CHAPTER I

PRELIMINARIES

A semigroup S is called a left [right] zero semigroup if ab = a [ab = b] for all a, b ε S.

A semigroup S with zero 0 is called a Kronecker semigroup if

$$ab = \begin{cases} a & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

for all a, b ε S.

By a subgroup of a semigroup S, we mean a subsemigroup of S which is also a group. Note that if a semigroup S is a union of subgroups of S and S contains a unique idempotent, then S is a group.

A semigroup S is said to be a left [right] group if S is a union of subgroups of S and the set of all idempotents of S forms a left [right] zero subsemigroup of S. Hence every left [right] zero semigroup is a left [right] group. It is clearly seen that a left [right] group which contains a zero element is trivial.

Let S be a semigroup and let 0 be a symbol not representing any element of S. Extend the given binary operation on S to one in $S \cup \{0\}$ by defining $00 = 0a = a0 = 0$ for all a ε S. Then $S \cup \{0\}$ is a semigroup with zero 0. Let

$$S^{0} = \begin{cases} S & \text{if } S \text{ has a zero element and contains more than one element,} \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

A semigroup S is said to admit a ring structure if there exists an operation + on S^o such that $(S^o,+,.)$ is a ring where . is the operation of S^o.

For a set A, let $|A|$ denote the cardinality of A.

For a set A, let P(A) denote the power set of A and let $P^*(A) = P(A)\setminus\{\emptyset\}$.

A hyperoperation or a multioperation o on a nonempty set H is a mapping of H × H into $P^*(H)$.

A hypergroupoid is a system (H,o) consisting of a nonempty set H together with a hyperoperation o on H. We shall usually write H instead of (H,o) when there is no danger of ambiguity.

Let (H,o) be a hypergroupoid. For nonempty subsets A, B of H, let

$$AoB = \bigcup_{\substack{a\in A\\ b\in B}} (aob)$$

and let $Aox = Ao\{x\}$ and $xoA = \{x\}oA$ for all $x \in H$. An element e of H is called an identity of (H,o) if $x \in (xoe)\cap(eox)$ for all $x \in H$. A hypergroupoid can have more than one identity. See Example 1 on page 9 as an example. An element e of H is called a scalar identity of (H,o) if $xoe = eox = \{x\}$ for all $x \in H$. If e is a scalar identity of (H,o), then e is the unique identity of (H,o). To prove this, let e be a scalar identity and f an identity of (H,o). Since f is an identity of (H,o), we have that $e \in eof$. It follows from the fact that e is a scalar identity of (H,o), we get $eof = \{f\}$. Then $e \in \{f\}$ which implies that $e = f$.

The hyperoperation o of a hypergroupoid (H,o) is said to be commutative if xoy = yox for all x, y ε H and it is said to be associative if (xoy)oz = xo(yoz) for all x, y, z ε H.

A semihypergroup is a hypergroupoid (H,o) such that the hyperoperation o is associative. A semihypergroup (H,o) is called a hypergroup if Hox = xoH = H for all x ε H. A hypergroup need not contain an identity. An example of such hypergroups is given in Example 3, page 11. A nonempty subset K of a hypergroup H is said to be a subhypergroup of H if K forms a hypergroup under the hyperoperation of H.

An element x of a semihypergroup (H,o) is said to be an inverse of an element y in (H,o) if there exists an identity e of (H,o) such that e ε (xoy)$\cap$(yox), that is, (xoy)$\cap$(yox) contains at least one identity of (H,o). Then every identity of a semihypergroup (H,o) is an inverse of itself since e ε eoe for every identity e of (H,o). Example 4 on page 12 shows that an element of a hypergroup containing some identities need not have an inverse.

A hypergroup H is said to be regular if H has at least one identity and every element of H has at least one inverse in H.

A regular hypergroup (H,o) is said to be reversible if for x, y, z ε H, x ε yoz implies z ε uox and y ε xov for some inverse u of y and for some inverse v of z in (H,o).

A canonical hypergroup is a commutative reversible hypergroup which has a scalar identity and every element has a unique inverse. Hence a hypergroup (H,o) is a canonical hypergroup if and only if

1. (H,o) is commutative,
2. (H,o) has a scalar identity,
3. every element of H has a unique inverse in (H,o) and

4. for a, x, y ε H, y ε aox implies x ε a'oy where a' denotes the unique inverse of a in (H,o).

Throughout the paper, the notation x' will denote the unique inverse of the element x in a canonical hypergroup. Observe that every abelian group is a canonical hypergroup.

Example 4 on page 12 shows that an inverse of an element of a hypergroup containing a scalar identity need not be unique. A subhypergroup of a canonical hypergroup need not be canonical. This is shown by Example 5 on page 13.

A hyperring is a system (A,+,.) such that

1. (A,+) is a canonical hypergroup,
2. (A,.) is a semigroup in which the scalar identity 0 of (A,+) is a zero of (A,.) and
3. x.(y + z) = x.y + x.z and (x + y).z = x.z + y.z for all x, y, z ε A.

The operations + and . of a hyperring (A,+,.) are called the addition and multiplication of A, respectively. We shall usually write A instead of (A,+,.) when there is no danger of ambiguity.

Let (A,+,.) be a hyperring. The scalar identity of the hypergroup (A,+) which is the zero of the semigroup (A,.) is usually called the zero of the hyperring (A,+,.) and it is usually denoted by 0. By an identity of the hyperring (A,+,.), we mean an identity of the semigroup (A,.) which is usually denoted by 1. For x, y ε A and n a positive integer, let x' denote the unique inverse of x in the canonical hypergroup (A,+), xy denote x.y and x^n denote xx...x (n times). It then follows easily that.

1. $0' = 0$,

2. $(x')' = x$ for all $x \in A$,

3. $x'y = (xy)' = xy'$ for all $x, y \in A$ and

4. $x'y' = xy$ for all $x, y \in A$.

A <u>commutative</u> <u>hyperring</u> A is a hyperring such that $xy = yx$ for all $x, y \in A$.

A commutative hyperring A is called a <u>hyperintegral</u> <u>domain</u> if for $x, y \in A$, $xy = 0$ implies $x = 0$ or $y = 0$.

A commutative hyperring $(A,+,.)$ is called a <u>hyperfield</u> if $(A \setminus \{0\},.)$ is a group.

A nonempty subset B of a hyperring $(A,+,.)$ is called a <u>subhyperring</u> of $(A,+,.)$ if B forms a hyperring under the same addition and multiplication of $(A,+,.)$, that is, B is a canonical subhypergroup of the hypergroup $(A,+)$ and B is a subsemigroup of the semigroup $(A,.)$. A subhyperring of a hyperring A need not contain the zero of A. See Example 6 on page 16 as an example.

A nonempty subset I of a hyperring $(A,+,.)$ is called a <u>left</u> [right] <u>hyperideal</u> of A if I is a subhypergroup of the hypergroup $(A,+)$ and $AI \subseteq I$ [$IA \subseteq I$]. A nonempty subset I of a hyperring A is called a <u>hyperideal</u> of A if it is both a left and a right hyperideal of A.

A hyperideal I of a hyperring A is called a <u>prime</u> <u>hyperideal</u> if for $x, y \in A$, $xy \in I$ implies $x \in I$ or $y \in I$.

A hyperideal I of a hyperring A is called a <u>maximal</u> <u>hyperideal</u> if $I \neq A$ and for a hyperideal K of A, $I \subseteq K \subseteq A$ implies $I = K$ or $K = A$.

A hyperring A in which $x^2 = x$ for all $x \in A$ is called a <u>Boolean</u> <u>hyperring</u>.

Let A be a hyperring and I a hyperideal of A. Then the following statements hold :

(1) $0 \in I$.

(2) For every $x \in I$, $x' \in I$.

(3) For any $x, y \in A$, either $(x + I) \cap (y + I) = \phi$ or $x + I = y + I$.

Hence I is a subhyperring of A and for $x, y \in A$, $x \in y + I$ if and only if $x + I = y + I$.

Let $A/I = \{x + I \mid x \in A\}$. Then A/I becomes a hyperring under the addition and the multiplication defined as follows :

$$(x + I) + (y + I) = \{z + I \mid z \in x + y\}$$

and

$$(x + I).(y + I) = xy + I$$

for all $x, y \in A$. The hyperring A/I has I as its zero and for $x \in A$, $x' + I$ is the unique inverse of $x + I$ under addition. The hyperring A/I is called a <u>quotient hyperring</u> of the hyperring A.

Let A and B be hyperrings. A map $\varphi : A \to B$ is called a <u>homomorphism</u> if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$. If φ is a homomorphism of A into B, the <u>kernel</u> of φ, denoted by ker φ is defined by

$$\ker \varphi = \{x \in A \mid \varphi(x) = 0\}.$$

A 1-1 homomorphism of A onto B is called an <u>isomorphism</u>. If there exists a 1-1 homomorphism of A into B, then A is said to be <u>embedded</u> in B. The hyperring A is said to be <u>isomorphic</u> to the hyperring B, written by $A \cong B$ if there is an isomorphism of A onto B.

If I is a hyperideal of a hyperring A, then the map $\varphi : A \to A/I$ defined by $\varphi(x) = x + I$ $(x \in A)$ is an onto homomorphism.

A semigroup S is said to admit a hyperring structure if there exists a hyperoperation + on S^o such that $(S^o,+,.)$ is a hyperring where . is the operation of S^o. Note that if a semigroup S admits a ring structure, then S admits a hyperring structure. The converse is not generally true. By Example 7 on page 18, the multiplicative semigroup [0,1] admits a hyperring structure but it is known from [6] that it does not admit a ring structure. By Example 8 on page 21, every group admits a hyperring structure. By [9], the symmetric group of degree n admits a ring structure if and only if $n \leq 2$. Hence for $n \geq 3$, the symmetric group of degree n admits a hyperring structure which does not admit a ring structure.

The following examples give various examples of hypergroups and hyperrings. Included in each example, we point out some of its important properties.

Example 1. Let A be a nonempty set. Define

$$x o y = A$$

for all $x, y \in A$. Then (A,o) is a hypergroup with the following properties :

1. For $x \in A$, x is an identity of (A,o).

2. For $x, y \in A$, x and y are inverses of each other in (A,o).

The hypergroup (A,o) is usually called a total hypergroup.

Example 2. Let (G,.) be a group with identity e and N a normal subgroup of G. Define

$$x o y = Nxy$$

for all x, y ε G. Since N is a normal subgroup of G, we have that for x, y, z ε G,

$$(xoy)oz = \bigcup_{a \varepsilon xoy} (aoz) = \bigcup_{a \varepsilon Nxy} Naz = NNxyz = Nxyz$$

and

$$xo(yoz) = \bigcup_{b \varepsilon yoz} (xob) = \bigcup_{b \varepsilon Nyz} Nxb = NxNyz = Nxyz.$$

Moreover for x ε G,

$$Gox = \bigcup_{g \varepsilon G} (gox) = \bigcup_{g \varepsilon G} Ngx = NGx = Gx = G$$

and

$$xoG = \bigcup_{g \varepsilon G} (xog) = \bigcup_{g \varepsilon G} Nxg = NxG = NG = G.$$

Therefore (xoy)oz = xo(yoz) for all x, y, z ε G and Gox = G = xoG for all x ε G. Hence (G,o) is a hypergroup. This hypergroup has the following important properties :

(1) N is the set of all identities of the hypergroup (G,o).

(2) For each x ε G, Nx^{-1} is the set of all inverses of x in (G,o).

From (1) and (2), we have that if G is abelian, then (G,o) is canonical if and only if N = {e}.

To prove (1), let a ε N. Then Na = N = aN. Thus for each x ε G, x ε Nx = Nx$\cap$xN = Nax$\cap$xaN = Nax$\cap$Nxa = (aox)$\cap$(xoa). Hence a is an identity of (G,o). Let e^* be an identity of (G,o). Then e ε $(eoe^*)\cap(e^*oe) = Nee^*\cap Ne^*e = Ne^*\cap e^*Ne = Ne^*\cap e^*N = Ne^*$ since N is normal in (G,.), so $N = Ne = Ne^*$. Thus e^* ε N. This proves that N is the set of all identities of (G,o) as required.

To prove (2), let x ε G. Let y ε Nx^{-1}. Then xy, yx ε N, so xyyx ε N. By (1), xyyx is an identity of (G,o). Since xy, yx ε N, it follows that $Nxyyx = Nyx$ and $xyyxN = xyN$ which imply that $xyyx \in Nyx \cap xyN = (yox) \cap (xoy)$. Therefore y is an inverse of x. Conversely, let z be an inverse of x in (G,o). Then there exists an element u in N such that $u \in (xoz) \cap (zox)$, so $u \in Nxz \cap Nzx$. Therefore $N = Nzx$ which implies that $zx \in N$. Thus $z = ze = zxx^{-1} \in Nx^{-1}$. Hence Nx^{-1} is the set of all inverses of x.

<u>Example 3</u>. For each $x, y \in [0,\infty)$, define

$$xoy = \begin{cases} \{\min\{x,y\}\} & \text{if } x \neq y, \\ [x,\infty) & \text{if } x = y \end{cases}$$

where $\min\{x,y\}$ denotes the minimum element of $\{x,y\}$. Then $([0,\infty),o)$ is a commutative hypergroupoid. It follows easily from the definition of the hyperoperation o on $[0,\infty)$ that for $x, y \in [0,\infty)$,

$$[x,\infty)oy = yo[x,\infty) = \begin{cases} [x,\infty) & \text{if } x \leq y, \\ \{y\} & \text{if } x > y. \end{cases}$$

Hence $[0,\infty)ox = xo[0,\infty) = [0,\infty)$ for all $x \in [0,\infty)$. Then for $x, y \in [0,\infty)$, we have

$$(xox)oy = [x,\infty)oy = \begin{cases} [x,\infty) & \text{if } x \leq y, \\ \{y\} & \text{if } x > y \end{cases}$$

and

$$xo(xoy) = \begin{cases} xox = [x,\infty) & \text{if } x < y, \\ xo[x,\infty) = [x,\infty) & \text{if } x = y, \\ xoy = \{y\} & \text{if } x > y \end{cases}$$

which imply that

$$(xox)oy = xo(xoy) \qquad \text{.......... } (*)$$

for all $x, y \in [0,\infty)$. In particular, $(xox)ox = xo(xox)$ for all $x \in [0,\infty)$. It then follows from (*) and the commutativity of o on $[0,\infty)$ that for $x, y \in [0,\infty)$,

$$(yox)ox = (xoy)ox = xo(xoy) = (xox)oy = yo(xox)$$

and

$$(xoy)ox = (yox)ox = xo(yox).$$

By the definition of o on $[0,\infty)$, we have that for distinct elements x, y and z in $[0,\infty)$,

$$(xoy)oz = \{\min\{x,y,z\}\} = xo(yoz).$$

Now, we have that

$$(xoy)oz = xo(yoz)$$

for all $x, y, z \in [0,\infty)$ and for $x \in [0,\infty)$,

$$[0,\infty)ox = [0,\infty) = xo[0,\infty).$$

Hence $([0,\infty),o)$ is a hypergroup.

For any $x \in [0,\infty)$, we have $x + 1 \in [0,\infty)$ and $xo(x + 1) = \{x\}$, so $x + 1 \notin xo(x + 1)$. Hence for $x \in [0,\infty)$, x is not an identity of the hypergroup $([0,\infty),o)$, that is, $([0,\infty),o)$ has no identity.

Example 4. Let H = {e, a, b, c}. Define the hyperoperation o on H as follows :

o	e	a	b	c
e	{e}	{a}	{b}	{c}
a	{a}	{b,c}	H	{a,b,c}
b	{b}	{a,b,c}	H	H
c	{c}	H	H	H

It follows from the first row and the first column of the table that

$$eox = \{x\} = xoe \qquad .... (*)$$

for all $x \in H$. Note that the hyperoperation o is not commutative on H since $aoc = \{a,b,c\}$ but $coa = H$. Since the union of the sets in each row or of the sets in each column is equal to H, we have that

$$Hox = H = xoH \qquad .... (**)$$

for all $x \in H$. It is clearly seen from (*) that for x, y, $z \in H$, if at least one of them is e, then $(xoy)oz = xo(yoz)$. We have from the given table that for $x, y \in H \setminus \{e\}$, $b, c \in xoy$. Since $coa = cob = coc = H$ and $aob = bob = cob = H$, we get that $(xoy)oz = H = xo(yoz)$ for all $x, y, z \in H \setminus \{e\}$. Hence

$$(xoy)oz = xo(yoz) \qquad .... (***)$$

for all $x, y, z \in H$. From (*), (**) and (***), (H,o) is a hypergroup having e as a scalar identity and the hypergroup (H,o) has the following properties :

1. The element a of H has no inverse in (H,o) since e does not belong to any of the sets aoe, aoa, boa and aoc.

2. b and c are both inverses of b and c since $bob = boc = cob = coc = H$. Thus an inverse of b is not unique and so is an inverse of c.

<u>Example 5</u>. Let $H = \{e, a, b, c, d\}$. Define the hyperoperation o on H as follows :

o	e	a	b	c	d
e	{e}	{a}	{b}	{c}	{d}
a	{a}	{a,b}	{a,b}	H	H ∖ {e}
b	{b}	{a,b}	{a,b}	H ∖ {e}	H
c	{c}	H	H ∖ {e}	{c,d}	{c,d}
d	{d}	H ∖ {e}	H	{c,d}	{c,d}

Observe that the hyperoperation o is commutative on H. From the first row and the first column, we have that

$$eox = \{x\} = xoe \qquad \text{..........} (1)$$

for all x ε H. Since the union of the sets in each row or of the sets in each column is equal to H, it follows that

$$Hox = H = xoH \qquad \text{..........} (2)$$

for all x ε H. We have easily from (1) that for x, y, z ε H, at least one of them is e implies that (xoy)oz = xo(yoz). The following statements are obtained easily from the table :

(i) xoy = {a,b} if x, y ε {a,b}.

(ii) xoy = {c,d} if x, y ε {c,d}.

(iii) {a,b}ox = xo{a,b} = H if x ε {c,d}.

(iv) {c,d}ox = xo{c,d} = H if x ε {a,b}.

(v) H ∖ {e} ⊆ xoy = yox if x ε {a,b} and y ε {c,d}.

(vi) (H ∖ {e})ox = xo(H ∖ {e}) = H for all x ε H ∖ {e}.

Hence for x, y, z ε H ∖ {e}, we have from (i) and (ii), respectively that

(xoy)oz = {a,b} = xo(yoz) if x, y, z ε {a,b}

and

$$(xoy)oz = \{c,d\} = xo(yoz) \quad \text{if } x, y, z \in \{c,d\},$$

and if exactly two elements of x, y and z are in either {a,b} or {c,d}, then from (i) - (vi), we get

$$(xoy)oz = H = xo(yoz).$$

Now, we obtain

$$(xoy)oz = xo(yoz) \qquad \text{..........} \ (3)$$

for all $x, y, z \in H$. From (1), (2) and (3), (H,o) is a hypergroup having e as a scalar identity. Then e is the unique inverse of e in (H,o). Since $e \in H = aoc = coa$, $e \notin aox$ for all $x \in H \setminus \{c\}$ and $e \notin cox$ for all $x \in H \setminus \{a\}$, we have that a and c are unique inverses of each other in (H,o). Similarly, b and d are unique inverses of each other in (H,o). For $x \in H$, let x' denote the unique inverse of x in (H,o).

To show that (H,o) is reversible, let $x, y, z \in H$ be such that $x \in yoz$. If $x = e$, then $e \in yoz$, so $z = y' \in y'oe = y'ox$. If $y = e$, then $x \in eoz = \{z\}$, so $z = x \in eox = e'ox = y'ox$. If $z = e$, then $x = y$ and hence $z = e \in y'oy = y'ox$. Therefore we prove that if at least one of x, y,and z is e, then $z \in y'ox$.

Assume that $x, y, z \in H \setminus \{e\}$.

<u>Case 1</u> : $x = a$. Since $a' = c$, $b' = d$, $c' = a$ and $d' = b$, we have from the table that

$$y'ox = y'oa = \begin{cases} H & \text{if } y = a, \\ H \setminus \{e\} & \text{if } y = b, \\ \{a,b\} & \text{if } y = c, \\ \{a,b\} & \text{if } y = d. \end{cases} \qquad \text{..........} \ (*)$$

If $y \in \{c,d\}$, it follows from the table and $a \in yoz$ that $z = a$ or $z = b$. By (*), $z \in y'oa = y'ox$.

Case 2 : x = b. Then

$$y^{'}ox = y^{'}ob = \begin{cases} H\setminus\{e\} & \text{if } y = a, \\ H & \text{if } y = b, \\ \{a,b\} & \text{if } y = c, \\ \{a,b\} & \text{if } y = d. \end{cases} \qquad \text{..........} \ (**)$$

If $y \in \{c,d\}$, from $b \in yoz$, we have that $z = a$ or $z = b$. By (**), $z \in y^{'}ob = y^{'}ox$.

Case 3 : x = c. The proof of this case is similar to that of Case 1.

Case 4 : x = d. The proof of this case is similar to that of Case 2.

Hence (H,o) is a canonical hypergroup. From (i), {a,b} is a subhypergroup of (H,o) and a and b are both identities of this subhypergroup. Then {a,b} is a subhypergroup of (H,o) which is not canonical.

Example 6. Define the hyperoperation $\oplus$ on $\mathbb{Z}_3$ as follows :

$\oplus$	0	1	2
0	{0}	{1}	{2}
1	{1}	{1}	$\mathbb{Z}_3$
2	{2}	$\mathbb{Z}_3$	{2}

Then $(\mathbb{Z}_3,\oplus)$ is a commutative hypergroupoid. It follows from the first row and the first column of the table that

$$0 \oplus x = \{x\} = x \oplus 0 \qquad \text{..........} \ (1)$$

for all $x \in \mathbb{Z}_3$. Since the union of the sets in each row or of the sets in each column is equal to $\mathbb{Z}_3$, we have that

$$\mathbb{Z}_3 \oplus x = \mathbb{Z}_3 = x \oplus \mathbb{Z}_3 \qquad \text{..........} \quad (2)$$

for all $x \in \mathbb{Z}_3$. It is clearly seen from (1) that for $x, y, z \in \mathbb{Z}_3$, if at least one of them is 0, then $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. The following statements are obtained easily from the table :

(i) $x \oplus x = \{x\}$ for all $x \in \mathbb{Z}_3$.

(ii) $x \oplus y = \mathbb{Z}_3$ if $x, y \in \mathbb{Z}_3 \setminus \{0\}$ and $x \neq y$.

Hence for $x, y, z \in \mathbb{Z}_3 \setminus \{0\}$, we have from (i) that

$$(x \oplus x) \oplus x = \{x\} = x \oplus (x \oplus x)$$

and if at least two elements of x, y and z are distinct, then from (i), (ii) and (2), we get

$$(x \oplus y) \oplus z = \mathbb{Z}_3 = x \oplus (y \oplus z).$$

Hence

$$(x \oplus y) \oplus z = x \oplus (y \oplus z) \qquad \text{..........} \quad (3)$$

for all $x, y, z \in \mathbb{Z}_3$. From (1), (2) and (3), $(\mathbb{Z}_3, \oplus)$ is a hypergroup having 0 as a scalar identity. Since $0 \in \mathbb{Z}_3 = 1 \oplus 2 = 2 \oplus 1$ and 0 is not an element of any of the sets $1 \oplus 0$, $1 \oplus 1$, $2 \oplus 0$ and $2 \oplus 2$, we have that 1 and 2 are unique inverses of each other in $(\mathbb{Z}_3, \oplus)$. For $x \in \mathbb{Z}_3$, let x' denote the unique inverse of x in $(\mathbb{Z}_3, \oplus)$. Note that $0' = 0$, $1' = 2$ and $2' = 1$.

To show that $(\mathbb{Z}_3, \oplus)$ is reversible, let $x, y, z \in \mathbb{Z}_3$ be such that $x \in y \oplus z$. If $x = 0$, then $0 \in y \oplus z$, so $z = y' \in y' \oplus 0 = y' \oplus x$. If $y = 0$, then $x \in 0 \oplus z = \{z\}$, so $z = x \in 0 \oplus x = 0' \oplus x = y' \oplus x$. If $z = 0$, then $x = y$ and hence $z = 0 \in y' \oplus y = y' \oplus x$. Therefore we prove that if at least one of x, y and z is 0, then $z \in y' \oplus x$.

Assume that $x, y, z \in \mathbb{Z}_3 \setminus \{0\}$. Then

$$y' \oplus x = \begin{cases} \mathbb{Z}_3 & \text{if } x = 1 \text{ and } y = 1, \\ \{1\} & \text{if } x = 1 \text{ and } y = 2, \\ \{2\} & \text{if } x = 2 \text{ and } y = 1, \\ \mathbb{Z}_3 & \text{if } x = 2 \text{ and } y = 2. \end{cases} \qquad \ldots\ldots\ldots\ldots (*)$$

If $x = 1$ and $y = 2$, from $x \in y \oplus z$, we have that $z = 1$. If $x = 2$ and $y = 1$, from $x \in y \oplus z$, we have that $z = 2$. By (*), $z \in y' \oplus x$.

Hence $(\mathbb{Z}_3,\oplus)$ is a canonical hypergroup and $\{1\}$ is a canonical subhypergroup of $(\mathbb{Z}_3,\oplus)$ which does not contain 0.

Next, we shall show that the usual multiplication . on $\mathbb{Z}_3$ is distributive over $\oplus$, that is, $x.(y \oplus z) = (x.y) \oplus (x.z)$ for all $x, y, z \in \mathbb{Z}_3$. Let $x, y, z \in \mathbb{Z}_3$. It is clear from the given table that if at least one of x, y and z is 0, then $x.(y \oplus z) = (x.y) \oplus (x.z)$. If $x, y, z \in \mathbb{Z}_3\setminus\{0\}$, then from (i) and (ii), we get

$$x.(y \oplus z) = \begin{cases} \{x.y\} = (x.y) \oplus (x.z) & \text{if } y = z, \\ \mathbb{Z}_3 = (x.y) \oplus (x.z) & \text{if } y \neq z. \end{cases}$$

This proves that $x.(y \oplus z) = (x.y) \oplus (x.z)$ for all $x, y, z \in \mathbb{Z}_3$. Since $(\mathbb{Z}_3\setminus\{0\},.)$ is an abelian group, $(\mathbb{Z}_3,\oplus,.)$ is a hyperfield.

Since $1 \oplus 1 = 1$ and $1.1 = 1$, $\{1\}$ is a subhyperring of the hyperfield $(\mathbb{Z}_3,\oplus,.)$ which does not contain 0. The map $\varphi : \mathbb{Z}_3 \to \mathbb{Z}_3$ defined by $\varphi(x) = 1$ for all $x \in \mathbb{Z}_3$ is a homomorphism from the hyperfield $(\mathbb{Z}_3,\oplus,.)$ into itself but $\varphi(0) \neq 0$. Observe that $\ker \varphi = \phi$.

<u>Example 7</u>. For each $x, y \in [0,1]$, define

$$x \oplus y = \begin{cases} \{\max\{x,y\}\} & \text{if } x \neq y, \\ [0,x] & \text{if } x = y \end{cases}$$

where $\max\{x,y\}$ denotes the maximum element of $\{x,y\}$. Then $([0,1],\oplus)$

is a commutative hypergroupoid. It follows easily from the definition of the hyperoperation $\oplus$ on $[0,1]$ that for $x, y \varepsilon [0,1]$,

$$[0,x] \oplus y = y \oplus [0,x] = \begin{cases} \{y\} & \text{if } x < y, \\ [0,x] & \text{if } x \geq y. \end{cases}$$

Hence $[0,1] \oplus x = x \oplus [0,1] = [0,1]$ for all $x \varepsilon [0,1]$. Then for $x, y \varepsilon [0,1]$, we get

$$(x \oplus x) \oplus y = [0,x] \oplus y = \begin{cases} \{y\} & \text{if } x < y, \\ [0,x] & \text{if } x \geq y. \end{cases}$$

and

$$x \oplus (x \oplus y) = \begin{cases} x \oplus y = \{y\} & \text{if } x < y, \\ x \oplus [0,x] = [0,x] & \text{if } x = y, \\ x \oplus x = [0,x] & \text{if } x > y \end{cases}$$

which imply that

$$(x \oplus x) \oplus y = x \oplus (x \oplus y) \qquad \text{.......... } (*)$$

for all $x, y \varepsilon [0,1]$. In particular, $(x \oplus x) \oplus x = x \oplus (x \oplus x)$ for all $x \varepsilon [0,1]$. It then follows from (*) and the commutativity of $\oplus$ on $[0,1]$ that for $x, y \varepsilon [0,1]$,

$$(y \oplus x) \oplus x = (x \oplus y) \oplus x = x \oplus (x \oplus y) = (x \oplus x) \oplus y = y \oplus (x \oplus x)$$

and

$$(x \oplus y) \oplus x = (y \oplus x) \oplus x = x \oplus (y \oplus x).$$

By the definition of $\oplus$ on $[0,1]$, we have that for distinct elements x, y and z in $[0,1]$,

$$(x \oplus y) \oplus z = \{\max\{x,y,z\}\} = x \oplus (y \oplus z).$$

Now, we obtain

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

for all $x, y, z \in [0,1]$. Hence $([0,1],\oplus)$ is a hypergroup.

Since $0 \oplus x = \{x\} = x \oplus 0$ for all $x \in [0,1]$, 0 is a scalar identity of the hypergroup $([0,1],\oplus)$. Since $0 \in [0,x] = x \oplus x$ for all $x \in [0,1]$, we have that for $x \in [0,1]$, x is an inverse of x in $([0,1],\oplus)$. Since 0 is the scalar identity of $([0,1],\oplus)$, 0 is the unique inverse of 0 in $([0,1],\oplus)$. For $x \in (0,1]$, x is the unique inverse of x in $([0,1],\oplus)$ since for every $y \in [0,1]\setminus\{x\}$, $0 \notin x \oplus y$ $(= \{\max\{x,y\}\})$. For $x \in [0,1]$, let x' denote the unique inverse of x in $([0,1],\oplus)$. Hence $x' = x$ for all $x \in [0,1]$.

To show that $([0,1],\oplus)$ is reversible, let $x, y, z \in [0,1]$ be such that $x \in y \oplus z$. Since

$$y \oplus z = \begin{cases} \{\max\{y,z\}\} & \text{if } y \neq z, \\ [0,y] & \text{if } y = z, \end{cases}$$

we have that $x = y > z$, $x = z > y$, $x < y = z$ or $x = y = z$. Each case gives $z \in y' \oplus x$ as follows :

$$x = y > z \Longrightarrow z \in [0,x] = y \oplus x = y' \oplus x,$$

$$x = z > y \Longrightarrow z \in \{\max\{y,x\}\} = y \oplus x = y' \oplus x,$$

$$x < y = z \Longrightarrow z \in \{\max\{y,x\}\} = y \oplus x = y' \oplus x \text{ and}$$

$$x = y = z \Longrightarrow z \in [0,y] = y \oplus x = y' \oplus x.$$

This proves that $([0,1],\oplus)$ is a canonical hypergroup.

Next, we shall show that $x.(y \oplus z) = (x.y) \oplus (x.z)$ for all $x, y, z \in [0,1]$ where . is the usual multiplication on $[0,1]$. Let $x, y, z \in [0,1]$. Then

$$x.(y \oplus z) = \begin{cases} \{x.z\} = (x.y) \oplus (x.z) & \text{if } y < z, \\ x.[0,y] = [0,x.y] = (x.y) \oplus (x.z) & \text{if } y = z, \\ \{x.y\} = (x.y) \oplus (x.z) & \text{if } y > z. \end{cases}$$

Hence $([0,1],\oplus,.)$ is a hyperring with identity 1. Since for $x, y \in [0,1]$, $x.y = 0$ implies $x = 0$ or $y = 0$, it follows that $([0,1],\oplus,.)$ is a hyperintegral domain and $\{0\}$ is a proper prime hyperideal of $([0,1],\oplus,.)$. The hyperring $([0,1],\oplus,.)$ is not a hyperfield since $((0,1],.)$ is not a group. Also, $[0,1)$ is a maximal hyperideal of $([0,1],\oplus,.)$ since $[0,1)$ is a subhypergroup of $([0,1],\oplus)$ and $[0,1].[0,1) \subseteq [0,1)$.

Example 8. Let $(G,.)$ be a group. For $x, y \in G^{o}$, define

$$x + y = \begin{cases} \{x\} & \text{if } y = 0, \\ \{y\} & \text{if } x = 0, \\ G^{o} \setminus \{x\} & \text{if } x = y \neq 0, \\ \{x,y\} & \text{if } x \neq y,\ x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Then $(G^{o},+)$ is a commutative hypergroupoid and

$$0 + x = \{x\} = x + 0 \qquad \text{..........} (*)$$

for all $x \in G^{o}$. Note that $G^{o} = G \cup \{0\}$ and $G^{o}.x = x.G^{o} = G^{o}$ for all $x \in G$ since $(G,.)$ is a group. First, we claim that for distinct elements $a, b \in G$,

$$(a + a) + b = G^{o} = a + (a + b). \qquad \text{..........} (**)$$

Let $a, b \in G$ be such that $a \neq b$. Then

$$\begin{aligned} (a + a) + b &= (G^{o} \setminus \{a\}) + b \\ &\supseteq (0 + b) \cup (b + b) \\ &= \{b\} \cup (G^{o} \setminus \{b\}) \\ &= G^{o} \end{aligned}$$

and

$$a + (a + b) = a + \{a,b\}$$
$$= (a + a) \cup (a + b)$$
$$= (G^o \setminus \{a\}) \cup \{a,b\}$$
$$= G^o.$$

Hence $(a + a) + b = G^o = a + (a + b)$.

To show that + is associative on G^o, let $x, y, z \in G^o$. If at least one of x, y and z is 0, it follows from (*) that $(x + y) + z = x + (y + z)$. Now, assume that $x, y, z \in G$. If $x = y = z$, then $(x + y) + z = (x + x) + x = x + (x + x) = x + (y + z)$ since + is commutative on G^o. If x, y and z are all distinct, we get

$$(x + y) + z = \{x,y\} + z$$
$$= (x + z) \cup (y + z)$$
$$= \{x,z\} \cup \{y,z\}$$
$$= \{x,y,z\}$$
$$= \{x,y\} \cup \{x,z\}$$
$$= (x + y) \cup (x + z)$$
$$= x + \{y,z\}$$
$$= x + (y + z).$$

It follows from (**) and the commutativity of + on G^o that if exactly two elements of x, y and z are equal, then $(x + y) + z = G^o = x + (y + z)$. This proves that $(x + y) + z = x + (y + z)$ for all $x, y, z \in G^o$.

For $x \in G^o$, if $x = 0$, then $G^o + x = G^o + 0 = G^o$ and if $x \neq 0$, then $G^o + x \supseteq (x + x) \cup (0 + x) = (G^o \setminus \{x\}) \cup \{x\} = G^o$. Then $G^o + x = x + G^o = G^o$ for all $x \in G^o$.

Now, we have that $(G^o,+)$ is a hypergroup and 0 is the scalar identity of $(G^o,+)$. It follows from the definition of + on G^o that for $x, y \in G^o$, $0 \in x + y$ if and only if $x = y$. Hence for $x \in G^o$, x is the unique inverse of x in $(G^{o},+)$. For $x \in G^{o}$, let x' denote the unique inverse of x in $(G^{o},+)$.

To show that $(G^{o},+)$ is reversible, let $x, y, z \in G^{o}$ be such that $x \in y + z$. If $x = 0$, then $0 \in y + z$, so $z = y' \in y' + 0 = y' + x$. If $y = 0$, then $x \in 0 + z = \{z\}$, so $z = x \in 0 + x = y' + x$. If $z = 0$, then $x = y$, so $z = 0 \in y' + y = y' + x$. Therefore we prove that if at least one of x, y and z is 0, then $z \in y' + x$. Assume that $x, y, z \in G$.

<u>Case 1</u> : $y = z$. Then $x \in z + z = G^{o} \setminus \{z\}$. Therefore $x \neq z$, and so $z \in \{z,x\} = z + x = y + x = y' + x$.

<u>Case 2</u> : $y \neq z$. Then $x \in y + z = \{y,z\}$ and $z \in G^{o} \setminus \{y\} = y + y$. If $x = y$, then $z \in y + y = y' + x$. If $x = z$, then $z \in \{y,x\} = y + x = y' + x$.

Hence $(G^{o},+)$ is a canonical hypergroup.

Next, we shall show that $x.(y + z) = (x.y) + (x.z)$ and $(y + z).x = (y.x) + (z.x)$ for all $x, y, z \in G^{o}$. It is clear from (*) that if at least one of x, y and z is 0, then $x.(y + z) = (x.y) + (x.z)$ and $(y + z).x = (y.x) + (z.x)$. Assume that $x, y, z \in G$. If $y = z$, then

$$\begin{aligned} x.(y + z) &= x.(y + y) \\ &= x.(G^{o} \setminus \{y\}) \\ &= x.G^{o} \setminus \{x.y\} \\ &= G^{o} \setminus \{x.y\} \\ &= (x.y) + (x.y) \\ &= (x.y) + (x.z) \end{aligned}$$

and similarly, $(y + z).x = (y.x) + (z.x)$. If $y \neq z$, then $x.y \neq x.z$, and hence

$$\begin{aligned} x.(y + z) &= x.\{y,z\} \\ &= \{x.y,x.z\} \\ &= (x.y) + (x.z) \end{aligned}$$

and similarly, $(y + z).x = (y.x) + (z.x)$.

Hence $(G^{o},+,.)$ is a hyperring.

Remark : We have from Example 8 that every group admits a hyperring structure.

Let X be a set. A partial transformation of X is a map from a subset of X into X. The empty transformation of X is the partial transformation of X with empty domain and it is denoted by 0. For a partial transformation α of X, the domain and range of α are denoted by $\Delta\alpha$ and $\nabla\alpha$, respectively. Let P_X be the set of all partial transformations of X (including 0). For $\alpha,\beta \in P_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \phi$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let $\alpha\beta = (\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}})(\beta|_{\nabla\alpha \cap \Delta\beta})$ (the composition of the maps $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$) where $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha \cap \Delta\beta}$ denote the restrictions of α and β to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$ and $\nabla\alpha \cap \Delta\beta$, respectively. Then P_X is a semigroup having 0 and 1_X as its zero and identity, respectively where 1_X is the identity map on X. The semigroup P_X is called the partial transformation semigroup on X. Observe that $\alpha,\beta \in P_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$.

By a transformation semigroup on X, we mean a subsemigroup of P_X.

Let I_X be the set of all 1-1 partial transformations of X. Then I_X is a subsemigroup of P_X and it is called the 1-1 partial transformation semigroup or the symmetric inverse semigroup on X.

By a transformation of X, we mean a map of X into itself.

Let T_X be the set of all transformations of X. Then T_X is a subsemigroup of P_X with identity 1_X and it is called the full transformation semigroup on X. Let

G_X = the symmetric group on X,

M_X = the set of all 1-1 transformations of X

and

E_X = the set of all onto transformations of X.

Then M_X and E_X are subsemigroups of T_X containing G_X.

For $\alpha \in T_X$, $x \in X$, α is said to be 1-1 at x if $(x\alpha)\alpha^{-1} = \{x\}$. For $\alpha \in T_X$, α is said to be almost 1-1 if the set $\{x \in X \mid \alpha \text{ is not 1-1 at } x\}$ is finite. Let AM_X be the set of all almost 1-1 transformations of X. Then AM_X is a subsemigroup of T_X containing M_X (see [10]).

For $\alpha \in T_X$, α is said to be almost onto if $X \setminus \nabla\alpha$ is finite. Let AE_X be the set of all almost onto transformations of X. Then AE_X is a subsemigroup of T_X containing E_X (see [10]).

The shift of a partial transformation α of X is defined to be the set

$$S(\alpha) = \{x \in \Delta\alpha \mid x\alpha \neq x\}.$$

A partial transformation α of X is said to be almost identical if the shift of α is finite. Let

U_X = the set of all almost identical partial transformations of X,

V_X = the set of all almost identical transformations of X

and

W_X = the set of all almost identical 1-1 partial transformations of X.

Then U_X, V_X and W_X are subsemigroups of P_X, T_X and I_X, respectively. Let

CP_X = the set of all constant partial transformations of X (including 0)

and

CT_X = the set of all constant transformations of X.

Then CP_X and CT_X are subsemigroups of P_X and T_X, respectively.