สมการเชิงฟังก์ชันกำลังสองเชิงสี่มิติแบบเพกซิเดอร์

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# 4-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATIONS OF PEXIDER TYPE

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$$\sum_{i=1}^{2^{n-1}} f_i\left(\sum_{j=1}^{n} \sigma_{ij} x_j\right) = 2^{n-1} \sum_{j=1}^{n} g_j(x_j) \qquad (1)$$

โดยที่ $n \in \mathbb{N} - \{1\}$ และ $\sigma_{ij} = (-1)^{\left\lfloor \frac{i-1}{2^{n-j}} \right\rfloor}, i = 1,2,\ldots,2^{n-1}, j = 1,2,\ldots,n$

โดยเฉพาะเมื่อ $n = 3$ และ $n = 4$ สมการจะอยู่ในรูป

$$f_1(x_1+x_2+x_3)+f_2(x_1+x_2-x_3)+f_3(x_1-x_2+x_3)+f_4(x_1-x_2-x_3)=4g_1(x_1)+4g_2(x_2)+4g_3(x_3)$$

และ

$$\begin{aligned} &f_1(x_1+x_2+x_3+x_4)+f_2(x_1+x_2+x_3-x_4)+f_3(x_1+x_2-x_3+x_4)+\\ &f_4(x_1+x_2-x_3-x_4)+f_5(x_1-x_2+x_3+x_4)+f_6(x_1-x_2+x_3-x_4)+\\ &f_7(x_1-x_2-x_3+x_4)+f_8(x_1-x_2-x_3-x_4)=8g_1(x_1)+8g_2(x_2)+8g_3(x_3)+8g_4(x_4)\end{aligned}$$

ตามลำดับ

ในงานวิจัยนี้ เราเริ่มด้วยการหาผลเฉลยทั่วไปของสมการ (1) เมื่อ $n = 3$ โดยการประยุกต์ใช้ผลที่ได้จากกรณี $n = 3$ เราสามารถหาผลเฉลยทั่วไปของสมการ (1) ในกรณี $n = 4$




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The multidimensional quadratic functional equations of Pexider type in the form

$$\sum_{i=1}^{2^{n-1}} f_i\Big(\sum_{j=1}^{n} \sigma_{ij}x_j\Big) = 2^{n-1}\sum_{j=1}^{n} g_j(x_j) \tag{1}$$

where $n \in \mathbb{N} - \{1\}$ and $\sigma_{ij} = (-1)^{\lfloor \frac{i-1}{2^{n-j}} \rfloor}, i = 1, 2, \ldots, 2^{n-1}, j = 1, 2, \ldots, n$ will be considered. In particular when $n = 3$ and $n = 4$, we have respectively the equations

$$\begin{aligned} f_1(x_1+x_2+x_3) + f_2(x_1+x_2-x_3) + f_3(x_1-x_2+x_3) + f_4(x_1-x_2-x_3) \\ = 4g_1(x_1) + 4g_2(x_2) + 4g_3(x_3), \end{aligned} \tag{2}$$

and

$$\begin{aligned} & f_1(x_1+x_2+x_3+x_4) + f_2(x_1+x_2+x_3-x_4) + f_3(x_1+x_2-x_3+x_4) + \\ & f_4(x_1+x_2-x_3-x_4) + f_5(x_1-x_2+x_3+x_4) + f_6(x_1-x_2+x_3-x_4) + \\ & f_7(x_1-x_2-x_3+x_4) + f_8(x_1-x_2-x_3-x_4) \\ & = 8g_1(x_1) + 8g_2(x_2) + 8g_3(x_3) + 8g_4(x_4). \end{aligned} \tag{3}$$

In this thesis, we first solved the equation (1) when $n = 3$. By applying this result, we obtained the general solutions for the case $n = 4$.


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# CHAPTER I
# INTRODUCTION

A functional equation is an equation including one or more unknown functions with prescribed domain and range. Solving a functional equation means to find all functions which satisfy it identically. Functional equations have substantially grown to become an important branch of mathematics. Particularly during the last two decades, with its special methods, there are numbers of interesting results and several applications. The most famous functional equation, namely the additive Cauchy equation, often simply called the Cauchy equation defined as follows:

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

A function that satisfies the equation (1.1) will be call an **additive function**. The classical quadratic functional equation is the equation of the form

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

Any solution of (1.2) will be refered as a **quadratic function** (See [1]). The Pexider type of the equation (1.1) is the equation of the form

$$f_1(x+y) = f_2(x) + f_3(y).$$

Similarly, the Pexider type of (1.2) is the equation of the form

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + 2f_4(y). \tag{1.3}$$

In 1999, Soon-Mo Jung [2] gave the general solutions of the equation (1.3). Specifically, he proved the following theorem:

**Theorem 1.1.** *Let $X$ and $Y$ be vector spaces over fields of characteristic different from 2, respectively. The functions $f_1, f_2, f_3, f_4 : X \to Y$ satisfy the functional equation (1.3) for all $x, y \in X$ if and only if there exist a quadratic function $Q : X \to Y$, additive functions $a_1, a_2 : X \to Y$, and constants $c_1, c_2, c_3, c_4 \in Y$ such that*

$$\begin{aligned} f_1(x) &= Q(x) + a_1(x) + a_2(x) + c_1, \\ f_2(x) &= Q(x) + a_1(x) - a_2(x) + c_2, \\ f_3(x) &= Q(x) + a_1(x) + c_3, \\ f_4(x) &= Q(x) + a_2(x) + c_4 \end{aligned}$$

*with*

$$c_1 + c_2 = 2c_3 + 2c_4.$$

Motivated by Theorem 1.1, we consider the quadratic functional equations of Pexider type of the form

$$\sum_{i=1}^{2^{n-1}} f_i\Big(\sum_{j=1}^{n} \sigma_{ij} x_j\Big) = 2^{n-1} \sum_{j=1}^{n} g_j(x_j) \tag{1.4}$$

where $n \in \mathbb{N} - \{1\}$ and $\sigma_{ij} = (-1)^{\lfloor \frac{i-1}{2^{n-j}} \rfloor}, i = 1, 2, \ldots, 2^{n-1}, j = 1, 2, \ldots, n$. In particular, when $n = 3$ and $n = 4$, we have respectively the equations

$$\begin{aligned} f_1(x_1 + x_2 + x_3) + f_2(x_1 + x_2 - x_3) + f_3(x_1 - x_2 + x_3) + f_4(x_1 - x_2 - x_3) \\ = 4g_1(x_1) + 4g_2(x_2) + 4g_3(x_3) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &f_1(x_1 + x_2 + x_3 + x_4) + f_2(x_1 + x_2 + x_3 - x_4) + f_3(x_1 + x_2 - x_3 + x_4) + \\ &f_4(x_1 + x_2 - x_3 - x_4) + f_5(x_1 - x_2 + x_3 + x_4) + f_6(x_1 - x_2 + x_3 - x_4) + \\ &f_7(x_1 - x_2 - x_3 + x_4) + f_8(x_1 - x_2 - x_3 - x_4) \\ &= 8g_1(x_1) + 8g_2(x_2) + 8g_3(x_3) + 8g_4(x_4). \end{aligned} \tag{1.6}$$

Note that the work of Soon-Mo Jung [2] in Theorem 1.1 implies the existence of the general solutions of the equation (1.4) for the case $n = 2$. In this thesis, we would like to extend his result by solving for the general solutions for the equations (1.5) and (1.6) which are corresponding to the cases $n = 3$ and $n = 4$.

# CHAPTER II

# SOLUTIONS OF 3-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION OF PEXIDER TYPE

In this chapter, we consider the equation (1.4)

$$\sum_{i=1}^{2^{n-1}} f_i(\sum_{j=1}^{n} \sigma_{ij}x_j) = 2^{n-1}\sum_{j=1}^{n} g_j(x_j)$$

when n=3, i.e. the equation takes the form

$$f_1(x_1+x_2+x_3)+f_2(x_1+x_2-x_3)+f_3(x_1-x_2+x_3)+f_4(x_1-x_2-x_3)$$
$$=4f_5(x_1)+4f_6(x_2)+4f_7(x_3). \quad (2.1)$$

(Here, we replace $g_1$ with $f_5$, $g_2$ with $f_6$, and $g_3$ with $f_7$ for the ease of the notation indexing in the proof.)

In order to solve for the general solutions, we first make some certain substitutions to find the relations between those $f_i$'s. Afterward, the appropriate arrangements allow us to apply the result of Soon-Mo Jung. By proving Theorem 2.1, the general solutions to the equation (2.1) will be guaranteed. This result will in turn be crucial for solving the equation (1.4) in the case $n = 4$.

**Theorem 2.1.** *Let $X$ and $Y$ be vector spaces over fields of characteristic different from 2. The functions $f_i : X \to Y (i = 1, \ldots, 7)$ satisfy the functional equation (2.1) for all $x_1, x_2, x_3 \in X$ if and only if there exist a quadratic function $Q : X \to Y$, additive functions $a_i : X \to Y (i = 1, \ldots, 4)$, and constants $c_i \in Y (i = 1, \ldots, 7)$*

*such that*

$$
\begin{aligned}
f_1(x) &= Q(x) + a_1(x) + a_2(x) + a_3(x) + a_4(x) + c_1,\\
f_2(x) &= Q(x) + a_1(x) + a_2(x) - a_3(x) - a_4(x) + c_2,\\
f_3(x) &= Q(x) + a_1(x) - a_2(x) + a_3(x) - a_4(x) + c_3,\\
f_4(x) &= Q(x) + a_1(x) - a_2(x) - a_3(x) + a_4(x) + c_4,\\
f_5(x) &= Q(x) + a_1(x) + c_5,\\
f_6(x) &= Q(x) + a_2(x) + c_6,\\
f_7(x) &= Q(x) + a_3(x) + c_7
\end{aligned} \tag{2.2}
$$

*with*

$$c_1 + c_2 + c_3 + c_4 = 4c_5 + 4c_6 + 4c_7. \tag{2.3}$$

*Proof.* First, assume that $f_i$'s are solutions of (2.1). Define $c_i = f_i(0)$ for $i = 1, \ldots, 7$. By substituting $x_1 = x_2 = x_3 = 0$ in (2.1), we see that the $c_i$'s satisfy the relation (2.3). For $i = 1, \ldots, 7$, let $F_i(x) = f_i(x) - c_i$. It is clear from (2.1) and (2.3) that the $F_i$'s satisfy the functional equation (2.1) with $F_i(0) = 0$.

Denoted by $F_i^e(x) = \frac{F_i(x)+F_i(-x)}{2}$ and $F_i^o(x) = \frac{F_i(x)-F_i(-x)}{2}$ the even part and the odd part of $F_i(x)$, respectively. It is easy to see that the $F_i^o$'s and the $F_i^e$'s also satisfy (2.1). Next, we consider (2.1) for the $F_i^o$'s:

$$
\begin{aligned}
F_1^o(x_1 + x_2 + x_3) + F_2^o(x_1 + x_2 - x_3) &+ F_3^o(x_1 - x_2 + x_3) + F_4^o(x_1 - x_2 - x_3)\\
&= 4F_5^o(x_1) + 4F_6^o(x_2) + 4F_7^o(x_3).
\end{aligned} \tag{2.4}
$$

Put $x_3 = 0$ in (2.4) to obtain a quadratic equation of Pexider type,

$$(F_1^o + F_2^o)(x_1 + x_2) + (F_3^o + F_4^o)(x_1 - x_2) = 2(2F_5^o)(x_1) + 2(2F_6^o)(x_2). \tag{2.5}$$

By Theorem 1.1 and $F_i^o$'s are odd functions, there exist additive functions $a_1, a_2 : X \to Y$ such that

$$F_1^o + F_2^o = 2a_1 + 2a_2, F_3^o + F_4^o = 2a_1 - 2a_2, F_5^o = a_1, F_6^o = a_2. \tag{2.6}$$

Then put $x_2 = 0$ in (2.4), we have

$$(F_1^o + F_3^o)(x_1 + x_3) + (F_2^o + F_4^o)(x_1 - x_3) = 2(2F_5^o)(x_1) + 2(2F_7^o)(x_3). \quad (2.7)$$

Similarly, by Theorem 1.1, there exists an additive function $a_3 : X \to Y$ such that

$$F_1^o + F_3^o = 2a_1 + 2a_3, F_2^o + F_4^o = 2a_1 - 2a_3, F_7^o = a_3. \quad (2.8)$$

Analogously, putting $x_3 = -x_2$ in (2.4) gives

$$F_1^o(x_1) + F_2^o(x_1 + 2x_2) + F_3^o(x_1 - 2x_2) + F_4^o(x_1) = 4F_5^o(x_1) + 4F_6^o(x_2) + 4F_7^o(-x_2).$$

Since $F_5^o = a_1$, $F_6^o = a_2$, and $F_7^o = a_3$, we have

$$F_1^o(x_1) + F_2^o(x_1 + 2x_2) + F_3^o(x_1 - 2x_2) + F_4^o(x_1) = 4a_1(x_1) + 4a_2(x_2) + 4a_3(-x_2).$$

Now, we rearrange the previous equation to the equation

$$F_2^o(x_1 + 2x_2) + F_3^o(x_1 - 2x_2) = 4a_1(x_1) + 4a_2(x_2) - 4a_3(x_2) - F_1^o(x_1) - F_4^o(x_1).$$

From this, we obtain the equation

$$F_2^o(x_1 + 2x_2) + F_3^o(x_1 - 2x_2) = (4a_1 - F_1^o - F_4^o)(x_1) + (4a_2 - 4a_3)(x_2).$$

Hence, we get the equation

$$F_2^o(x_1 + 2x_2) + F_3^o(x_1 - 2x_2) = 2(2a_1 - \frac{F_1^o}{2} - \frac{F_4^o}{2})(x_1) + 2(a_2 - a_3)(2x_2).$$

Applying Theorem 1.1 again, there exists an additive function $a_4 : X \to Y$ such that

$$F_2^o = a_1 - a_4 + a_2 - a_3, F_3^o = a_1 - a_4 - a_2 + a_3, 2a_1 - \frac{F_1^o}{2} - \frac{F_4^o}{2} = a_1 - a_4. \quad (2.9)$$

From (2.6), (2.8), and (2.9), we have

$$F_1^o = a_1 + a_2 + a_3 + a_4, F_2^o = a_1 + a_2 - a_3 - a_4, F_3^o = a_1 - a_2 + a_3 - a_4,$$
$$F_4^o = a_1 - a_2 - a_3 + a_4, F_5^o = a_1, F_6^o = a_2, F_7^o = a_3.$$

Now, we consider (2.1) for the $F_i^e$'s:

$$F_1^e(x_1+x_2+x_3)+F_2^e(x_1+x_2-x_3)+F_3^e(x_1-x_2+x_3)+F_4^e(x_1-x_2-x_3) = 4F_5^e(x_1)+4F_6^e(x_2)+4F_7^e(x_3). \quad (2.10)$$

By letting $x_3=0$ in (2.10), we obtain

$$(F_1^e+F_2^e)(x_1+x_2)+(F_3^e+F_4^e)(x_1-x_2) = 2(2F_5^e)(x_1)+2(2F_6^e)(x_2).$$

Hence, by Theorem 1.1 again and since the $F_i^e$'s are even and $F_i^e(0)=0$, there exists a quadratic function $Q: X \to Y$ such that

$$F_1^e+F_2^e=2Q, F_3^e+F_4^e=2Q, 2F_5^e=2Q, 2F_6^e=2Q. \quad (2.11)$$

Then let $x_2=0$ in (2.10), and using Theorem 1.1, we get

$$F_1^e+F_3^e=2Q, F_2^e+F_4^e=2Q, 2F_7^e=2Q. \quad (2.12)$$

Analogously, by letting $x_1=0$ in (2.10), we have

$$F_1^e+F_4^e=2Q, F_2^e+F_3^e=2Q. \quad (2.13)$$

Thus, from (2.11), (2.12), (2.13), we get

$$F_1^e=F_2^e=F_3^e=F_4^e=F_5^e=F_6^e=F_7^e=Q.$$

Conversely, if there exist a quadratic function $Q: X \to Y$, additive functions $a_i : X \to Y (i=1,\ldots,4)$, and constants $c_i \in Y (i=1,\ldots,7)$ which satisfy (2.2) and (2.3), it is obvious that $f_i$'s satisfy the functional equation (2.1). □

# CHAPTER III

# SOLUTIONS OF 4-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION OF PEXIDER TYPE

In this chapter, we consider the equation (1.4)

$$\sum_{i=1}^{2^{n-1}} f_i(\sum_{j=1}^{n} \sigma_{ij}x_j) = 2^{n-1}\sum_{j=1}^{n} g_j(x_j)$$

when n=4, i.e. the equation takes the form

$$\begin{aligned}
&f_1(x_1+x_2+x_3+x_4)+f_2(x_1+x_2+x_3-x_4)+f_3(x_1+x_2-x_3+x_4)+\\
&f_4(x_1+x_2-x_3-x_4)+f_5(x_1-x_2+x_3+x_4)+f_6(x_1-x_2+x_3-x_4)+\\
&f_7(x_1-x_2-x_3+x_4)+f_8(x_1-x_2-x_3-x_4)\\
&=8f_9(x_1)+8f_{10}(x_2)+8f_{11}(x_3)+8f_{12}(x_4). \qquad (3.1)
\end{aligned}$$

(Here, we replace $g_1$ with $f_9$, $g_2$ with $f_{10}$, $g_3$ with $f_{11}$, and $g_4$ with $f_{12}$ for the ease of the notation indexing in the proof.)

It is interesting that the case $n = 4$ poses much harder difficulties not seen in the previous case. In particular, there are even part of $f_i$'s, for some $i$, that do not directly satisfy the quadratic equation. But the enough relations between them, we are able to resolve the problem in Lemma 3.2. Again with the appropriate substitutions, Lemma 3.2 and Theorem 2.1, we finally proved the Theorem 3.1 and therefore obtained the general solutions for the equation (1.4) in the case $n = 4$.

**Theorem 3.1.** *Let $X$ and $Y$ be vector spaces over fields of characteristic different from 2. The functions $f_i : X \to Y (i = 1, \ldots, 12)$, satisfy the functional equation (3.1) for all $x_1, x_2, x_3, x_4 \in X$ if and only if there exist quadratic functions $Q_1, Q_2 : X \to Y$, additive functions $a_i : X \to Y (i = 1, \ldots, 8)$, and constants $c_i \in Y (i = 1, \ldots, 12)$ such that*

$$\begin{aligned}
f_1(x) &= Q_1(x) + a_1(x) + a_2(x) + a_3(x) + a_4(x) + a_5(x) + a_6(x) + a_7(x) + a_8(x) + c_1, \\
f_2(x) &= Q_2(x) + a_1(x) + a_2(x) + a_3(x) - a_4(x) - a_6(x) - a_7(x) - a_8(x) + c_2, \\
f_3(x) &= Q_2(x) + a_1(x) + a_2(x) - a_3(x) + a_4(x) - a_5(x) + a_6(x) - a_7(x) - a_8(x) + c_3, \\
f_4(x) &= Q_1(x) + a_1(x) + a_2(x) - a_3(x) - a_4(x) - a_6(x) + a_7(x) + a_8(x) + c_4, \\
f_5(x) &= Q_2(x) + a_1(x) - a_2(x) + a_3(x) + a_4(x) - a_6(x) - a_7(x) + a_8(x) + c_5, \\
f_6(x) &= Q_1(x) + a_1(x) - a_2(x) + a_3(x) - a_4(x) - a_5(x) + a_6(x) + a_7(x) - a_8(x) + c_6, \\
f_7(x) &= Q_1(x) + a_1(x) - a_2(x) - a_3(x) + a_4(x) - a_6(x) + a_7(x) - a_8(x) + c_7, \\
f_8(x) &= Q_2(x) + a_1(x) - a_2(x) - a_3(x) - a_4(x) + a_5(x) + a_6(x) - a_7(x) + a_8(x) + c_8, \\
f_9(x) &= \frac{Q_1}{2}(x) + \frac{Q_2}{2}(x) + a_1(x) + c_9, \qquad (3.2) \\
f_{10}(x) &= \frac{Q_1}{2}(x) + \frac{Q_2}{2}(x) + a_2(x) + c_{10}, \\
f_{11}(x) &= \frac{Q_1}{2}(x) + \frac{Q_2}{2}(x) + a_3(x) + c_{11}, \\
f_{12}(x) &= \frac{Q_1}{2}(x) + \frac{Q_2}{2}(x) + a_4(x) + c_{12}
\end{aligned}$$

*with*

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 = 8c_9 + 8c_{10} + 8c_{11} + 8c_{12}. \qquad (3.3)$$

Before proving Theorem 3.1, we need the following lemma:

**Lemma 3.2.** *Let $X$ and $Y$ be vector spaces over fields of characteristic different from 2. Let $Q : X \to Y$ be a quadratic function. The even functions $F_1$ and*

*$F_2 : X \to Y$ such that $F_1(0) = 0 = F_2(0)$ and satisfy the equations*

$$\begin{aligned}
&F_1(x_1 + x_2 + x_3 + x_4) + F_2(x_1 + x_2 + x_3 - x_4) + F_2(x_1 + x_2 - x_3 + x_4) + \\
&F_1(x_1 + x_2 - x_3 - x_4) + F_2(x_1 - x_2 + x_3 + x_4) + F_1(x_1 - x_2 + x_3 - x_4) + \\
&F_1(x_1 - x_2 - x_3 + x_4) + F_2(x_1 - x_2 - x_3 - x_4) \\
&= 8Q(x_1) + 8Q(x_2) + 8Q(x_3) + 8Q(x_4) \qquad (3.4)
\end{aligned}$$

*and*

$$F_1 + F_2 = 2Q \qquad (3.5)$$

*for all $x_1, x_2, x_3, x_4 \in X$ if and only if there exist quadratic functions $Q_1, Q_2 : X \to Y$ such that $F_1 = Q_1$, $F_2 = Q_2$ and $Q_1 + Q_2 = 2Q$.*

*Proof.* First, suppose that $F_1, F_2$ satisfy the equations (3.4) and (3.5). Consider the equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y). \qquad (3.6)$$

If we let $x = 0 = y$ in (3.6), we get $Q(0) = 0$. And if we let $x = y$ in (3.6), we get $Q(2x) = 4Q(x)$. Putting $x_1 = x_2 = x_3 = x_4 = \frac{x}{2}$ in (3.4) gives

$$F_1(2x) + 4F_2(x) = 32Q(\frac{x}{2}).$$

Replacing $F_2$ by $2Q - F_1$ in the previous equation, we have

$$F_1(2x) - 4F_1(x) = 32Q(\frac{x}{2}) - 8Q(x). \qquad (3.7)$$

From (3.7) and the fact that $Q(2x) = 4Q(x)$, we get

$$F_1(2x) = 4F_1(x). \qquad (3.8)$$

Now, putting $x_1 = x, x_2 = y, x_3 = \frac{x+y}{2}, x_4 = \frac{x+y}{2}$ in (3.4) yields

$$\begin{aligned}
F_1(2x + 2y) + 2F_1(x - y) + 2F_2(x + y) + F_2(2x) + F_2(-2y)& \\
= 8Q(x) + 8Q(y) + 16Q(\frac{x + y}{2}).& \qquad (3.9)
\end{aligned}$$

Substituting $F_2$ by $2Q - F_1$ in (3.9), we have

$$F_1(2x+2y) + 2F_1(x-y) + 4Q(x+y) - 2F_1(x+y) + 2Q(2x) - F_1(2x) + \\ 2Q(-2y) - F_1(-2y) = 8Q(x) + 8Q(y) + 16Q(\frac{x+y}{2}). \quad (3.10)$$

Applying (3.8) and properties of $Q$ to the equation (3.10), we obtain

$$4F_1(x+y) + 2F_1(x-y) + 4Q(x+y) - 2F_1(x+y) + 8Q(x) - 4F_1(x) + \\ 8Q(y) - 4F_1(y) = 8Q(x) + 8Q(y) + 4Q(x+y).$$

From the previous equation, we get

$$2F_1(x+y) + 2F_1(x-y) = 4F_1(x) + 4F_1(y). \quad (3.11)$$

Divide (3.11) by 2, we have

$$F_1(x+y) + F_1(x-y) = 2F_1(x) + 2F_1(y).$$

Thus $F_1$ is a quadratic function. Since $F_2 = 2Q - F_1$, we have $F_2$ is also a quadratic function. Therefore, there exist quadratic functions $Q_1, Q_2$ such that $F_1 = Q_1$ and $F_2 = Q_2$.

Conversely, if there exist quadratic functions $Q_1, Q_2 : X \to Y$ such that $F_1 = Q_1$, $F_2 = Q_2$, and $Q_1 + Q_2 = 2Q$, it is not hard to see that $Q_i$'s satisfy the equations (3.4) and (3.5), and $F_1(0) = 0 = F_2(0)$.

□

Now, we can prove Theorem 3.1 as follows:

*Proof.* First, assume that $f_i$'s are solutions of (3.1). Define $c_i = f_i(0)$ for $i = 1, \ldots, 12$. By substituting $x_1 = x_2 = x_3 = x_4 = 0$ in (3.1), we see that the $c_i$'s satisfy the relation (3.3). For $i = 1, \ldots, 12$, let $F_i(x) = f_i(x) - c_i$. It is clear from (3.1) and (3.3) that the $F_i$'s satisfy the functional equation (3.1) with $F_i(0) = 0$.

Again, denoted by $F_i^e(x)$ and $F_i^o(x)$ the even part and the odd part of $F_i(x)$, respectively. It is easy to see that the $F_i^o$'s and the $F_i^e$'s also satisfy (3.1). Next, we consider (3.1) for the $F_i^o$'s:

$$F_1^o(x_1+x_2+x_3+x_4)+F_2^o(x_1+x_2+x_3-x_4)+F_3^o(x_1+x_2-x_3+x_4)+$$
$$F_4^o(x_1+x_2-x_3-x_4)+F_5^o(x_1-x_2+x_3+x_4)+F_6^o(x_1-x_2+x_3-x_4)+$$
$$F_7^o(x_1-x_2-x_3+x_4)+F_8^o(x_1-x_2-x_3-x_4)$$
$$=8F_9^o(x_1)+8F_{10}^o(x_2)+8F_{11}^o(x_3)+8F_{12}^o(x_4). \tag{3.12}$$

Put $x_4=0$ in (3.12) to obtain a quadratic equation of Pexider type,

$$(F_1^o+F_2^o)(x_1+x_2+x_3)+(F_3^o+F_4^o)(x_1+x_2-x_3)+(F_5^o+F_6^o)(x_1-x_2+x_3)+$$
$$(F_7^o+F_8^o)(x_1-x_2-x_3)=4(2F_9^o)(x_1)+4(2F_{10}^o)(x_2)+4(2F_{11}^o)(x_3)$$

By Theorem 2.1 and since $F_i^o$'s are odd functions, there exist additive functions $a_1,a_2,a_3,a_5:X\to Y$ such that

$$F_1^o+F_2^o=2a_1+2a_2+2a_3+a_5, F_3^o+F_4^o=2a_1+2a_2-2a_3-a_5,$$
$$F_5^o+F_6^o=2a_1-2a_2+2a_3-a_5, F_7^o+F_8^o=2a_1-2a_2-2a_3+a_5,$$
$$F_9^o=a_1, F_{10}^o=a_2, F_{11}^o=a_3. \tag{3.13}$$

By putting $x_3=0$ in (3.12) and applying Theorem 2.1 again , there exist additive functions $a_4,a_6:X\to Y$ such that

$$F_1^o+F_3^o=2a_1+2a_2+2a_4+2a_6, F_2^o+F_4^o=2a_1+2a_2-2a_4-2a_6,$$
$$F_5^o+F_7^o=2a_1-2a_2+2a_4-2a_6, F_6^o+F_8^o=2a_1-2a_2-2a_4+2a_6,$$
$$F_{12}^o=a_4. \tag{3.14}$$

Similarly, by letting $x_1=0$, there exists an additive function $a_7:X\to Y$ such that

$$F_1^o-F_8^o=2a_2+2a_3+2a_4+2a_7, F_2^o-F_7^o=2a_2+2a_3-2a_4-2a_7,$$
$$F_3^o-F_6^o=2a_2-2a_3+2a_4-2a_7, F_4^o-F_5^o=2a_2-2a_3-2a_4+2a_7. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$F_1^o+F_6^o=2a_1+2a_3+2a_6+2a_7 \quad and, \quad F_3^o+F_8^o=2a_1-2a_3+2a_6-2a_7. \tag{3.16}$$

Then putting $x_4 = -x_2$ in (3.12), we get

$$\begin{aligned}&F_1^o(x_1+x_3)+F_2^o(x_1+2x_2+x_3)+F_3^o(x_1-x_3)+F_4^o(x_1+2x_2-x_3)+\\&F_5^o(x_1-2x_2+x_3)+F_6^o(x_1+x_3)+F_7^o(x_1-2x_2-x_3)+F_8^o(x_1-x_3)\\&=8F_9^o(x_1)+8F_{10}^o(x_2)+8F_{11}^o(x_3)+8F_{12}^o(-x_2).\end{aligned}$$

Next, we rearrange the previous equation to the equation

$$\begin{aligned}&F_2^o(x_1+2x_2+x_3)+F_4^o(x_1+2x_2-x_3)+F_5^o(x_1-2x_2+x_3)+\\&F_7^o(x_1-2x_2-x_3)=8F_9^o(x_1)+8F_{10}^o(x_2)+8F_{11}^o(x_3)+\\&8F_{12}^o(-x_2)-(F_1^o+F_6^o)(x_1+x_3)-(F_3^o+F_8^o)(x_1-x_3).\end{aligned}$$

By using (3.16), we get

$$\begin{aligned}&F_2^o(x_1+2x_2+x_3)+F_4^o(x_1+2x_2-x_3)+F_5^o(x_1-2x_2+x_3)+F_7^o(x_1-2x_2-x_3)\\&=8F_9^o(x_1)+8F_{10}^o(x_2)+8F_{11}^o(x_3)+8F_{12}^o(-x_2)-(2a_1+2a_3+2a_6+2a_7)(x_1+x_3)-\\&(2a_1-2a_3+2a_6-2a_7)(x_1-x_3).\end{aligned}\tag{3.17}$$

Now, we can transform (3.17) to the equation

$$\begin{aligned}&F_2^o(x_1+2x_2+x_3)+F_4^o(x_1+2x_2-x_3)+F_5^o(x_1-2x_2+x_3)+F_7^o(x_1-2x_2-x_3)\\&=4(a_1-a_6)(x_1)+4(a_2-a_4)(2x_2)+4(a_3-a_7)(x_3).\end{aligned}\tag{3.18}$$

By Theorem 2.1, there exists an additive function $a_8 : X \to Y$ such that

$$\begin{aligned}&F_2^o=a_1-a_6+a_2-a_4+a_3-a_7-a_8, F_4^o=a_1-a_6+a_2-a_4-a_3+a_7+a_8,\\&F_5^o=a_1-a_6-a_2+a_4+a_3-a_7+a_8, F_7^o=a_1-a_6-a_2+a_4-a_3+a_7-a_8\end{aligned}\tag{3.19}$$

From (3.13), (3.14), and (3.19), we obtain

$$\begin{aligned}&F_1^o=a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8, F_2^o=a_1+a_2+a_3-a_4-a_6-a_7-a_8,\\&F_3^o=a_1+a_2-a_3+a_4-a_5+a_6-a_7-a_8, F_4^o=a_1+a_2-a_3-a_4-a_6+a_7+a_8,\\&F_5^o=a_1-a_2+a_3+a_4-a_6-a_7+a_8, F_6^o=a_1-a_2+a_3-a_4-a_5+a_6+a_7-a_8,\\&F_7^o=a_1-a_2-a_3+a_4-a_6+a_7-a_8, F_8^o=a_1-a_2-a_3-a_4+a_5+a_6-a_7+a_8,\\&F_9^o=a_1, F_{10}^o=a_2, F_{11}^o=a_3, F_{12}^o=a_4.\end{aligned}$$

Now, we consider (3.1) for the $F_i^e$'s:

$$\begin{aligned}
&F_1^e(x_1+x_2+x_3+x_4)+F_2^e(x_1+x_2+x_3-x_4)+F_3^e(x_1+x_2-x_3+x_4)+\\
&F_4^e(x_1+x_2-x_3-x_4)+F_5^e(x_1-x_2+x_3+x_4)+F_6^e(x_1-x_2+x_3-x_4)+\\
&F_7^e(x_1-x_2-x_3+x_4)+F_8^e(x_1-x_2-x_3-x_4)\\
&=8F_9^e(x_1)+8F_{10}^e(x_2)+8F_{11}^e(x_3)+8F_{12}^e(x_4). \qquad (3.20)
\end{aligned}$$

Since the $F_i^e$'s are even and $F_i^e(0)=0$, by letting $x_4=0$ in (3.20) and using Theorem 2.1, there exists a quadratic function $Q: X \to Y$ with

$$\begin{aligned}
&F_1^e+F_2^e=2Q, F_3^e+F_4^e=2Q, F_5^e+F_6^e=2Q, F_7^e+F_8^e=2Q,\\
&2F_9^e=2Q, 2F_{10}^e=2Q, 2F_{11}^e=2Q. \qquad (3.21)
\end{aligned}$$

Put $x_3=0$ in (3.20) and using Theorem 2.1 again, we get

$$F_1^e+F_3^e=2Q, F_2^e+F_4^e=2Q, F_5^e+F_7^e=2Q, F_6^e+F_8^e=2Q, 2F_{12}^e=2Q. \quad (3.22)$$

Similarly, letting $x_2=0$ in (3.20) gives

$$F_1^e+F_5^e=2Q, F_2^e+F_6^e=2Q, F_3^e+F_7^e=2Q, F_4^e+F_8^e=2Q. \qquad (3.23)$$

Analogously, putting $x_1=0$ in (3.20) yields

$$F_1^e+F_8^e=2Q, F_2^e+F_7^e=2Q, F_3^e+F_6^e=2Q, F_4^e+F_5^e=2Q. \qquad (3.24)$$

From the equations (3.21), (3.22), (3.23), and (3.24), we obtain

$$\begin{aligned}
&F_1^e=F_4^e=F_6^e=F_7^e, F_2^e=F_3^e=F_5^e=F_8^e,\\
&F_1^e+F_2^e=2Q, F_9^e=Q, F_{10}^e=Q, F_{11}^e=Q, F_{12}^e=Q. \qquad (3.25)
\end{aligned}$$

From (3.20) and (3.25), we now can apply Lemma 3.2 to get

$$F_1^e=F_4^e=F_6^e=F_7^e=Q_1 \quad \text{and} \quad F_2^e=F_3^e=F_5^e=F_8^e=Q_2$$

where $Q_1, Q_2 : X \to Y$ are quadratic functions such that $Q_1+Q_2=2Q$.

Conversely, if there exist quadratic functions $Q_1, Q_2 : X \to Y$, additive functions $a_i : X \to Y (i=1,\ldots,8)$, and constants $c_i \in Y (i=1,\ldots,12)$ which satisfy (3.2) and (3.3), it is obvious that $f_i$'s satisfy the functional equation (3.1). □

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