CHAPTER III

NUMERICAL SCHEME

Boundary Discretization

The problem has been formulated in the form of integral equations in the previous chapter as expressed in (19),(20),(21),(22) and (23). To solve these equations numerically , as shown in Fig.6, the boundary of plate may be divided into a finite number, N, of intervals, γ_j, j = 1,2,...,N each with length $2\Delta_j$ and the center of each intervals is considered to be the nodal point being at $(\bar{\xi}_j,\bar{\eta}_j)$ in which the unknown functions, deflection $w(\bar{\xi},\bar{\eta})$ and slope $\partial w(\bar{\xi},\bar{\eta})/\partial n$, are discretely assumed to be constant values, w_j and $\partial w_j/\partial n$, in each intervals. Thus the integrals in (19),(20),(21),(22) and (23) can be replaced approximately by equivalent summations. Thereafter, we define the replaced algebraic equations of (19) and (20) at the centers of the above intervals , $(\bar{x}_i,\bar{y}_i)$, i = 1,2,...,N and at each corners of the plate where i = N+1,N+2,...,N+K. From eq.(21), (22) and (23) we also define at each supports inside the plate domain where i = N+K+1,N+K+2,...,N+K+L.

Consequently, equations (19),(20),(21),(22) and (23) may be replaced numerically by a set of (2N+K+3L) algebraic equations in (2N+K+3L) unknowns which are w_j , j = 1,2,...,N+K, $\partial w_j/\partial n$, j = 1,2,...,N and w_j , $\partial w_j/\partial \xi$, $\partial w_j/\partial \eta$, j = N+K+1,...,N+K+L.

Rewriting of eq.(19),(20),(21),(22) and (23) yields

$$\frac{\phi_i w_i}{2\pi} - \sum_{j=1,2}^{N} \left\{ \frac{\partial w_j}{\partial n} \int_{\Gamma_j} M_n^*(\bar{x}_i,\bar{y}_i;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=1,2}^{N} \left\{ w_j \int_{\Gamma_j} V_n^*(\bar{x}_i,\bar{y}_i;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=N+1}^{N+K} R^*(\bar{x}_i,\bar{y}_i;\bar{\xi}_j,\bar{\eta}_j) w_j + \sum_{j=N+K+1}^{N+K+L} K_{aj} w^*(\bar{x}_i,\bar{y}_i;\xi_j,\eta_j) w_j$$

$$+ \sum_{j=N+K+1}^{N+K+L} K_{rj}^{y} \frac{\partial w^*}{\partial \xi}(\bar{x}_i,\bar{y}_i;\xi_j,\eta_j) \frac{\partial w_j}{\partial \xi} + \sum_{j=N+K+1}^{N+K+L} K_{rj}^{x} \frac{\partial w^*}{\partial \eta}(\bar{x}_i,\bar{y}_i;\xi_j,\eta_j) \frac{\partial w_j}{\partial \eta}$$

$$= \int_{\Omega} q(\xi,\eta) w^*(\bar{x}_i,\bar{y}_i;\xi,\eta)\, d\Omega(\xi,\eta) \ , \quad i = 1,2,..,N+K \qquad (24)$$

$$\frac{\phi_i}{2\pi}\frac{\partial w_i}{\partial n} - \sum_{j=1,2}^{N} \left\{ \frac{\partial w_j}{\partial n} \int_{\Gamma_j} \frac{\partial M_n^*}{\partial n}(\bar{x}_i,\bar{y}_i;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=1,2}^{N} \{ w_j \int_{\Gamma_j} \frac{\partial V_n^*}{\partial n}(\bar{x}_i,\bar{y}_i;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \}$$

$$+ \sum_{j=N+1}^{N+K} \frac{\partial R^*}{\partial n}(\bar{x}_i,\bar{y}_i;\bar{\xi}_j,\bar{\eta}_j)\, w_j + \sum_{j=N+K+1}^{N+K+L} K_{aj} \frac{\partial w^*}{\partial n}(\bar{x}_i,\bar{y}_i;\xi_j,\eta_j)\, w_j$$

$$+ \sum_{j=N+K+1}^{N+K+L} K^{y}_{rj} \frac{\partial^2 w^*}{\partial n \partial \xi}(\bar{x}_i,\bar{y}_i;\xi_j,\eta_j) \frac{\partial w_j}{\partial \xi} + \sum_{j=N+K+1}^{N+K+L} K^{x}_{rj} \frac{\partial^2 w^*}{\partial n \partial \eta}(\bar{x}_i,\bar{y}_i;\xi_j,\eta_j) \frac{\partial w_j}{\partial \eta}$$

$$= \int_{\Omega} q(\xi,\eta) \frac{\partial w^*}{\partial n}(\bar{x}_i,\bar{y}_i;\xi,\eta)\, d\Omega(\xi,\eta) \;, \quad i = 1,2,..,N \qquad (25)$$

$$w_i \quad - \quad \sum_{j=1,2}^{N} \{ \frac{\partial w_j}{\partial n} \int_{\Gamma_j} M_n^*(x_i,y_i;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \}$$

$$+ \sum_{j=1,2}^{N} \{ w_j \int_{\Gamma_j} V_n^*(x_i,y_i;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \}$$

$$+ \sum_{j=N+1}^{N+K} R^{*}(x_i,y_i;\bar{\xi}_j,\bar{\eta}_j)w_j \;+\; \sum_{j=N+K+1}^{N+K+L} K_{aj} w^{*}(x_i,y_i;\xi_j,\eta_j)w_j$$

$$+ \sum_{j=N+K+1}^{N+K+L} K^{y}_{rj}\frac{\partial w^{*}}{\partial \xi}(x_i,y_i;\xi_j,\eta_j)\frac{\partial w_j}{\partial \xi} \;+\; \sum_{j=N+K+1}^{N+K+L} K^{x}_{rj}\frac{\partial w^{*}}{\partial \eta}(x_i,y_i;\xi_j,\eta_j)\frac{\partial w_j}{\partial \eta}$$

$$= \int_{\Omega} q(\xi,\eta) w^{*}(x_i,y_i;\xi,\eta)\,d\Omega(\xi,\eta)\,, \quad i = 1,2,..,L \qquad (26)$$

$$\frac{\partial w_i}{\partial x} \;-\; \sum_{j=1,2}^{N} \left\{ \frac{\partial w_j}{\partial n} \int_{\Gamma_j} \frac{\partial M^{*}_{n}}{\partial x}(x_i,y_i;\bar{\xi},\bar{\eta})\,d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=1,2}^{N} \left\{ w_j \int_{\Gamma_j} \frac{\partial V^{*}_{n}}{\partial x}(x_i,y_i;\bar{\xi},\bar{\eta})\,d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=N+1}^{N+K} \frac{\partial R^{*}}{\partial x}(x_i,y_i;\bar{\xi}_j,\bar{\eta}_j)w_j \;+\; \sum_{j=N+K+1}^{N+K+L} K_{aj}\frac{\partial w^{*}}{\partial x}(x_i,y_i;\xi_j,\eta_j)w_j$$

$$+ \sum_{j=N+K+1}^{N+K+L} K^{y}_{rj}\frac{\partial^2 w^{*}}{\partial x \partial \xi}(x_i,y_i;\xi_j,\eta_j)\frac{\partial w_j}{\partial \xi} \;+\; \sum_{j=N+K+1}^{N+K+L} K^{x}_{rj}\frac{\partial^2 w^{*}}{\partial x \partial \eta}(x_i,y_i;\xi_j,\eta_j)\frac{\partial w_j}{\partial \eta}$$

$$= \int_{\Omega} q(\xi,\eta)\frac{\partial w^{*}}{\partial x}(x_i,y_i;\xi,\eta)\,d\Omega(\xi,\eta) \;, \quad i = 1,2,..,L \qquad (27)$$

$$\frac{\partial w_i}{\partial y} - \sum_{j=1,2}^{N} \left\{ \frac{\partial w_j}{\partial n} \int_{\Gamma_j} \frac{\partial M_n^{*}}{\partial y}(x_i,y_i;\bar{\xi},\bar{\eta})\,d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=1,2}^{N} \left\{ w_j \int_{\Gamma_j} \frac{\partial V_n^{*}}{\partial y}(x_i,y_i;\bar{\xi},\bar{\eta})\,d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$+ \sum_{j=N+1}^{N+K} \frac{\partial R^{*}}{\partial y}(x_i,y_i;\bar{\xi}_j,\bar{\eta}_j)\,w_j + \sum_{j=N+K+1}^{N+K+L} K_{aj}\frac{\partial w^{*}}{\partial y}(x_i,y_i;\xi_j,\eta_j)\,w_j$$

$$+ \sum_{j=N+K+1}^{N+K+L} K^{y}_{rj}\frac{\partial^2 w^{*}}{\partial y\partial \xi}(x_i,y_i;\xi_j,\eta_j)\frac{\partial w_j}{\partial \xi} + \sum_{j=N+K+1}^{N+K+L} K^{x}_{rj}\frac{\partial^2 w^{*}}{\partial y\partial \eta}(x_i,y_i;\xi_j,\eta_j)\frac{\partial w_j}{\partial \eta}$$

$$= \int_{\Omega} q(\xi,\eta)\frac{\partial w^{*}}{\partial y}(x_i,y_i;\xi,\eta)\,d\Omega(\xi,\eta) \;, \quad i = 1,2,..,L \qquad (28)$$

Evaluation of the Domain Integrals

The domain integrals which appear in the right hand side of equation (24), (25), (26), (27) and (28) take the general form :

$$I \quad = \quad \int_{\Omega} q(\xi,\eta)f^{*}(x,y;\xi,\eta)d\Omega(\xi,\eta) \quad . \qquad (29)$$

In the case of singular load, P , acting at (ξ,η), this load can be merely replaced by a Dirac delta function , $\delta(\xi,\eta)$, for which

$$\int_{\Omega} \delta(\xi,\eta)f^{*}(x,y;\xi_{o},\eta_{o})d\Omega(\xi_{o},\eta_{o}) \quad = \quad f^{*}(x,y;\xi,\eta) \quad ,$$

and equation (29) becomes

$$I \quad = \quad P(\xi,\eta)f^{*}(x,y;\xi,\eta) \quad . \qquad (30)$$

As mentioned above, the uniformly distributed load , $q(\xi,\eta)$, may be treated by dividing the loaded area into M finite strips, each with width $\Delta\eta$ and in which assume an the equivalent line load , p_i ,

$$p_i(\xi,\eta) \quad = \quad q_i(\xi,\eta)\Delta\eta_i \qquad (31)$$

acting at the center of each strip. Therefore, equation (29) can be replaced approximately by

$$I = \sum_{i=1,2}^{M} \left\{ p_i(\xi,\eta) \int_{\xi_i} f^*(x,y;\xi,\eta)\,d\Omega(\xi) \right\} . \qquad (32)$$

Treatment of Singularities

In eq.(24) and (25), the integration of function M_n, V_n, $\partial M_n/\partial n$ and $\partial V_n/\partial n$ on the plate boundary when $(\bar{x}_i,\bar{y}_i)$ and $(\bar{\xi}_j,\bar{\eta}_j)$ are coincident (i.e. $r = 0$) introduces some problems since these functions have term $\ln(r)$, $1/r$, and $1/r^2$ which are singular at $r = 0$. Tottenham (6) introduces the simple way to investigate these integrals to avoid the problem of singularities of these functions by using the equations of equilibrium.

From the fundamental solution (eq.(17)) , the expression of bending moment, twisting moment and shear in polar co-ordinates can be shown to be :

$$M_r^* = -D\left[\frac{\partial^2 w^*}{\partial r^2} + \nu \left\{ \frac{1}{r}\frac{\partial w^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^*}{\partial \theta^2} \right\} \right] \qquad (33a)$$

$$M_{r\theta}^* = D(1-\nu)\,\frac{\partial \left(\frac{1}{r}\frac{\partial w^*}{\partial \theta} \right)}{\partial r} \qquad (33b)$$

$$Q_r^* = -D \frac{\partial(\nabla^2 w^*)}{\partial r} \tag{33c}$$

$$V_r^* = Q_r^* + \frac{1}{r}\frac{\partial M_{\theta r}^*}{\partial \theta} \tag{33d}$$

$$\nabla^2 w^* = \frac{\partial^2 w^*}{\partial r^2} + \frac{1}{r}\frac{\partial w^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^*}{\partial \theta^2} \tag{33e}$$

Thus, from (17) and (33),

$$M_r^* = \frac{-1}{8\pi} [\, 2(1+\nu)\ln r + 3 + \nu \,]$$

$$Q_r^* = -\frac{1}{2\pi r}$$

$$M_{r\theta}^* = 0$$

$$V_r^* = -\frac{1}{2\pi r} = Q_r^*$$

Consider the equilibrium of a semi-circular disc (Fig.7a), center of circle at (0,0) of which the unit load is applied. The total vertical shear along the straight edge can be found from the condition of zero vertical force as

$$\int_{-\Delta}^{+\Delta} V_n^{*}\, dt + \frac{1}{2} + \int_{\pi/2}^{3\pi/2} (V_r^{*})_{r=\Delta}\Delta d\theta + R^{*}_{t=\Delta} + R^{*}_{t=-\Delta} = 0$$

$$\text{or} \int_{-\Delta}^{+\Delta} V_n^{*}\, dt = \frac{-1}{2} - \int_{\pi/2}^{3\pi/2} \frac{(-1\ )}{2\pi\Delta}\Delta d\theta = 0 . \qquad (34)$$

The total normal bending moment along the same edge can be obtained by considering the moment equilibrium as

$$\int_{-\Delta}^{+\Delta} M_n^{*}\, dt + \int_{\pi/2}^{3\pi/2} (M_r^{*})_{r=\Delta}\cos\theta\ \Delta d\theta$$

$$- \int_{\pi/2}^{3\pi/2} (V_r^{*})_{r=\Delta}\Delta^{2}\cos\theta\ d\theta = 0$$

$$\text{Thus} \int_{-\Delta}^{+\Delta} M_n^{*}\, dt = \frac{\Delta}{4\pi}\left[\ 1 - \nu - 2(1 + \nu)\ln\Delta\ \right] . \qquad (35)$$

Similarly for the case which corresponds to a unit couple applied at the origin (Fig.7b), deflection, w_z , and corresponding stress resultants may be expressed as

$$w_z = \frac{\partial w^*}{\partial n} = \frac{-r\cos\theta}{8\pi D}\left[1 + 2\ln r \right]$$

$$M_{rz} = \frac{(1+\nu)\cos\theta}{4\pi r}$$

$$M_{r\theta z} = \frac{(1-\nu)\sin\theta}{4\pi r}$$

$$Q_{rz} = -\frac{\cos\theta}{2\pi r^2}$$

$$V_{rz} = -\frac{(3-\nu)\cos\theta}{4\pi r^2}$$

Using the same procedure as before to evaluate these integrals , it can be shown that

$$\int_{-\Delta}^{+\Delta} V_{n2}\, dt + \int_{\pi/2}^{3\pi/2} (V_{r2})_{r=\Delta} \Delta d\theta + R_{2\,n=0,\,t=\Delta} + R_{2\,n=0,\,t=-\Delta} = 0$$

$$\text{or} \quad \int_{-\Delta}^{+\Delta} \frac{\partial V_n^*}{\partial n} dt = \int_{-\Delta}^{+\Delta} V_{n2}\, dt = -\frac{(1+\nu)}{2\pi\Delta} \qquad (36)$$

$$\int_{-\Delta}^{+\Delta} \frac{\partial M_n^*}{\partial n} dt = \int_{-\Delta}^{+\Delta} M_{n2}\, dt = 0 \,. \qquad (37)$$

Consider the corner force, R^*, in eq.(24) where $(\bar{x},\bar{y})$ and $(\bar{\xi},\bar{\eta})$ coincide. The corner force , R^* , can be found to be zero by using equilibrium condition of vertical forces of the free-body circular sector element of the plate corner (Fig.7c). Let ϵ be the radius of the circular sector. The portion of applied unit force, resulting shears along edges and the corner force must be in equilibrium :

$$R^* + \int_{-\epsilon}^{0} V_{n1}^*\, dt_1 + \int_{0}^{\epsilon} V_{n2}^*\, dt_2 + \frac{\phi}{2\pi}$$

$$+ \int_{\pi-\phi/2}^{\pi+\phi/2} (V_r^*)_{r=\epsilon}\, \epsilon d\theta = 0 \,.$$

$$\text{Thus} \qquad R^{*} + \int_{-\epsilon}^{0} V^{*}_{n1}\, dt_1 + \int_{0}^{\epsilon} V^{*}_{n2}\, dt_2 = 0 \,.$$

From symmetry it can be shown that the shearing forces that act on the diametral sections of the element must be zero (10).

$$\text{Therefore,} \qquad R^{*} = 0 \,. \qquad (38)$$

<u>Domain Solution</u>

The deflection function at any point, (x,y), as written in (18) can be computed numerically in the form :

$$w(x,y) = \sum_{j=1,2}^{N} \left\{ \frac{\partial w_j}{\partial n} \int_{\Gamma_j} M^{*}_{n}(x,y;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$- \sum_{j=1,2}^{N} \left\{ w_j \int_{\Gamma_j} V^{*}_{n}(x,y;\bar{\xi},\bar{\eta})\, d\Gamma(\bar{\xi},\bar{\eta}) \right\}$$

$$- \sum_{j=N+1}^{N+K} R^{*}(x,y;\bar{\xi}_j,\bar{\eta}_j)\, w_j \quad - \sum_{j=N+K+1}^{N+K+L} K_{aj} w^{*}(x,y;\xi_j,\eta_j)\, w_j$$

$$- \sum_{j=N+K+1}^{N+K+L} K^{y}_{rj} \frac{\partial w^{*}}{\partial \xi}(x,y;\xi_j,\eta_j) \frac{\partial w_j}{\partial \xi} - \sum_{j=N+K+1}^{N+K+L} K^{x}_{rj} \frac{\partial w^{*}}{\partial \eta}(x,y;\xi_j,\eta_j) \frac{\partial w_j}{\partial \eta}$$

$$+ \int_{\Omega} q(\xi,\eta) w^{*}(x,y;\xi,\eta) d\Omega(\xi,\eta) \quad . \qquad (39)$$

Finally, the desired stress resultants inside the domain can be obtained by appropriate differentiating of the influence functions in eq.(39) as the case may be.