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FIXED POINTS AND BEST PROXIMITY POINTS OF A CYCLIC MAP

Mr. Annop Kaewkhao

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By	Mr. Annop Kaewkhao
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Thesis Advisor	Professor Kritsana Neammanee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

S. Hannongbua .... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

Imchit Termwuttipong .... Chairman

(Associate Professor Imchit Termwuttipong, Ph.D.)

Kritsana Neammanee .... Thesis Advisor

(Professor Kritsana Neammanee, Ph.D.)

Phichet Chaoha .... Examiner

(Associate Professor Phichet Chaoha, Ph.D.)

ทรงเกียรติ สุเมธกิจการ .... Examiner

(Assistant Professor Songkiat Sumetkijakan, Ph.D.)

Suthep Suantai .... External Examiner

(Professor Suthep Suantai, Ph.D.)

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วิทยานิพนธ์ฉบับนี้ประกอบด้วยสองส่วน

ในส่วนแรก เราตอบคำถามของ Al-Thagafi และ Shahzad เกี่ยวกับการมีจุดใกล้เคียงที่สุดของการส่งหดตัวชนิดพีแบบวนบนปริภูมิบานาคแบบสะท้อน นอกจากนั้นเราขยายผลทฤษฎีบทของ Al-Thagafi และ Shahzad ไปสู่ปริภูมิอิงระยะทางที่สอดคล้องกับสมบัติ UC

ส่วนที่สอง เราพิสูจน์ว่า สำหรับการส่งหดตัวอย่างอ่อนชนิดหลายค่านั้น ลำดับของการทำซ้ำเชิงภาพฉายแบบ Picard ลู่เข้าสู่จุดตรึง และเราให้อัตราการลู่เข้าไว้ด้วย และเราพิสูจน์ทฤษฎีบท Collage แบบวางนัยทั่วไป นอกจากนี้เราขยายแนวคิดเงื่อนไขแบบวนไปยัง การส่งหลายค่า พร้อมทั้งศึกษาการมีจุดตรึงและการมีจุดใกล้เคียงที่สุด ยิ่งไปกว่านั้นเราศึกษาปัญหาความไม่เป็นอิสระของข้อมูล สำหรับการส่งหลายค่าชนิดพิเศษบางกลุ่ม

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This thesis contains two parts.

In the first part, we provide a positive answer to a question raised by Al-Thagafi and Shahzad on the existence of a best proximity point for a cyclic φ-contraction map in a reflexive Banach space. Moreover, we extend the Al-Thagafi and Shahzad theorem to metric spaces with property UC.

In the second part, we prove that the Picard projection iteration sequence converges to a fixed point and the rate of convergence is given. We also prove the generalize Collage Theorem for a multi-valued weak contraction map.
In addition, we extend the cyclic condition to multi-valued maps and study the existence of a fixed point and the existence of a best proximity point for the map having been extended. Moreover, we study data dependence problem for a special class of multi-valued maps.

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CONTENTS

CHAPTER I

INTRODUCTION

Let (X, d) be a metric space and $T : X \to X$ be a self map. We say that $x \in X$ is a fixed point of T if $x = Tx$. A fundamental result in fixed point theory is the Banach contraction principle which ensures that a contraction map on a complete metric space has a fixed point. Several extensions of this result have appeared in subsequent papers.

In 2003, one of the interesting extensions was given by Kirk, Srinivasan, and Veeramani [17]. They extended the fixed point theory from a self map to a map defined on the union of two subsets A and B of a metric space and satisfying the cyclic condition, i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$. Their result showed that, if there exists $k \in (0, 1)$ such that the map T satisfies $d(Tx, Ty) \le kd(x, y)$ for all $x \in A,\ y \in B$, then T has a unique fixed point in $A \cap B$.

In 2006, Eldred and Veeramani [13] extended the theorem of [17] to the case where $A \cap B = \varnothing$ and T is a cyclic contraction map by using the concept of best proximity points. They showed that, on a nonempty closed and convex subset of a uniformly convex Banach space, a cyclic contraction map T has a unique best proximity point. And they raised a question that, does the conclusion of their result still hold or not where X is a reflexive Banach space.

In 2009, Al-Thagafi and Shahzad [1] provided a positive answer of this question by adding some conditions. Moreover, they introduced a new class of maps, called cyclic φ-contractions, which contains the cyclic contraction mapping as a subclass. The existence and convergence results of a best proximity point are obtained on a nonempty closed and convex subsets of a uniformly convex Banach space. They also raised the question that, in reflexive Banach space the existence of best proximity points still hold for the case of a cyclic φ-contraction mapping or not.

In the first part of this thesis, we extend the result of Al-Thagafi and Shahzad to a metric space with the property UC (introduced in 2009 by Suzuki, Kikkawa and Vetro [31]) and provide a positive answer by adding some conditions.

In the second part, we work on multi-valued maps. Let $\mathcal{CB}(X)$ be the family of all nonempty closed bounded subsets of X and let $T : X \to \mathcal{CB}(X)$ be a multi-valued map. An element $x \in X$ satisfying $x \in Tx$ is called a fixed point of multi-valued map T . We denote by H the Hausdorff metric on $\mathcal{CB}(X)$ induced by metric d. The study of fixed point theorems for multi-valued mapping has been initiated by Markin [20] and Nadler [23]. The result usually referred to as Nadler's fixed point theorem and extended the Banach contraction principle from single valued maps to multi-valued maps. Since then, extensive literatures have been developed. There consist of many theorems dealing with fixed points for multi-valued mappings, see [2, 3, 10, 11, 22, 24, 26, 27]. Most of these cases require the image of each point to be closed and bounded. In others words, to be compact. In this part, we study Theorems on multi-valued maps. We extend the cyclic condition to multi-valued maps defined on the union of two subsets A and B of a metric space, i.e., $Tx \in \mathcal{CB}(B)$ and $Ty \in \mathcal{CB}(A)$ for all $x \in A$ and $y \in B$, and gave the existence of a best proximity point for multi-valued cyclic map.

This study was organized into 6 chapters as follows: Chapter I is an introduction to the research problem. Chapter II is concerned with some well known definitions and useful results that will be used in our research.

In Chapter III, we extend the Theorem of Al-Thagafi and Shahzad to a metric space with property UC and provide a positive answer for the question of them by adding some conditions.

In Chapter IV, we prove that the Picard projection iteration sequence converges to a fixed point, with a rate of convergence is given. The generalize Collage Theorem for a special class of multi-valued mappings is also proved.

In Chapter V, we extend the cyclic condition to multi-valued mappings and study the existence of a fixed point and the existence of a best proximity point.

In Chapter VI, we study data dependence problem for a special class of multi-valued mappings.

CHAPTER II

PRELIMINARIES

In this chapter, we collect information that will be needed for understanding of the research work. Almost all of them are merely stated without proof, since if can be found in many standard text book, for example in [15, 32].

Let X be a nonempty set. A ***metric*** on X is a real function $d : X \times X \to [0, \infty)$ which satisfies the following three conditions : For any x, y, z in X

1. $d(x, y) \geq 0$, and $d(x, y) = 0 \;\Leftrightarrow\; x = y$;

2. $d(x, y) = d(y, x)$;

3. $d(x, y) \leq d(x, z) + d(z, y)$.

The space X with a metric d define on $X \times X$ is called a ***metric space***. Let X be a metric space with metric d. If x_0 is a point of X and r is a positive real number, the ***open ball*** $B_r(x_0)$ with ***center*** x_0 and ***radius*** r is the subset of X define by

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}.$$

The ***closed ball*** $B_r[x_0]$ is define by

$$B_r[x_0] = \{x \in X : d(x, x_0) \leq r\}.$$

A subset G of the metric space X is called an ***open set***, if given any point x in G, there exists a positive real number r such that $B_r(x) \subset G$. A subset F of X is called a ***closed set*** if the complement F^c of F is open. A subset C of X is called a ***bounded set*** if there exists a positive real number M such that $d(x, y) \leq M$ for all $x, y \in C$.

Let X be a metric space with metric d, and let $\{x_n\}$ be a sequence of points in X. We say that $\{x_n\}$ is ***convergent*** if there exists a point x in X such that for each $\epsilon > 0$, we can find a positive integer n_0 with

$$n \geq n_0 \Rightarrow d(x, x_n) < \epsilon.$$

We usually symbolize this by writing $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, and express it verbally by saying that $\{x_n\}$ ***converges to*** x. The point x is called the ***limit*** of the sequence $\{x_n\}$.

Theorem 2.1. *A subset F of a metric space X is closed if and only if*

$$\{x_n\} \subset F \text{ and } \lim_{n\to\infty} x_n = x \Rightarrow x \in F.$$

A sequence $\{x_n\}$ in X is a ***Cauchy sequence*** if for each $\epsilon > 0$, there exists a positive integer n_0 such that

$$m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon.$$

It is obvious that every convergent sequence is a Cauchy sequence. A ***complete metric space*** is a metric space in which every Cauchy sequence is convergent. The following fact is quite useful.

Theorem 2.2. *Every closed subspace of a complete metric space is itself complete.*

Let X and Y be metric spaces with metrics d_1 and d_2, respectively, and let T be a mapping of X into Y. T is said to be ***continuous at a point*** x_0 in X if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_1(x, x_0) < \delta \Rightarrow d_2(Tx, Tx_0) < \epsilon.$$

A mapping of X into Y is said to be ***continuous*** if it is continuous at each point in its domain X. A mapping T of X into Y is called ***nonexpansive*** if

$$d_2(Tx, Ty) \leq d_1(x, y) \text{ for all } x, y \in X.$$

A mapping T of X into Y is called ***contractive*** if

$$d_2(Tx, Ty) < d_1(x, y) \text{ for all } x, y \in X.$$

A mapping T of X into Y is called c-***contraction*** if there exists a positive number $c < 1$ with the property that

$$d_2(Tx, Ty) \leq c d_1(x, y) \text{ for all } x, y \in X.$$

It is obvious that such mappings are continuous.

Let (X,d) be a metric space and let $T : X \to X$ be a map. We say that $x \in X$ is a ***fixed point*** of T if $Tx = x$. There are some well-known results on fixed point theorems for a mapping on a complete metric space. For instance,

Theorem 2.3. *[4] (Banach Contraction Theorem) Let (X,d) be a complete metric space and $T : X \to X$ be a c-contraction mapping. Then T has a unique fixed point.*

Theorem 2.4. *[5, 6] (Collage Theorem) Let (X,d) be a complete metric space and $T : X \to X$ be a c-contraction mapping. Then for any $x \in X$,*

$$d(x,x^*) \le \frac{1}{1-c} d(x,Tx),$$

where x^* is the fixed point of T.

Theorem 2.5. *[8](Continuity of Fixed Points) Let (X,d) be a complete metric space and $T_1, T_2 : X \to X$ be contraction mappings with contraction factors c_1 and c_2 and fixed points x_1^* and x_2^*, respectively. Then*

$$d(x_1^*, x_2^*) \le \frac{1}{1-\min\{c_1,c_2\}} d_\infty(T_1,T_2),$$

where $d_\infty(T_1,T_2) = \sup\limits_{x\in X} d(T_1x, T_2x)$.

Theorem 2.6. *[14] (Kannan's Theorem) Let (X,d) be a complete metric space and $T : X \to X$. If there exists a real number b with $0 \le b < \frac{1}{2}$ such that, $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$ for all $x, y \in X$. Then T has a unique fixed point.*

Theorem 2.7. *[9] (Chatterjea's theorem) Let (X,d) be a complete metric space and $T : X \to X$. If there exists a real number c with $0 \le c < \frac{1}{2}$ such that, $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$ for all $x, y \in X$. Then T has a unique fixed point.*

Let X be a nonempty set. A class $\mathcal{F}$ of subsets of X is called ***topology*** on X if it satisfies the following conditions:

1. $X \in \mathcal{F}$ and $\varnothing \in \mathcal{F}$;

2. the union of every class of sets in $\mathcal{F}$ is a set in $\mathcal{F}$;

3. the intersection of every finite class of sets in $\mathcal{F}$ is in $\mathcal{F}$.

A ***topological space*** consists of two objects: A nonempty set X and a topology $\mathcal{F}$ on X. The set in the class $\mathcal{F}$ are called the ***open sets*** of the topological space $(X, \mathcal{F})$. It is customary to denote the topological space $(X, \mathcal{F})$ by the symbol X which is used for its underlying set of points. A ***closed set*** in a topological space is a set whose complement is open. Let (X, d) be any metric space, and let the topology be the class of all subsets of X which are open in the sense of the definition in the metric space. This is called the ***usual topology*** on a metric space, and we say that these sets are the ***open sets generated by the metric*** d on the space. Let X be a topological space and let A be a subset of X. A ***closure*** $\overline{A}$ of a subset A is defined to be the intersection of all closed subsets of X which contains A. A class $\{\mathcal{F}_\mu : \mu \in D\}$ of open subsets of X is said to be an ***open cover*** of A if each point in A belongs to at least one of $\mathcal{F}_\mu$'s, that is, $\bigcup\limits_{\mu \in D} \mathcal{F}_\mu \supset A$. A subclass of an open cover which is itself an open cover is called a ***subcover***. A subset A of a topological space X is called ***compact*** if every open cover of A has a finite subcover and it is called ***relatively compact*** if $\overline{A}$ is compact . In particular, if X is compact, X is said to be a ***compact space***. This definition applies to metric space as well. In this case, there is a characterization of compactness.

Theorem 2.8. *A subset A of a metric space X is compact if and only if any sequence $\{x_n\}$ of points of A has a subsequence $\{x_{n_k}\}$ which converges to a point of A.*

A ***normed linear space*** is a linear space X in which to each vector x there corresponds a real number, denoted by $\|x\|$ and called the ***norm*** of x, in such a manner that

1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$, for any x and y in X and for any $\alpha \in \mathbb{R}$.

It is easy to verify that the normed linear space X is a metric space with respect to the metric d defined by $d(x, y) = \|x - y\|$. A subset C of a normed linear space is said to be ***convex*** if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and each scalar $\lambda \in [0, 1]$. A ***Banach space*** is a complete normed linear space.

Let X be a Banach space, and let $T : X \to \mathbb{R}$ be a ***linear functional***, that is, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. In addition, T is called ***bounded*** if there exists a real number $K \geq 0$ with the property that $|T(x)| \leq K\|x\|$ for all $x \in X$. It is not difficult to see that if T is continuous then T is bounded but the converse is not true. If T is a linear functional, then the following conditions on T are all equivalent to the other.

1. T is continuous;

2. T is bounded;

3. T is continuous at the origin.

We now denote the set of all bounded (or continuous) linear functionals of X by X^*. For $x \in X$ and $f \in X^*$ defined $i(x)(f) = f(x)$. It is easily seen that $i(x) \in X^{**}$ and that, in fact, the map $i : X \to X^{**}$ is an isometric isomorphism, called the canonical embedding of X into X^{**}. If $i(X) = X^{**}$, then X is said to be ***reflexive***. The ***weak topology*** on X is the smallest topology such that each bounded linear functional on X is continuous. We say that a sequence $\{x_n\}$ in X ***converges weakly*** to x if $\{x_n\}$ converges to x in the weak topology, that is $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all $f \in X^*$, and denoted by $x_n \rightharpoonup x$. A subset C of X is ***weakly closed*** if it is closed in the weak topology. The ***weakly open*** sets are now taken as those sets whose complements are weakly closed. Sets which are compact in this topology are said to be ***weakly compact***.

Theorem 2.9. *Let C be convex subset of a Banach space X. Then C is closed in the norm topology if and only if C is weakly closed.*

Theorem 2.10. *Let $\{x_n\}$ be a sequence in a Banach space X. If $x_n \rightharpoonup x_0$, then $\{x_n\}$ is a bounded set.*

Theorem 2.11. *Let $\{x_n\}$ be a sequence in a Banach space X. If $x_n \rightharpoonup x_0$, then $\|x_0\| \leq \liminf \|x_n\|$.*

Theorem 2.12. *Let X be a Banach space. Then X is reflexive if and only if every bounded closed convex subset of X is weakly compact.*

Convexity of Banach Spaces : Let X be a Banach space. Then X is ***strictly convex*** if for all $x, y, p \in X$ and $R > 0$,

$$\|x-p\| \leq R, \|y-p\| \leq R, \ x \neq y \Rightarrow \|\frac{x+y}{2}\| < R.$$

This condition is equivalent to the following:

$$\|x\| = \|y\| = 1, \ x \neq y \Rightarrow \|x+y\| < 2.$$

A Banach space X is ***uniformly convex*** if for any two sequence $\{x_n\}, \{y_n\}$ in X such that

$$\|x_n\| = \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$. We first have the following:

Theorem 2.13. *Let X be a Banach space. Then the following conditions are equivalent:*

1. *X is uniformly convex;*
2. *if for any two sequences $\|x_n\|, \ \|y_n\|$ in X,*

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$;

3. *for any ϵ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ depending only on ϵ such that*

$$\|x-y\| \leq 2(1-\delta)$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \epsilon$.

As a direct consequence of the previous theorem, we have the following:

Theorem 2.14. *If a Banach space X is uniformly convex, then X is strictly convex.*

Remark 2.15. *The converse of Theorem 2.14 is not true (see [21] in page 451).*

Let X be a Banach space. Then we define a function $\delta : [0,2] \to [0,1]$ called the ***modulus of convexity*** of X as follows:

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}.$$

It is obvious that δ is a nondecreasing function. That is, if $\epsilon_1 \leq \epsilon_2$, then $\delta(\epsilon_1) \leq \delta(\epsilon_2)$. We also have the following:

Theorem 2.16. *Let X be a Banach space. Then E is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon > 0$.*

The following theorem is really useful.

Theorem 2.17. *Let X be a uniformly convex Banach space. Then, for any r and ϵ with $0 < \epsilon \leq r$, the inequalities $\|x\| \leq r, \|y\| \leq r$ and $\|x - y\| \geq \epsilon$ imply $\delta(\frac{\epsilon}{r}) > 0$ and*

$$\frac{\|x+y\|}{2} \leq \left\{1 - \delta(\frac{\epsilon}{r})\right\},$$

where δ is the modulus of convexity of X.

Theorem 2.18. *If a Banach space X is uniformly convex, then X is reflexive.*

CHAPTER III

CYCLIC φ-CONTRACTION MAPPING

In this chapter, we study theorems on a cyclic φ-contraction mapping. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to X$ is said to be a ***cyclic*** mapping if $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2003, Kirk, Srinivasan, and Veeramani [17] extended the Banach contraction theorem for self map to a map T defined on the union of two subsets A and B of a metric space and satisfying the cyclic condition. The result ensure that $A \cap B \neq \varnothing$.

Theorem 3.1. *[17] Let A and B be nonempty closed subsets of a complete metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic mapping. If there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all} \ \ x \in A \text{ and } y \in B. \tag{3.1}$$

Then $A \cap B \neq \varnothing$ and T has a unique fixed point in $A \cap B$.

In 2006, Eldred and Veeramani[13] extended this result to the case where $A \cap B = \varnothing$ and T is a cyclic contraction mapping by using the concept of a best proximity point. A mapping $T : A \cup B \to A \cup B$ is said to be a ***cyclic contraction*** if T is a cyclic and there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B) \ \text{ for all } \ x \in A \ \text{ and } \ y \in B. \tag{3.2}$$

A point x in $A \cup B$ is called a ***best proximity point*** of T if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) \ : \ x \in A, y \in B\}$. Obvious that in case of $A \cap B \neq \varnothing$, (3.1) and (3.2) are equivalent and a best proximity point of T is a fixed point of T. Theorem 3.2 is an extension of Theorem 3.1.

Theorem 3.2. *[13] Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and $T : A \cup B \to A \cup B$ be a cyclic contraction mapping. Then* *for every $x_0 \in A$, there exists a unique $z \in A$ such that z is a best proximity point and $x_{2n} \to z$ where $x_{n+1} = Tx_n$.*

Question 1 : In their paper, Eldred and Veeramani raised the problem that the conclusion of Theorem 3.2 still hold or not if X is a reflexive Banach space.

In 2009, Suzuki, Kikkawa and Vetro[31] prove the existence of best proximity points for a special map which contains the cyclic contraction mappings as a subclass on a metric space with the property UC. Let A and B be nonempty subsets of a metric space (X, d). (A, B) is said to satisfy the ***property UC*** if the following holds:

If $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(x'_n, y_n) = d(A, B)$, then $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

Example 3.3. *[31] Let A and B be nonempty subsets of a metric space (X, d).*

1. *Let $d(A, B) = 0$. Then (A, B) satisfies the property UC.*
2. *Let A' and B' be nonempty subsets $A \subseteq A', B \subseteq B'$ and $d(A, B) = d(A', B')$. If (A', B') satisfies the property UC, then (A, B) also satisfies the property UC.*
3. *Let X be a uniformly convex Banach space. Assume that A is convex. Then (A, B) has the property UC.*
4. *Let X be a strictly convex Banach space X. Assume that A is convex and relatively compact, and the closure of B is weakly compact. Then (A, B) has the property UC.*

Theorem 3.4 is a result of [31].

Theorem 3.4. *[31] Let (X, d) be a metric space and let A and B be nonempty subsets of X such that (A, B) satisfies the property UC. Let T be a cyclic mapping on $A \cup B$ and there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \le r \max\{d(x, y), d(x, Tx), d(y, Ty)\} + (1 - r)d(A, B) \tag{3.3}$$

for all $x \in A$ and $y \in B$. Fixed $x_0 \in A$ and define $x_{n+1} = Tx_n$ for each $n \ge 0$. If A is complete, then there exists a unique $z \in A$ such that $x_{2n} \to z$, $T^2 z = z$ and z is a best proximity point.

Al-Thagafi and Shahzad[1] provided a positive answer of question 1 by adding some conditions.

Theorem 3.5. *[1] Let A and B be nonempty closed convex subsets of a reflexive strictly convex Banach space X, and $T : A \cup B \to A \cup B$ be a cyclic contraction mapping. If $(A-A)\cap(B-B) = \{0\}$, then there exists a unique $z \in A$ such that z is a best proximity point and $T^2 z = z$.*

Theorem 3.6. *[1] Let A and B be nonempty subsets of a reflexive Banach space X such that A is closed convex and let $T : A\cup B \to A\cup B$ be a cyclic contraction mapping. Then there exists a best proximity point z in A if one of the following conditions is satisfied:*

(a) T is weakly continuous on A.

*(b) T satisfies the **proximal property**, i.e.,*

if $x_n \rightharpoonup x \in A \cup B$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = d(A, B)$, then $\|x - Tx\| = d(A, B)$.

Furthermore, this point is unique if X is strictly convex.

Moreover, Al-Thagafi and Shahzad introduced a new class of maps, called cyclic φ-contraction, which contains the cyclic contraction mapping as a subclass.

Definition 3.7. *Let A and B be nonempty subsets of a metric space (X, d). Let $T : A\cup B \to A\cup B$ be cyclic and $\varphi : [0,\infty) \to [0,\infty)$ be strictly increasing. The mapping T is said to be **cyclic φ-contraction** if*

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$.

Since φ is strictly increasing and T is cyclic, we have the Proposition 3.8.

Proposition 3.8. *Let A and B be nonempty subsets of a metric space (X, d). If $T : A\cup B \to A\cup B$ is a cyclic φ-contraction mapping, then $d(T^2x, Tx) \le d(Tx, x)$ for all $x \in A\cup B$. Furthermore, $d(Tx, Ty) \le d(x, y)$ for all $x \in A$ and $y \in B$.*

Convergence and existence criterion of best proximity point for cyclic φ-contraction mappings are obtained in Theorem 3.9.

Theorem 3.9. *[1] Let A and B be nonempty subsets of a uniformly convex Banach space X such that A is closed and convex and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction mapping. Let $x_0 \in A$ and $x_{n+1} = Tx_n$. Then there exists a unique $z \in A$ such that $x_{2n} \to z$, $T^2z = z$ and z is a best proximity point of T.*

Al-Thagafi and Shahzad[1] also raised the following question:

Question 2. Do the Theorem 3.5 and 3.6 still hold for the case of a cyclic φ-contraction mapping ?

In this chapter, we extend Theorem 3.9 to a metric space with the property UC in the Theorem 3.13. In addition, we give an existence of a best proximity point in a reflexive Banach space in Theorem 3.19 and Theorem 3.22. This implies a positive answer of Question 2.

3.1 Cyclic φ-contraction maps with property UC

In this section, we extend Theorem 3.9 to a metric space with the property UC in the Theorem 3.13 . We begin with some lemmas which are related to the context of our results.

Lemma 3.10. *Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the property UC and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. Let $z \in A$. Then the followings are equivalent:*

(i) z is a best proximity point of T.

(ii) z is a fixed point of T^2.

In this case, Tz is a best proximity point of T in B.

Proof. We first show that (i) implies (ii). Assume that z is a best proximity point of T. Since φ is strictly increasing and T is cyclic, by Proposition 3.8, we have

$$\begin{aligned} d(A, B) &\leq d(T^2z, Tz) \\ &\leq d(Tz, z) \\ &= d(A, B). \end{aligned}$$

Thus, $d(T^2z, Tz) = d(A, B) = d(Tz, z)$. This implies that Tz is a best proximity point of T in B and by the property UC, we have $d(T^2z, z) = 0$. Hence $T^2z = z$, that is z is a fixed point of T^2.

To prove (ii) implies (i), we assume that z is a fixed point of T^2 and z is not a best proximity point of T. Hence

$$d(A, B) < d(z, Tz). \tag{3.4}$$

By (3.4) and the facts that φ is strictly increasing and T is φ-contraction, we have

$$\begin{aligned} d(z, Tz) &= d(T^2z, Tz) \\ &\leq d(Tz, z) - \varphi(d(Tz, z)) + \varphi(d(A, B)) \\ &< d(Tz, z), \end{aligned}$$

which is a contradiction. Hence z is a best proximity point of T. □

Lemma 3.11. *[1] Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. Then, for any fixed $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$.*

Lemma 3.12. *Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the property UC and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. Then, for any fixed $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$, we have $\{x_{2n}\}$ is a Cauchy sequence.*

Proof. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$. To show that $\{x_{2n}\}$ is a Cauchy sequence, suppose the contrary. Then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there is $m_k > n_k \geq k$ satisfying $d(x_{2n_k}, x_{2m_k}) > \epsilon_0$.

By Lemma 3.11, we have $\lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B)$, $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B)$ and

$$\lim_{k\to\infty} d(x_{2n_k}, x_{2n_k+1}) = d(A, B). \tag{3.5}$$

From the property UC of (A, B), we obtain

$$\lim_{n\to\infty} d(x_{2n+2}, x_{2n}) = 0. \tag{3.6}$$

From (3.5), (3.6) and the fact that

$$\begin{aligned} d(A,B) &\leq d(x_{2n_k+1}, x_{2m_k}) \\ &\leq d(x_{2n_k+1}, x_{2n_k+2}) + d(x_{2n_k+2}, x_{2n_k+4}) + \cdots \\ &\quad \cdots + d(x_{2m_k-4}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2m_k}). \end{aligned}$$

Thus

$$d(A,B) \leq \lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) \leq d(A,B). \tag{3.7}$$

Hence $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = d(A,B)$.

From this fact, the equality (3.5) and the UC-property of (A,B), we have

$$\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k}) = 0.$$

This contradicts the assumption that $d(x_{2n_k}, x_{2m_k}) > \epsilon_0$. Therefore $\{x_{2n}\}$ is Cauchy. $\square$

Theorem 3.13. *Let A and B be nonempty subsets of a metric space (X,d) such that (A,B) satisfies the property UC and A is complete. Let $T : A\cup B \to A\cup B$ be a cyclic φ-contraction map. Then the followings hold:*

(i) T has a unique best proximity point z in A.

(ii) z is a unique fixed point of T^2 in A.

(iii) T has at least one best proximity point in B.

(iv) For every $x_0 \in A$ and $x_{n+1} = Tx_n$, $\{x_{2n}\}$ converges to z.

Proof. (i) Fix $x_0 \in A$ and let $x_{n+1} = Tx_n$. By Lemma 3.12, $\{x_{2n}\}$ is a Cauchy sequence in A. Since A is complete, there exists $z \in A$ such that $\lim_{n\to\infty} d(z, x_{2n}) = 0$. Hence, by Lemma 3.11 and the fact that

$$d(A,B) \leq d(z, x_{2n-1}) \leq d(z, x_{2n}) + d(x_{2n}, x_{2n-1}),$$

we have

$$\lim_{n\to\infty} d(z, x_{2n-1}) = d(A,B). \tag{3.8}$$

Since $\{x_{2n}\}$ belongs to A, $Tz \in B$ and Proposition 3.8, we have

$$\begin{aligned} d(A,B) &\leq d(x_{2n},Tz) \\ &= d(Tx_{2n-1},Tz) \\ &\leq d(x_{2n-1},z). \end{aligned} \tag{3.9}$$

Therefore, letting $n \to \infty$ in (3.9) and (3.8), we have

$$d(z,Tz) = d(A,B), \tag{3.10}$$

i.e., z is a best proximity point of T in A.

In order to show the uniqueness, we let z' be another best proximity point of T in A. By Lemma 3.10, we have $z' = T^2z'$. From this fact and Proposition 3.8, we have

$$\begin{aligned} d(Tz,z') &= d(Tz,T^2z') \\ &\leq d(z,Tz') \\ &= d(T^2z,Tz') \\ &\leq d(Tz,z') \end{aligned}$$

Thus

$$d(z',Tz) = d(z,Tz'). \tag{3.11}$$

Next we will show that

$$d(z',Tz) = d(A,B). \tag{3.12}$$

Suppose that $d(z',Tz) > d(A,B)$. Then

$$\begin{aligned} d(Tz',z) &= d(Tz',T^2z) \\ &\leq d(z',Tz) - \varphi(d(z',Tz)) + \varphi(d(A,B)) \\ &< d(z',Tz), \end{aligned}$$

which contradictions to (3.11). Hence (3.12) holds. It follows from (3.10), (3.12) and the property UC of (A,B) that $d(z,z') = 0$. Thus $z = z'$.

The conditions (ii) and (iii) follow from (i) and Lemma 3.10.

(iv) is obtained in the proof of (i). □

Remark 3.14. *In case of $d(A,B) = 0$, we know that (A,B) satisfies the property UC. Hence we obtain the existence and uniqueness of the fixed point of a cyclic φ -contraction map.*

Remark 3.15. *For nonempty subsets A and B of a uniformly convex Banach space X, we know that (A,B) has the property UC if A is convex. By this fact, it is obvious that Theorem 3.9 is a consequence of Theorem 3.13. But the converse is not true (see [21] in page 451).*

The following examples show that there exist cyclic φ-contraction maps which do not satisfy (3.3).

Example 3.16.

1. *Let $X := \mathbb{R}$ with the usual metric. For $A = B = [0,1]$, define $T : A \cup B \to A \cup B$ by $Tx := \dfrac{x}{1+x}$. If $\varphi(t) := \dfrac{t^2}{1+t}$ for $t \geq 0$, then T is a cyclic φ-contraction map ([1]). We see that T does not satisfy (3.3). Indeed, suppose there exists $r \in [0,1)$ such that*

$$d(Tx,Ty) \leq r \max\{d(x,y), d(x,Tx), d(y,Ty)\} + (1-r)d(A,B).$$

If $y = Tx$, then we have

$$d(Tx,T^2x) \leq r \max\{d(x,Tx), d(x,Tx), d(Tx,T^2x)\} + (1-r)d(A,B).$$

Thus

$$d(Tx,T^2x) \leq rd(x,Tx) + (1-r)d(A,B),$$

i.e.,

$$\left|\frac{x}{1+x} - \frac{x}{1+2x}\right| \leq r\left|x - \frac{x}{1+x}\right| + (1-r)\cdot 0.$$

This implies $\dfrac{1}{1+2x} \leq r$ for every $x \in (0,1]$.

Hence, as $x \to 0$, $r \geq 1$ which is a contradiction.

2. *Let $X := \mathbb{R}$ with the usual metric. For $A = [0,1]$ and $B = [-1,0]$, define $T : A \cup B \to A \cup B$ by $Tx := \dfrac{-x}{1+x}$ if $x \in A$ and $Tx := \dfrac{-x}{1-x}$ if $x \in B$. If $\varphi(t) := \dfrac{t^2}{1+t}$ for $t \geq 0$, then T is a cyclic φ-contraction map ([1]) and by the same argument of the above example, we see that T does not satisfy (3.3).*

It follows from the above examples that, there exist pairs of (A, B) with the property UC such that the existence of a best proximity point cannot be obtained from Theorem 3.4.

3.2 Best proximity point on reflexive Banach spaces

In this section, we give the existence of a best proximity point for a map on a reflexive Banach space in Theorem 3.19 and Theorem 3.22. This implies a positive answer of Question 2. To do this, we begin with the following lemmas.

Lemma 3.17. *Let A and B be nonempty subsets of a strictly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic map such that*

$$\|T^2x - Tx\| \leq \|Tx - x\| \quad \textit{for all} \quad x \in A \cup B.$$

Assume that A is convex and $d(A, B) > 0$. If x is a best proximity point of T in A, then $x = T^2x$.

Proof. Let x is a best proximity point of T in A i.e, $\|x - Tx\| = d(A, B)$.
Since $d(A, B) \leq \|T^2x - Tx\| \leq \|x - Tx\| = d(A, B)$, then $\|T^2x - Tx\| = d(A, B)$. Suppose that $T^2x \neq x$, then $T^2x - Tx \neq x - Tx$. Since A is convex, $\dfrac{T^2x + x}{2} \in A$. It follows from the strict convexity of X that

$$\begin{aligned} d(A, B) &\leq \left\|\frac{T^2x + x}{2} - Tx\right\| \\ &= \left\|\frac{T^2x - Tx}{2} + \frac{x - Tx}{2}\right\| \\ &< d(A, B), \end{aligned}$$

which is a contraction. Thus $T^2x = x$. $\square$

Lemma 3.18. *Suppose that A and B are nonempty subsets of a strictly convex Banach space X such that A is convex and $d(A,B) > 0$. Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map, $x \in A$. Then the followings are equivalent:*

(i) x is a proximity point of T.

(ii) x is a fixed point of T^2.

Moreover, such point x is unique.

Proof. By Lemma 3.17, (i) implies (ii). To prove (ii) implies (i), we can use the same argument of Lemma 3.10.

To show the uniqueness, let x' be another best proximity point of T in A i.e., $\|x'-Tx'\| = d(A,B)$. Then by Lemma 3.17, we have $x = T^2x$ and $x' = T^2x'$.

From this, we have

$$\begin{aligned} \|Tx - x'\| &= \|Tx - T^2x'\| \\ &\leq \|x - Tx'\| \\ &= \|T^2x - Tx'\| \\ &\leq \|Tx - x'\|. \end{aligned}$$

Thus

$$\|x' - Tx\| = \|x - Tx'\|. \tag{3.13}$$

Next we will show that

$$\|x' - Tx\| = d(A,B). \tag{3.14}$$

Suppose that $\|x' - Tx\| > d(A,B)$. Then

$$\begin{aligned} \|Tx' - x\| &= \|Tx' - T^2x\| \\ &\leq \|x' - Tx\| - \varphi(\|(x', Tx\|) + \varphi(d(A,B)) \\ &< \|x' - Tx\|, \end{aligned}$$

which contradiction to (3.13). Hence (3.14) holds. Since A is convex, $\frac{x+x'}{2} \in A$. It follows from the strict convexity of X that

$$\begin{aligned} d(A,B) &\leq \|\frac{x+x'}{2} - Tx\| \\ &= \|\frac{x-Tx}{2} + \frac{x'-Tx}{2}\| \\ &< d(A,B), \end{aligned}$$

which is a contradiction. Hence $x' = x$. □

Theorem 3.19. *Let A and B be nonempty closed convex subsets of a reflexive Banach space X such that A is bounded, $d(A,B) > 0$. And let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. Then T has a best proximity point in A if one of the following conditions is satisfied:*

(i) T is weakly continuous on A.

(ii) T satisfies the proximal property.

Furthermore, the best proximity point of T is unique if X is strictly convex.

Proof. For $x_0 \in A$, let $x_{n+1} = Tx_n$ for each $n \geq 0$. Since A is bounded, the sequence $\{x_{2n}\}$ also bounded. As X is reflexive and A is closed convex, we have $\{x_{2n}\}$ has a subsequence $\{x_{2n_i}\}$ converges weakly to $x \in A$.

(i) Since T is weakly continuous on A and $T(A) \subseteq B$, $\{x_{2n_i+1}\}$ converges weakly to $Tx \in B$. Therefore $x_{2n_i} - x_{2n_i+1}$ converge weakly to $x - Tx$. Hence $\|x - Tx\| \leq \liminf \|x_{2n_i} - x_{2n_i+1}\| = d(A,B)$, so $\|x - Tx\| = d(A,B)$.

(ii) Since $x_{2n_i} \rightharpoonup x$ and by Remark 3.11, we have $\lim\limits_{n_i \to \infty} d(x_{2n_i}, x_{2n_i+1}) = d(A,B)$. Hence by the the proximal property, we get that $\|x - Tx\| = d(A,B)$.
When X is strictly convex. The uniqueness follows from Lemma 3.18. □

Remark 3.20. *In case $d(A,B) = 0$, the result of Theorem 3.19 follows from Theorem 3.13.*

To prove the Theorem 3.22 we need the following lemma.

Lemma 3.21. *[18] Let A and B be nonempty closed convex subsets of a reflexive Banach space X. If A is bounded, then there exists $(x,y) \in A \times B$ such that $\|x - y\| = d(A,B)$.*

Theorem 3.22. *Let A and B be nonempty closed convex subsets of a reflexive strictly convex Banach space X such that A is bounded, $d(A,B) > 0$ and let $T : A \cup B \to A \cup B$ be a cyclic and*

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all} \quad x \in A \quad \text{and} \quad y \in B. \tag{3.15}$$

If $(A-A) \cap (B-B) = \{0\}$, then T has a unique best proximity point $x \in A$. Furthermore x is a fixed point of T^2.

Proof. By Lemma 3.21, there exists $(x,y) \in A \times B$ such that $\|x - y\| = d(A,B)$. Hence

$$\begin{aligned} d(A,B) &\le \|Tx - Ty\| \\ &\le \|x - y\| \\ &= d(A,B). \end{aligned}$$

This implies

$$\|Tx - Ty\| = d(A,B). \tag{3.16}$$

Suppose that there exists $(x', y') \in A \times B$ such that $(x', y') \neq (x, y)$ and $\|x' - y'\| = d(A,B)$.

Case I. $x - y = x' - y'$. Then we have $x - x' = y - y' \in (A - A) \cap (B - B) = \{0\}$. This show that $x - x' = y - y' = 0$. Hence $x = x'$ and $y = y'$, i.e., $(x,y) = (x', y')$ which is a contradiction.

Case II. $x - y \neq x' - y'$. Since A and B are convex, $\dfrac{x + x'}{2} \in A$ and $\dfrac{y + y'}{2} \in B$. It follows from the strict convexity of X that

$$\begin{aligned} d(A,B) &\le \left\|\frac{x + x'}{2} - \frac{y + y'}{2}\right\| \\ &= \left\|\frac{x - y}{2} + \frac{x' - y'}{2}\right\| \\ &< d(A,B), \end{aligned}$$

which is a contradiction.

Thus there exists a unique order pair $(x,y) \in A \times B$ such that $\|x - y\| = d(A,B)$. By this fact and (3.16), we have $x = Ty$ and $y = Tx$. Hence $\|x - Tx\| = d(A,B)$ and $T^2 x = x$. $\square$

Corollary 3.23. *Let A and B be nonempty closed convex subsets of a reflexive strictly convex Banach space X such that A is bounded, $d(A,B) > 0$ and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. If $(A-A) \cap (B-B) = \{0\}$, then T has a unique best proximity point $x \in A$. Furthermore x is a fixed point of T^2.*

Remark 3.24. *In case $d(A,B) = 0$, the result of Corollary 3.23 follows from Theorem 3.13.*

Remark 3.25. *In case of A and B are unbounded, the following example shows that the conclusion of Theorem 3.22 may be not true. Let*

$A = \{(x,y) : 1 \le x < \infty,\ 1 + \frac{1}{x} \le y \le 2\}$ and

$B = \{(x,y)\ :\ 1 \le x < \infty,\ -1 \le y \le -\frac{1}{x}\}$. We can see that $d(A,B) = 1$ and there exist no points $a \in A, b \in B$ such that $\|a-b\| = d(A,B)$.

Theorem 3.26. *Let A and B be nonempty closed convex subsets of a reflexive Banach space X, and let $T : A \cup B \to A \cup B$ be a cyclic map. Assume that for fixed $x_0 \in A$, a sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ has the following properties :*

(1) $\{x_{2n}\}$ has a weakly convergent subsequence,

(2) $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = d(A,B)$.

Then there exists $(x,y) \in A \times B$ such that $\|x-y\| = d(A,B)$.

Proof. Fix $x_0 \in A$, and define $x_{n+1} := Tx_n$. From (1) and (2), there exists $\{x_{2n_i}\}$ converges weakly to $x \in A$ and $\lim\limits_{n_i\to\infty} \|x_{2n_i} - x_{2n_i+1}\| = d(A,B)$. From this fact and the inequality

$$\|x_{2n_i+1}\| \le \|x_{2n_i} - x_{2n_i+1}\| + \|x_{2n_i}\|,$$

we obtain that $\{x_{2n_i+1}\}$ is bounded. Thus there exists subsequence $\{x_{2n_{i_j}+1}\}$ converges weakly to $y \in B$. Therefore $x_{2n_{i_j}} - x_{2n_{i_j}+1}$ converge weakly to $x-y$. Hence $\|x-y\| \le \liminf \|x_{2n_{i_j}} - x_{2n_{i_j}+1}\| = d(A,B)$, so $\|x-y\| = d(A,B)$. □

CHAPTER IV

MULTI-VALUED WEAK CONTRACTION MAPPING

Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X and let T be a multi-valued mapping, i.e., $T : X \to \mathcal{P}(X)$. An element $x \in X$ such that $x \in Tx$ is called a ***fixed point*** of T. We denote by $\mathcal{CB}(X)$ the family of all nonempty closed bounded subsets of X, $\mathcal{K}(X)$ the class of all nonempty compact subsets of X and F_T the set of all fixed points of T, i.e., $F_T = \{x \in X : x \in Tx\}$. Let (X, d) be a metric space. If x_0 is a point of X and A, B are nonempty subsets of X, the ***distance between*** x ***and*** A defined by

$$d(x, A) = \inf\{d(x, a) : a \in A\},$$

the ***distance between*** A ***and*** B defined by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Let $h(A, B) = \sup\{d(a, B) : a \in A\}$, the ***Hausdorff-Pompeiu generalized function on*** $\mathcal{P}(X)$ ***induced by*** d defined by

$$H(A, B) = \max\{h(A, B), h(B, A)\}.$$

For $x, y \in X$, and nonempty subsets A, B, C of X, the followings are obvious ;

1. if $A \subseteq B$, then $d(A, C) \geq d(B, C)$ and $h(A, C) \leq h(B, C)$ and $h(C, A) \geq h(C, B)$,
2. $d(x, A) \leq d(x, y) + d(y, A)$,
3. $d(x, A) \leq d(x, y) + d(y, B) + h(B, A)$,
4. $d(x, A) \leq d(x, B) + h(B, A)$,
5. if $x \in A$, then $d(x, B) \leq H(A, B)$.

The following property of the functional H are well-known.

Lemma 4.1. *Let (X,d) be a metric space, $A,B \in \mathcal{P}(X)$ and $q \in \mathbb{R}, q > 1$ be given. Then for every $a \in A$ there exists $b \in B$ such that $d(a,b) \leq qH(A,B)$.*

Theorem 4.2. *Let X be a metric space, then H is a metric on $\mathcal{CB}(X)$.*

Theorem 4.3 extends the Banach Contraction Theorem from single-valued maps to multi-valued maps.

A multi-valued mapping $T : X \to P(X)$ is called a ***c-contraction mapping*** if there exists a constant $c \in (0,1)$ such that

$$H(Tx, Ty) \leq cd(x,y) \text{ for all } x, y \in X.$$

Theorem 4.3. *[23](Nadler Theorem) Let (X,d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued c-contraction map. Then T has at least one fixed point.*

Given a point $x \in X$ and a compact set $A \subset X$ we know that there exist $a^* \in A$ such that $d(x,a^*) = d(x,A)$. We call a^* the ***metric projection*** of the point x on the set A and denote it by $a^* = P_x A$. Obviously $P_x A$ is not unique but we choose one of it.

Let $T : X \to \mathcal{P}(X)$ be a multi-valued such that Tx is nonempty and compact for all $x \in X$. We define the projection associated with a multi-valued T by $Px = P_x Tx$. For $x_0 \in X$, we define $x_{n+1} = Px_n$, $n = 0, 1, 2, \ldots$ and we call the sequence $\{x_n\}_{n=0}^{\infty}$ a ***Picard projection iteration sequence of*** T ***at a point*** x_0.

In 2007, Kunze, La Torre and Vrscay [19] extended Theorem 2.3-Theorem 2.5 to a compact multi-valued c-contraction mapping.

Theorem 4.4-Theorem 4.6 are results of [19].

Theorem 4.4. *[19] Let (X,d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multi-valued c-contraction mapping. Then for any $x_0 \in X$, the Picard projection iteration sequence $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in F_T$.*

Theorem 4.5. *[19] Let (X,d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multi-valued c-contraction mapping. Then*

$$d(x_0, F_T) \leq \frac{1}{1-c} d(x_0, Tx_0),$$

for all $x_0 \in X$.

Theorem 4.6. *[19] Let (X,d) be a complete metric space and $T_1, T_2 : X \to \mathcal{K}(X)$ be multi-valued contraction mappings with contraction factors c_1 and c_2, respectively. If F_{T_1} and F_{T_2} are compact, then*

$$H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \min\{c_1, c_2\}},$$

where $d_\infty(T_1, T_2) = \sup_{x \in X} H(T_1 x, T_2 x)$.

4.1 Multi-Valued Zamfirescu Mapping

In 1972, Zamfirescu[33] introduced a class of mappings need not continuous called Zamfirescu mapping. We give here the definition.

Definition 4.7. *[33] Let (X,d) be a metric space, $T : X \to X$ is called **Zamfirescu mapping** if there exist positive real numbers a, b and c satisfying $a < 1$, $b < \frac{1}{2}$ and $c < \frac{1}{2}$, such that, for each $x, y \in X$ at least one of the followings is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Theorem 4.8 is the main result of [33].

Theorem 4.8. *[33] Let (X,d) be a complete metric space and T be a self map on X. If T is a Zamfirescu mapping, then T has a unique fixed point.*

Zamfirescu' theorem (Theorem 4.8) is a generalization of Banach's theorem (Theorem 2.3), Kannan' s theorem (Theorem 2.6) and Chatterjea's theorem (Theorem 2.7).
In this section, we extend the definition of Zamfirescu mappings for multi-valued mappings.

Definition 4.9. *Let (X,d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a **multi-valued Zamfirescu** mapping if there exist positive real numbers a, b and c satisfying $a < 1$, $b < \frac{1}{2}$ and $c < \frac{1}{2}$, such that, for each $x, y \in X$ at least one of the followings is true:*

($\tilde{z}_1$) $H(Tx,Ty) \leq ad(x,y)$,

($\tilde{z}_2$) $H(Tx,Ty) \leq b[d(x,Tx)+d(y,Ty)]$,

($\tilde{z}_3$) $H(Tx,Ty) \leq c[d(x,Ty)+d(y,Tx)]$.

Theorem 4.12 and Theorem 4.13 are main results of this section. It basically shows that any multi-valued Zamfirescu mapping has a fixed point. To do this, we begin with the following lemmas.

Lemma 4.10. *Let (X,d) be a complete metric space and $\{x_n\}$ be a sequence in X. If there exists a positive number $\alpha<1$ such that $d(x_n,x_{n+1}) \leq \alpha d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges to some $u \in X$ which the following estimates*

(1) $d(x_n,u) \leq \dfrac{\alpha^n}{1-\alpha} d(x_0,x_1), \qquad n=0,1,2,\ldots$

(2) $d(x_n,u) \leq \dfrac{\alpha}{1-\alpha} d(x_{n-1},x_n), \qquad n=1,2,,3,\ldots$

hold.

Proof. Let $\{x_n\}$ be a sequence in X such that there exists a positive number $\alpha<1$ such that $d(x_n,x_{n+1}) \leq \alpha d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$.
By mathematical induction, we have

$$d(x_n,x_{n+1}) \leq \alpha^n d(x_0,x_1), \quad n \in \mathbb{N} \tag{4.1}$$

and, respectively,

$$d(x_{n+k-1},x_{n+k}) \leq \alpha^k d(x_{n-1},x_n), \qquad k \in \mathbb{N}\cup\{0\}. \tag{4.2}$$

So, for any $n,p \in \mathbb{N}$, by (4.1), we obtain

$$\begin{aligned} d(x_n,x_{n+p}) &\leq \sum_{k=n}^{n+p-1} d(x_k,x_{k+1}) \\ &\leq \sum_{k=n}^{n+p-1} \alpha^k d(x_0,x_1) \\ &= \frac{\alpha^n(1-\alpha^p)}{1-\alpha} d(x_0,x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} d(x_0,x_1). \end{aligned} \tag{4.3}$$

Since $0 < \alpha < 1$, it yields that $\alpha^n \to 0$ (as $n \to \infty$), which together with (4.3) implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (X, d) is complete, $\{x_n\}_{n=0}^{\infty}$ converges to some $u \in X$.

By (4.2), we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \sum_{k=1}^{p} d(x_{n+k-1}, x_{n+k},) \\ &\leq \sum_{k=1}^{p} \alpha^k d(x_{n-1}, x_n) \\ &= \frac{\alpha(1-\alpha^p)}{1-\alpha} d(x_{n-1}, x_n) \\ &\leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n). \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), by letting $p \to \infty$, we have the estimations (1) and (2). □

Lemma 4.11. *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping and let $\{x_n\}$ be a sequence in X such that for all $n = 0, 1, 2, \ldots,$ $x_{n+1} \in Tx_n$. If $\lim_{n\to\infty} x_n = u$, then u is a fixed point of T.*

Proof. Let $\lim_{n\to\infty} x_n = u$. We have

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &\leq d(u, x_{n+1}) + H(Tx_n, Tu). \end{aligned}$$

In case x_n, u satisfy $(\tilde{z}_1)$, we have

$$d(u, Tu) \leq d(u, x_{n+1}) + ad(x_n, u). \tag{4.5}$$

If x_n, u satisfy $(\tilde{z}_2)$, then

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + b[d(x_n, Tx_n) + d(u, Tu)] \\ &\leq d(u, x_{n+1}) + b[d(x_n, x_{n+1}) + d(u, Tu)]. \end{aligned} \tag{4.6}$$

Suppose that x_n, u satisfy $(\tilde{z}_3)$. Then

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + c[d(x_n, Tu) + d(u, Tx_n)] \\ &\leq d(u, x_{n+1}) + c[d(x_n, Tu) + d(u, x_{n+1})]. \end{aligned} \tag{4.7}$$

Therefore, letting $n \to \infty$ in (4.5), (4.6, (4.7), we get $d(u, Tu) = 0$.

Since Tu is closed, $u \in Tu$. Hence u is a fixed point of T. □

Theorem 4.12. *Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping. Then T has at least one fixed point.*

Proof. Choose q be such that $1 < q < \min\{\frac{1}{a}, \frac{1}{2b}, \frac{1}{2c}\}$ and $\alpha = \max\{qa, \frac{qb}{1-qb}, \frac{qc}{1-qc}\}$. Note that $0 < \alpha < 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$.

If $H(Tx_0, Tx_1) = 0$ then $Tx_0 = Tx_1$, i.e., $x_1 \in Tx_1$. This means that $F_T \neq \varnothing$.

Suppose that $H(Tx_0, Tx_1) > 0$. By Lemma 4.1 there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq qH(Tx_0, Tx_1).$$

For x_1 and x_2, if $H(Tx_1, Tx_2) = 0$ then $Tx_1 = Tx_2$, i.e., $x_2 \in Tx_2$. Hence T has a fixed point. Let $H(Tx_1, Tx_2) > 0$. Again by Lemma 4.1 there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq qH(Tx_1, Tx_2).$$

By this procedure, we have two cases.

Firstly, if there is k^{th} procedure such that $H(Tx_{k-1}, Tx_k) = 0$, then x_k is a fixed point.

Secondly, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq qH(Tx_{n-1}, Tx_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

If x_{n-1}, x_n satisfy $(\tilde{z}_1)$, then we have

$$d(x_n, x_{n+1}) \leq qad(x_{n-1}, x_n). \tag{4.8}$$

If x_{n-1}, x_n satisfy $(\tilde{z}_2)$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq qb[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\leq qb[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Hence

$$d(x_n, x_{n+1}) \leq (\frac{qb}{1-qb})d(x_{n-1}, x_n). \tag{4.9}$$

For the last case, if x_{n-1}, x_n satisfy $(\tilde{z}_3)$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq qc[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &= qcd(x_{n-1}, Tx_n) \\ &\leq qcd(x_{n-1}, x_{n+1}) \\ &\leq qc[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Thus

$$d(x_n, x_{n+1}) \leq (\frac{qc}{1-qc})d(x_{n-1}, x_n). \tag{4.10}$$

From (4.8), (4.9), (4.10), we have

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n), \qquad n = 1, 2, , 3, \ldots .$$

Hence by this fact and Lemma 4.10, $\{x_n\}$ converges to some $u \in X$. By Lemma 4.11, u is a fixed point of T. □

The next theorem shows that the Picard projection iteration sequence of a multi-valued Zamfirescu mapping with compact valued converges to its fixed point.

Theorem 4.13. *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multi-valued Zamfirescu mapping. Then*

(1) for any $x_0 \in X$, the Picard projection iteration sequence $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in F_T$;

(2) The following estimates

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \qquad n = 0, 1, 2, \ldots$$
$$d(x_n, x^*) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n), \qquad n = 1, 2, , 3, \ldots$$

hold, for a certain constant $0 < \alpha < 1$.

Proof. Let T be a multi-valued Zamfirescu mapping, i.e., for each $x, y \in X$ satisfying at least one condition $(\tilde{z}_1), (\tilde{z}_2)$ or $(\tilde{z}_3)$, and Tx is compact for all $x \in X$.

(1) Let $x_0 \in X$ be arbitrary and let $\{x_n\}_{n=0}^{\infty}$ be the Picard projection iteration sequence.

For x_1, x_2 we see that

$$d(x_1, x_2) = d(x_1, Px_1) = d(x_1, Tx_1) \leq h(Tx_0, Tx_1) \leq H(Tx_0, Tx_1).$$

If x_0, x_1 satisfy $(\tilde{z}_1)$, then $d(x_1, x_2) \leq ad(x_0, x_1)$.

In case that x_0, x_1 satisfy $(\tilde{z}_2)$, then

$$\begin{aligned} d(x_1, x_2) &\leq b[d(x_0, Tx_0) + d(x_1, Tx_1)] \\ &= b[d(x_0, x_1) + d(x_1, x_2)]. \end{aligned}$$

Hence $$d(x_1,x_2) \;\leq\; \frac{b}{1-b}d(x_0,x_1).$$

Suppose that x_0, x_1 satisfy $(\tilde{z}_3)$. Thus

$$\begin{aligned} d(x_1,x_2) &\leq c[d(x_0,Tx_1)+d(x_1,Tx_0)] \\ &= cd(x_0,Tx_1) \\ &\leq c[d(x_0,x_1)+d(x_1,Tx_1)] \\ &= c[d(x_0,x_1)+d(x_1,x_2)]. \end{aligned}$$

This implies $$d(x_1,x_2) \;\leq\; \frac{c}{1-c}d(x_0,x_1).$$

Therefore, in all cases, we have

$$d(x_1,x_2) \leq \alpha d(x_0,x_1), \quad \text{where} \quad \alpha = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}.$$

Consequently, for each $n \in \mathbb{N}$ we have

$$d(x_n,x_{n+1}) \leq \alpha d(x_{n-1},x_n).$$

Hence by this fact and Lemma 4.10, $\{x_n\}$ converges to some $u \in X$. Thus by Lemma 4.11, we have u is a fixed point of T. The condition (2) holds from Lemma 4.10. □

In example 4.14, we give a mapping T such that Nadler's fixed point theorem can not be applied while we can use Theorem 4.12 to assure that T has a fixed point.

Example 4.14. *Let $X = [0,1]$ and $T : X \to \mathcal{CB}(X)$ be defined by*

$$Tx = \begin{cases} [0,\frac{1}{4}] & ; x \in [0,1) \\ [0,\frac{1}{2}] & ; x = 1. \end{cases}$$

Then T satisfies $(\tilde{z}_3)$ by choosing $c = \frac{1}{3}$. To show this, it suffices to show in case $x = 1$ and $y \in [0,1)$. Note that $Tx = [0,\frac{1}{2}]$ and $Ty = [0,\frac{1}{4}]$. Thus $d(x,Ty) = d(1,[0,\frac{1}{4}]) = \frac{3}{4}$ and $H(Tx,Ty) = H([0,\frac{1}{2}],[0,\frac{1}{4}]) = \frac{1}{4}$.

Hence

$$\begin{aligned} H(Tx,Ty) &= \tfrac{1}{4} \\ &= \tfrac{1}{3}[\tfrac{3}{4}] \\ &\leq \tfrac{1}{3}[d(x,Ty)+d(y,Tx)]. \end{aligned}$$

Therefore T is a a multi-valued Zamfirescu mapping. By Theorem 4.12, T has a fixed point. For $y = 1, x \in [\frac{3}{4}, 1)$, we observe that $H(Tx, Ty) = \frac{1}{4} \geq |x - y|$. This implies that T is not a multi-valued c-contraction for any $c \in (0, 1)$. Then we can not apply Nadler's theorem with this example.

Proposition 4.15. *Let (X,d) be a complete metric space. If T is a multi-valued Zamfirescu mapping then F_T is complete.*

Proof. Let $\{x_n\}_{n=0}^{\infty}$ be a Cauchy sequence in F_T. Since X is complete then there exist $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Hence by Lemma 4.11, u is a fixed point of T. Therefore we have the theorem. □

4.2 Multi-valued Weak Contraction

In 2007, one of interesting generalization of the Nadler's fixed point theorem was given by Berinde [7].

Definition 4.16. *[7] Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a **multi-valued weak contraction** or **multi-valued** (θ, L)-**weak contraction** if there exist constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(x, Ty), \text{ for all } x, y \in X.$$

Remark 4.17. *It is known that a c-contraction mapping is a weak contraction.*

Theorem 4.18. *[7] Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued weak contraction mapping. Then T has at least one fixed point.*

In this section, we prove that the Picard projection iteration sequence converges to a fixed point, give a rate of convergence and generalize Collage Theorem (Theorem 2.4 and Theorem 4.5) for the case of multi-valued weak contraction.

Lemma 4.19. *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multi-valued (θ, L)-weak contraction mapping and let $\{x_n\}$ be a sequence in X such that for all $n = 0, 1, 2, \ldots$, $x_{n+1} \in Tx_n$. If $\lim_{n\to\infty} x_n = x^*$, then x^* is a fixed point of T.*

Proof. Assume that $\lim_{n\to\infty} x_n = x^*$. Thus we have

$$\begin{aligned} d(x^*, Tx^*) &\le d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &\le d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + \theta d(x_n, x^*) + Ld(x^*, Tx_n) \\ &\le d(x^*, x_{n+1}) + \theta d(x_n, x^*) + L[d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n)] \\ &= d(x^*, x_{n+1}) + \theta d(x_n, x^*) + Ld(x^*, x_{n+1}), \end{aligned}$$

for all $n = 0, 1, 2, ...$

Hence we have $d(x^*, Tx^*) = 0$ by letting $n \to \infty$. Since Tx^* is closed, $x^* \in Tx^*$. Therefore x^* is a fixed point of T. □

The next Proposition is immediately obtained from Lemma 4.19.

Proposition 4.20. *Let (X,d) be a complete metric space. If T is a multi-valued weak contraction mapping then F_T is complete.*

Theorem 4.21. *Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multi-valued (θ, L)-weak contraction. Then*

(1) for any $x_0 \in X$, the Picard projection iteration sequence $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in F_T$;

(2) the following estimates

$$d(x_n, x^*) \le \frac{\theta^n}{1-\theta} d(x_0, x_1), \qquad n = 0, 1, 2, \ldots$$

$$d(x_n, x^*) \le \frac{\theta}{1-\theta} d(x_{n-1}, x_n), \qquad n = 1, 2, , 3, \ldots$$

hold.

Proof. Fix $x_0 \in X$ and $\{x_n\}_{n=0}^{\infty}$ the Picard projection iteration.

For each $n \in \mathbb{N}$ we see that,

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, Px_n) \\
&= d(x_n, Tx_n) \\
&\leq H(Tx_{n-1}, Tx_n) \\
&\leq \theta d(x_{n-1}, x_n) + Ld(x_n, Tx_{n-1}) \\
&= \theta d(x_{n-1}, x_n). \qquad (4.11)
\end{aligned}
$$

By this fact and Lemma 4.10, $\{x_n\}$ converges to some $x^* \in X$. Thus by Lemma 4.19, u is a fixed point of T and (2) holds by Lemma 4.10.

□

Corollary 4.22. *(Generalized Collage Theorem) Let (X, d) be a complete metric space and $T : X \to \mathcal{K}(X)$ be a multi-valued (θ, L)-weak contraction mapping. Then*

$$d(x_0, F_T) \leq \frac{1}{1-\theta} d(x_0, Tx_0),$$

for all $x_0 \in X$.

Proof. Let $x_0 \in X$. By Theorem 4.21(1), there exists a point $x^* \in F_T$ such that the Picard projection iteration $\{x_n\}_{n=0}^{\infty}$ converges to x^*. Hence, by Theorem 4.21(2), we have

$$d(x_0, F_T) \leq d(x_0, x^*) \leq \frac{1}{1-\theta} d(x_0, x_1) = \frac{1}{1-\theta} d(x_0, Tx_0).$$

□

Theorem 4.23. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to \mathcal{K}(X)$ be multi-valued weak contractions with parameters (θ_1, L_1) and (θ_2, L_2), respectively. If F_{T_1} and F_{T_2} are closed and bounded, then*

$$H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \max\{\theta_1, \theta_2\}},$$

where $d_\infty(T_1, T_2) = \sup_{x \in X} H(T_1 x, T_2 x)$.

Proof. Let $x \in F_{T_1}$. By Corollary 4.22, we have

$$(1-\theta_2)d(x, F_{T_2}) \leq d(x, T_2x) \leq H(T_1x, T_2x) \leq d_\infty(T_1, T_2).$$

Take the supremum with respect to $x \in F_{T_1}$, we get

$$(1-\theta_2)h(F_{T_1}, F_{T_2}) \leq d_\infty(T_1, T_2).$$

Next, upon interchanging F_{T_1} with F_{T_2} we obtain

$$(1-\theta_1)h(F_{T_2}, F_{T_1}) \leq d_\infty(T_1, T_2).$$

. Hence

$$H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1-\max\{\theta_1, \theta_2\}}.$$

□

Remark 4.24. Since a c-contraction is a $(\theta, 0)$-weak contraction, Theorem 4.4 can be deduced from Theorem 4.21 and Theorem 4.5 can be deduced from Corollary 4.22.

The next corollary is immediate using Theorem 4.23.

Corollary 4.25. *Let (X, d) be a complete metric space and $T_n : X \to \mathcal{K}(X)$ be a sequence of multi-valued weak contractions with weak contractivity constants θ_n such that $\sup_n \theta_n = \theta < 1$. Suppose that $T_n \to T$ with the metric d_∞ and T is a compact multi-valued weak contraction. Then $F_{T_n} \to F_T$ in the Hausdorff metric.*

Corollary 4.25 implies the following corollary.

Corollary 4.26. *[19] Let (X, d) be a complete metric space and $T_n : X \to \mathcal{K}(X)$ be a sequence of multi-valued contractions with contractivity constants a_n such that $\sup_n a_n = a < 1$. Suppose that $T_n \to T$ with the metric d_∞ and T is a compact multi-valued contraction. Then $F_{T_n} \to F_T$ in the Hausdorff metric.*

The next theorem shows that a multi-valued Zamfirescu mapping is a multi-valued weak contraction. Hence Theorem 4.13 follows immediately from Theorem 4.21.

Theorem 4.27. *Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping. Then T is a multi-valued weak contraction.*

Proof. Let T be a multi-valued Zamfirescu mapping and $x, y \in X$. Then at least one of $(\tilde{z_1})$, $(\tilde{z_2})$ or $(\tilde{z_3})$ is true with parameters a, b and c, respectively.

If x and y satisfy $(\tilde{z_1})$, then $H(Tx, Ty) \leq ad(x, y)$.

In case that x and y satisfy $(\tilde{z_2})$ we see that

$$\begin{aligned} H(Tx, Ty) &\leq b[d(x, Tx) + d(y, Ty)] \\ &\leq b[[d(x, Ty) + H(Tx, Ty)] + [d(y, x) + d(x, Ty)]] \\ &= bd(x, y) + 2bd(x, Ty) + bH(Tx, Ty). \end{aligned}$$

Hence $H(Tx, Ty) \leq \dfrac{b}{1-b} d(x, y) + \dfrac{2b}{1-b} d(x, Ty)$.

If x and y satisfy $(\tilde{z_3})$, then we have

$$\begin{aligned} H(Tx, Ty) &\leq c[d(x, Ty) + d(y, Tx)] \\ &\leq c[d(x, Ty) + [d(y, x) + d(x, Ty) + H(Tx, Ty)]] \\ &= cd(x, y) + 2cd(x, Ty) + cH(Tx, Ty). \end{aligned}$$

Thus $H(Tx, Ty) \leq \dfrac{c}{1-c} d(x, y) + \dfrac{2c}{1-c} d(x, Ty)$.

Let

$$\theta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}.$$

Then we have $0 < \theta < 1$ and for all $x, y \in X$,

$$H(Tx, Ty) \leq \theta d(x, y) + 2\theta d(x, Ty).$$

Hence T is a multi-valued $(\theta, 2\theta)$-weak contraction. □

The following example shows that a multi-valued weak contraction may not be a Zamfirescu mapping.

Example 4.28. *Let $X = [0, 1]$ and $Tx = \{x\}$ for all $x \in X$. Then T is a multi-valued weak contraction and is not a Zamfirescu mapping.*

Proof. Recall that, for all $x, y \in X$,

$$\begin{aligned} H(Tx, Ty) &= H(\{x\}, \{y\}) \\ &= d(x, y) \\ &= \theta d(x, y) + (1 - \theta) d(x, y) \end{aligned}$$

for $\theta \in (0,1)$. Then T is a $(\theta, 1-\theta)$-weak contraction for all $\theta \in (0,1)$. If T is a multi-valued Zamfirescu mapping, Then there exist positive real numbers a, b and c satisfying $a < 1$, $b < \frac{1}{2}$ and $c < \frac{1}{2}$, such that, for each $x, y \in X$ at least one of the followings is true:

$(\tilde{z}_1)$ $H(Tx, Ty) \leq ad(x, y)$,

$(\tilde{z}_2)$ $H(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

$(\tilde{z}_3)$ $H(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Let $\theta = \max\{a, 2b, 2c\}$. Observe that $0 < \theta < 1$. Thus for all $x, y \in X$, we have

$$\begin{aligned} H(Tx, Ty) &\leq \theta \max\{d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\} \\ &= \theta|x - y| \\ &< \theta|x - y|. \end{aligned}$$

This is impossible, since $H(Tx, Ty) = |x - y|$. Hence T is not a multi-valued Zamfirescu mapping. □

CHAPTER V

CYCLIC MULTI-VALUED MAPPING

In this chapter, we extend the cyclic property of single-valued mapping to multi-valued mapping and investigate the existence of its fixed point and the existence of a best proximity point.

To establish our results, we introduce the following class of multi-valued mappings.

Definition 5.1. *Let A and B be nonempty subsets of a metric space (X, d). A multi-valued mapping T is said to be cyclic(on A and B) if $T : A \cup B \to \mathcal{P}(X)$ such that $Tx \subseteq B$, for all $x \in A$ and $Ty \subseteq A$ for all $y \in B$.*

5.1 Fixed point of cyclic multi-valued mapping

The next theorem is the main result of this section. It basically shows that any cyclic multi-valued mappings which satisfy a contractive condition has at least one fixed point.

Theorem 5.2. *Let A and B be nonempty closed subsets of a metric space (X, d). Suppose T is a cyclic(on A and B) multi-valued mapping with nonempty closed bounded valued. If there exists a constant $k \in (0, 1)$ such that*

$$H(Tx, Ty) \le kd(x, y) \quad \text{for all } x \in A \text{ and } y \in B.$$

Then T has at lest one fixed point in $A \cap B$.

Proof. Let $\alpha > 1$ be such that $\alpha k < 1$. Let $x_0 \in A$ and $x_1 \in Tx_0 \subseteq B$.
If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, i.e., $x_1 \in Tx_1$ which actually means that $F_T \neq \varnothing$. Assume that $H(Tx_0, Tx_1) > 0$. By Lemma 4.1, there exists $x_2 \in Tx_1 \subseteq A$ such that $d(x_1, x_2) \le \alpha H(Tx_0, Tx_1)$. By the definition of T, we have $d(x_1, x_2) \le \alpha k d(x_0, x_1)$.
Next, if $H(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$, i.e., $x_2 \in Tx_2$. This means that $F_T \neq \varnothing$. If $H(Tx_1, Tx_2) > 0$, by Lemma 4.1, there exists $x_3 \in Tx_2 \subseteq B$ such that $d(x_2, x_3) \le$

$\alpha kd(x_1, x_2)$.

Continue this procedure, we have two cases.

Case I. There is k^{th} procedure such that $H(Tx_{k-1}, Tx_k) = 0$. Then x_k is a fixed point.

Case II. There exists a sequence $\{x_n\}$ such that $x_{2n} \in A,\ x_{2n+1} \in B$ and for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \alpha kd(x_{n-1}, x_n).$$

Since $0 < \alpha k < 1$, by Lemma 4.10, $\{x_n\}$ converges. Let $x := \lim x_n$. Note that

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n+1}) + d(x_{n+1}, Tx) \\ &\leq d(x, x_{n+1}) + H(Tx_n, Tx) \\ &\leq d(x, x_{n+1}) + kd(x_n, x). \end{aligned}$$

By letting $n \to \infty$, we obtain $d(x, Tx) = 0$. Since Tx is closed, $x \in Tx$. Therefore x is a fixed point of T. To show $x \in A \cap B$, we note that $x_{2n} \to x$, $x_{2n+1} \to x$. Since A and B are closed, $x \in A \cap B$. $\square$

Theorem 5.2 can be extended to a finite chain of nonempty closed subsets of X.

Theorem 5.3. *Let $\{A_i\}_{i=1}^n$ be a finite family of nonempty closed subsets of complete metric space. Suppose $T : \bigcup_{i=1}^n A_i \to \mathcal{P}(X)$ with closed bounded valued satisfying the following conditions (where $A_{n+1} = A_1$)*

(a) $Ta_i \subseteq A_{i+1}$, for $a_i \in A_i$ and $1 \leq i \leq n$;

(b) $\exists k \in (0, 1)$ such that $H(Tx, Ty) \leq kd(x, y)$ for all $x \in A_i$ and $y \in A_{i+1}$, for $1 \leq i \leq n$.

Then T has at lest one fixed point in $\bigcap_{i=1}^n A_i$.

The proof of Theorem 5.3 is similar to that of Theorem 5.2.

Similarly, the proof of Theorem 5.2 we can extend to next theorem.

Theorem 5.4. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose T is a cyclic(on A and B) multi-valued mapping. If there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx)$$

and

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(x, Ty\}$$

for all $x \in A$ *and* $y \in B$*, with closed bounded valued. Then* T *has at lest one fixed point in* $A \cap B$ *and* F_T *is complete.*

Proof. The proof of existence of a fixed point are essentially similar to those of Theorem 5.2 and, therefore, are omitted here.

To show that F_T is complete, it suffice to prove that F_T is closed. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in F_T, which $x_n \to u$. Since $F_T \subseteq A \cap B$ and $A \cap B$ is closed, we have $u \in A \cap B$.

Note that

$$\begin{aligned}
d(u, Tu) &\leq d(u, x_n) + d(x_n, Tx_n) + H(Tx_n, Tu) \\
&= d(u, x_n) + H(Tx_n, Tu) \\
&\leq d(u, x_n) + \theta d(x_n, u) + Ld(u, Tx_n) \\
&\leq d(u, x_n) + \theta d(x_n, u) + L(d(u, x_n) + d(x_n, Tx_n)) \\
&= d(u, x_n) + \theta d(x_n, u) + Ld(u, x_n).
\end{aligned}$$

Therefore, letting $n \to \infty$, we get $d(u, Tu) = 0$. Since Tu is closed, $u \in Tu$. Hence F_T is closed. □

The next Corollary 5.5 is immediately obtained from Theorem 5.4.

Corollary 5.5. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d)*. Suppose* T *is a cyclic multi-valued mapping. If there exist two constants* $\theta \in (0, 1)$ *and* $L \geq 0$ *such that* $H(Tx, Ty) \leq \theta d(x, y) + L \min\{d(y, Tx), d(x, Ty)\}$ *for all* $x \in A$ *and* $y \in B$*, with closed bounded valued. Then* T *has at lest one fixed point in* $A \cap B$*.*

Remark 5.6. *Theorem 4.18 is a consequence of Corollary 5.5.*

5.2 Best proximity point of cyclical multi-valued mapping

In this section, we extend the results of the Theorem 5.2 to the case of disjoint sets with property **UC**.

Definition 5.7. *Let A and B be nonempty subsets of a metric space (X, d), and T a multi-valued mapping cyclic on A and B. Then T is said to be k-contraction if there exists a constant $k \in (0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B) \quad \text{for all } x \in A \text{ and } y \in B.$$

To prove the main result of this section, we shall need a lemma.

Lemma 5.8. *Let A and B be nonempty subsets of a metric space (X, d) with the property UC and let $\{x_n\}$ be a sequence in A. If there exists a sequence $\{y_n\}$ in B such that $\lim\limits_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim\limits_{n\to\infty} d(x_{n+1}, y_n) = d(A, B)$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. By the property UC, we have $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Hence, there is $\epsilon > 0$ such that for each $k \in \mathbb{N}$, $\exists m_k > n_k > k$ such that

$$d(x_{n_k}, x_{m_k}) > \epsilon. \tag{5.1}$$

By the assumption, we have

$$\lim_{k\to\infty} d(y_{n_k}, x_{n_k}) = d(A, B). \tag{5.2}$$

Note that

$$\begin{aligned} d(A, B) &\leq d(y_{n_k}, x_{m_k}) \\ &\leq d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ &\leq d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + \cdots + d(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Letting $k \to \infty$, we have

$$\begin{aligned} d(A, B) &\leq \lim_{k\to\infty} d(y_{n_k}, x_{m_k}) \\ &\leq \lim_{k\to\infty} d(y_{n_k}, x_{n_k}) \\ &= d(A, B). \end{aligned}$$

Hence

$$\lim_{k\to\infty} d(y_{n_k}, x_{m_k}) = d(A, B). \tag{5.3}$$

From (5.2), (5.3) and property UC, we have $\lim\limits_{k\to\infty} d(x_{n_k}, x_{m_k}) = 0$. This contradicts to (5.1). $\square$

Theorem 5.9. *Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the property UC and A is complete. Let T be a cyclic(on A and B) k-contraction multi-valued mapping, with closed bounded valued. Then T has a best proximity point z in A.*

Proof. Fix $x_0 \in A$. Let $x_1 \in Tx_0 \subseteq B$. There exists $x_2 \in Tx_1 \subseteq A$ such that

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, Tx_1) + k \\ &\leq H(Tx_0, Tx_1) + k. \end{aligned}$$

Similarly, there exists $x_3 \in Tx_2 \subseteq B$ such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + k^2.$$

Continue this process, we have a sequence $\{x_n\}$ such that $\{x_{2n}\} \subseteq A$, $\{x_{2n+1}\} \subseteq B$, $x_{n+1} \in Tx_n$ and

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + k^n \\ &\leq kd(x_{n-1}, x_n) + (1-k)d(A, B) + k^n \\ &\leq k[H(Tx_{n-2}, Tx_{n-1}) + k^{n-1}] + (1-k)d(A, B) + k^n \\ &= k[kd(x_{n-2}, x_{n-1}) + (1-k)d(A, B) + k^{n-1}] + (1-k)d(A, B) + k^n \\ &= k^2 d(x_{n-2}, x_{n-1}) + (1-k^2)d(A, B) + 2k^n. \end{aligned}$$

Inductively, we have

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + (1-k^n)d(A, B) + nk^n.$$

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$. This implies

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B), \lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B).$$

Since $\{x_{2n}\} \subseteq A$, $\{x_{2n+2}\} \subseteq A$ and $\{x_{2n+1}\} \subseteq B$, by Lemma 5.8 we have $\{x_{2n}\}$ is a Cauchy sequence.

Since A is complete, there exists $z \in A$ such that $\lim_{n\to\infty} d(z, x_{2n}) = 0$.

From this fact and the fact that

$$\begin{aligned} d(A, B) &\leq d(z, x_{2n-1}) \\ &\leq d(z, x_{2n}) + d(x_{2n}, x_{2n-1}), \end{aligned}$$

we have $\lim_{n\to\infty} d(z, x_{2n-1}) = d(A, B)$.

Since

$$\begin{aligned} d(A,B) &\leq d(x_{2n}, Tz) \\ &\leq H(Tx_{2n-1}, Tz) \\ &\leq kd(x_{2n-1}, z) + (1-k)d(A,B) \\ &\leq kd(x_{2n-1}, z) + (1-k)d(x_{2n-1}, z) \\ &= d(x_{2n-1}, z), \end{aligned}$$

we have $d(z, Tz) = d(A, B)$, i.e. z is a best proximity point of T in A. □

Remark 5.10. *In view of the Example 3.3,*

1. *In case of $d(A, B) = 0$, (A, B) satisfies the property UC. By this fact we see that Theorem 5.9 is an extension of Theorem 5.2.*

2. *For nonempty subsets A and B of a uniformly convex Banach space X, we know that (A, B) has the property UC if A is convex. So Theorem 3.2 is a consequence of Theorem 5.9.*

CHAPTER VI

DATA DEPENDENCE

In this chapter, we study a problem of data dependence for a special class of multi-valued mappings. The following data dependence problem is well known

Problem : Let (X,d) be a metric space and T_1, T_2 be two multi-valued maps such that F_{T_1} and F_{T_2} are nonempty and there exists $\eta > 0$ with the property $H(T_1x, T_2x) \leq \eta$ for all $x \in X$. Data dependence problem is a problem of finding an upper bound of $H(F_{T_1}, F_{T_2})$.

In 2003, Rus, Petruşel and Sîntămărian[30] gave an important abstract notions as follows.

Definition 6.1. *Let (X,d) be a metric space and $T : X \to \mathcal{P}(X)$ be a multi-valued mapping. T is said to be a multi-valued weakly Picard (briefly MWP) mapping if for each $x \in X$ and any $y \in Tx$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that*

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in Tx_n$ *for all* $n = 0, 1, 2 \ldots$ *and*

(iii) $\{x_n\}_{n=0}^{\infty}$ *converges to a fixed point of* T.

Remark 6.2. *A sequence $\{x_n\}_{n=0}^{\infty}$ satisfying conditions (i) and (ii) in Definition 6.1 is also called a sequence of successive approximations of T starting from (x, y).*

Definition 6.3. *Let (X,d) be a metric space and $T : X \to \mathcal{P}(X)$ be a MWP mapping. Then we define the multi-valued mapping $T^{\infty} : G(T) \to \mathcal{P}(F_T)$ where $G(T) = \{(x,y) : x \in X, y \in Tx\}$, by the formula;*

$T^{\infty}(x,y) := \{z \in F_T|$ *there exists a sequence of successive approximations of T starting from (x,y) which converges to z* $\}$.

Definition 6.4. *Let (X,d) be a metric space, $T : X \to \mathcal{P}(X)$ be a MWP mapping and $c > 0$. Then T is a $c-$multi-valued weakly Picard (briefly c-MWP) mapping if for every $(x,y) \in G(T)$ there exists a selection $t^\infty(x,y)$ in $T^\infty(x,y)$ such that $d(x,t^\infty(x,y)) \leq cd(x,y)$.*

We shall present some examples of *c-MWP* mappings given in [30]. (See more details in ([11], [23]-[28]).

Example 6.5. *Let (X,d) be a metric space.*

1. *Let $T : X \to \mathcal{CB}(X)$ be a multi-valued a-contraction $(0 < a < 1)$. Then T is a c-MWP mapping with $c = (1-a)^{-1}$.*

2. *Let $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping for which there exist positive real numbers α, β and γ such that $\alpha+\beta+\gamma < 1$ and*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty),$$

   for all $x,y \in X$. Then T is a c-MWP mapping with $c = (1-\gamma)[1-(\alpha+\beta+\gamma)]^{-1}$.

3. *Let $T : X \to \mathcal{CB}(X)$ be a closed multi-valued mapping for which there exist positive real numbers α and β such that $\alpha+\beta < 1$ and*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta d(y,Ty),$$

   for all $x \in X$ and $y \in Tx$. Then T is a c-MWP mapping with $c = (1-\beta)[1-(\alpha+\beta)]^{-1}$.

4. *Let $x_0 \in X$ and $r > 0$. Let $T : \tilde{B}(x_0,r) \to \mathcal{CB}(X)$, where $\tilde{B}(x_0,r) = \{x \in X : d(x_0,x) \leq r\}$ be a multi-valued mapping for which there exist $\alpha,\beta,\gamma \in \mathbb{R}_+$ and $\alpha+\beta+\gamma < 1$ such that*

   (i) $H(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty)$, $\forall x,y \in \tilde{B}(x_0,r)$

   (ii) $\delta(x_0,Tx_0) < [1-(\alpha+\beta+\gamma)](1-\gamma)^{-1}r$.

   Then T is a c-MWP mapping with $c = (1-\gamma)[1-(\alpha+\beta+\gamma)]^{-1}$.

Now we introduce an important abstract notion of Collage mapping as follow.

Definition 6.6. *Let (X,d) be a metric space, $T : X \to \mathcal{P}(X)$, $F_T \neq \varnothing$. Then T is a* ***Collage*** *multi-valued mapping if there exists $c > 0$ such that $d(x, F_T) \leq cd(x, Tx)$ for all $x \in X$.*

Remark 6.7. *It is obvious that a c-MWP mapping(in Definition 6.1) is a Collage multi-valued mapping.*

The following example shows that a Collage multi-valued mapping may not be a c-MWP mapping.

Example 6.8. *Let $T : \{1,2,3\} \to \{1,2,3\}$ be such that $T(1) = 3$, $T(2) = 2$ and $T(3) = 1$. It obvious that T is not MWP mapping. We see that $F_T = \{2\}$, $d(x,Tx) = 2$ and $d(x, F_T) = 1$ where $x \neq 2$ and hence we can choose $c > 0$ with satisfying $d(x, F_T) \leq cd(x, Tx)$ for all $x \in \{1,2,3\}$. Therefore T is a Collage multi-valued mapping.*

Our main results are the followings.

Theorem 6.9. *Let (X,d) be a metric space and $T_1, T_2 : X \to \mathcal{P}(X)$ be two multi-valued mappings. Suppose that:*

(i) T_i is a c_i-Collage multi-valued mapping, for $i \in \{1,2\}$ and

(ii) there exists $\eta > 0$ such that $H(T_1x, T_2x) \leq \eta$, for all $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$.

Proof. We note that

$$H(F_{T_1}, F_{T_2}) = \max\{\sup_{a \in F_{T_2}} d(a, F_{T_1}), \sup_{b \in F_{T_1}} d(b, F_{T_2})\}.$$

Since T_i is a c_i-Collage multi-valued mappings, for $i \in \{1,2\}$,

$$H(F_{T_1}, F_{T_2}) \leq \max\{\sup_{a \in F_{T_2}} c_1 d(a, T_1 a), \sup_{b \in F_{T_1}} c_2 d(b, T_2 b)\},$$

that is

$$H(F_{T_1}, F_{T_2}) \leq \max\{c_1, c_2\} \max\{\sup_{a \in F_{T_2}} d(a, T_1 a), \sup_{b \in F_{T_1}} d(b, T_2 b)\}.$$

Since $a \in T_2a$ and $b \in T_2b$,

$$H(F_{T_1}, F_{T_2}) \leq \max\{c_1, c_2\} \max\{ \sup_{a \in F_{T_2}} H(T_2a, T_1a), \sup_{b \in F_{T_1}} H(T_1b, T_2b)\}.$$

Therefore, by assumption we have $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$. Then we complete the proof. □

Corollary 6.10. *Let (X, d) be a complete metric space and $T_n : X \to \mathcal{CB}(X)$ be a sequence of Collage multi-valued with Collage constants c_n such that $\sup_n c_n < \infty$. Suppose that $T_n \to T$ with the metric d_∞ and T is a Collage multi-valued mapping. Then $F_{T_n} \to F_T$ with the Hausdorff metric.*

Proof. This is obvious by Theorem 6.9. □

Let us consider several consequences of this result which follow by Example 6.5, Remark 6.7 and Theorem 6.9.

Corollary 6.11. *Let (X, d) be a metric space and $T_1, T_2 : X \to \mathcal{CB}(X)$ be two multi-valued mappings for which there exist positive real numbers α_i, β_i and γ_i, $\alpha_i + \beta_i + \gamma_i < 1$ such that*

$$H(T_ix, T_iy) \leq \alpha_i d(x, y) + \beta_i d(x, Tx) + \gamma_i d(y, Ty),$$

for all $x, y \in X$ and $i \in \{1, 2\}$.

Suppose that there exists $\eta > 0$ such that $H(T_1x, T_2x) \leq \eta$ for all $x \in X$.

Then

$H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$,

with $c_i = (1 - \gamma_i)[1 - (\alpha_i + \beta_i + \gamma_i)]^{-1}$, $i \in \{1, 2\}$.

Corollary 6.12. *Let (X, d) be a metric space and $T_1, T_2 : X \to \mathcal{CB}(X)$ be two multi-valued mappings for which there exist positive real numbers α_i and β_i, $\alpha_i + \beta_i + < 1$ such that*

$$H(T_ix, T_iy) \leq \alpha_i d(x, y) + \beta_i d(y, Ty),$$

for all $x \in X,\ y \in Tx$ *and* $i \in \{1, 2\}$.

Suppose that there exists $\eta > 0$ *such that* $H(T_1x, T_2x) \leq \eta$ *for all* $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$, *with* $c_i = (1 - \beta_i)[1 - (\alpha_i + \beta_i)]^{-1}$, $i \in \{1, 2\}$.

Corollary 6.13. *Let* $x_0 \in X$ *and* $r > 0$. *Let* $T : \tilde{B}(x_0, r) \to \mathcal{CB}(X)$, *where* $\tilde{B}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$ *be a multi-valued mapping for which there exist* $\alpha, \beta, \gamma \in \mathbb{R}_+$ *and* $\alpha + \beta + \gamma < 1$ *such that*

(i) $H(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty),\ \forall x, y \in \tilde{B}(x_0, r)$

(ii) $\delta(x_0, Tx_0) < [1 - (\alpha + \beta + \gamma)](1 - \gamma)^{-1} r$.

Suppose that there exists $\eta > 0$ *such that* $H(T_1x, T_2x) \leq \eta$ *for all* $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$, *with with* $c = (1 - \gamma)[1 - (\alpha + \beta + \gamma)]^{-1}$.

REFERENCES

[1] Al-Thangafi, M.A. and Shahzad, N., Convergence and existence results for best proximity points, *Nonlinear Analysis.* 70, 3665-3671 (2009).

[2] Assad, N.A., Fixed point theorems for set valued transformations on compact set, *Boll. Un. Mat. Ital.* 4, 1-7 (1973).

[3] Assad, N.A. and Kirk, W.A., Fixed point theorems for set-valued mapping of contractive type, *Pacific J.Math.* 43, 553-562 (1972).

[4] Banach, S., Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund, Math.* 3, 133-181 (1922).

[5] Barnsley, M.F., Fractals Everywhere, *Academic Press*, New York, 1989.

[6] Barnsley, M.F., Ervin, V., Hardin, D.& Lancaster, J., Solution of an inverse problem for fractals and other sets, *Proc. Natl. Acad. Sci.* 83, 1975-1977 (1985).

[7] Berinde, M. and Berinde, V., On general class of multi-valued weakly Picard mapping, *J. Math. Anal. Appl.* 326, 772-782 (2007).

[8] Centore, P., Vrscay, E.R., Continuity of fixed points for attractors and invariant measures for iterated function systems, *Canad. Math. Bull.* 37, 315-329 (1994).

[9] Chatterjea, S.K., Fixed-point theorems, *C.R. Acad. Bulgare Sci.* 25, 727-730 (1972).

[10] Ciric, L.B., Fixed Point Theory. Contraction Mapping Principle, FME Press, Beograd, 2003.

[11] Covitz, H. and Nadler Jr., S.B., Multi-valued contraction mappings in generalized metric spaces, *Isr. J. Math.* 8, 5-11 (1970).

[12] Di Bari, C., Suzuki, T., Vetro, C., Best proximity points for cyclic Meir-Keeler contractions, *Nonlinear Anal.* 69, 3790-3794 (2008).

[13] Eldred, A.A. and Veeramani, P., Existence and convergence of best proximity points, *J. Math. Anal. Appl.* 323, 1001-1006 (2006).

[14] Kannan, R., Some results on fixed points, *Bull. Calcutta Math. Soc.* 10, 71-76 (1968).

[15] Khamsi, M.A., and Kirk, W.A., *An In troduction to Metric Spaces and Fixed Point Theory*, John Wiley & Sons, Inc., New York, 2001.

[16] Kirk, W.A., Contraction mappings and extensions, in: W.A. Kirk, B. Sims (Eds.), Handbook of Metric Fixed Point Theory, Kluwer Academic Publishers, Dordrecht, 2001, pp. 1-34.

[17] Kirk, W.A., Srinivasan, P.S., Veeramani, P., Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory.* 4, 79-89 (2003).

[18] Kirk, W.A., Reich, S., Veeramani, P., Proximinal retracts and best proximity pair theorems, *Numer. Funct. Anal. Optim.* 24, 851-862 (2003).

[19] Kunze, H.E., La Torre D., & Vrscay, E.R., Contractive multifunctions, fixed point inclusions and iterated multifunction systems, *J. Math. Anal. Appl.* 330, 159-173 (2007).

[20] Markin, J.T., A fixed point theorem for set-valued mappings, *Bull. Amer. Math. Soc.* 74, 639-640 (1968).

[21] Megginson, R.E., *An Introduction to Banach Space Theory*, Springer, New York, 1998.

[22] Mizoguchi, N. and Takahashi, W., Fixed point theorems for multi-valued mappings on complete metric spaces, *J. Math. Anal. Appl.* 141, 177-188 (1989).

[23] Nadler, S.B., Multi-valued contraction mappings, *Pac. J. Math.* 30, 475-488 (1969).

[24] Petruşel, A., On Frigon-Grans-type multifunctions, *Nonlinear Anal. Forum.* 7, 113-121 (2002).

[25] Reich, S., Kannan's fixed point theorem, *Boll. Un. Mat. Ital.* 4, 1-11 (1971).

[26] Reich, S., A fixed point theorem for locally contractive multi-valued functions, *Rev. Roum. Math. Pures. Appl.* 17, 569-572 (1972).

[27] Reich, S., Fixed point of contractive functions, *Boll. Un. Mat. Ital.* 5, 26-42 (1972).

[28] Rus, I.A., Basic Problems of the metric fixed point theory revisited(II), *Stud. Univ. Babeş-Bolyai.* 36, 81-99 (1991).

[29] Rus, I.A., Generalized Contractions, Cluj University Press, Cluj-Napoca, 2001.

[30] Rus, I.A., Petruşel, A., Sîntămărian, A., Data dependence of the fixed point set of some multi-valued weakly Picard operators, *Nonlinear Anal.* 52, 1947-1959 (2003).

[31] Suzuki, T., Kikkawa, M., Vetro, C., The existence of best proximity points in metric spaces with the property UC, *Nonlinear Analysis.* 71, 2918-2926 (2009).

[32] Takahashi, W., *Nonlinear Functional Analysis : Fixed Point Theory and it Applications*, Yokohama Publishers, Inc., Japan, 2000.

[33] Zamfirescu, T., Fix point theorems in metric spaces, *Arch. Math. (Basel)* 23, 292-298 (1972).

VITA

Mister Annop Kaewkhao was born on December 14, 1978 in Chonburi, Thailand. He got a Bachelor of Education in Mathematics in 2003 from Burapha University and got a Master of Science in Mathematics in 2005 from Chulalongkorn University. After two years, he furthered study for the Ph.D. program at Chulalongkorn University.