^*_{(a,b)} \subseteq T^*_{(m,n)}$.

Conversely, suppose that $T^*_{(a,b)} \subseteq T^*_{(m,n)}$. Since $(a, b) \in T^*_{(a,b)} \subseteq T^*_{(m,n)}$, $(a, b) =$

$((-1)k_1m, (-1)k_2n)$ for some $k_1, k_2 \in \mathbb{Z}_0^-$. Therefore $a = (-1)k_1m$ and $b = (-1)k_2n$. Since $m \neq 0$, $m|a$. Since $n \neq 0$, $n|b$. $\square$

From Chapter II, a k-ideal is defined that an ideal I of a ternary semiring S is called a k-ideal if $x + y \in I; x \in S, y \in I$ imply that $x \in I$.

Theorem 4.6. *Let $(m, n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then $T^*_{(m,n)}$ is a k-ideal of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.*

Proof. Let $(a, b) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $(t_1, t_2) \in T^*_{(m,n)}$.Then $(t_1, t_2) = ((-1)k_1m, (-1)k_2n)$ for some $k_1, k_2 \in \mathbb{Z}_0^-$. Suppose that $(a, b) + (t_1, t_2) \in T^*_{(m,n)}$ and $(a, b) \notin T^*_{(m,n)}$. Since $(a, b) \notin T^*_{(m,n)}$, $a \neq (-1)k_3m$ for all $k_3 \in \mathbb{Z}_0^-$ or $b \neq (-1)k_4m$ for all $k_4 \in \mathbb{Z}_0^-$. Without loss of generality , we assume $a \neq (-1)k_3m$ for all $k_3 \in \mathbb{Z}_0^-$. Since $(a, b) + (t_1, t_2) \in T^*_{(m,n)}$, $(a, b) + (t_1, t_2) = ((-1)k_5m, (-1)k_6n)$ for some $k_5, k_6 \in \mathbb{Z}_0^-$. Then we have

$$\begin{aligned}
(a, b) + (t_1, t_2) &= ((-1)k_5m, (-1)k_6n) \\
(a, b) + ((-1)k_1m, (-1)k_2n) &= ((-1)k_5m, (-1)k_6n) \\
(a + (-1)k_1m, b + (-1)k_2n) &= ((-1)k_5m, (-1)k_6n).
\end{aligned}$$

Thus we get $a + (-1)k_1m = (-1)k_5m$ and $b + (-1)k_2n = (-1)k_6n$. If $m = 0$, then $(-1)k_1m = 0 = (-1)k_5m$. Thus we have $a = 0$, so we can write $a = (-1)(0)m$. This is a contradiction with $a \neq (-1)k_3m$ for all $k_3 \in \mathbb{Z}_0^-$. Next, suppose that $m \neq 0$. Since $m|(-1)k_5m$, $m|(-1)k_1m$ and $a + (-1)k_1m = (-1)k_5m$, $m|a$. Since $a, m \in \mathbb{Z}_0^-$ and $m|a$, $a = (-1)lm$ for some $l \in \mathbb{Z}_0^-$. This is a contradiction with $a \neq (-1)k_3m$ for all $k_3 \in \mathbb{Z}_0^-$. Therefore we can conclude that $T^*_{(m,n)}$ is a k-ideal. $\square$

Note. For $(m, n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Now, we have that $T^*_{(m,n)}$ is a k-ideal but $T_{(m,n)}$ is not a k-ideal.

Theorem 4.7. *$\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ satisfies the ascending chain condition on $T^*_{(m,n)}-$ideals.*

Proof. Let $\{T^*_{(m_i,n_i)} \mid (m_i,n_i) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $i \in \mathbb{N}\}$ be an ascending chain of $T^*_{(m,n)}-$ideals of $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Thus

$$T^*_{(m_1,n_1)} \subseteq T^*_{(m_2,n_2)} \subseteq T^*_{(m_3,n_3)} \subseteq \cdots$$

By Theorem 4.5, we have that

$$(m_1,n_1) \leq (m_2,n_2) \leq (m_3,n_3) \leq \cdots .$$

Since $(m_i,n_i) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ for all $i \in \mathbb{N}$, the increasing sequence $\{(m_i,n_i)\}$ of ordered pairs of negative integers is finite. Thus there exists $j \in \mathbb{N}$ be such that $T^*_{(m_i,n_i)} = T^*_{(m_j,n_j)}$ for each $i \geq j$. Therefore $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ satisfies the ascending chain condition on $T^*_{(m,n)}$-ideals. $\square$

For $(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ and $(a,b) \leq (m,n)$, the notation $S((a,b),(m,n))$ will be used to denote the set $\{(t_1,t_2) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^- \mid (a,b) \leq (t_1,t_2) \leq (m,n)\}$.

Theorem 4.8. *Let $(a,b),(m,n) \in \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ be such that $(a,b) \leq (m,n)$. Then $S((a,b),(m,n))$ is a basis for $T^*_{(a,b)}$ and $T^*_{(m,n)}$.*

Proof. Let $(t_1,t_2) \in T^*_{(a,b)}$. Then $(t_1,t_2) = ((-1)k_1a,(-1)k_2b)$ for some $k_1,k_2 \in \mathbb{Z}_0^-$. Therefore we can write

$$(t_1,t_2) = (-1,-1)(k_1,k_2)(a,b).$$

Since $(a,b) \in S((a,b),(m,n))$, $S((a,b),(m,n))$ is a basis for $T^*_{(a,b)}$.

Similarly, we have that $S((a,b),(m,n))$ is a basis for $T^*_{(m,n)}$. $\square$

Theorem 4.9. *Let $a_1,a_2,b_1,b_2 \in \mathbb{Z}_0^-$ where $a_1,a_2,b_1,b_2 \neq 0$. If $-d_1$ is the greatest common divisor of a_1 and b_1, $-d_2$ is the greatest common divisor of a_2 and b_2, then there exists $s_1,s_2,t_1,t_2 \in \mathbb{Z}_0^-$ such that*

$(-1,-1)(s_1,s_2)(a_1,a_2) = (-1,-1)(t_1,t_2)(b_1,b_2) + (d_1,d_2)$ *or*

$(-1,-1)(t_1,t_2)(b_1,b_2) = (-1,-1)(s_1,s_2)(a_1,a_2) + (d_1,d_2)$ *or*

$(-1,-1)(s_1,t_2)(a_1,b_2) = (-1,-1)(t_1,s_2)(b_1,a_2) + (d_1,d_2)$ *or*
$(-1,-1)(t_1,s_2)(b_1,a_2) = (-1,-1)(s_1,t_2)(a_1,b_2) + (d_1,d_2)$.

Proof. Since $-d_1$ is the greatest common divisor of a_1 and b_1, it is well known that $-d_1 = s_1'(-a_1) + t_1'(-b_1)$ for some integers s_1' and t_1'. Since $0 \leq -d_1 \leq -a_1, 0 \leq -d_1 \leq -b_1$ and $(-a_1),(-b_1)$ and $(-d_1)$ are all positive, it follows that $s_1' \geq 0$ and $t_1' \leq 0$ or $s_1' \leq 0$ and $t_1' \geq 0$. If $s_1' \geq 0$ and $t_1' \leq 0$, then

$$\begin{aligned} -d_1 &= s_1'(-a_1) + t_1'(-b_1) \\ (-1)s_1'(-a_1) &= (-1)t_1'b_1 + d_1 \\ (-1)(-s_1')a_1 &= (-1)t_1'b_1 + d_1. \end{aligned}$$

Thus $(-1)s_1a_1 = (-1)t_1b_1 + d_1$ where $s_1 = -s_1' \leq 0$ and $t_1 = t_1' \leq 0$.
On the other hand, if $s_1' \leq 0$ and $t_1' \geq 0$, then

$$\begin{aligned} -d_1 &= s_1'(-a_1) + t_1'(-b_1) \\ (-1)t_1'(-b_1) &= (-1)s_1'a_1 + d_1 \\ (-1)(-t_1')b_1 &= (-1)s_1'a_1 + d_1. \end{aligned}$$

Thus $(-1)t_1b_1 = (-1)s_1a_1 + d_1$ where $s_1 = s_1' \leq 0$ and $t_1 = -t_1' \leq 0$.
Hence we have that $(-1)s_1a_1 = (-1)t_1b_1 + d_1$ or $(-1)t_1b_1 = (-1)s_1a_1 + d_1$.

Since $-d_2$ is the greatest common divisor of a_2 and b_2, prove in the same way, we get that $(-1)s_2a_2 = (-1)t_2b_2 + d_2$ or $(-1)t_2b_2 = (-1)s_2a_2 + d_2$.
Therefore, now we have

$$(-1)s_1a_1 = (-1)t_1b_1 + d_1 \quad \text{or} \quad (-1)t_1b_1 = (-1)s_1a_1 + d_1$$

and

$$(-1)s_2a_2 = (-1)t_2b_2 + d_2 \quad \text{or} \quad (-1)t_2b_2 = (-1)s_2a_2 + d_2.$$

Hence

$$(-1)s_1a_1 = (-1)t_1b_1 + d_1 \quad \text{and} \quad (-1)s_2a_2 = (-1)t_2b_2 + d_2 \tag{1}$$

or

$$(-1)t_1b_1 = (-1)s_1a_1 + d_1 \quad \text{and} \quad (-1)t_2b_2 = (-1)s_2a_2 + d_2 \tag{2}$$

or

$$(-1)s_1a_1 = (-1)t_1b_1 + d_1 \quad \text{and} \quad (-1)t_2b_2 = (-1)s_2a_2 + d_2 \tag{3}$$

or

$$(-1)t_1b_1 = (-1)s_1a_1 + d_1 \quad \text{and} \quad (-1)s_2a_2 = (-1)t_2b_2 + d_2. \tag{4}$$

From (1), we have $((-1)s_1a_1, (-1)s_2a_2) = ((-1)t_1b_1 + d_1, (-1)t_2b_2 + d_2)$. Then

$$(-1,-1)(s_1,s_2)(a_1,a_2) = (-1,-1)(t_1,t_2)(b_1,b_2) + (d_1,d_2).$$

From (2), we have $((-1)t_1b_1, (-1)t_2b_2) = ((-1)s_1a_1 + d_1, (-1)s_2a_2 + d_2)$. Thus

$$(-1,-1)(t_1,t_2)(b_1,b_2) = (-1,-1)(s_1,s_2)(a_1,a_2) + (d_1,d_2).$$

From (3), we have $((-1)s_1a_1, (-1)t_2b_2) = ((-1)t_1b_1 + d_1, (-1)s_2a_2 + d_2)$. So

$$(-1,-1)(s_1,t_2)(a_1,b_2) = (-1,-1)(t_1,s_2)(b_1,a_2) + (d_1,d_2).$$

From (4), we have $(-1,-1)(t_1,s_2)(b_1,a_2) = (-1,-1)(s_1,t_2)(a_1,b_2)+(d_1,d_2)$. Hence

$$(-1,-1)(t_1,s_2)(b_1,a_2) = (-1,-1)(s_1,t_2)(a_1,b_2) + (d_1,d_2).$$

Therefore the proof of the theorem follows. $\square$

From Chapter II, the identity of the ternary semiring S is defined that if there exists an element $e \in S$ such that $eea = aee = eae = a$ for all $a \in S$, then e is called the identity of the ternary semiring S.

Remarks. (i) 1 is an identity of the ternary semiring $\mathbb{Z}_0^+$.

(ii) -1 is an identity of the ternary semiring $\mathbb{Z}_0^-$.

(iii) 1 and -1 are identities of the ternary semiring $\mathbb{Z}$.

(iv) $(1,1)$ is an identity of the ternary semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

(v) $(-1,-1)$ is an identity of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$.

(vi) $(1,1), (1,-1), (-1,1)$ and $(-1,-1)$ are identities of the ternary semiring $\mathbb{Z} \times \mathbb{Z}$.

(vii) The ternary semiring $\mathbb{Z}^n$ has 2^n identities.

Note. Let $n \in \mathbb{N}$. If we define the additive and the ternary multiplicative operator on $\mathbb{Z}^n$ or $(\underbrace{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots}_{\text{n copies}})$, as follows

$$(a_1, a_2, a_3, \cdots, a_n) + (b_1, b_2, b_3, \cdots, b_n) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots, a_n + b_n)$$

and

$$(a_1, a_2, a_3, \cdots, a_n)(b_1, b_2, b_3, \cdots, b_n)(c_1, c_2, c_3, \cdots, c_n) = (a_1b_1c_1, a_2b_2c_2, a_3b_3c_3, \cdots, a_nb_nc_n),$$

then it is easy to show that $\mathbb{Z}^n$ is a ternary semiring.

From Chapter II, a homomorphism from a ternary semiring S to a ternary semiring T is defined that a homomorphism from S to T is a map $\varphi : S \to T$ which satisfies

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$$

for all $a, b, c \in S$.

A monomorphism is a homomorphism which is one-to-one.

An epimorphism is a homomorphism which is onto.

An isomorphism is a homomorphism which is one-to-one and onto.

We said that a ternary semiring S and a ternary semiring T are *isomorphic*, if there exists an isomorphism $\phi : S \to T$ or there exists an isomorphism $\psi : T \to S$. In this case, we will use the notation $S \cong T$.

Remark. *If a map $\varphi : S \to T$ is an isomorphism, e_S is an identity of the ternary semiring S, then $\varphi(e_s)$ is an identity of the ternary semiring T.*

Theorem 4.10. *The ternary semirings $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ are isomorphic.*

Proof. Define a map $\varphi : \mathbb{Z}_0^+ \to \mathbb{Z}_0^-$ by $\varphi(n) = -n$ for all $n \in \mathbb{Z}_0^+$. Let $a, b, c \in \mathbb{Z}_0^+$, consider

$$\varphi(a+b) = -(a+b) = (-a) + (-b) = \varphi(a) + \varphi(b) \quad \text{and}$$
$$\varphi(abc) = -(abc) = (-a)(-b)(-c) = \varphi(a)\varphi(b)\varphi(c).$$

Hence φ is a homomorphism. By the definition of the map φ, it is easy to see that φ is one-to-one onto mapping. Consequently, we can conclude that $\mathbb{Z}_0^+ \cong \mathbb{Z}_0^-$. $\square$

Theorem 4.11. *The ternary semirings $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ are isomorphic.*

Proof. Define a map $\varphi : \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \to \mathbb{Z}_0^- \times \mathbb{Z}_0^-$ by

$$\varphi(x, y) = (-x, -y) \quad \text{for all } x, y \in \mathbb{Z}_0^+.$$

To show that φ is a homomorphism, let $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Consider,

$$\begin{aligned}
\varphi((x_1, y_1) + (x_2, y_2)) &= \varphi(x_1 + x_2, y_1 + y_2) \\
&= (-(x_1 + x_2), -(y_1 + y_2)) \\
&= ((-x_1) + (-x_2), (-y_1) + (-y_2)) \\
&= (-x_1, -y_1) + (-x_2, -y_2) \\
&= \varphi(x_1, y_1) + \varphi(x_2, y_2).
\end{aligned}$$

Then φ is a homomorphism. By the definition of the map φ, it is easy to show that φ is one-to-one and onto mapping. Consequently, we can conclude that $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \cong \mathbb{Z}_0^- \times \mathbb{Z}_0^-$. $\square$

Theorem 4.12. *The ternary semirings $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ are not isomorphic.*

Proof. Suppose that $\mathbb{Z}_0^+ \cong \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Then there exists a map $\varphi : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ be such that φ is an one-to-one and onto homomorphism. Since φ is one-to-one and onto, there exist $x, y \in \mathbb{Z}_0^+$ with $x \neq y$ such that

$$\varphi(x) = (0, 1) \quad \text{and} \quad \varphi(y) = (1, 0)$$

Since φ is an isomorphism and we know that 1 is the only identity of the ternary semiring $\mathbb{Z}_0^+$ and $(1, 1)$ is the only identity of the ternary semiring $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$, we have that $\varphi(1) = (1, 1)$.

Let $n \in \mathbb{Z}_0^+$. Then

$$\begin{aligned}
\varphi(n) &= \varphi((n)(1)) \\
&= \varphi(\underbrace{1+1+1+\cdots+1}_{\text{n copies}}) \\
&= \underbrace{\varphi(1)+\varphi(1)+\varphi(1)+\cdots+\varphi(1)}_{\text{n copies}} \\
&= n\varphi(1) \\
&= n(1,1) \\
&= (n,n)
\end{aligned}$$

Hence $\varphi(n) = (n,n)$ for all $n \in \mathbb{Z}_0^+$. There is no $x, y \in \mathbb{Z}_0^+$ satisfying $\varphi(x) = (0,1)$ and $\varphi(y) = (1,0)$. Therefore $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ are not isomorphic. $\square$

Theorem 4.13. *The ternary semirings $\mathbb{Z}_0^-$ and $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ are not isomorphic.*

Proof. This proof is similar to the proof of the Theorem 4.12. $\square$

Note. If we prove in the same way, we get that $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ are not isomorphic, as well as $\mathbb{Z}_0^-$ and $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ are not isomorphic.

A well-known result states that if $\varphi : \mathbb{Z} \to \mathbb{Z}$ is a ring isomorphism, then $\varphi(x) = x$ for all $x \in \mathbb{Z}$. In the case of a ternary semiring isomorphism, we obtain an additional solution of φ, which is $\varphi(x) = -x$.

Theorem 4.14. *If a map $\phi : \mathbb{Z} \to \mathbb{Z}$ is a ternary semiring isomorphism, then $\phi(x) = x$ or $\phi(x) = -x$ for all $x \in \mathbb{Z}$.*

Proof. Suppose that a map $\phi : \mathbb{Z} \to \mathbb{Z}$ is a ternary semiring isomorphism. Since 1 and -1 are identities of the ternary semiring $\mathbb{Z}$, thus $\phi(1)$ and $\phi(-1)$ are identities of the ternary semiring $\mathbb{Z}$. Since 1 and -1 are the identities of the ternary semiring $\mathbb{Z}$ and $\phi(1)$ is the identity of the ternary semiring $\mathbb{Z}$, $\phi(1) = 1$ or $\phi(1) = -1$.

Case 1: $\phi(1) = 1$, let $n \in \mathbb{Z}_0^+$

$$\begin{aligned}
\phi(n) &= \phi((n)(1)) \\
&= \phi(\underbrace{1+1+1+\cdots+1}_{\text{n copies}}) \\
&= \underbrace{\phi(1)+\phi(1)+\phi(1)+\cdots+\phi(1)}_{\text{n copies}} \\
&= n\phi(1) \\
&= n(1) \\
&= n.
\end{aligned}$$

Thus $\phi(n) = n$ for all $n \in \mathbb{Z}_0^+$.

Since ϕ is an isomorphism and $\phi(1) = 1$, $\phi(-1) = -1$.

Let $m \in \mathbb{Z}^-$. Then

$$\begin{aligned}
\phi(m) &= \phi((-m)(-1)) \\
&= \phi(\underbrace{(-1)+(-1)+(-1)+\cdots+(-1)}_{\text{-m copies}}) \\
&= \underbrace{\phi(-1)+\phi(-1)+\phi(-1)+\cdots+\phi(-1)}_{\text{-m copies}} \\
&= (-m)\phi(-1) \\
&= (-m)(-1) \\
&= m.
\end{aligned}$$

Then $\phi(m) = m$ for all $m \in \mathbb{Z}^-$.

Therefore, we obtain $\phi(x) = x$ for all $x \in \mathbb{Z}$.

Case 2: $\phi(1) = -1$, let $n \in \mathbb{Z}_0^+$. Then

$$\begin{aligned}
\phi(n) &= \phi((n)(1)) \\
&= \phi(\underbrace{1+1+1+\cdots+1}_{\text{n copies}}) \\
&= \underbrace{\phi(1)+\phi(1)+\phi(1)+\cdots+\phi(1)}_{\text{n copies}} \\
&= n\phi(1) \\
&= n(-1) \\
&= -n.
\end{aligned}$$

That is $\phi(n) = -n$ for all $n \in \mathbb{Z}_0^+$.

Since ϕ is an isomorphism and $\phi(1) = 1$, $\phi(-1) = 1$.

Let $m \in \mathbb{Z}_0^-$. Then

$$\begin{aligned}
\phi(m) &= \phi((-m)(-1)) \\
&= \phi(\underbrace{(-1)+(-1)+(-1)+\cdots+(-1)}_{\text{-m copies}}) \\
&= \underbrace{\phi(-1)+\phi(-1)+\phi(-1)+\cdots+\phi(-1)}_{\text{-m copies}} \\
&= (-m)\phi(-1) \\
&= (-m)(1) \\
&= -m.
\end{aligned}$$

Hence $\phi(m) = -m$ for all $m \in \mathbb{Z}_0^-$.

Therefore,we have that $\phi(x) = -x$ for all $x \in \mathbb{Z}$.

By any cases, we can conclude that $\phi(x) = x$ or $\phi(x) = -x$ for all $x \in \mathbb{Z}$. □

Remark. *If S and T are ternary semirings with m and n identities, respectively, where $m \neq n$, then S and T are not isomorphic.*

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