Chapter 3

THE PROPOSED ITERATIVE TECHNIQUE

In this chapter is presented a new iterative technique for convergence improvement of the numerical solution for highly geometric nonlinear problems. The scheme proposes a subdivision procedure for the first trial displacement to get a good "trial" equilibrium state at the end of the first cycle of iteration. For completeness the iterative scheme of the Newton-Raphson method and some of its modified versions are also briefly outlined.

Newton-Raphson Method

Consider the load-displacement curve of a nonlinear function $P_o = f(u)$ for a single degree of freedom system in figure 3.1. Let P and u_o be the known values of the load and the resulting displacement at the initial equilibrium state, respectively. The load P_o is now considered to be increased to P, and it is required to determine the corresponding displacement u. One approximate method to solve this nonlinear problem is to expand the solution about some starting point, based on Taylor series expansion of function P = f(u) about u_o with the higher order terms truncated. Thus,

$$f(u_o + \Delta u) = f(u_o) + \left(\frac{df}{du}\right)_o (\Delta u) + \text{higher order terms} \qquad (3.1)$$

where $f(u_o + \Delta u)$ is P, $f(u_o)$ is the load P_o and $(df/du)_o$ is the slope of the load-displacement curve at point 0 which is the tangent stiffness K_o. Due to the linearization process, one seeks Δu_1 instead of Δu, and the incremental equation for solving Δu_1 is sought

$$K_o (\Delta u_1) = P - P_o \tag{3.2}$$

where $P - P_o$ can be interpreted as incremental load (i.e. the difference between the applied load and the resisting force of the structure).

The first aprroximation of the displacement can thus be obtained by adding the incremental displacement Δu_1 to the initial displacement u_o as

$$u_1 = u_o + \Delta u_1 \tag{3.3}$$

The next step is to evaluate the tangent stiffness K_1 and the resisting force P_1 corresponding to u_1. The next incremental displacement Δu_2 is then determined from solving

$$K_1 (\Delta u_2) = P - P_1 \tag{3.4}$$

The value of $P - P_1$ is called the unbalanced force. The iteration is repeated until the unbalanced force is negligibly small. The final displacement u is simply the summation of the initial and the incremental displacements, i.e.

$$u = u_o + \Delta u_1 + \Delta u_2 + \ldots\ldots \quad (3.5)$$

In summary, the recursion equations for the iterative process are as follows:

at the iteration step i

$$K_{i-1}\, \Delta u_i = P - P_{i-1} \quad (3.5)$$

$$u_i = u_{i-1} + \Delta u_i \quad (3.6)$$

Underrelaxation Method (1)

As shown in figure 3.2, it is appropriate to use the underrelaxation method to scale down the iterative incremental displacement for solving a hardening structure, whose tangent stiffness matrix increases with increasing displacement. The procedure of this method is similar to the Newton-Raphson method except that the total displacement u_i will be multiplied by the underrelaxation factor β ($\beta < 1$) before the next iteration step is computed. Therefore, the displacement u_i in every cycle is equal to $\beta\,(u_{i-1} + \Delta u_i)$.

Averaging Procedure (see eg.(2))

This procedure is similar to the underrelaxation method except that the underrelaxation factor, β, is selected as 0.5 as many times as necessary; otherwise, the factor of unity will be used. This method is used in order to avoid any biased judgement of the selection of the underrelaxation factor, β.

Krishna Modified Newton-Raphson Method (3)

In this method, the displacement u_i, for the i^{th} cycle of iteration is taken as $u_i = [\, u_i + u_{i+1} \,]/2$ in which u_{i+1} results from the same load causing u_i but applied to the configuration corresponding to u_i. The same modification is also applied to the incremental displacement Δu.

Kar Modified Newton-Raphson Method (4)

Similar to the previous methods, the appropriate value of the modifying factor, γ, is proposed by Kar. He suggests the factor, γ as the ratio of the lastest applied load at any cycle of iteration to its corresponding equilibrium load calculated on the basis of the linearized solution. This factor is less than unity for the first cycle, but it may be larger than one in another cycle. As shown in figure 3.3, the modifying factor in the first cycle is P/P_1, which is less than one. In the second cycle the modifying factor can be calculated from $[P_2 - P]/[P_2 - P_3]$, which is larger than one.

Since the displacement may be reduced by the modifying factor (when it is very small) to such a way that it is much smaller than the correct displacement, this factor should be limited to some reasonable value. Kar suggests the value of 0.25 as the lower limit (4) which is also adopt in this study.

The Proposed Iterative Technique

As depicted in figure 3.1, the displacement u_1 computed from the linearized equations of equilibrium is usually much larger than the correct one by several orders of magnitude. Consequently, the use of the displacement u_1 as the initial trial displacement often leads to slow convergence or even numerical difficulties. In the modified versions of the Newton-Raphson method, the displacement is scaled down in order to accerelate the convergence. Even so, the corresponding equilibrium load, P_1, may still be much overestimated. To overcome this problem, a new iterative technique is proposed in order to improve the solution convergence and stability of highly geometric nonlinear problems.

The proposed technique features a simple procedure to assess a good trial equilibrium state at the first cycle of iteration since a poor estimate of the state may lead to slow convergence or even numerical overflow in a highly nonlinear problem. In order to understand this technique clearly, the procedure will be described step by step as follows (see figure 3.4) :

Step 1 Based on the linearized equations of equilibrium the displacement u_1 is calculated from the applied load, P.

Step 2 The displacement u_1 is divided equally into n segments of Δu. This process will be hereafter referred to as subdivision of the 1^{st} trial displacement.

Step 3 The displacement $u_{1,i}$ is then computed based on the relation $u_{1,i} = u_{1,i-1} + \Delta u$ in which $u_{1,o}$ equals zero.

Step 4 The corresponding equilibrium load $P_{1,i}$ is computed. If it is less than the applied load, P, step 3 and 4 will be repeated.

Step 5 The estimated displacement for the first iterative cycle, u_1, is calculated by interpolation. Thus,

$$u'_1 = u_{1,i-1} + \Delta u \, [P - P_{1,i-1}] \, / \, [P_{1,i} - P_{1,i-1}] \qquad (3.8)$$

where $P_{1,i-1} < P < P_{1,i}$

Step 6 The equilibrium load, P'_1, associated with the displacement u'_1 so obtained is then calculated.

Step 7 The Newton-Raphson iterative procedure is applied thereafter until the final solution is obtained.

It should be noted that the iterative procedure above is described with reference to a single degree of freedom system. For a multi-degree of freedom system, the same procedure still applies, with the factor $[P - P_{1,i-1}] \, / \, [P_{1,i} - P_{1,i-1}]$ in equation (3.8) evaluated for the largest applied load component and used for all other degrees of freedom.

Convergence Criteria

In this study it is assumed that a converged solution is achieved when the ratio of the Euclidean norm of the incremental nodal displacements in any cycle of iteration to that of the total displacements at the previous cycle is less than the prescribed tolerance value. Depending on the degree of nonlinearity, a tolerance limit of 10^{-3} - 10^{-6} may be specified, with the samller value appropriate for highly nonlinear problems.