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A Spatial Distance Explanation of the Relationship Between Class Size and Achievement

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ABSTRACT

*Glass and Smith (1979) and Glass, Cahen, Smith and Filby (1982), in their meta-analysis of class size and academic achievement, documented the contention that as the class size increases, the achievement of students decreases. To explain this finding, Preece (1987) presented a theoretical model based on the assumption that a teacher adjusts the teaching to the student who is least able in class. The present paper proposed an alternative mathematical theory based on the spatial distance between students and teacher. This "spatial distance" theory was shown to be quite effective in explaining the relationship between class size and student achievement. As a result of this theoretical work, there now exist at least two viable explanations for the meta-analytical data on class size and students' achievement. So, the stage has been set for future research that will try to resolve Preece's **teacher adjustment** hypothesis vs. the present study's **spatial distance** hypothesis.*

A Spatial Distance Explanation of the Relationship Between Class Size and Achievement

Glass and Smith (1979) and Glass, Cahen, Smith and Filby (1982), in their meta-analysis of studies on the relationship between class size and achievement, found that as the class size increases, the achievement of the students decreases. To explain this class size effect, Preece (1987) presented a mathematical theoretical model based on the assumption that a teacher adjusts the teaching to the student who is least able in class. A larger class size tends to be more heterogeneous and likely to include students who are less able, resulting in smaller class achievement.

The purpose of this paper is to present an alternative mathematical theory to explain the class size effect. Contrary to Preece's theory which is based on the assumption that a teacher adjusts the teaching to the least able student in class, this theory is based on the **different** assumption that a student's academic achievement is a function of how far the student is physically away from the teacher. This assumption has support from a number of studies that found course grade decreased as a function of physical distance from the instructor (Becker, Sommer, Bee, & Oxley, 1973; Holliman & Anderson, 1986). It will be shown that this theory not only could account for past studies showing the effect of class size on student academic achievement, but also could predict outcomes from possible future empirical studies **not** involving class size as a variable of study.

The Postulate of the Theory

As mentioned, a number of studies have indicated that academic achievement is a function of spatial distance from the teacher--the further the student is from the teacher the lower the achievement. However, the form of this function has never been made explicit in the literature. The present paper attempts to specify the form of this function. There are some good reasons to believe that this function is not linear, but rather, curvilinear and decelerating.

First, instructional information is generally presented visually and auditorily. It is a well-known fact that visual information and auditory information become more degraded as the receiver of the information moves further away from the sources. Actually, in physics, the intensity of light varies inversely as a squared distance from the source (Hecht, 1987, p. 45). The functional relationship between intensity of light and distance from the source is thus curvilinear and decelerating. The intensity of a sound wave also varies inversely as the squared distance from the source (Jones & Childers, 1993, p. 411).

Second, past research indicated that class participation drops more from the center of first row to the center of second row than from the center of second row to the center of further row (Sommer, 1967), suggesting a curvilinear and decelerating relationship. Actually Sommer (1967) found that participation dropped 7% (61% to 54%) from the center of the first row to the center of the second row, and dropped only 3% from the center of the second row to the center of the third row. Given that there is a direct relationship between class participation and achievement (Finn & Cox, 1992; Kennedy, 1992), it would be reasonable to conjecture that the functional relationship between achievement and distance from teacher is also curvilinear and decelerating.

Third, Holliman and Anderson (1986) presented data which supported the contention that achievement is a curvilinear and decelerating function of distance from the teacher. In their study, the drop of the average achievement score from the first two rows of students to the second two rows was about six points (81.88 to 75.55) whereas the drop from the second two rows to the next two rows was only about one point (75.55 to 74.91).

The three considerations above strongly suggest that academic achievement is a **curvilinear** and **decelerating** function of distance from the teacher. Such a function can be represented well by a decreasing exponential function which will be taken as the **postulate** of the present mathematical theory. The function is:

$$A = Re^{-s(D-1)} + M \qquad (1)$$

where A = Academic achievement of a student.

R = Range (or difference) of achievement between the student who is seated nearest to the teacher and the student who is seated furthest.

e = exponential (approximate value = 2.7183)

s = steepness of the decaying exponential function.

D = Distance (physical) of the student from teacher.

M = Minimum achievement (of the student seated furthest from the teacher).

Achievement A is measured by any raw test score or its linear transformation such as a standard score. Range R is the **difference** in achievement (measured in raw score or its linear transformation) between the student who is seated nearest to the teacher and the student who is seated furthest. Distance D is measured in terms of multiples of a "comfortable distance" between the teacher and the nearest student. D must be one or greater. In a typical classroom that holds 30-35 students, this "comfortable distance" could be about one yard. The steepness of the decaying exponential function s is a nonnegative real number that has to be determined empirically. The power component "D-1" (D minus one) of e has one subtracted from D so that when D=1 the achievement is maximum.

As an example, let's assume that the score of a student when seated furthest from the teacher in a particular classroom, is 72 and the score when the student is seated closest to the teacher is 82. Let's further assume that the decelerating factor s is 1.00. Then, Equation 1 becomes:

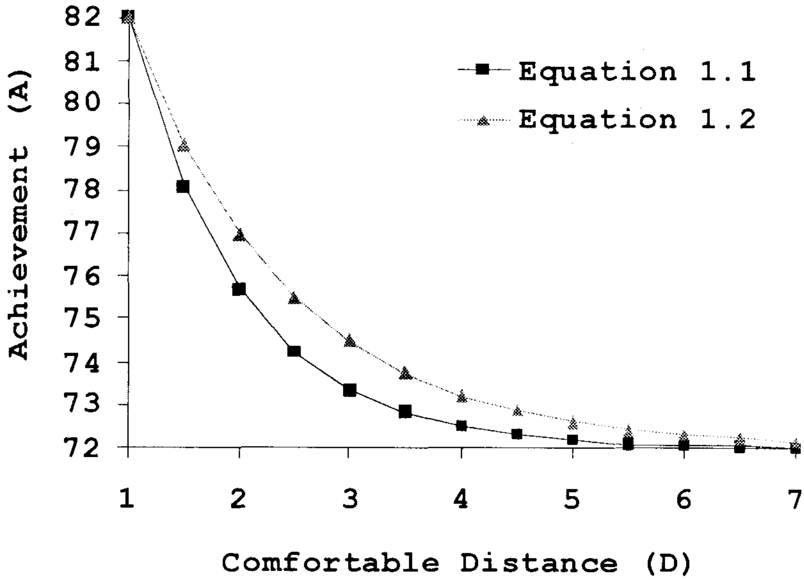
$$A = 10e^{-(D-1)} + 72 \qquad (1.1)$$

The graph of Equation 1.1 is shown in Figure 1.

As another example, let's assume that the R and the M in Equation 1 remain the same but that the decelerating factor s is 0.70 rather than 1.00. In such a case, Equation 1 becomes:

$$A = 10e^{-0.7\,(D-1)} + 72 \qquad (1.2)$$

The graph of Equation 1.2 is superimposed on that of Equation 1.1 and both are shown together in Figure 1.



Note. Equation 1.1: $A = 10e^{-(D-1)} + 72$

Equation 1.2: $A = 10e^{-0.7\,(D-1)} + 72$

Figure 1. Achievement as a function of physical Distance from the teacher.

The postulate of the present theory as shown in Equation 1 pertains to an individual student. Thus, *A* is the achievement of an **individual** student, and *D* is the distance of the **individual** student from the teacher. However, this postulate could be extended to accommodate a group or "class" of students. In this context, we can refer to "**class** achievement" and "**class** distance" (from the teacher). The extended form of the postulate is expressed as:

$$A_c = R_c\, e^{-s(D_c - 1)} + M_c \qquad (2)$$

where

A_c = Academic achievement of a class (**class achievement**).

R_c = Range (or difference) of achievement between the class physically located nearest to the teacher and the class located furthest.

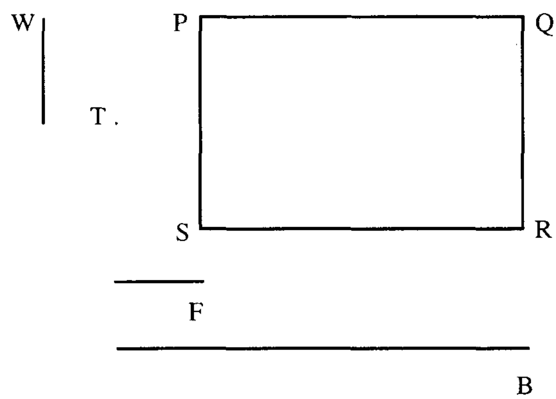
D_c = Distance (physical) of the class from teacher (**class distance**).

M_c = Minimum achievement (of the class located furthest from the teacher).

Class achievement A_c as opposed to the achievement of an individual student could be measured in terms of the "average test score" of all students in class. When the achievement of this whole class is compared with that of a "reference" or "control" class, it could be expressed as an "effect size." Effect size is computed by first subtracting the average score of the control group (or class) from that of the experimental group, and then dividing the result by the control group's standard deviation (Glass, McGaw & Smith, 1981; Light & Pillemer, 1984). Thus, an "effect size" is simply a linear transformation of the average test score of all students in class.

The distance of the whole class from the teacher D_c can be represented by the average of the distances of all the students in the class from the teacher. D_c's unit of measurement is the same as that of D in Equation 1, i.e., in multiples of a comfortable distance between two people. It will be shown that D_c increases as class size increases. The computational formula for D_c is developed in the following paragraphs.

Consider a class of students represented by the rectangle PQRS with a teacher represented as a point T a distance of F from the front of the class and a distance of B from the back of the class (see Figure 2). Also let W be the distance from the center of the class to the side of the class.



Note. PQRS = a class of students
T = Teacher
F = distance of teacher from Front of class
B = distance of teacher from Back of class
W = Width from center to side of class

Figure 2. Line map of a typical classroom

Imagine a class (rectangle PQRS) filled uniformly with students. The average distance of the students from the teacher D_c is then approximately equal to the average distance of all possible points in the rectangle from the point T. "The average distance of all possible points in the rectangle PQRS from the point T" can be found from the following mathematical expression:

$$\frac{1}{W(B-F)}\int_0^W \int_F^B \sqrt{x^2+y^2}\,dxdy$$

Equating D_c to this expression, we have:

$$D_c = \frac{1}{W(B-F)}\int_0^W \int_F^B \sqrt{x^2+y^2}\,dxdy \qquad (3)$$

where x moves horizontally from front (F) of class to back (B) of class (boundary of the inner integral) and y moves vertically from center (0) of class to side (W) of class (boundary of the outer integral). The point T serves as origin for both x and y. Note that $\sqrt{x^2+y^2}$ is the distance of any point (student) in the rectangle PQRS from the point T (teacher). Note also in Equation 3 that the inner range of integration (from F to B) and the outer range of integration (from 0 to W) indicate that the Equation pertains to only the left half (from the teacher T's view) of the class. Equation 3 by itself therefore represents the average distance of the left side of the class from the teacher. However, because of the symmetry of the class, Equation 3 also represents the average distance of the right side of the class from the teacher. Therefore, Equation 3 represents the average distance of the whole class from the teacher. The detailed evaluation of the double integral in Equation 3 is carried out in the Appendix. The final result is shown below:

$$D_c = \frac{1}{W(B-F)}\int_0^W \int_F^B \sqrt{x^2+y^2}\,dxdy$$

$$D_c = \frac{1}{W(B-F)}\left[(P_1+P_2)-(P_3+P_4)\right] \qquad (4)$$

where

$$P_1 = \frac{B^3}{6}\left[\frac{\tan\theta_1}{\cos\theta_1} + \ln\tan\left(\frac{\pi}{4}+\frac{\theta_1}{2}\right)\right] \qquad (4.1)$$

$$P_2 = \frac{W^3}{6}\left[\frac{\tan\theta_2}{\cos\theta_2} + \ln\tan\left(\frac{\pi}{4}+\frac{\theta_2}{2}\right)\right] \qquad (4.2)$$

$$P_3 = \frac{F^3}{6}\left[\frac{\tan\theta_3}{\cos\theta_3} + \ln\tan\left(\frac{\pi}{4}+\frac{\theta_3}{2}\right)\right] \qquad (4.3)$$

$$P_4 = \frac{W^3}{6}\left[\frac{\tan\theta_4}{\cos\theta_4} + \ln\tan\left(\frac{\pi}{4} + \frac{\theta_4}{2}\right)\right] \qquad (4.4)$$

where

$$\theta_1 = \arctan\frac{W}{B} \qquad (5.1)$$

$$\theta_2 = \arctan\frac{B}{W} \qquad (5.2)$$

$$\theta_3 = \arctan\frac{W}{F} \qquad (5.3)$$

$$\theta_4 = \arctan\frac{F}{W} \qquad (5.4)$$

For an example, let's imagine a rectangular class PQRS uniformly filled by 35 students (seven rows of five students each) with line PS measuring 5 yards and PQ measuring 7 yards (see Figure 2). Let's further imagine that the teacher T stands half a yard from line PS. Note that the first row of students will not be exactly on line PS but maybe half a yard away. (Consider line PS an invisible wall.) Thus, the actual distance between the teacher and the student in the center of row 1 is more like a yard rather than half a yard. In terms of the parameters shown in Figure 2, we then have the following:

B = 7.5 yards
W = 2.5 yards
F = 0.5 yards

Substituting these into Equations 5.1-5.4 yields:

θ_1 = 0.32
θ_2 = 1.25
θ_3 = 1.37
θ_4 = 0.20

Substituting the above seven values into Equations 4.1-4.4 and eventually into Equation 4 yields a D_c of 4.32 for our hypothetical classroom.

With a mechanism to calculate D_c in place, we are able to examine how this D_c changes as class size increases. Let's examine Figure 2. Assuming again that line PS is 5 yards long and the center of it is 0.5 yard from teacher T. A gradual increase of class size can be represented by imagining line QR as being originally on line PS and then gradually moving away from it. As line QR moves away from line PS, the corresponding D_c as calculated above will also change. The D_c change as the center of line QR moves away from point T is shown in Figure 3. Note that when line QR is superimposed on line PS, the double integral in Equation 3 turns into a single integral shown in Equation 6 below:

$$D_c = \frac{1}{W}\int_0^W \sqrt{X^2 + y^2}\,dy \qquad (6)$$

where X is a constant representing the distance of line PS from T (Teacher).

Note that D_c increases almost linearly (actually with very little increasing acceleration) as the center of the line QR moves from 0.5 yard to 7.5 yards away from T (see Figure 3).

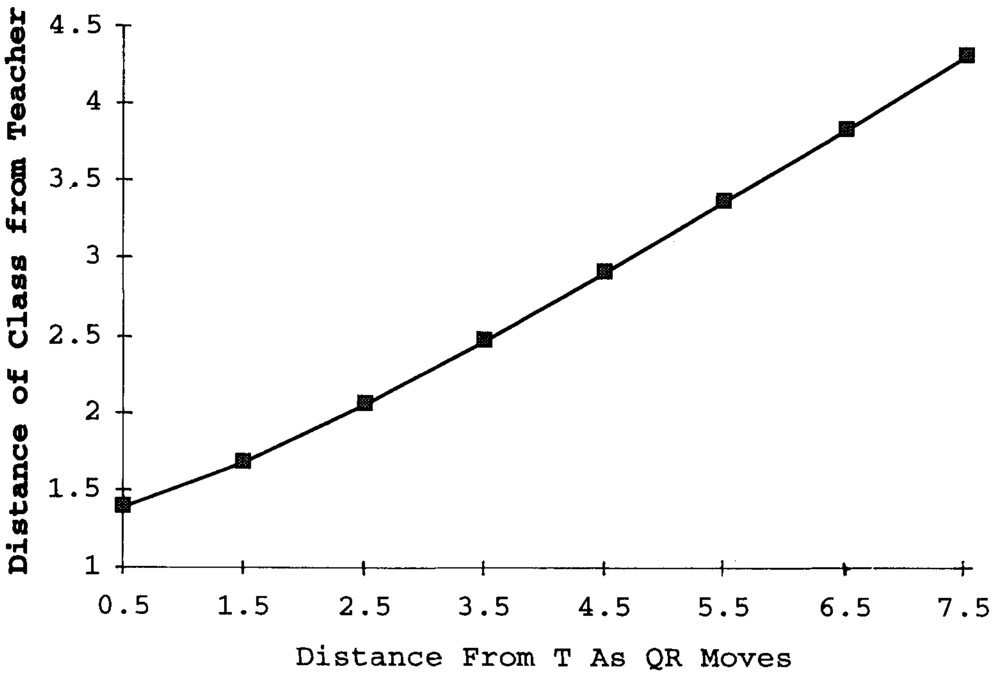


Figure 3. Change in Distance of Class (D_c) away from the teacher as class becomes bigger

Consider a typical physical classroom that can house a class size of from 1 student to a maximum of 35 students. Let's now figure out the distances (D_c's) from the teacher of classes of sizes 1, 5, 10, 15, 20, 25, 30, 35. Assume there are five students in each row, and rows closest to the teacher are filled first. Assume next that each student in a row is one comfortable yard from the next person in the row and each row is one comfortable yard from the next row. Assume further that the student in class size of 1 sits in the center of row one. Assume finally that the teacher's typical position is one yard away from the center of the first row. The distance from the teacher of class size of 1 is then 1 yard. Since a class of five students fills the first row and this first row occupies the space from line PS = 0.5 to line QR = 1.5 yards from teacher T (see Figure 2), the distance of this class from the teacher could be read off from Figure 3 at YARD = 1.5 on the horizontal axis. D_c for class sizes of 10, 15, 20, 25, 30, 35 could be read off Figure 3 at

YARDS = 2.5, 3.5, 4.5, 5.5, 6.5, 7.5 on the horizontal axis respectively. The eight D_c values were computed and shown in Table 1.

Table 1. Distances from Teacher of Classes of Various Sizes

Class size	Distance of class (D_c) from teacher (in yards)
1	1.00
5	1.69
10	2.07
15	2.48
20	2.92
25	3.38
30	3.85
35	4.32

Explaining Glass et al.'s Class-size Effect by the Present Theory

Glass, Cahen, Smith, and Filby (1982) found in their meta-analysis of research studies on class size and achievement that as class size increases the students' achievement decreases. This relationship was particularly evident in a subset of 14 studies reviewed in which subjects were randomly assigned. Because of the random assignment these 14 studies were considered to have good internal validity. The results showed achievement to be a **curvilinear decelerating** function of class size. The present paper will demonstrate that this relationship between class size and achievement can be predicted from or explained by the mathematical theory described in the previous section,

The 14 studies reviewed by Glass et al. yielded 30 "Effect Sizes" of smaller classes compared with larger classes. The regression equation relating these effect sizes to class sizes was found to be:

$$ES = 0.26 \ln(L/S) \qquad (7)$$

where

ES = estimated Effect Size of smaller compared with larger class

ln = natural logarithm

S = size of Smaller class

L = size of Larger class

When we set L = 35, the effect sizes of classes of sizes 1, 5, 10, 15, 20, 25, 30, 35 can be calculated from the above equation and shown in Table 2 (see "Glass's" column in Table 2).

Note "Glass's" column in Table 2 that, in terms of academic achievement measured in effect size, the largest class size (35) has an achievement index of zero because the largest class size is compared with itself. What this means for the present theory is that the "M_c" term in Equation 2 is also zero since the largest class size of 35 is furthest from the teacher compared with the smaller ones. Thus, Equation 2 in this special case reduces to:

$$A_c = R_c e^{-s(D_c - 1)} \qquad (8)$$

Recall that R_c (see Equation 2) is the **difference** in achievement between the class nearest to the teacher and the class furthest from the teacher. Thus, R_c in Equation 8 represents the difference in achievement (measured in effect size) between classes of size 1 and size 35. "Glass's" column in Table 2 shows this difference to be 0.92 minus 0.00 = 0.92. Substituting R_c = 0.92 into Equation 8, we have

$$A_c = 0.92 e^{-s(D_c - 1)} \qquad (8.1)$$

Effect sizes of classes of sizes 1, 5, 10, 15, 20, 25, 30, 35 compared with class size of 35 could be predicted from the present mathematical theory using Equation 8.1. Setting "s" in Equation 8.1 to a value of 1 yields the predicted Achievement (A_c) in terms of "effect sizes" as a function of distance (D_c) of the class away from the teacher as shown in Table 2 (see "Present theory's" column in Table 2).

Table 2 Estimated Effect Sizes of Various Class Sizes Compared with Class Size of 35, Using Glass's Empirically Derived Regression Equation vs. Equation 8 of the Present Theory

Class size	Distance of class (D_c) from teacher	Estimated effect size	
		Glass's	Present theory's
1	1.00	0.92	0.92
5	1.69	0.51	0.46
10	2.07	0.33	0.32
15	2.48	0.22	0.21
20	2.92	0.15	0.13
25	3.38	0.09	0.09
30	3.85	0.04	0.05
35	4.32	0.00	0.03

Glass et al.'s effect sizes in Table 2 are estimated from the empirical data reviewed. The present theory's effect sizes in Table 2 are predicted from the present mathematical theory. The two sets of effect sizes fit quite well together when shown together graphically in Figure 4.

This good match between Glass et al.'s empirically generated curve and the present theory's generated curve lends support to the present theory. It also lends support to setting the parameter s in Equation 1 to the value of 1 as done earlier. Setting $s=1$ is both parsimonious and real world fitting.

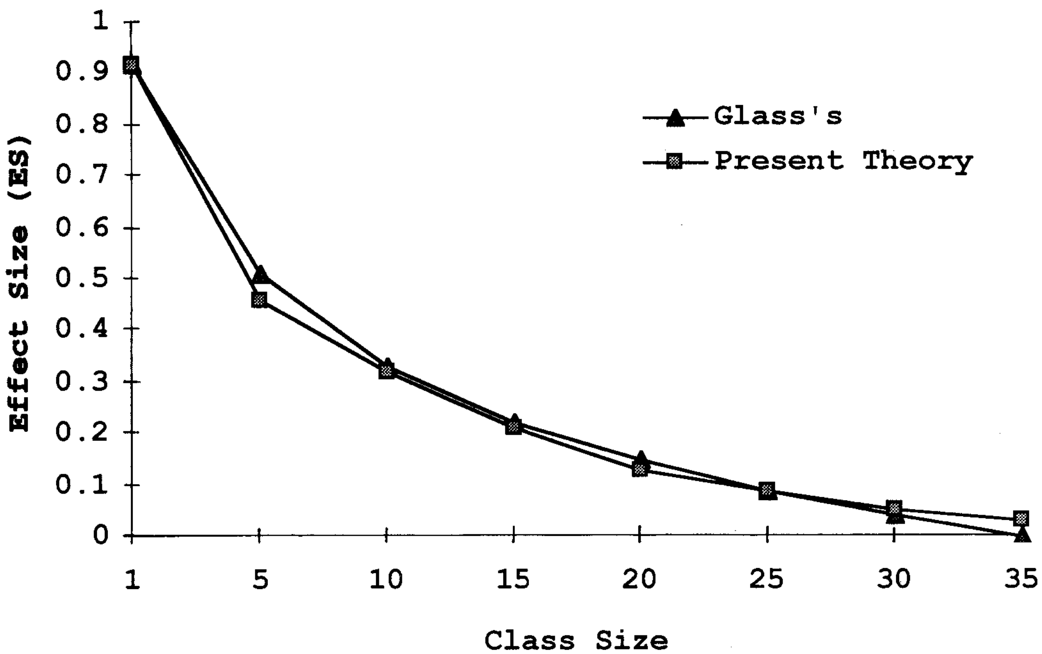


Figure 4. Effect size as a function of class size: Glass's curve derived from empirical findings vs. the curve generated by the present theory.

The importance of the s parameter in Equation 8 may emerge in future investigation of class size and achievement. It is possible that this parameter may be tied to quality of teaching. It is conceivable that when the quality of teaching is high the "achievement as a function of class size" curve may not be as steep as the one shown in Figure 4. The s parameter in this case would be "less than one." On the other hand, if the quality of teaching is very poor, the curve may be even steeper than the one shown in Figure 4. The s parameter in this case would then be "greater than one." This consideration is possible in the present theory's Equation 8 while it is not dealt with directly in Glass et al's empirically generated equation--Equation 7.

Explaining more research results by the present theory

As demonstrated above, the present theory explains, in terms of physical distance between students and teacher, the effect of class size on students' academic achievement as indicated by Glass et al's (1982) meta-analysis of research studies done on class size. The present theory could also explain why Finn and

Achilles' (1990) study obtained an average effect size of 0.14 among white students in favor of a smaller class. This study was a large-scale (statewide, Tennessee) well-controlled experiment in which the students and teachers were randomly assigned to small (13-17 pupils) and large (22-25 pupils) classes. The effect size of 0.14 was obtained by averaging the five effect sizes computed for various achievement tests among white students (Finn & Achilles, 1990, p. 567). A glance at Table 2 in the present paper indicated that the present theory estimates the effect size difference between class size of 15 and class size of 25 to be 0.21-0.09 = 0.12--a close match with Finn and Achilles's figure of 0.14. The Finn and Achilles's study was part of Tennessee's project STAR (Student-Teacher Achievement Ratio) that was well described and summarized in Mosteller (1995) and Achilles (1996).

The present study could further explain why Indiana's Prime Time project which also aimed at reducing class size achieved a smaller effect size than Tennessee's STAR project. Tilliski (1990) reported an effect size of only 0.013 for Indiana's Prime Time project when comparing smaller classes (averaging 19.1 students per class) against larger classes (averaging 26.9 students per class) on "composite" (reading, math, and writing) achievement measure (p. 27). A glance at Table 2 in the present paper indicated that the present theory estimates the effect size difference between class size of 20 and class size of 25 to be 0.13-0.09 = 0.04--a fair match with Tillitski's figure of 0.013.

The match might have been better had Indiana's PRIME TIME project employed "random assignment" of students to small and large classes. Past research indicated that the effect of a smaller class on achievement was more pronounced in studies where students were "randomly" assigned to small and large classes (Glass & Smith 1979, Figure 4, p. 15). Indiana's PRIME TIME project, unlike Tennessee's STAR project, did not employ random assignment of students to small vs. large classes.

Discussion

The present mathematical theory explains and predicts the class size effect. So does Preece's (1987) theoretical model. However, the present theory is based on the notion of the physical distance between the students and the teacher. The further the student is from the teacher physically, the lower the academic achievement becomes. On the other hand, Preece's theory is based on the assumption that a teacher adjusts the teaching to the student who is least able in class. A larger class size tends to be more heterogeneous and likely to include students who are less able, resulting in smaller class achievement. Is there a way to further evaluate the merits of these two competing theories?

According to the diagram in Figure 2, the teacher T faces the shorter side of the rectangular class PQRS. If the teacher faces the longer side of the rectangular class, would there be a difference in students' achievement according to the present theory? The distance of the class (D_c) from the teacher in a class of 35 with the teacher facing the shorter side of the rectangular class was calculated earlier to be 4.32. If the teacher moves to face the class along the longer side of the rectangle, Equation 4 will give a D_c of only 3.67, a much shorter average distance. Thus, the present theory predicts that class achievement will be greater if the teacher faces the longer side of the rectangular class than the shorter side. Preece's (1987) theoretical model would predict no difference in class achievement between the two physical positions of the teacher. Such a study could be carried out in the future.

Alternatively, a study could be conducted to compare two physically identical classrooms--one containing fewer students sitting farther apart (sparsely populated), the other containing more students sitting closer together (densely populated). Care is taken to ensure that the average distance between the students and the teacher is the same for both classes and that the students are randomly assigned to the two classrooms. The present theory predicts that achievement will be the same for both classes. Preece's (1987) theoretical model would predict better achievement for the class with fewer students.

Future studies such as those described above would shed some light on the relative merits of the present theory vs. that of Preece's. Depending on the outcomes of such studies, it is possible that the "spatial distance" factor (present theory) and "teacher adjustment" factor (Preece's theory) are both operating.

References

Achilles, C. M. (1996). Students achieve more in smaller classes. ***Educational Leadership***, *53*(5), 76-77.

Becker, F. D., Sommer, R., Bee, J., & Oxley, B. (1973). College classroom ecology. ***Sociometry***, *36*(4), 514-525.

Finn, J. D., & Achilles, C. M. (1990). Answers and questions about class size: a statewide experiment. ***American Educational Research Journal***, **27**(3), 557-577.

Finn, J. D., & Cox, D. (1992). Participation and withdrawal among fourth-grade pupils. ***American Educational Research Journal***, **29**(1), 141-162.

Glass, G. V., Cahen, L. S., Smith, M. L., & Filby, N. N. (1982). ***School class size***. Beverly Hills, CA: Sage.

Glass, G. V., McGaw, B., & Smith, M. L. (1981). ***Meta-analysis of social research***. Beverly Hills, CA: Sage.

Glass, G. V., & Smith, M. L. (1979). Meta-analysis of research on class size and achievement. ***Educational Evaluation and Policy Analysis***, **1**(1), 2-16.

Hecht, E. (1987), ***Optics***. Reading, MA: Addison-Wesley.

Holliman, W. B., & Anderson, H. N. (1986). Proximity and student density as ecological variables in a college classroom. ***Teaching of Psychology***, **13**(4), 200-203.

Jones, E. R., & Childers, R. L. (1993). ***Contemporary college physics***. Reading, MA: Addison-Wesley.

Kennedy, E. (1992). A multilevel study of elementary male black students and white students. ***Journal of Educational Research***, *86*(2), 105-110.

Light, R. J., & Pillemer, D.B. (1984). ***Summing up: the science of reviewing research***. Cambridge MA: Harvard University Press.

Mosteller, F. (1995). The Tennessee study of class size in the early school grades. ***Critical Issues for Children and Youths***, **5**(2), 113-127.

Preece, P. F. W. (1987). Class size and learning: a theoretical model. ***Journal of Educational Research***, *80*(6), 377-379.

Sommer, R. (1967). Classroom ecology. ***The Journal of Applied Behavioral Science***, *3*(4), 489-503.

Taylor, A. E., & Mann, W. R. (1972). ***Advanced calculus*** (2nd ed.). Lexington, MA: Xerox College Publishing.

Tillitski, C. (1990). The longitudinal effect size of PRIME TIME, Indiana's state sponsored reduced class size program. *Contemporary Education, 62*(1), 24-27.

Appendix

Evaluation of the Double Integral $\int_0^W \int_F^B \sqrt{x^2 + y^2}\,dxdy$

Let's examine Figure A1 which is a redrawing of the left side (from teacher T's view) of the class shown in Figure 2. Rectangle PQBF (see Figure A1) represents the left half of the class. T (teacher's position) is the origin of the X and Y coordinate system. Line TF (length F) is the distance of the teacher from the front of class. Line TB (length B) is the distance of the teacher from the back of class. Line TW (length W) is half of the width of class.

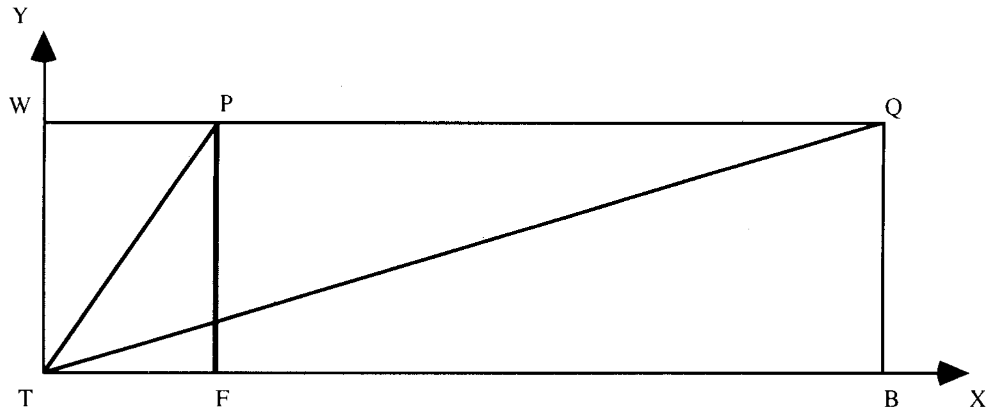


Figure A1. **Line map of the left half of classroom**

The double integral $\int_0^W \int_F^B \sqrt{x^2 + y^2}\,dxdy$ represents the sum of the distances of all points in the rectangle PQBF from the teacher T. Evaluation of this double integral runs into difficulty if one proceeds with rectangular coordinates (Taylor & Mann, 1972). Polar coordinates present a simpler solution. To utilize the polar coordinates, it is convenient to work with triangles. In terms of triangles, the "sum of the distances of all points in the rectangle PQBF from the teacher T" is equivalent to the sum of all points in triangle QTB away from T (call it P_1, or part one), plus the sum in triangle QTW (P_2, or part two), minus the sum in triangle PTF (P_3, or part three), minus the sum in triangle PTW (P_4, or part 4).

Let P_1 stand for the sum of all points in triangle QTB away from T. In terms of polar coordinates (see Taylor & Mann, 1972, p. 397):

$$P_1 = \int_0^{\theta_1} \int_0^{B \sec \theta} r^2 dr d\theta$$

$$P_1 = \frac{B^3}{3} \int_0^{\theta_1} \sec^3 \theta d\theta$$

$$P_1 = \frac{B^3}{6} \left[\frac{\tan \theta_1}{\cos \theta_1} + \ln \tan\left(\frac{\pi}{4} + \frac{\theta_1}{2} \right) \right] \qquad \text{(A1)}$$

where:

θ_1 = Angle QTB = $\arctan \frac{W}{B}$

B = *length TB*

W = *length TW or QB*

r = distance from T to any point in triangle QTB

Using similar procedure, P_2, the sum of all points in triangle QTW away from T, has the following solution:

$$P_2 = \frac{W^3}{6} \left[\frac{\tan \theta_2}{\cos \theta_2} + \ln \tan\left(\frac{\pi}{4} + \frac{\theta_2}{2} \right) \right] \qquad \text{(A2)}$$

where:

θ_2 = Angle QTW = $\arctan \frac{B}{W}$

W = length TW

Similarly, P_3, the sum of all points in triangle PTF away from T, has the following solution:

$$P_3 = \frac{F^3}{6}\left[\frac{\tan\theta_3}{\cos\theta_3} + \ln\tan\left(\frac{\pi}{4} + \frac{\theta_3}{2}\right)\right] \qquad \text{(A3)}$$

where:

θ_3 = Angle PTF = $\arctan\frac{W}{F}$

F = length TF

And finally, P_4, the sum of all points in triangle PTW away from T, has the following solution:

$$P_4 = \frac{W^3}{6}\left[\frac{\tan\theta_4}{\cos\theta_4} + \ln\tan\left(\frac{\pi}{4} + \frac{\theta_4}{2}\right)\right] \qquad \text{(A4)}$$

where:

θ_4 = Angle PTW = $\arctan\frac{F}{W}$

W = length TW