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VIRTUALLY STABLE MAPS AND THEIR CONVERGENCE SETS

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We generalize the concept of a virtually nonexpansive selfmap of a metric space by introducing the notion of a virtually stable selfmap of a Hausdorff space. We also prove that every virtually nonexpansive selfmap is virtually stable, and the fixed point set of a virtually stable selfmap is a retract of the convergence set. Some properties of the convergence set and the fixed point set of a virtually stable selfmap are also investigated.

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CONTENTS

CHAPTER I

INTRODUCTION

In [1], P. Chaoha introduced the notion of virtually nonexpansive selfmaps of a metric space and proved that various types of nonexpansive maps are virtually nonexpansive. Moreover, for any virtually nonexpansive map f, $F(f)$ is a retract of $C(f)$. Hence, we can predict some topological properties of $F(f)$ if we know the properties of $C(f)$. In particular, it has been shown in [2] that, for certain kinds of virtually nonexpansive maps, their convergence sets are star-convex and hence their fixed point sets are contractible.

In this thesis, we will extend the notion of virtually nonexpansive maps to virtually stable maps in a more general setting and explore some properties of their convergence sets to obtain topological properties of their fixed point sets.

CHAPTER II
PRELIMINARIES

In this chapter, we recall some basic terminology and concepts used throughout the work. For more details, please consult [3], [4], [5], [6], [7], and [8].

Definition 2.1. *Let X be a nonempty set and $\mathcal{T}$ a collection of subsets of X such that*

1. *$X \in \mathcal{T}$, and $\phi \in \mathcal{T}$;*
2. *any union of members of $\mathcal{T}$ is also a member of $\mathcal{T}$;*
3. *any finite intersection of members of $\mathcal{T}$ is also a member of $\mathcal{T}$.*

Then $\mathcal{T}$ is called a **topology** *on X, elements of $\mathcal{T}$ are called* **open** *sets of X, and $(X, \mathcal{T})$ is called a* **topological space**. *Sometimes we omit specific mention of $\mathcal{T}$ if no confusion will arise. A subset A of a topological space X is said to be* **closed** *in X if the set $X - A$ is open in X. And, for $x \in X$, a* **neighborhood** *of x is an open set containing x.*

Definition 2.2. *Let X be a nonempty set and $\mathcal{B}$ a collection of subsets of X such that*

1. *for each $x \in X$, there is $U \in \mathcal{B}$ such that $x \in U$;*
2. *for all $U, V \in \mathcal{B}$, if $x \in U \cap V$, then there is $G \in \mathcal{B}$ such that $x \in G \subseteq U \cap V$.*

Then $\mathcal{B}$ is called a **basis** *for a topology on X, and the set*
$\{G \subseteq X : \forall g \in G, \exists U \in \mathcal{B}$ such that $g \in U \subseteq G\}$, denoted by $< \mathcal{B} >$, is a topology on X and we call it the **topology generated by** *$\mathcal{B}$.*

Given a subset A of a topological space X, the **interior** of A, denoted by **IntA**, is defined as the union of all open sets contained in A, and the **closure** of A, denoted by $\overline{\mathbf{A}}$, is defined as the intersection of all closed sets containing A. Obviously, Int A is an open set and $\overline{A}$ is a closed set; furthermore, Int $A \subseteq A \subseteq \overline{A}$. If A is an open set, $A = \text{Int } A$; while if A is closed, $A = \overline{A}$.

Definition 2.3. *Let $(X, \mathcal{T})$ be a topological space and Y a subset of X. Then the collection $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is a topology on Y, and we call it the* **subspace topology**. *In addition, $(Y, \mathcal{T}_Y)$ is called a* **subspace** *of X.*

Definition 2.4. *A* **metric** *on a set X is a function $d : X \times X \to X$ such that*

1. *$d(x, y) \geq 0$ for all $x, y \in X$, and the equality holds if and only if $x = y$;*

2. *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

3. *$d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.*

Given a metric d on X and $x, y \in X$, $d(x, y)$ is often called the **distance** between x and y with respect to the metric d. Given $\epsilon > 0$, the set $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ is called the $\epsilon -$ **ball** centered at x. Let $\mathcal{T}_d = \{G \subseteq X : \forall a \in G, \exists \epsilon > 0 \, such\, that\, B_d(a, \epsilon) \subseteq G\}$. Then $\mathcal{T}_d$ is a topology on X. The topology $\mathcal{T}_d$ is called the **topology induced by the metric** d and (X, d) is called a **metric space**. Sometimes we omit specific mention of d if no confusion will arise.

Definition 2.5. *A topological space $(X, \mathcal{T})$ is said to be* **metrizable** *if there exists a metric d on X such that $\mathcal{T}$ is a topology induced by d, and in this case we can denote $(X, \mathcal{T})$ by (X, d).*

Example 2.6. *The* **usual metric** *on $\mathbb{R}^n$ is the metric d defined by*
$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, *for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{R}^n$.*

Definition 2.7. *Let X and Y be topological spaces. We say that a function $f : X \to Y$ is* **continuous at a point** *x in X if for each neighborhood U of $f(x)$*

there is neighborhood V of x such that $f(V) \subseteq U$. If f is continuous at every point x in $A \subseteq X$, then f is said to be **continuous on A**. *If f is continuous on X, then we simply say that f is* **continuous**.

Theorem 2.8 (Intermediate value theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on a closed interval $[a, b]$ where $a, b \in \mathbb{R}$ and $a \leq b$. If N is a real number between $f(a)$ and $f(b)$, then there is $c \in [a, b]$ such that $f(c) = N$.*

Definition 2.9. *Let X be a topological space and Y a nonempty subset of X. We say that Y is* **connected** *if and only if there is no pair of subsets U, V of X such that*

1. *$U \cup V = Y$;*
2. *$U \cap Y \neq \phi$, and $V \cap Y \neq \phi$;*
3. *$\overline{U} \cap V = \phi$, and $U \cap \overline{V} = \phi$.*

Definition 2.10. *Let X be a topological space and Y a nonempty subset of X. We say that Y is* **path connected** *if for each pair of points x, y in Y, there is a continuous map $f : [a, b] \to X$ of some closed interval $[a, b]$ in the real line into the subspace Y of X such that $f(a) = x$ and $f(b) = y$.*

Remark 2.11. *Let X be a topological space. Then the relation on X defined by*

$$x \sim y \text{ if } x \text{ and } y \text{ belong to the same (path-)connected subset of } X$$

is an equivalence relation. The equivalence classes of this relation are called the **(path−)components** *of X.*

Theorem 2.12. *Let X be a topological space. Then every connected component of X is closed.*

Theorem 2.13. *Let X and Y be topological spaces, and $f : X \to Y$ a continuous map. If A is a (path-)connected subspace of X, then $f(A)$ is (path-)connected.*

Definition 2.14. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A family* $\mathcal{F}$ *of continuous maps on* X *to* Y *is said to be* **equicontinuous** *at* $x \in X$ *if for each* $\epsilon > 0$*, there is* $\delta > 0$ *such that for every* $u \in X$ *and* $f \in \mathcal{F}$*,* $d_Y(f(x), f(u)) < \epsilon$ *whenever* $d_X(x, u) < \delta$*.*

Definition 2.15. *A subset* A *of a topological space* X *is said to be* **dense** *in* X *if* $\overline{A} = X$*.*

Example 2.16. *The set* $\mathbb{Q}$ *of all rational numbers is dense in the space* $\mathbb{R}$*.*

Definition 2.17. *A subset* A *of a topological space* X *is called a* $\mathbf{G}_\delta-$**set** *in* X *if it is an intersection of a countable collection of open subsets of* X*.*

Definition 2.18. *Given a set* X*, we define a* **sequence** *in* X *to be a function* $\mathbf{x} : \mathbb{N} \to X$*. We often denote the value of* x *at* i *by* x_i *rather than* $\mathbf{x}(i)$*, and denote* $\mathbf{x}$ *itself by the symbol* (x_n)*.*

Definition 2.19. *A sequence* (x_n) *of real numbers is called a strictly increasing sequence if* $x_n < x_{n+1}$ *for all* $n \in \mathbb{N}$*, and it is called a strictly decreasing sequence if* $x_n > x_{n+1}$ *for all* $n \in \mathbb{N}$*.*

Definition 2.20. *Let* X *be a topological space. A sequence* (x_n) *in* X *is said to* **converge** *to a point* y *in* X *if for each neighborhood* U *of* y*, there is* $N \in \mathbb{N}$ *such that* $x_n \in U$ *whenever* $n \geq N$*. We denote this by* $\lim_{n\to\infty} x_n = y$ *or* $x_n \to y$*.*

Theorem 2.21. *Let* X *and* Y *be topological spaces, and* $f : X \to X$ *a continuous map. If* (x_n) *is a sequence in* X *such that* $x_n \to x'$ *for some* $x' \in X$*. Then* $f(x_n) \to f(x')$*.*

Definition 2.22 (Monotone convergence theorem)**.** *Let* (x_n) *be a strictly increasing sequence or a strictly decreasing sequence in an interval* $[a, b] \subseteq \mathbb{R}$ *where* $a \leq b$*. Then* (x_n) *converges to some element in* $[a, b]$*.*

Definition 2.23. *Let* (X, d) *be a metric space. A sequence* (x_n) *in* X *is said to be a* **Cauchy sequence** *if for each* $\epsilon > 0$*, there is* $N \in \mathbb{N}$ *such that* $d(x_n, x_m) < \epsilon$ *whenever* $n, m \geq N$*.*

Definition 2.24. *A metric space* (X, d) *is said to be* **complete** *if every Cauchy sequence in* X *converges(to a point) in* X.

Example 2.25. *The space* $\mathbb{R}$ *with the usual metric is a complete metric space, but its subspace* $\mathbb{Q}$ *is not.*

Definition 2.26. *A topological space* X *is called a* **Hausdorff space** *if for each pair of distinct points* x, y *in* X, *there exist open sets* U_x *and* U_y *such that* $x \in U_x$, $y \in U_y$, *and* $U_x \cap U_y = \phi$.

Definition 2.27. *Let* X *be a topological space such that one-point sets are closed in* X. *Then* X *is said to be* **regular** *if for each pair of a point* $x \in X$ *and a closed set* $B \subseteq X$ *disjoint from* x, *there exist disjoint open sets containing* x *and* B, *respectively.*

Theorem 2.28 (Urysohn metrization theorem)**.** *Every regular topological space* X *with a countable basis is metrizable.*

Definition 2.29. *Let* X *be a topological space. We say that* X *is* **contractible** *if there exist* $x \in X$ *and a continuous map* $H : X \times [0, 1] \to X$ *such that*

1. $H(y, 0) = y$ *for all* $y \in X$;

2. $H(y, 1) = x$ *for all* $y \in X$.

Definition 2.30. *Let* X *be a topological space and* $A \subseteq X$. *A* **retraction** *of* X *onto* A *is a continuous map* $r : X \to A$ *such that* $r|_A$ *is the identity map of* A. *If such a map* r *exists, we say that* A *is a* **retract** *of* X.

Definition 2.31. *Let* X *be a topological space and* $f : X \to X$ *a selfmap.*
The **convergence set** *of* f *is defined to be the set*

$$C(f) = \{x \in X : \text{the sequence } (f^n(x)) \text{ converges in } X\}$$

and the **fixed point set** *of* f *is defined to be the set* $F(f) = \{p \in X : f(p) = p\}$. *We call* $p \in F(f)$ *a* **fixed point** *of* f. *If* $p \in F(f)$, *then we define*
$C_p(f) = \{x \in X : f^n(x) \to p\}$. *Moreover, for each* $x \in C(f)$, *the continuity of* f *implies that*

$$f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f(f^n(x)) = \lim_{n\to\infty} f^n(x).$$

That is $\lim_{n\to\infty} f^n(x) \in F(f)$ *and hence we naturally obtain a well-defined map* $f^\infty : C(f) \to F(f)$ *given by* $f^\infty(x) = \lim_{n\to\infty} f^n(x)$ *for each* $x \in C(f)$.

Note that for a Hausdorff space X, a continuous selfmap $f : X \to X$ and a fixed point p, $F(f)$ is closed in X, but $C(f)$ and $C_p(f)$ need not be closed in X. For example, consider the map $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x^3$ for all $x \in \mathbb{R}$, and the map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2$ for all $x \in \mathbb{R}$. Then $C(g) = (-1, 1]$ and $C_0(h) = (-1, 1)$. Since we always have $f^\infty(x) = x$ for any $x \in F(f)$, the map f^∞ will be a retraction whenever it is continuous. Moreover, when f^∞ is continuous, any retraction from a superset of $C(f)$ onto $F(f)$ that satisfies a certain condition is simply a continuous extension of f^∞ by the following theorem.

Theorem 2.32. *Let* X *be a topological space and* $f : X \to X$ *a continuous selfmap. Suppose* f^∞ *is continuous and* $R : C(f) \to F(f)$ *is any retraction. If* $R \circ f = R$, *then* $R = f^\infty$.

Proof. See Theorem 1.1 in [1]. □

Definition 2.33. *Let* (X, d) *be a metric space, and* $f : X \to X$ *a continuous selfmap.*

- f *is called* **nonexpansive** *if* $d(f(x), f(y)) \leq d(x, y)$ *for any* $x, y \in X$.
- f *is called* **quasi − nonexpansive** *if* $d(f(x), p) \leq d(x, p)$ *for any* $x \in X$ *and* $p \in F(f)$.
- f *is called* **asymptotically nonexpansive** *if there is a sequence* (k_n) *of real numbers converging to 1 such that* $d(f^n(x), f^n(y)) \leq k_n d(x, y)$ *for any* $x, y \in X$ *and* $n \in \mathbb{N}$.
- f *is called* **asymptotically quasi − nonexpansive** *if there is a sequence* (k_n) *of real numbers converging to 1 such that* $d(f^n(x), p) \leq k_n d(x, p)$ *for any* $x \in X$, $p \in F(f)$ *and* $n \in \mathbb{N}$.

- *f is called* **virtually nonexpansive** *if $\{f^n : n \in \mathbb{N}\}$ is equicontinuous on $C(f)$.*

- *f is called* **periodic** *if $f^n = 1_X$ for some $n \in \mathbb{N}$.*

- *f is called* **recurrent** *if for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $x \in X$, $d(f^n(x), x) < \epsilon$.*

Notice that

1. Nonexpansive maps, quasi-nonexpansive maps and asymptotically nonexpansive maps are asymptotically quasi-nonexpansive;

2. Periodic maps are recurrent.

Theorem 2.34. *An asymptotically quasi-nonexpansive map is virtually nonexpansive.*

Proof. See Theorem 1.9 in [1]. □

Definition 2.35. *A* **topological $\mathbb{R}$ − linear space** *V is a vector space $(V, +, \cdot)$ over a topological field $\mathbb{R}$ which is endowed with a Hausdorff topology such that, the vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$ are continuous functions.*

Definition 2.36. *Let V be a topological $\mathbb{R}$-linear space, $v \in V$ and A a nonempty subset of V. We define $A - v = \{a - v : a \in A\}$.*

Definition 2.37. *Let V be a topological $\mathbb{R}$-linear space, X a nonempty subset of V and $x_0 \in X$. We say that X is* **$\mathbf{x_0}$ − star − convex** *if for each $x \in X$,*

$$\{tx + (1-t)x_0 : t \in [0, 1]\} \subseteq X$$

Definition 2.38. *[2] Let X be a 0-star-convex subset of a topological $\mathbb{R}$-linear space, $f : X \to X$ and $\phi : [0,1] \to [0,1]$ continuous selfmaps. We will call f a* $\phi -$ **homogeneous map**, *if for each $t \in [0,1]$ and $x \in X$,*

$$f(tx) = \phi(t)f(x).$$

Proposition 2.39. *Let X be a 0-star-convex subset of a topological $\mathbb{R}$-linear space, and $f : X \to X$ a non-constant ϕ-homogeneous map. Then we have the followings:*

1. $\phi(st) = \phi(s)\phi(t)$ *for all $s, t \in [0,1]$,*

2. $\{0,1\} \subseteq F(\phi)$,

3. $0 \in F(f)$.

Proof. See Proposition 2.4 in [2]. □

Theorem 2.40. *Let X be a 0-star-convex subset of a topological $\mathbb{R}$-linear space, and $f : X \to X$ a ϕ-homogeneous map with $C(\phi) = [0,1]$. Then $C(f)$ is 0-star-convex.*

Proof. See Theorem 2.5 in [2]. □

CHAPTER III

VIRTUALLY STABLE MAPS

From now on, if not otherwise state, X is a nonempty Hausdorff space and $f : X \to X$ a continuous selfmap.

Definition 3.1. *A fixed point x of f is said to be* **virtually f − stable** *if for each neighborhood U of x, there exists a strictly increasing sequence of natural numbers (k_n) and a neighborhood V of x satisfying $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. We simply call f* **virtually stable** *if every fixed point of f is virtually f-stable.*

Definition 3.2. *A fixed point x of f is said to be* **uniformly virtually f − stable** *if there exists a strictly increasing sequence of natural numbers (k_n) such that for each neighborhood U of x, there exists a neighborhood V of x with $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. We simply call f* **uniformly virtually stable** *if every fixed point of f is uniformly virtually f-stable with respect to the same (k_n).*

Notice that a map f whose every fixed point is uniformly virtually f-stable may not be uniformly virtually stable, and any uniformly virtually stable map is virtually stable. Moreover, it is easy to see that a periodic map is uniformly virtually stable while a virtually nonexpansive map is uniformly virtually stable.

Proposition 3.3. *A recurrent selfmap of a metric space is always uniformly virtually stable.*

Proof. Let (X, d) be a metric space and $f : X \to X$ a recurrent map. Since f is recurrent, the set

$$\left\{k \in \mathbb{N} : d(f^k(x), x) < \tfrac{1}{n} \text{ for all } x \in X\right\}$$

is infinite for each $n \in \mathbb{N}$. Hence there is a strictly increasing sequence of natural numbers (k_n) such that $d(f^{k_n}(x), x) < \frac{1}{n}$ for all $x \in X$ and $n \in \mathbb{N}$. Let $x \in F(f)$,

$m \in \mathbb{N}$ and $y \in B_d(x, \frac{1}{2(m+1)})$. We will show that $f^{k_n}(B_d(x, \frac{1}{2(m+1)})) \subseteq B_d(x, \frac{1}{m})$ for all $n \geq 2(m+1)$. For each $n \geq 2(m+1)$, $d(f^{k_n}(y), x) \leq d(f^{k_n}(y), y) + d(y, x) \leq \frac{1}{n} + \frac{1}{2(m+1)} \leq \frac{1}{2(m+1)} + \frac{1}{2(m+1)} < \frac{1}{m}$. Since f^{k_i} is continuous for $i = 1, ..., 2m+1$, there exists a neighborhood U of x such that $f^{k_i}(U) \subseteq B_d(x, \frac{1}{m})$ for $i = 1, ..., 2m+1$. Hence, $B_d(x, \frac{1}{2(m+1)}) \cap U$ is a neighborhood of x such that $f^{k_n}(B_d(x, \frac{1}{2(m+1)}) \cap U) \subseteq B_d(x, \frac{1}{m})$ for all $n \in \mathbb{N}$. We have that x is uniformly virtually f-stable with respect to (k_n) and f is uniformly virtually stable with respect to (k_n) as desired. □

The next example shows that there exists a virtually stable map (indeed a periodic map) that is not virtually nonexpansive.

Lemma 3.4. *Suppose X is a topological space whose topology is generated by a basis $\mathcal{A}$. If every element of $\mathcal{A}$ is closed in X, then X is regular.*

Proof. Let F be a closed subset of X and $x \in F^c$. Then there exists $P \in \mathcal{A}$ such that $x \in P \subseteq F^c$. By assumption P is open and closed, it follows that P and P^c are disjoint neighborhoods of x and F, respectively. Hence X is regular. □

Example 3.5. *Let $\mathcal{A} = \{[p, q) \subseteq \mathbb{R} : p, q \in \mathbb{Q}\, and\, p < q\}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -x$ for all $x \in \mathbb{R}$.*

We will show that $(\mathbb{R}, <\mathcal{A}>)$ is metrizable by showing that $\mathcal{A}$ is a countable basis for a topology on $\mathbb{R}$, and $(\mathbb{R}, <\mathcal{A}>)$ is regular. Clearly, A is countable. For each $r \in \mathbb{R}$, there exist $p, q \in \mathbb{Q}$ such that $p < r < q$; i.e., $r \in [p, q) \in \mathcal{A}$. For each $x \in [p_1, q_1) \cap [p_2, q_2)$, where $[p_1, q_1)$, $[p_2, q_2) \in \mathcal{A}$, we have $x \in [p_1, q_1) \cap [p_2, q_2) = [max\{p_1, p_2\}, min\{q_1, q_2\}) \in \mathcal{A}$. Thus $\mathcal{A}$ is a countable basis for a topology on $\mathbb{R}$. Clearly, one-pointed sets are closed. Let $[p, q) \in \mathcal{A}$ and $x \notin [p, q)$.

If $x \in (-\infty, p)$, then there exist $a, b \in \mathbb{Q}$ such that $a < x < b < p$; i.e., $x \in [a, b) \subseteq (-\infty, p)$.

If $x \in [q, \infty)$, then there exists $c \in \mathbb{Q}$ such that $x < c$; i.e., $x \in [q, c) \subseteq [q, \infty)$. Therefore, $[p, q)^C$ is open and, by Lemma 3.4, $(\mathbb{R}, <\mathcal{A}>)$ is regular. By Urysohn metrization theorem, $(\mathbb{R}, <\mathcal{A}>)$ is metrizable.

It is clear that f is periodic and hence recurrent. Thus f is uniformly virtually stable. We will prove that f is not virtually nonexpansive by showing that

$\{f^n : n \in \mathbb{N}\}$ is not equicontinuous at 0. Let $E \in \mathcal{A}$ be a neighborhood of 0 and $k \in E$ for some $k \in \mathbb{R}^+$. Let $q \in \mathbb{Q}^+$. Since $f^{2n+1}(k) = -k \notin [0, q)$ for all $n \in \mathbb{N}$, $f^{2n+1}(E) \nsubseteq [0, q)$ for all $n \in \mathbb{N}$. Thus $\{f^n : n \in \mathbb{N}\}$ is not equicontinuous at 0.

The next theorem guarantees that f^∞ is always a retraction of $C(f)$ onto $F(f)$ whenever X is regular and f is virtually stable.

Theorem 3.6. *Suppose X be a regular space and f a virtually stable selfmap. Then $f^\infty : C(f) \to F(f)$ is continuous and hence $F(f)$ is a retract of $C(f)$.*

Proof. Let $x \in C(f)$ and U a neighborhood of $f^\infty(x)$ in $F(f)$. Since X is regular, so is $F(f)$. Then, there is a neighborhood W of $f^\infty(x)$ in X such that $W \cap F(f) \subseteq \overline{W} \cap F(f) \subseteq U$. Now, by virtual stability, there exist a neighborhood V of $f^\infty(x)$ in X and a strictly increasing sequence (k_n) of positive integers such that $f^{k_n}(V) \subseteq W$ for all $n \in \mathbb{N}$. Since V is a neighborhood of $f^\infty(x)$, there is $N \in \mathbb{N}$ such that $f^N(x) \in V$. Let $K = f^{-N}(V) \cap C(f)$. Then K is a neighborhood of x in $C(f)$ such that

$$\begin{aligned} f^\infty(K) &= \left\{\lim_{n\to\infty} f^n(x) : x \in K\right\} \\ &= \left\{\lim_{n\to\infty} f^n(f^N(x)) : x \in K\right\} \\ &\subseteq \left\{\lim_{n\to\infty} f^n(x) : x \in V \cap C(f)\right\} \\ &= \left\{\lim_{n\to\infty} f^{k_n}(x) : x \in V \cap C(f)\right\} \\ &\subseteq \overline{W} \cap F(f) \\ &\subseteq U. \end{aligned}$$

Thus f^∞ is continuous and $F(f)$ is a retract of $C(f)$. □

To explore the connectedness of convergence sets and fixed point sets of virtually stable maps, we begin with Corollary 3.7.

Corollary 3.7. *Let X be a regular space. If f is virtually stable and $C(f)$ is (path-)connected, then $F(f)$ is (path-)connected.*

Proof. By Theorem 3.6, $f^\infty : C(f) \to F(f)$ is continuous. Then $F(f)$ is (path-)connected by Theorem 2.13. □

Corollary 3.8. *Let X be a regular space and f virtually stable. If $F(f)$ is a finite set, then $C(f)$ is disconnected.*

Proof. Let $F(f)$ be a finite set. Since X is a Hausdorff space, $F(f)$ is disconnected. If X is a finite set, then we are done. Now we consider the case that X is an infinite set. Suppose that $C(f)$ is connected. By Theorem 3.6, $F(f)$ is connected, which is a contradiction. Hence, $C(f)$ is disconnected. □

The next example show that if f is not virtually stable, then the condition $F(f)$ is a finite set does not guarantee that $C(f)$ is disconnected.

Example 3.9. *Consider $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $f(x) = x^2$. It is easy to show that 1 is not a virtually f-stable fixed point, $C(f) = [0, 1]$, and $F(f) = \{0, 1\}$.*

By considering the next example, we will face the fact that although f is nonexpasive, neither $C(f)$ nor $F(f)$ must be connected. Moreover, $C_p(f)$, where $p \in F(f)$, may not be connected.

Example 3.10. *Let $L = \mathbb{R}^2 - \{(0, 0), (0, 1), (0, -1)\}$ and L be equiped with the usual metric. Now we consider the map $g : L \to L$ defined by*

$$g(x, y) = \begin{cases} (-x, y), & \text{if } y < 0, \\ (-x, -y), & \text{if } y > 0 \end{cases}, \text{ for all } (x, y) \in L.$$

It is easy to obtain the results that g is nonexpansive,
$C(g) = \{(x, y) \in L : x = 0\}$, $F(g) = \{(x, y) \in L : x = 0$ and $y < 0\}$,
and $C_{(x,y)}(g) = \{(0, y), (0, -y)\}$ for all $(x, y) \in F(g)$.

The next theorem provides sufficient conditions that can guarantee the connectedness of convergence sets.

Theorem 3.11. *Let $f : X \to X$ be a continuous map satisfying one of the following conditions:*

1. *$C_p(f)$ is connected for all $p \in F(f)$,*

2. *for each component A of $C(f)$ there is $h_A \in \mathbb{N}$ such that $f^{h_A}(A) \cap A \neq \phi$.*

If $F(f)$ is connected, then $C(f)$ is connected.

Proof. Let f satisfies (1) and $F(f)$ is connected. Suppose on the contrary that $C(f)$ is not connected. Then $F(f) \subseteq A$ for some component A of $C(f)$ and there exists component B of $C(f)$ such that $B \cap A = \phi$. Let $x \in B$. Then $\lim\limits_{n\to\infty} f^n(x) = p$ for some $p \in F(f)$. Since $C_p(f) \cap A \neq \phi$ and $C_p(f) \cap B \neq \phi$, then $C_p(f)$ is not connected. This contradicts to the assumption.

Assume that (2) is true and $F(f)$ is connected. Since $F(f)$ is connected, $F(f) \subseteq A$ for some component A of $C(f)$. Suppose that $C(f)$ is not connected. Hence, there exists the component B of $C(f)$ such that $B \cap A = \phi$. Since $f^{h_B}(B) \cap B \neq \phi$ and $f^{h_B}(B)$ is connected, we get that $f^{nh_B}(B) \subseteq B$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} f^{nh_B}(x) = \lim\limits_{n\to\infty} f^n(x) \in F(f) \subseteq C(f)$ for all $x \in B$ and, by Theorem 2.12, B is closed in $C(f)$, we have $\lim\limits_{n\to\infty} f^n(x) = \lim\limits_{n\to\infty} f^{nh_B}(x) \in B$ for all $x \in B$. Hence $F(f) \cap B \neq \phi$, which is a contradiction. □

Lemma 3.12. *Let $f : X \to X$ be continuous. If $C_p(f)$ is path connected for all $p \in F(f)$ and $F(f)$ is path connected, then $C(f)$ is path connected.*

Proof. Suppose on the contrary that $C(f)$ is not path connected. Then $F(f) \subseteq A$ for some path component A of $C(f)$ and there exists path component B of $C(f)$ such that $B \cap A = \phi$. Let $x \in B$. Then $\lim\limits_{n\to\infty} f^n(x) = p$ for some $p \in F(f)$. Since $C_p(f) \cap A \neq \phi$ and $C_p(f) \cap B \neq \phi$, then $C_p(f)$ is not path connected. This contradicts to the assumption. □

Some properties of convergence sets of virtually stable maps can be seen in the following results.

Lemma 3.13. *For each $x \in X$, we have $x \in C(f)$ if and only if the sequence $(f^n(x))$ has a subsequence $(f^{n_k}(x))$ converging to a fixed point of f and*

$$sup\ \{n_{k+1} - n_k : k \in \mathbb{N}\} < \infty.$$

Proof. ($\Rightarrow$)Obvious.

($\Leftarrow$)Let sup $\{n_{k+1} - n_k : k \in \mathbb{N}\} = h$ and $1 \leq i \leq h$. Suppose $\lim\limits_{k\to\infty} f^{n_k}(x) = p \in F(f)$. Since f is continuous, we have $p = f^i(p) = f^i(\lim\limits_{k\to\infty} f^{n_k}(x)) = \lim\limits_{k\to\infty} f^{i+n_k}(x)$. To show that $f^n(x) \to p$, we let U be a neighborhood of p. Since $\lim\limits_{k\to\infty} f^{i+n_k}(x) = p$ and $\lim\limits_{k\to\infty} f^{n_k}(x) = p$, there exists $N \in \mathbb{N}$ such that $f^{i+n_l}(x) \in U$ and $f^{n_l}(x) \in U$ for all $l \geq N$. Let $j \geq n_{N+1}$. Then there exists $r' \geq N + 1$ such that $n_{N+1} \leq n_{r'} \leq j \leq n_{r'+1}$. Hence $j = n_{r'} + s$ for some $0 \leq s \leq h$ and we have $f^j(x) \in U$ for all $j \geq n_{N+1}$. □

Theorem 3.14. *Let p be a uniformly virtually f-stable fixed point with respect to (k_n) and $x \in X$. Suppose there exist $r, h \in \mathbb{N}$ with $k_{r+i} = k_i + h; i \in \mathbb{N}$. Then $x \in C_p(f)$ if and only if there exists a sequence of natural numbers (r_n) such that for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with*
$r_n = k_{i+m-1} + r_1 - k_i, \forall i \in \mathbb{N}$ and $f^{r_n}(x) \to p$.

Proof. ($\Rightarrow$) Since $f^n(x) \to p$, $f^{nh}(x) \to p$. By the assumption, $k_i + h = k_{i+r}$ for all $i \in \mathbb{N}$. Hence $k_i + nh = k_{i+(n-1)r} + h$ for all $i \in \mathbb{N}$. By letting, $r_n = nh$ and $m = (n-1)r + 1$, we are done.

($\Leftarrow$) To show that $f^n(x) \to p$, let U be a neighborhood of p. Then there exists a neighborhood V of p such that $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$. Since $f^{r_n}(x) \to p$, there exists $N \in \mathbb{N}$ such that $f^{r_N}(x) \in V$. Thus $f^{r_N+k_i}(x) \in U$ for all $i \in \mathbb{N}$. By the assumption, there exists $M \in \mathbb{N}$ such that $k_i + r_N = k_{i+M-1} + r_1$ for all $i \in \mathbb{N}$. Then $f^{k_i+r_1}(x) \in U$ for all $i \geq M$. Hence $f^{k_n+r_1}(x) \to p$. Since $k_{i+nr} = k_i + nh$ for all $i > 0$, $n \geq 0$, we have $k_{nr+i+1} - k_{nr+i} = k_{i+1} - k_i$ for all $1 \leq i \leq r$ and $n \geq 0$. Thus sup $\{k_{n+1} - k_n : n \in \mathbb{N}\} =$ sup $\{k_{n+1} - k_n : 1 \leq n \leq r\}$. By Lemma 3.13, $f^n(x) \to p$. □

Corollary 3.15. *Suppose (X,d) is a metric space, f is virtually nonexpansive, $x \in X$ and $p \in F(f)$. Then, $x \in C_p(f)$ if and only if $(f^n(x))$ has a subsequence converging to p. Hence, $C_p(f) = \{x \in X : d(O(f,x),p) = 0\}$, where $O(f,x) = \{f^n(x) : n \in \mathbb{N}\}$.*

Proof. All notations follow Theorem 3.14. Since f is virtually nonexpansive, we can set $k_n = n$ for all $n \in \mathbb{N}$, $h = 1$ and $r = 1$. Let (r_n) be any strictly increasing sequence of natural numbers such that $f^{r_n}(x) \to p$ and $n' \in \mathbb{N}$. Then $r_{n'} = i + (r_{n'} - r_1 + 1) - 1 + r_1 - i = k_{i+(r_{n'}-r_1+1)-1} + r_1 - k_i$ for all $i \in \mathbb{N}$. Hence we get the result by Theorem 3.14. □

Theorem 3.16. *Suppose (X,d) is a metric space, p is a uniformly virtually f-stable fixed point with respect to the sequence (nh) for some $h \in \mathbb{N}$ and $x \in C_p(f)$. Then for each $\epsilon > 0$, there exists $\delta > 0$ with $f^{nh}(B_d(x,\delta)) \subseteq B_d(f^{nh}(x),\epsilon)$ for all $n \in \mathbb{N}$.*

Proof. Let $\epsilon > 0$. By uniformly virtual stability, there exists $r \in (0, \frac{\epsilon}{2})$ such that, for each $n \in \mathbb{N}$, $f^{nh}(B_d(p,r)) \subseteq B_d(p, \frac{\epsilon}{2})$. Since $x \in C_p(f)$, there exists $N \in \mathbb{N}$ such that $f^{nh}(x) \in B_d(p,r)$ for all $n \geq N$. By the continuity of $f^h, ..., f^{Nh}$, there exists $\delta > 0$ such that $f^{Nh}(B_d(x,\delta)) \subseteq B_d(p,r)$ and $f^{nh}(B_d(x,\delta)) \subseteq B_d(f^{nh}(x),\epsilon)$ for $n \leq N$. Thus, for each $n \in \mathbb{N}$ and $y \in B_d(x,\delta)$, we consider the following 2 cases:

Case 1 : $n \leq N$.

By the property of δ above, we have $f^{nh}(B_d(x,\delta)) \subseteq B_d(f^{nh}(x),\epsilon)$.

Case 2 : $n > N$.

Suppose $n = N + i$ for some $i \in \mathbb{N}$. Then

$$\begin{aligned} d(f^{nh}(y), f^{nh}(x)) &= d(f^{(N+i)h}(y), f^{(N+i)h}(x)) \\ &\leq d(f^{(N+i)h}(y), p) + d(p, f^{(N+i)h}(x)). \end{aligned}$$

Since $f^{Nh}(x)$, $f^{Nh}(y) \in B_d(p,r)$, $d(f^{nh+Nh}(y),p) < \frac{\epsilon}{2}$ for each $n \in \mathbb{N}$.

Thus $d(f^{(N+i)h}(x),p) + d(p, f^{(N+i)h}(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence, we get the result. □

The next theorem generalizes Theorem 1.2 in [2].

Theorem 3.17. *Suppose (X, d) is a complete metric space and f is uniformly virtually stable with respect to the sequence (nh) for some $h \in \mathbb{N}$. Then $C(f)$ is a G_δ-set.*

Proof. By Theorem 3.16, for every $x \in C(f)$ and $m \in \mathbb{N}$, there exists $\delta_{x,m} > 0$ such that $f^{nh}(B_d(x, \delta_{x,m})) \subseteq B_d(f^{nh}(x), \frac{1}{m})$ for every $n \in \mathbb{N}$. Let $K = \bigcap_{m\in\mathbb{N}} \bigcup_{x\in C(f)} B_d(x, \delta_{x,m})$. Clearly, K is a G_δ-set. We will show that $K = C(f)$. It is clear that $C(f) \subseteq K$. To show that $K \subseteq C(f)$, we let $k \in K$ and $n \in \mathbb{N}$. Then there exist $x \in C(f)$ and $\delta_{x,4n} > 0$ such that $d(k, x) < \delta_{x,4n}$. Hence, $d(f^{mh}(k), f^{mh}(x)) < \frac{1}{4n}$ for all $m \in \mathbb{N}$. Since $x \in C(f)$, there is $p_n \in F(f)$ and $N_n \in \mathbb{N}$ such that $d(f^{mh}(x), p_n) < \frac{1}{4n}$ for all $m > N_n$. Then $d(f^{mh}(k), p_n) \leq d(f^{mh}(k), f^{mh}(x)) + d(f^{mh}(x), p_n) < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n}$ for every $m > N_n$, and $d(f^{m'h}(k), f^{mh}(k)) \leq d(f^{m'h}(k), p_n) + d(p_n, f^{mh}(k)) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ for all $m', m > N_n$. Hence, $(f^{nh}(k))_{n\in\mathbb{N}}$ is a Cauchy sequence and $f^{nh}(k) \to p'$ for some $p' \in X$. We will prove that $p' \in F(f)$ by showing that $p_n \to p'$. Let $n \in \mathbb{N}$ and $l \geq n$. Then $d(f^{mh}(k), p_l) < \frac{1}{2l}$ for all $m \geq N_l$. Since $f^{nh}(k) \to p'$, there is $M_l \in \mathbb{N}$ such that $M_l > N_l$ and $d(f^{M_l h}(k), p') < \frac{1}{2l}$. Hence,

$$d(p_l, p') \leq d(p_l, f^{M_l h}(k)) + d(f^{M_l h}(k), p') < \frac{1}{2l} + \frac{1}{2l} = \frac{1}{l} \leq \frac{1}{n}.$$

Since $F(f)$ is closed, $p' \in F(f)$. By Lemma 3.13, $k \in C(f)$. □

Corollary 3.18. *Let (X, d) be a complete metric space and f virtually nonexpansive. Then $C(f)$ is a G_δ-set.*

Theorem 3.19. *Suppose (X, d) is a complete metric space and f is assymptotically nonexpansive. Then $C(f)$ is closed.*

Proof. To show that $C(f)$ is closed, let $x \in X$ and (x_n) be a sequence in $C(f)$ such that $x_n \to x$. We will show that $(f^n(x))$ is a Cauchy sequence. Let $m \in \mathbb{N}$. Since f is assymptotically nonexpansive, there exists $K > 1$ such that $d(f^n(y), f^n(z)) < Kd(y, z)$ for all $n \in \mathbb{N}$ and $y, z \in X$. Since $x_n \to x$, there exists $M \in \mathbb{N}$ such that

$d(x, x_M) < \frac{1}{K3m}$. Since $x_M \in C(f)$, $(f^n(x_M))$ is a Cauchy sequence. Hence there exists $M' \in \mathbb{N}$ such that $d(f^h(x_M), f^k(x_M)) < \frac{1}{3m}$ for all $h, k \geq M'$.
Then, for $h, k \geq M'$,

$$\begin{aligned} d(f^h(x), f^k(x)) &\leq d(f^h(x), f^h(x_M)) + d(f^h(x_M), f^k(x_M)) + d(f^k(x_M), f^k(x)) \\ &< K\frac{1}{K3m} + \frac{1}{3m} + K\frac{1}{K3m} \\ &= \frac{1}{m}. \end{aligned}$$

Thus $x \in C(f)$ and $C(f)$ is closed. □

CHAPTER IV
APPLICATION TO
ϕ-HOMOGENEOUS MAPS

In this chapter, we will investigate some properties of ϕ-homogeneous maps, their convergence sets, and their fixed point sets. We begin this chapter by extending some notions introduced in [2].

Definition 4.1. *Let X be an x_0-star-convex subset of a topological $\mathbb{R}$-linear space, and $f : X \to X$ and $\phi : [0,1] \to [0,1]$ continuous selfmaps. We will call f a* $\phi -$ **homogeneous map with respect to $\mathbf{x_0}$**, *if for each $t \in [0,1]$ and $x \in X$*

$$f(tx + (1-t)x_0) = \phi(t)f(x) + (1-\phi(t))x_0.$$

Example 4.2. *Note that $\mathbb{C}$ is a topological $\mathbb{R}$-linear space. So it is certainly 1-star-convex. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z-1|^2 + 1$. Then $f(tz+(1-t)) = |(tz+(1-t))-1|^2+1 = |tz-t|^2+1 = t^2|z-1|^2+1 = t^2(|z-1|^2+1)+(1-t^2) = t^2 f(z) + (1-t^2)$. Hence, f is a ϕ-homogeneous map with respect to 1 where $\phi : [0,1] \to [0,1]$ is a continuous selfmap defined by $\phi(x) = x^2$ for all $x \in [0,1]$.*

Example 4.3. *Let $A, B \in \mathbb{R}$ and $A \neq 1$. Since $\mathbb{R}$ is a topological $\mathbb{R}$-linear space, it is certainly $\frac{B}{1-A}$-star-convex. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = Ax + B$. Then f is a ϕ-homogeneous map with respect to $\frac{B}{1-A}$ where ϕ is the identity map.*

Example 4.4. *Consider $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $f(x) = x^3$. It is easy to show that f is a ϕ-homogeneous map with respect to 0 where $\phi(t) = t^3$, f is not virtually stable, $C(f) = [0,1]$, and $F(f) = \{0,1\}$ which is not 0-star-convex.*

From Definition 4.1, notice that

1. When $x_0 = 0$, the definition coincides with ϕ-homogeneous map in [2];

2. If ϕ is the identity map, then $F(f)$ is x_0-star-convex;

3. Although ϕ is the identity map and X is a topological $\mathbb{R}$-linear space, f need not be linear.

Example 4.5. *Since $\mathbb{R}$ is a topological $\mathbb{R}$-linear space, it is certainly 1-star-convex. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x-1|+1$. We have that $f(tx+(1-t)) = |tx+(1-t)-1|+1 = t|x-1|+1 = t|x-1|+t-t+1 = t(|x-1|+1)+(1-t) = tf(x)+(1-t)$ for all $t \in [0,1]$. Then f is a ϕ-homogeneous map with respect to 1. But f is not linear since $f(1-1) = f(0) = 2 \neq 4 = f(1)+f(-1)$.*

Example 4.6. *Recall that $L^2([-1,1])$ is a topological $\mathbb{R}$-linear space. So it is 0-star-convex. Consider $T : L^2([-1,1]) \to L^2([-1,1])$ defined by*

$$T(f)(x) = \sqrt{\int_{-1}^{x} (f(y))^2 dy} \ .$$

It is easy to show that T is a ϕ-homogeneous map with respect to 0.
Let g be the identity map on $[-1,1]$. Then $-g \in L^2([-1,1])$ and
$T(g+(-g))(x) = \sqrt{\int_{-1}^{x} (y-y)^2 dy} = 0$, *but*
$(T(g)+T(-g))(x) = \sqrt{\int_{-1}^{x} (y)^2 dy} + \sqrt{\int_{-1}^{x} (-y)^2 dy} = 2\sqrt{\frac{x^3+1}{3}}$.
Hence, T is not linear.

From now on, let X be an x_0-star-convex subset of a topological $\mathbb{R}$-linear space, $f : X \to X$ and $\phi : [0,1] \to [0,1]$ continuous selfmaps. Furthermore, we define $f' : X - x_0 \to X - x_0$ by $f'(x) = f(x+x_0) - x_0$ for all $x \in X - x_0$.

Lemma 4.7. *Let $f : X \to X$ be a ϕ-homogeneous map with respect to x_0. Then $f' : X - x_0 \to X - x_0$ is a ϕ-homogeneous map, $C(f') + x_0 = C(f)$, and $F(f') + x_0 = F(f)$.*

Proof. We will show that f' is a ϕ-homogeneous map. Clearly, $X - x_0$ is an

0-star-convex set. Let $t \in [0,1]$ and $x \in X - x_0$. Then

$$\begin{aligned} f'(tx) &= f(tx + x_0) - x_0 \\ &= f(t(x + x_0) + (1-t)x_0) - x_0 \\ &= (\phi(t)f(x + x_0) + (1 - \phi(t))x_0) - x_0 \\ &= \phi(t)(f(x + x_0) - x_0) \\ &= \phi(t)f'(x). \end{aligned}$$

To show that $C(f') + x_0 = C(f)$, we observe that $(f')^n(x) = (f')^{n-1}(f(x + x_0) - x_0) = (f')^{n-2}(f^2(x + x_0) - x_0) = ... = f^n(x + x_0) - x_0$ for all $x \in X - x_0$ and $n \geq 2$. Hence, for each $x \in X - x_0$, we have $x \in C(f')$ if and only if $x + x_0 \in C(f)$. Moreover, by the definition of f', $x \in F(f')$ if and only if $x + x_0 \in F(f)$. □

Lemma 4.8. *If $f : X \to X$ is a non-constant ϕ-homogeneous map with respect to x_0, we have the followings:*

1. *$\phi(st) = \phi(s)\phi(t)$ for all $s, t \in [0,1]$;*

2. *$\{0,1\} \subseteq F(\phi)$;*

3. *$x_0 \in F(f)$;*

4. *If $\phi(x) = 0$, then $x = 0$;*

5. *If $\phi(x) = 1$, then $x = 1$;*

6. *ϕ is a strictly increasing function;*

7. *$C(\phi) = [0,1]$.*

Proof. By Lemma 4.7, f' is a ϕ-homogeneous map. Then (1), (2), and the fact that $0 \in F(f')$ are results from Proposition 2.39. Again, by Lemma 4.7, $x_0 \in F(f)$.

We will show (4). By (2), $t = \sup \{x \in [0,1] : \phi(x) = 0\} < 1$ exists. Since ϕ is continuous, $\phi(t) = 0$. Suppose on the contrary that $t > 0$. Then, by (1), $\phi(t) = \phi(\sqrt{t}\sqrt{t}) = \phi(\sqrt{t})\phi(\sqrt{t})$. Since $\sqrt{t} > t$, we have $\sqrt{t} > 0$. Hence, $0 = \phi(t) =$

$\phi(\sqrt{t})\phi(\sqrt{t}) > 0$, which is a contradiction.

We will prove (5). It is similar to (4) that there exists $t = \inf\{x \in [0,1] : \phi(x) = 1\}$ such that $\phi(t) = 1$. Suppose on the contrary that $t < 1$. Since $t^2 < t$, $\phi(t^2) < 1$. Then $1 = \phi(t)\phi(t) = \phi(t^2) < 1$, which is a contradiction.

To show (6), let $s, t \in [0,1]$ be such that $s < t$. Then $\frac{s}{t} < 1$, so $\phi(\frac{s}{t}) < 1$. We get that $\phi(s) = \phi(t(\frac{s}{t})) = \phi(t)\phi(\frac{s}{t}) < \phi(t)$.

To show (7), let $x \in [0,1]$. If $x = \phi(x)$, we are done. If $x < \phi(x)$, then for each $n \in \mathbb{N}$, $\phi^n(x) < \phi^{n+1}(x)$ by (6). We obtain that $(\phi^n(x))$ is a strictly increasing sequence in $[0,1]$. The monotone convergence theorem guarantees that $x \in C(\phi)$. If $\phi(x) < x$, we get that $(\phi^n(x))$ is a strictly decreasing sequence in $[0,1]$. Again, we have $x \in C(\phi)$. □

Theorem 4.9. *Let $f : X \to X$ be a ϕ-homogeneous map with respect to x_0. Then $C(f)$ is x_0-star-convex.*

Proof. If f is a constant function, we are done. Otherwise, by Theorem 2.40 and Lemma 4.8, $C(f')$ is 0-star convex. Moreover, by Lemma 4.7, $C(f) = C(f') + x_0$. Hence, $C(f)$ is x_0-star-convex. □

The next example shows that the fixed point set of a ϕ-homogeneous map with respect to x_0 need not be x_0-star-convex.

Example 4.10. *Consider $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $g(x) = x^2$. It is easy to show that g is a ϕ-homogeneous map with respect to 0 where $\phi(t) = t^2$, $C(g) = [0,1]$, and $F(g) = \{0,1\}$ which is not 0-star-convex.*

The next theorem improves and generalizes Theorem 3.3 in [2].

Theorem 4.11. *If $f : X \to X$ is a virtually stable ϕ-homogeneous map with respect to x_0 that fixes more than one point, then $\phi(t) = t$ for all $t \in [0,1]$.*

Proof. By the assumption and Lemma 4.8, there is $x_1 \in F(f) - \{x_0\}$. Suppose on the contrary that there exists $t_0 \in (0,1)$ such that $\phi(t_0) \neq t_0$. Then we consider the following 2 cases:

Case 1 : $\phi(t_0) > t_0$.

By Lemma 4.8, $0 \in F(\phi)$. Thus sup $\{t \in [0, t_0) : \phi(t) = t\}$ exists. Let $t' =$ sup $\{t \in [0, t_0) : \phi(t) = t\}$. Since $F(\phi)$ is closed, t' must be a fixed point. Then $\phi(t') = t' < t_0 < \phi(t_0)$. By intermediate value theorem and the property of t', there exists $t_1 \in (t', t_0)$ such that $\phi(t') = t' < t_1 < \phi(t_1) = t_0$. Similarly, there exists $t_2 \in (t', t_1)$ such that $\phi(t') = t' < t_2 < \phi(t_2) = t_1$. By continuing this process, we obtain, for each $n \in \mathbb{N}$, there exists $t_n \in (t', t_{n-1})$ such that $\phi(t') = t' < t_n < \phi(t_n) = t_{n-1}$ and $\phi^n(t_n) = t_0$. Hence, (t_n) is a strictly decreasing sequence in $[t', t_0]$. By monotone convergence theorem, there exists $t'' \in [t', t_0]$ such that $t_n \to t''$. Since ϕ is continuous, $\phi(t'') = \phi(\lim_{n\to\infty}(t_n)) = \lim_{n\to\infty} \phi(t_n) = \lim_{n\to\infty} t_{n-1} = t''$. Therefore, $t'' = t'$. Because $x_1 \in F(f)$ and $t' \in F(\phi)$, we have $f(t'x_1 + (1-t')x_0) = \phi(t')f(x_1) + (1-\phi(t'))x_0 = t'x_1 + (1-t')x_0$, that is $t'x_1 + (1-t')x_0 \in F(f)$. Moreover, $t'x_1 + (1-t')x_0 \neq \phi(t_0)x_1 + (1-\phi(t_0))x_0$ since $\phi(t_0) > t'$ and $x_1 \neq x_0$. Then there is a neighborhood U of $t'x_1 + (1-t')x_0$ such that $\phi(t_0)x_1 + (1-\phi(t_0))x_0 \notin U$. We will show that the fixed point $t'x_1 + (1-t')x_0$ is not virtually f-stable and obtain a contradiction. Since $t_n \to t'$, we have $t_nx_1 + (1-t_n)x_0 \to t'x_1 + (1-t')x_0$. It follows that, for each neighborhood V of $t'x_1 + (1-t')x_0$, there exists $N \in \mathbb{N}$ such that $t_nx_1 + (1-t_n)x_0 \in V$ for all $n \geq N$. Then $\phi(t_0)x_1 + (1-\phi(t_0))x_0 = \phi^{n+1}(t_n)x_1 + (1-\phi^{n+1}(t_n))x_0 = f^{n+1}(t_nx_1 + (1-t_n)x_0) \in f^{n+1}(V)$ for all $n \geq N$. Thus, $f^{n+1}(V)$ can not be a subset of U for each $n \geq N$. Hence, U is a neighborhood of $t'x_1 + (1-t')x_0$ having the property that for all neighborhood V of $t'x_1 + (1-t')x_0$, there is no strictly increasing sequence of natural numbers (k_n) that $f^{k_n}(V) \subseteq U$ for all $n \in \mathbb{N}$.

Case 2 : $\phi(t_0) < t_0$.

By Lemma 4.8, $1 \in F(\phi)$. We can let $t' = \inf\{t \in (t_0, 1] : \phi(t) = t\} \in F(\phi)$. It is similar to case 1 that there exists a strictly increasing sequence (t_n) in $[t_0, t']$ such that $t_n \to t'$ and $\phi^n(t_n) = t_0$ for all $n \in \mathbb{N}$. Moreover, by imitating the process in case 1, we obtain a contradiction that f is not virtually stable.

Hence, $\phi(t) = t$ for all $t \in [0, 1]$. $\square$

Corollary 4.12. *If $f : X \to X$ is a virtually stable ϕ-homogeneous map with respect to x_0, then $F(f)$ is x_0-star-convex.*

Proof. If f has only one fixed point, then we are done. Otherwise, by Theorem 4.11, we immediately have $f(tx + (1-t)x_0) = tf(x) + (1-t)x_0 = tx + (1-t)x_0$ for all $t \in [0, 1]$ and $x \in F(f)$. Therefore $F(f)$ is x_0-star-convex as desired. □

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