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RELATIONSHIP BETWEEN THE CHARACTERS AND THE ELEMENTARY SYMMETRIC SUMS OF WEIGHTS

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กำหนดให้ Φ เป็นระบบราก $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ เป็นฐานของ Φ Λ เป็นแลตทิซน้ำหนักของ Φ $\lambda_1, \ldots, \lambda_n$ เป็นน้ำหนักหลักมูล $\mathcal{W}$ คือ กรุปไวล์ของ Φ $\mathbb{Z}[\Lambda]$ เป็นกรุปริงของ Λ เหนือ $\mathbb{Z}$ และ $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ เป็นเซตของสมาชิกใน $\mathbb{Z}[\Lambda]$ ที่ไม่แปรเปลี่ยนภายใต้ $\mathcal{W}$ สำหรับน้ำหนัก μ เรานิยาม**ผลบวกสมมาตรมูลฐาน** $S(e^\mu)$ ของ μ **ผลบวกสลับมูลฐาน** $A(e^\mu)$ ของ μ และ **แคแรกเตอร์** χ_μ ของ μ ตามลำดับดังนี้

$$S(e^\mu) = \sum_{\beta \in \mathcal{W}\mu} e^\beta \qquad A(e^\mu) = \sum_{w \in \mathcal{W}} \det(w) e^{w(\mu)} \qquad \text{และ} \qquad \chi_\mu = \frac{A(e^{\mu+\delta})}{A(e^\delta)}$$

โดยที่ δ คือกึ่งผลรวมของรากบวก กำหนดให้ $S = \{S(e^{\lambda_i}) : 1 \le i \le n\}$ และ $\chi = \{\chi_{\lambda_i} : 1 \le i \le n\}$ เป็นเซตของผลบวกสมมาตรมูลฐานของน้ำหนักหลักมูลและเซตของแคแรกเตอร์ของน้ำหนักหลักมูล ตามลำดับ เป็นที่รู้กันดีว่า ทั้ง S และ χ ต่างเป็นฐานสำหรับ $\mathbb{Z}$-โมดูล $\mathbb{Z}[\Lambda]^{\mathcal{W}}$

ในงานวิจัยนี้เราสนใจหาความสัมพันธ์ระหว่างสมาชิกในเซต S และ χ ในกรณีของระบบรากที่มีแผนภาพดิงคินเป็น $\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n$ และ G_2 เมื่อ n เป็นจำนวนนับที่เหมาะสม

ภาควิชา**คณิตศาสตร์**	ลายมือชื่อนิสิต . . . .
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Let Φ be a root system, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a base of Φ, Λ the weight lattice of Φ, $\lambda_1, \ldots, \lambda_n$ fundamental weights, $\mathcal{W}$ the Weyl group of Φ, $\mathbb{Z}[\Lambda]$ the group ring of Λ over $\mathbb{Z}$ and $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ the set of elements in $\mathbb{Z}[\Lambda]$ which are invariant under $\mathcal{W}$. For a weight μ, we define the *elementary symmetric sum* $S(e^\mu)$ of μ, the *elementary alternating sum* $A(e^\mu)$ of μ and the *character* χ_μ of μ as follows:

$$S(e^\mu) = \sum_{\beta \in \mathcal{W}\mu} e^\beta, \qquad A(e^\mu) = \sum_{w \in \mathcal{W}} \det(w) e^{w(\mu)} \qquad \text{and} \qquad \chi_\mu = \frac{A(e^{\mu+\delta})}{A(e^\delta)}$$

respectively, where δ is the half sum of all positive roots. Let $S = \left\{S(e^{\lambda_i}) : 1 \leq i \leq n\right\}$ and $\chi = \left\{\chi_{\lambda_i} : 1 \leq i \leq n\right\}$ be the set of elementary symmetric sums of fundamental weights and set of characters of fundamental weights, respectively. It is well-known that both S and χ are bases for $\mathbb{Z}$-module $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.

In this research, we are interested in finding relations between elements in the sets S and χ in the case of root systems whose Dynkin diagrams are $\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n$ and G_2 for appropriate integers n.

Department **Mathematics**	Student's signature..............................
Field of study **Mathematics**	Advisor's signature..............................
Academic year **2004**	Co-advisor's signature -

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CONTENTS

CHAPTER I

PRELIMINARIES

In this chapter, we summarize basic concepts of root systems, abstract theory of weights, Lie algebra, and representation theory which are related to this thesis in Sections 1.1 – 1.4. Moreover, our motivation for this thesis and key theorems which will play major roles in the following chapters are stated in Section 1.5. The main references of these are from [2], [3] and [4].

1.1. Root Systems

First, we fix V to be a finite-dimensional Euclidean space with an inner product $(\ ,\ )$. We introduce a reflection in V which is used for defining a root system in V. Then, we define a base of a root system in V which is a basis of V with a certain property. Finally, we bring in Dynkin diagrams which lead to catagorizing root systems.

Definition 1.1.1. [4] A *reflection* in V is an invertible linear transformation leaving pointwise fixed some *hyperplane* (subspace of codimension one) and sending any vector orthogonal to that hyperplane into its negative.

Any nonzero vector α determines a reflection σ_α, with *reflecting hyperplane* $P_\alpha = \big\{\beta \in V : (\beta, \alpha) = 0\big\}$. Of course, nonzero vectors proportional to α yield the same reflection. Evidently, a reflection is *orthogonal*, i.e., $\big(\sigma_\alpha(\beta), \sigma_\alpha(\gamma)\big) = (\beta, \gamma)$ for all $\alpha, \beta, \gamma \in V$. There is an explicit formula for a reflection σ_α $(\alpha \in V)$ as follows:

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \quad \text{for all } \beta \in V.$$

Since the number $2\dfrac{(\beta,\alpha)}{(\alpha,\alpha)}$ occurs frequently, for our convenience, we denote it by $(\beta,\alpha^\vee)$ where $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$, so that $\sigma_\alpha(\beta) = \beta - (\beta,\alpha^\vee)\alpha$.

Definition 1.1.2. [4] A subset Φ of V is called a *root system* in V if the following axioms are satisfied:

- **(R1)** Φ is finite, spans V, and does not contain 0.
- **(R2)** If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
- **(R3)** If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.
- **(R4)** If $\alpha, \beta \in \Phi$, then $(\beta,\alpha^\vee) \in \mathbb{Z}$.

For each root system Φ, the elements of Φ are called *roots*, and the *rank* of Φ is the dimension of V.

Axiom (R4) in Definition 1.1.2 limits severely the possible angles occuring between pairs of roots. Recall that the cosine of the angle θ between vectors $\alpha,\beta \in V$ is given by the formula $\|\alpha\|\|\beta\|\cos\theta = (\alpha,\beta)$. Therefore, $(\beta,\alpha^\vee) = 2\dfrac{(\beta,\alpha)}{(\alpha,\alpha)} = 2\dfrac{\|\beta\|}{\|\alpha\|}\cos\theta$ and $(\alpha,\beta^\vee)(\beta,\alpha^\vee) = 4\cos^2\theta$. This last number is a nonnegative integer; but $0 \le \cos^2\theta \le 1$, and $(\alpha,\beta^\vee), (\beta,\alpha^\vee)$ have like sign. The following possibilities of θ and $\|\beta\|^2/\|\alpha\|^2$ are the only ones when $\alpha \neq \pm\beta$ and $\|\beta\| \ge \|\alpha\|$ are given.

$(\alpha,\beta^\vee)$	$(\beta,\alpha^\vee)$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Definition 1.1.3. [4] A subset Δ of a root system Φ in V is called a *base* if

(B1) Δ is a basis of V,

(B2) each root β can be written as $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with integral coefficients k_α all nonnegative or nonpositive.

The roots in a base are called *simple.* Moreover, σ_α, where α is a simple root, is called a *simple reflection.*

Definition 1.1.4. [4] Let Φ be a root system, Δ a base of Φ and $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ a root. If all $k_\alpha \geq 0 (k_\alpha \leq 0)$, we call β *positive* (*negative*) and write $\beta \succ 0$ ($\beta \prec 0$). The set of all positive and negative roots (relative to Φ) is denoted by Φ^+ and Φ^-, respectively.

Note 1.1.5. Let Φ be a root system and Δ a base of Φ. It is obvious that $\Delta \subseteq \Phi^+$ and $\Phi^- = -\Phi^+$.

Definition 1.1.6. [4] Let Φ be a root system in V. The *Weyl group* $\mathcal{W}$ of Φ is the subgroup of invertible linear transformations on V generated by reflections σ_α where $\alpha \in \Phi$.

Example 1.1.7. We consider $\mathbb{R}^2$ as a vector space with the usual inner product. Let $\Phi = \{\pm(0,1), \pm(1,0)\}$. Then Φ is a root system in $\mathbb{R}^2$ with the base $\Delta = \{(0,1), (1,0)\}$ and the Weyl group $\mathcal{W} = \{i_{\mathbb{R}^2}, \sigma_{(0,1)}, \sigma_{(1,0)}, \sigma_{(0,1)}\sigma_{(1,0)}\}$ where

$$\sigma_{(0,1)}(0,1) = -(0,1) \qquad \sigma_{(0,1)}(1,0) = (1,0)$$

$$\sigma_{(1,0)}(0,1) = (0,1) \qquad \sigma_{(1,0)}(1,0) = -(1,0).$$

It is not obvious that (B2) guarantees the existence of a base of a root system. However, the following thoerem assures that. In fact, a base is not unique but there is a relation between these bases.

Theorem 1.1.8. [4] *For each root system Φ in V, bases exist but are not unique. Moreover, if Δ and Δ' are two bases of Φ, then $w(\Delta') = \Delta$ for some $w \in \mathcal{W}$.*

Definition 1.1.9. [4] Let Φ be a root system in V and $\mathcal{W}$ the Weyl group of Φ. Let α and β be any elements of V. We say that α is *conjugate* to β or α and β are $\mathcal{W}$- *conjugate* if there exists $w \in \mathcal{W}$ such that $w(\beta) = \alpha$.

Proposition 1.1.10. *Let $\mathcal{W}$ be the Weyl group of a root system and α, β the same length roots. Then α and β are conjugate under $\mathcal{W}$ via $w \in \mathcal{W}$ such that $w(\beta) = \alpha$ and*

$$w = \begin{cases} \sigma_{\beta-\alpha}, & \text{if } (\alpha, \beta) = 0, \\ \sigma_\alpha \sigma_\beta \sigma_\alpha, & \text{if } (\alpha, \beta) = 1, \\ \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta, & \text{if } (\alpha, \beta) = -1. \end{cases}$$

Note also that if $(\alpha, \beta) \neq 0$, then $(\alpha^\vee, \beta) = (\alpha, \beta^\vee) = \pm 1$.

Proof. This is obvious. □

Theorem 1.1.11. [4] *Let Φ be a root system in V, Δ a base of Φ and $\mathcal{W}$ the Weyl group of Φ.*

(a) *If α is any root, there exists $w \in \mathcal{W}$ such that $w(\alpha) \in \Delta$.*

(b) *$\mathcal{W}$ is generated by the $\sigma_\alpha (\alpha \in \Delta)$.*

We can see from Theorem 1.1.11 (b) that the Weyl group of a root system is generated by the simple reflections which are relative to a base.

From Example 1.1.7, we obtain that $\mathcal{W} = \langle \sigma_{(0,1)}, \sigma_{(1,0)} \rangle$.

Definition 1.1.12. [4] Let Δ be a fixed base of root system Φ in V. We define a *partial order* $\prec$ on V as follows: for each $\lambda, \mu \in V$, $\mu \prec \lambda$ (or $\lambda \succ \mu$) if and only if

$\lambda - \mu$ is a sum of simple roots. In addition, we define $\mu \preceq \lambda$ if and only if $\mu \prec \lambda$ or $\mu = \lambda$.

The partial order $\prec$ (or $\succ$) has an eminent role for this thesis. We can see from Example 1.1.7 that $(-1,0) \prec (0,1)$ since $(0,1)-(-1,0) = (1,1) = (0,1)+(1,0)$ but $(1,0) \not\prec (0,1)$ since $(0,1)-(1,0) = (-1,1)$ is not a sum of $(1,0)$ and $(0,1)$.

Definition 1.1.13. [4] A root system Φ is called *irreducible* if it cannot be partition into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

Theorem 1.1.14. [4] *Let Φ be an irreducible root system. Then at most two root lengths occur in Φ, and all roots of a given length are conjugate under its Weyl group.*

Proposition 1.1.15. [4] *Let Φ be a root system in V, Δ a fixed base and $\mathcal{W}$ the Weyl group. Suppose $S \subseteq \Delta$. Let $V(S)$ be the vector subspace of V spanned by S, and let $\Phi(S) = \Phi \cap V(S)$. Then S is a base of the root system $\Phi(S)$. We also call $\Phi(S)$ a* subroot system *of Φ. Therefore, the Weyl group of $\Phi(S)$ is*

$$\mathcal{W}\big(\Phi(S)\big) = \langle \sigma_\alpha : \alpha \in S \rangle .$$

Proposition 1.1.16. [4] *Let Φ be a root system in V. Then Φ decomposes (uniquely) as the union of irreducible root system Φ_i (in subspace V_i of V) such that $V = V_1 \oplus \cdots \oplus V_t$ (orthogonal direct sum, i.e., each vector in V_i is orthogonal to each vector in the others) for some $t \in \mathbb{N}$.*

Example 1.1.17. We note from Example 1.1.7 that $\Phi = \big\{\pm(0,1)\big\} \oplus \big\{\pm(1,0)\big\}$ such that $\{\pm(0,1)\}$ is root system in $\big\langle(0,1)\big\rangle$ and $\{\pm(1,0)\}$ is root system in $\big\langle(1,0)\big\rangle$.

Proposition 1.1.16 shows that it is sufficient to classify only the irreducible root systems.

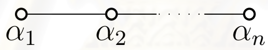
Definition 1.1.18. [4] For each root system with rank n and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, the *Dynkin diagram* of Δ is a graph having n vertices the i-th joined to the j-th, where $i \neq j$, by $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$ edges and adding an arrow pointing to the shorter of the two roots (in term of their lengths) whenever a double or triple edge occurs.

There is a natural notion of isomorphism between root systems.

Definition 1.1.19. [4] Let Φ and Φ' be root systems in respective finite-dimensional real vector spaces V and V'. We call that (Φ, V) and (Φ', V') are *isomorphic* if there exists a linear isomorphism $\phi : V \to V'$ sending Φ onto Φ' such that $\big(\phi(\beta), \phi(\alpha)^\vee\big) = (\beta, \alpha^\vee)$ for each pair of roots $\alpha, \beta \in \Phi$.

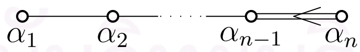
Theorem 1.1.20. [4] *If Φ is an irreducible root system of rank n, its Dynkin diagram is one of the following (n vertices in each case):*

A_n $(n \geq 1)$ **:**

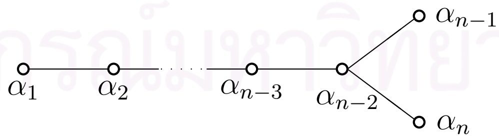


B_n $(n \geq 2)$ **:**

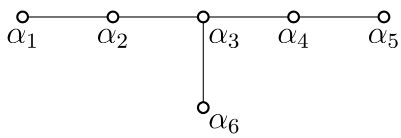
C_n $(n \geq 3)$ **:**



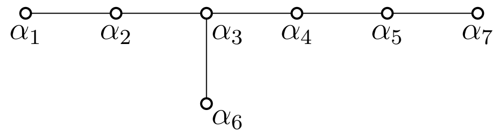
D_n $(n \geq 4)$ **:**



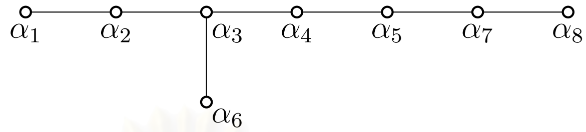
E_6 **:**



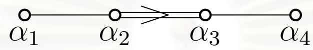
E_7 :



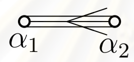
E_8 :



F_4 :



G_2 :



On the other hand, a root system can be constructed from each Dynkin diagram (unique upto isomorphism). Note also that root systems with Dynkin diagram C_2 and B_2 are isomorphic but root systems with Dynkin diagram C_3 and B_3 are not isomorphic. For further details, the reader can read in [4], p.63–65.

1.2. Abstract Theory of Weights

In this section, we provide definitions of a weight lattice, fundamental weights, saturated weights and some theorems of weights which are relevant to this thesis. Additionally, we define the group ring of the weight lattice. We focus on some particular elements of that group ring, namely, the elementary symmetric sums of fundamental weights and the characters of fundamental weights. We carry on using the notations $V, \Phi, \Delta, \mathcal{W}$ and others as in Section 1.1.

Definition 1.2.1. [4] We define the *weight lattice* Λ to be

$$\Lambda = \big\{\lambda \in V : (\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\big\},$$

and call its elements *weight*. For a fixed base $\Delta \subseteq \Phi$, we define $\lambda \in \Lambda$ to be *dominant* if all integers $(\lambda, \alpha^\vee)$ $(\alpha \in \Delta)$ are nonnegative, *strongly dominant* if these integers are positive. Let Λ^+ be the set of all dominant weights.

If $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, then the vectors $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$, again, form a basis of V.

Definition 1.2.2. [4] Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a fixed base of a root system Φ. The *fundamental weights* are $\lambda_1, \ldots, \lambda_n$ such that for all $1 \le i, j \le n$

$$(\lambda_i, \alpha_j^\vee) = \delta_{ij}.$$

We can see that the fundamental weights are dominant and

$$\sigma_{\alpha_i}(\lambda_j) = \begin{cases} \lambda_j, & \text{if } j \neq i, \\ \lambda_i - \alpha_i, & \text{if } j = i. \end{cases}$$

Proposition 1.2.3. [4] *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a fixed base of a root system Φ. The fundamental weights $\lambda_1, \ldots, \lambda_n$ form a $\mathbb{Z}$-basis for the weight lattice Λ. In addition, for each $\lambda \in \Lambda$,*

$$\lambda = \sum_{i=1}^{n} (\lambda, \alpha_i^\vee)\lambda_i.$$

Furthermore, $\lambda \in \Lambda^+$ if and only if $(\lambda, \alpha_i^\vee) \ge 0$ for all $1 \le i \le n$.

Remark 1.2.4. As a result of Proposition 1.2.3,

$$\Lambda = \left\{ \sum_{\alpha_i \in \Delta} k_{\alpha_i}\lambda_i : k_{\alpha_i} = (\lambda, \alpha_i^\vee) \in \mathbb{Z} \text{ for all } \alpha_i \in \Delta \right\}, \text{ and}$$

$$\Lambda^+ = \left\{ \sum_{\alpha_i \in \Delta} k_{\alpha_i}\lambda_i : k_{\alpha_i} = (\lambda, \alpha_i^\vee) \in \mathbb{Z}_0^+ \text{ for all } \alpha_i \in \Delta \right\}.$$

Since $\Lambda \subseteq V$ and $\prec$ is a partial order on V, we obtain a partial order on Λ. By Definition 1.1.12, for $\lambda, \mu \in \Lambda$

$$\mu \prec \lambda \text{ (or } \lambda \succ \mu) \quad \text{if and only if} \quad \lambda - \mu = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

where $k_\alpha \in \mathbb{Z}_0^+$ not all zeros for all simple roots α. Moreover, we define $\mu \preceq \lambda$ if and only if $\mu \prec \lambda$ or $\mu = \lambda$.

Definition 1.2.5. [4] Define δ to be the half sum of all positive roots, i.e.,

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Proposition 1.2.6. [4] *The half sum δ of all positive roots is a weight. In fact, $\delta = \sum_{i=1}^{n} \lambda_i$.*

Observe that δ is another example of dominant weights.

Theorem 1.2.7. [4] *Each weight is conjugate under $\mathcal{W}$ to one and only one dominant weight. If λ is dominant, then $w(\lambda) \preceq \lambda$ for all $w \in \mathcal{W}$ and if λ is strongly dominant, then $w(\lambda) = \lambda$ only when $w = 1$.*

Theorem 1.2.8. [4] *Let $\lambda \in \Lambda^+$, then the number of dominant weights μ such that $\mu \prec \lambda$ is finite.*

Definition 1.2.9. [4] We call a subset Π of the weight Λ *saturated* if for all $\lambda \in \Pi$, $\alpha \in \Phi$, and i between 0 and (λ, α) (inclusive), the weights $\lambda - i\alpha$ also lie in Π.

Notice that any saturated set is automatically stable under $\mathcal{W}$.

Definition 1.2.10. [4] Let $\lambda \in \Lambda^+$. We say that a saturated set Π has the *highest weight* λ if and only if $\lambda \in \Pi$ and $\mu \preceq \lambda$ for all $\mu \in \Pi$.

Theorem 1.2.11. [4] *A saturated set of weights having highest weight λ must be finite.*

Theorem 1.2.12. [4] *Let Π be a saturated set with highest weight λ. If $\mu \in \Lambda^+$ and $\mu \prec \lambda$, then $\mu \in \Pi$.*

Theorem 1.2.12 emerges a very clear picture of a saturated set Π having highest weight λ, i.e., Π consists of all dominant weights lower than or equal to λ in the partial ordering, along with their conjugates under $\mathcal{W}$. In particular, for given $\lambda \in \Lambda^+$, at most one such set Π can exist. Conversely, given $\lambda \in \Lambda^+$, we may simply define Π to be the set of all dominant weights below λ, along with their $\mathcal{W}$-conjugates. Since Π is stable under $\mathcal{W}$, it can be seen to be saturated, and Π has λ as a highest weight.

Definition 1.2.13. [4] For $\lambda \in \Lambda^+$, define $\Pi(\lambda)$ to be the smallest saturated subset of Λ that contains λ. (Since intersections of saturated sets are saturated, it is clear that the smallest saturated subset exists.)

Theorem 1.2.14. [7] *For $\lambda \in \Lambda^+$, we have*

$$\Pi(\lambda) = \Big\{\mu \in \Lambda : w(\mu) \preceq \lambda \textit{ for all } w \in \mathcal{W}\Big\} = \bigcup_{\substack{\mu\in\Lambda^+\\ \mu\preceq\lambda}} \mathcal{W}\mu.$$

Remark 1.2.15. For each $\lambda \in \Lambda^+$, we obtain that $\Pi(\lambda)$ is the smallest saturated set containing λ with highest weight λ.

Definition 1.2.16. [7] A dominant weight λ is *minuscule* if $\lambda \neq 0$ and $(\lambda, \alpha^\vee) \in \{0, \pm 1\}$ for all $\alpha \in \Phi$.

Theorem 1.2.17. [7] *A dominant weight λ is a minimal element of $(\Lambda^+, \prec)$ if and only if $\lambda = 0$ or λ is minuscule.*

Now, we give notion of a group ring which is mainly referred to [5]. Consider an arbitary family $\{G_i : i \in I\}$ (where I is an arbitary set) of groups. We define a binary operation on the Cartesian product $\prod_{i\in I} G_i$ as follows. If $f, g \in \prod_{i\in I} G_i$ $\left(\text{that is } f, g : I \to \bigcup_{i\in I} G_i\right)$, then $fg : I \to \bigcup_{i\in I} G_i$ is the function given by $i \mapsto f(i)g(i)$. The (external) *weak direct product* of groups $\{G_i : i \in I\}$, denoted $\prod_{i\in I}^{w} G_i$, is the set of all $f \in \prod_{i\in I} G_i$ such that $f(i) = e_i$, the identity in G_i, for all but a finite number of $i \in I$. If all the groups G_i are (additive) abelian, $\prod_{i\in I}^{w} G_i$ is usually called the (external) *direct sum* and is denoted $\sum_{i\in I} G_i$.

Let G be a (multiplicative) group and R a ring. Let $R(G)$ be the additive abelian group $\sum_{g\in G} R$ (one copy of R for each $g \in G$). It will be convenient to adopt a new notation for the element of $R(G)$, i.e.,

$$R(G) = \left\{ \sum_{i=1}^{n} r_i g_i : n \in \mathbb{N},\ r_i \in R \text{ and } g_i \in G \text{ for all } i \right\}.$$

We also allow the possibility that some of the r_i are zero or that some g_i are repeated, so that an element of $R(G)$ may be written in formally different ways (for example, $r_1g_1 + 0g_2 = r_1g_1$ or $r_1g_1 + s_1g_1 = (r_1 + s_1)g_1$). In this notation, addition in the group $R(G)$ is given by

$$\sum_{i=1}^{n} r_i g_i + \sum_{i=1}^{n} s_i g_i = \sum_{i=1}^{n} (r_i + s_i) g_i,$$

by inserting zero coefficients if necessary we can always assume that two formal sums involve exactly the same indices $1, \ldots, n$.

The group $R(G)$ becomes a commutative ring if a specific multiplication is provided.

Definition 1.2.18. [5] Let G be a (multiplicative) group and R a ring. Let $R(G)$ be the additive abelian group $\sum_{g\in G} R$ with addition described above. Define a multiplication in $R(G)$ by

$$\left(\sum_{i=1}^{n} r_i g_i\right)\left(\sum_{j=1}^{m} s_i h_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}(r_i s_j)(g_i h_j).$$

With these operations, $R(G)$ is a ring. Then $R(G)$ is called the *group ring* of G over R.

Let Λ be the weight lattice of a root system Φ and $G = \{e^{\lambda} : \lambda \in \Lambda\}$. Then G is a multiplicative group if we define the multiplication $e^{\lambda}e^{\alpha} = e^{\lambda+\alpha}$ for all $\lambda, \alpha \in G$ and extend it linearly. Then the group ring $\mathbb{Z}[G]$ of G over $\mathbb{Z}$ exists. In general, we use $\mathbb{Z}[\Lambda]$ in sense of $\mathbb{Z}[G]$ and call $\mathbb{Z}[\Lambda]$ the *group ring* of the weight lattice Λ over $\mathbb{Z}$, i.e.,

$$\mathbb{Z}[\Lambda] = \left\{\sum_{\lambda\in\Lambda} k_\lambda e^{\lambda} \,:\, k_\lambda \in \mathbb{Z} \text{ and } k_\lambda \neq 0 \text{ for all but finitely many } \lambda \in \Lambda\right\}.$$

Note that $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}$-module with basis element e^{λ} in one-to-one correspondence with the element λ of Λ.

Definition 1.2.19. [2] Let $x \in \mathbb{Z}[\Lambda]$, then x is *invariant* under $\mathcal{W}$ if $w(x) = x$ for all $w \in \mathcal{W}$ and x is *anti-invariant* under $\mathcal{W}$ if $w(x) = \det(w)x$ for all $w \in \mathcal{W}$. Moreover, $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ is defined as

$$\mathbb{Z}[\Lambda]^{\mathcal{W}} = \{x \in \mathbb{Z}[\Lambda] : w(x) = x \text{ for all } w \in \mathcal{W}\},$$

i.e., the set of elements of $\mathbb{Z}[\Lambda]$ which are invariant under $\mathcal{W}$.

Definition 1.2.20. [3] Let λ be a weight in the weight lattice Λ and $\mathcal{W}\lambda$ the orbit of λ under $\mathcal{W}$. The *elementary symmetric sum* $S(e^{\lambda})$ *of* λ and the *elementary* *alternating sum* $A(e^\lambda)$ *of* λ are given by

$$S(e^\lambda) = \sum_{\mu \in \mathcal{W}\lambda} e^\mu \qquad \text{and} \qquad A(e^\lambda) = \sum_{w \in \mathcal{W}} \det(w) e^{w(\lambda)},$$

respectively. Furthermore, if λ is dominant, the *character* χ_λ with highest weight λ is defined by

$$\chi_\lambda = \frac{A(e^{\lambda+\delta})}{A(e^\delta)}.$$

Remark 1.2.21. We observe that $\chi_\lambda \in \mathbb{Z}[\Lambda]^{\mathcal{W}}$ for all $\lambda \in \Lambda^+$. Moreover, for each $\lambda \in \Lambda^+$, we can see that $S(e^\lambda) \in \mathbb{Z}[\Lambda]^{\mathcal{W}}$ and

$$S(e^\lambda) = \sum_{\mathcal{W}(\lambda)w \in \mathcal{W}/\mathcal{W}(\lambda)} e^{w(\lambda)} = \frac{1}{|\mathcal{W}(\lambda)|} \sum_{w \in \mathcal{W}} e^{w(\lambda)},$$

where $\mathcal{W}(\lambda) = \{w \in \mathcal{W} : w(\lambda) = \lambda\}$. We give examples for these in following chapters.

Proposition 1.2.22. *Let* $\lambda \in \Lambda^+$*, then* $S(e^\lambda)$ *has* λ *as the highest weight.*

Proof. It follows from $\mu \in \mathcal{W}\lambda \subseteq \Pi(\lambda)$. $\square$

Proposition 1.2.23. [2] *Let* Φ *be a root system,* Λ *the weight lattice and* $\mathcal{W}$ *the Weyl group of* Φ*. Then the set*

$$\Big\{ S(e^\lambda) : \lambda \textit{ is a dominant weight} \Big\}$$

forms a basis for the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.

Definition 1.2.24. [5] Let R be a ring and denote

$$R[x_1, \ldots, x_n] = \big\{ f : \mathbb{N}^n \to R : f(u) \neq 0 \text{ for at most finite numbers of } u \in \mathbb{N}^n \big\}.$$

The ring $R[x_1, \ldots, x_n]$ is called the *ring of polynomials* in n determinates over R.

Theorem 1.2.25. [2] *Let Λ be the weight lattice, $\lambda_1, \dots, \lambda_n$ the fundamental weights, Φ a root system and $\mathcal{W}$ the Weyl group of Φ. We have*

$$\mathbb{Z}[x_1, x_2, \dots, x_n] \cong \mathbb{Z}[\Lambda]^{\mathcal{W}}$$

such that $x_i \in \mathbb{Z}[\Lambda]$ having highest weight λ_i for all $1 \leq i \leq n$.

Example 1.2.26. Let Φ be a root system, $\lambda_1, \dots, \lambda_n$ fundamental weights, Λ the weight lattice and $\mathcal{W}$ the Weyl group of Φ. Then, according to Definition 1.2.20, Proposition 1.2.22 and Theorem 1.2.25,

$$\mathbb{Z}[\Lambda]^{\mathcal{W}} \cong \mathbb{Z}[S(e^{\lambda_1}), \dots, S(e^{\lambda_n})] \qquad \text{and}$$
$$\mathbb{Z}[\Lambda]^{\mathcal{W}} \cong \mathbb{Z}[\chi_{\lambda_1}, \dots, \chi_{\lambda_n}].$$

We can see from Theorem 1.2.25 that the set $\{x_1, \dots, x_n\}$ is a basis for $\mathbb{Z}$-module $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. Moreover, Example 1.2.26 shows that for a fixed root system with fundamental weights $\lambda_1, \dots, \lambda_n$, the set $\left\{S(e^{\lambda_i}) : \text{for all } 1 \leq i \leq n\right\}$ and the set $\left\{\chi_{\lambda_i} : \text{for all } 1 \leq i \leq n\right\}$ are bases for $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.

In Sections 1.3–1.4, we introduce concept of Lie algebra and representation theory, respectively. In addition, this thesis focuses on semisimple Lie algebras, so we give notion of a representation of a semisimple Lie algebra L. Then there exist concepts of roots of L and weights of representation which can be viewed as roots and weights described in Sections 1.1–1.2.

1.3. Elementary Concepts of Lie Algebras

We give concepts of Lie algebras very briefly in this section. The further details can be read from [4].

Definition 1.3.1. [4] A vector space L over a field $\mathbb{F}$, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ and called the *bracket* or *commutator* of x and y, is called a *Lie algebra* over $\mathbb{F}$ if the following axioms are satisfied:

(L1) The bracket operation is bilinear.

(L2) $[x, x] = 0$ for all x in L.

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Axiom (L3) is called the *Jacobi identity.*

In this section we are concerned with a Lie algebra whose underlying vector space is finite-dimensional.

Definition 1.3.2. [4] Let L be a Lie algebra. A subspace K of L is called a (Lie) *subalgebra* of L if $[x, y] \in K$ for all $x, y \in K$.

Definition 1.3.3. [4] A subspace I of a Lie algebra L is called an *ideal* of L if $x \in L$ and $y \in I$ together imply $[x, y] \in L$.

Definition 1.3.4. [4] Let L and $L^{'}$ be Lie algebras over the same field. A linear map $\varphi : L \to L^{'}$ is called a *Lie algebra homomorphism* if $\varphi\big([x, y]\big) = \big[\varphi(x), \varphi(y)\big]$ for all $x, y \in L$.

Definition 1.3.5. [4] Define a sequence of ideals of a Lie algebra L the *derived series* by

$$L^{(0)} = L,\ L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}], \ldots,\ L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$$

for $i \in \mathbb{N}$. Moreover, L is called *solvable* if $L^{(n)} = 0$ for some $n \in \mathbb{N}$.

Proposition 1.3.6. [4] *Let L be a Lie algebra. If I and J are solvable ideals of L, then so is I+J.*

As a first application, let L be an arbitrary Lie algebra and S a maximal solvable ideal of L (i.e., one included in no larger solvable ideal). If I is any other solvable ideal of L, then Proposition 1.3.6 forces $S+I=S$ (by maximality), or $I \subseteq S$. This proves the existence of a unique maximal solvable ideal.

Definition 1.3.7. [4] Let L be a Lie algebra. The maximal solvable ideal is called the *radical* of L and denoted Rad L. In case Rad $L=0$, the Lie algebra L is called *semisimple.*

Definition 1.3.8. [4] Define a sequence of ideals of L the *descending central series* by

$$L^0 = L,\ L^1 = [L,L](=L^{(1)}),\ L^2 = [L,L^1],\dots,\ L^i = [L,L^{i-1}]$$

for $i \in \mathbb{N}$. Moreover, L is caled *nilpotent* if $L^n = 0$ for some $n \in \mathbb{N}$.

Definition 1.3.9. [4] The *normalizer* of a subalgebra K of L is defined by

$$N_L(K) = \big\{x \in L : [x,K] \subseteq K\big\}.$$

Definition 1.3.10. [4] A *Cartan subalgebra* (abbreviated CSA) of a Lie algebra L is a nilpotent subalgebra which equals its normalizer in L.

1.4. Elementary Representation Theory

In this section, we, first, provide the definition of a representation which leads to a decomposition of root spaces. Then we define an L-module where L is a semisimple Lie algebra, weight spaces and multiplicities of weights.

Definition 1.4.1. [4] Let L be a Lie algebra. A *representation* of L is a homomorphism

$$\phi : L \to gl(V)$$

for some vector space V where $gl(V) = \mathrm{End}(V)$.

Example 1.4.2. [4] Let L be a Lie algabra. We define the map $ad : L \to gl(L)$ where $gl(L) = \mathrm{End}(L)$ by

$$ad(x)y = [x,y] \quad \text{for all } x,y \in L.$$

Then ad is called an *adjoint representation* of L.

From now on, let L denote a semisimple Lie algebra over the algebrically closed field of characteristic 0 and H a fixed CSA of L. We are going to present the structure of L via its adjoint representation. Recall that H^* is the set of linear functionals on H. We consider

$$L_\alpha = \big\{x \in L : [h,x] = \alpha(h)x \text{ for all } h \in H\big\}$$

where α ranges over H^* and call $L_\alpha \neq \{0\}$ a *root space*. Actually, L is the direct sum of the subspaces $L_\alpha(\alpha \in H^*)$. The set of all nonzeros $\alpha \in H^*$ for which $L_\alpha \neq \{0\}$ is denoted by Φ and the elements of Φ are called the *roots* of L relative to H. With this notation we have a *root space decomposition*:

$$L = L_0 \oplus \coprod_{\alpha \in \Phi} L_\alpha.$$

Importantly, we can see that Φ is a root system in a real Euclidean space V as described in Section 1.1 since Φ is embedded in V with a bijection φ such that $\varphi(\Phi)$ is a root system in V.

Next, we provide the definition of L-module.

Definition 1.4.3. [4] A vector space V over a field $\mathbb{F}$, endowed with an operation $L \times V \to V$ (denoted $(x,v) \mapsto x \cdot v$), is called an *L-module* if the following conditions are satisfied: for all $x,y \in L$, $v,w \in V$, and $a,b \in \mathbb{F}$,

(M1) $(ax+by)\cdot v = a(x\cdot v) + b(y \cdot v)$,

(M2) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$,

(M3) $[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$.

It is convenient to use the language of modules along with the equivalent language of representations. For example, if $\phi : L \to gl(V)$ is a representation of L, then V may be viewed as an L-module via $x \cdot v = \phi(x)(v)$. Conversely, given an L-module V, this equation defines a representation $\phi : L \to gl(V)$.

Proposition 1.4.4. [4] *If V is a finite dimensional L-module, then $V = \coprod_{\lambda \in H^*} V_\lambda$, where*

$$V_\lambda = \big\{v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in H\big\}.$$

Definition 1.4.5. [4] If $V_\lambda \neq \{0\}$ $(\lambda \in H^*)$, we call V_λ a *weight space* and λ a *weight* of V.

Since the set Φ of all roots of L is a root system in a real Euclidean space and Theorem 1.1.8 guarantees that a base Δ of Φ exists, we obtain a *base* Δ in L by this way. In addition, we call an element of Δ in L a *simple root.* We define a partial ordering $\prec$ on H^* as follow: $\mu \prec \lambda$ if and only if $\lambda - \mu$ is a sum of simple roots in L where $\lambda, \mu \in H^*$. From here on, let Φ be a root system of L and Δ a fixed base of Φ.

Definition 1.4.6. A *maximal vector* of weight λ in a finite-dimensional L-module V is a nonzero vector $v^+ \in V_\lambda$ killed by all L_α $(\alpha \in \Delta)$ and we call λ the *highest weight* of V.

Part (a) of the following theorem justifies the terminology highest weight for λ.

Theorem 1.4.7. [4] *Let V be a finite-dimensional L–module with highest weight λ and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Then*

(a) *the weights* μ *of* V *are of the form*

$$\mu = \lambda - \sum_{i=1}^{n} k_i \alpha_i \ ,$$

where $k_i \in \mathbb{Z}^+$ *for all* i*, i.e., weights* μ *satisfy* $\mu \prec \lambda$*,*

(b) *for each* $\mu \in H^*$*,* $\dim V_\mu < \infty$*, and* $\dim V_\lambda = 1$*.*

A linear functional $\lambda \in H^*$ is called *integral* if all $\lambda(h_i)$ are integers where $h_i (1 \leq i \leq n)$ are basis elements of H. If all $\lambda(h_i)$ are nonnegative integers, then we call λ *dominant integral*. The set of all integral linear functionals, denoted Λ, is therefore a lattice in H^* (or equally well, in the real vector space generated by roots) and the set of dominant integral linear functions is denoted by Λ^+.

Recall that Φ is viewed as a root system in a Euclidean space V. Then the weights occuring in a finite-dimensional L-module are also weights in the sense of the abstract theory in Section 1.2. Actually, integral linear functionals in H^* and dominant integral linear functionals in H^* are 1-1 correspondence with weights and dominant weights in the language of Section 1.2 with respect to Φ, respectively, so all concerned results proved in Sections 1.1–1.2 are available from now on (more details can be seen in [4], Chapter VI p.112.).

Theorem 1.4.8. [4] *Let* $\lambda \in H^*$ *be dominant integral. Then there exists an irreducible* L*-module of highest weight* λ*. Denote by* $V(\lambda)$ *the irreducible* L*-module of highest weight* λ*.*

Definition 1.4.9. [4] If V is an L-module, let $\Pi(V) = \{\mu \in H^* : V_\mu \neq 0\}$ denote the set of all its weights. For $V = V(\lambda)$, write instead $\Pi(\lambda)$.

As we see from Theorem 1.4.8 that dominant integral λ is a highest weight of for some irreducible L–module $V(\lambda)$ so that $\Pi(\lambda)$ is definable.

Proposition 1.4.10. [4] *Let λ be dominant integral. Then $\Pi(\lambda)$ is saturated in the sense of Theorem 1.2.14.*

Theorem 1.4.11. [4] *If $\lambda \in H^*$ is dominant integral, then the irreducible L-module $V = V(\lambda)$ is finite-dimensional, and its set $\Pi(\lambda)$ of weights is permuted by $\mathcal{W}$, with $\dim V_\mu = \dim V_{\sigma\mu}$ for all $\sigma \in \mathcal{W}$.*

Theorem 1.4.12. [4] *The map $\lambda \mapsto V(\lambda)$ includes a one-to-one correspondence between the set of all dominant integral in H^* and the isomorphism classes of finite-dimensional irreducible L-modules.*

Definition 1.4.13. [4] Let $V = V(\lambda)$ $(\lambda \in \Lambda^+)$ be an L-module. If $\mu \in H^*$ is an integral linear functional, define the *multiplicity* of μ in $V(\lambda)$ to be $m_\lambda(\mu) = \dim V(\lambda)_\mu (= 0$ in case μ is not a weight of $V(\lambda))$.

1.5. Motivation

Let L be semisimple Lie algebras, H a fixed CSA of L, Φ the set of all roots of L, $\mathcal{W}$ the Weyl group of Φ. Now, Φ is a root system of a real Euclidean space and dominant integral and integral linear functionals are in 1-1 correspondence with dominant weights and weights with respect to Φ, respectively. Then the weight lattice Λ and the set of all dominant weights Λ^+ described in Definition 1.2.1 are the same as Λ the set of all integrals and Λ^+ the set of all integrals, respectively. Let $\lambda_1, \ldots, \lambda_n$ be fundamental weights with respect to Φ

In this thesis, we consider irreducible L such that the Dynkin diagrams of Φ are $\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n, \mathsf{D}_n$ and G_2 for appropriate n. Recall from Section 1.2 that for a fixed root system with fundamental weights $\lambda_1, \ldots, \lambda_n$, the sets

$$S = \left\{ S(e^{\lambda_i}) \,:\, \text{for all } 1 \le i \le n \right\} \quad \text{and} \quad \chi = \left\{ \chi_{\lambda_i} \,:\, \text{for all } 1 \le i \le n \right\}$$

are bases for the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. Thus, we are interested in finding a formative relation between the members of S and the members of χ.

In order to do that, the following facts are needed.

Theorem 1.5.1. [**Fruthenthal's formula**, [4]] *Let $V = V(\lambda)$ be an irreducible L-module of highest weight λ where $\lambda \in \Lambda^+$. If $\mu \in \Lambda$, then the multiplicity of μ in V is given recursively as follows:*

$$\big((\lambda+\delta,\lambda+\delta)-(\mu+\delta,\mu+\delta)\big)m_\lambda(\mu) = 2\sum_{\alpha\in\Phi^+}\sum_{i=1}^{\infty} m_\lambda(\mu+i\alpha)(\mu+i\alpha,\alpha). \quad (1.5.2)$$

Note 1.5.3. Let λ be a dominant weight. Then $V(\lambda)$ exists from Theorem 1.4.12 and

$$m_\lambda(\mu) = \begin{cases} \dim V(\lambda)_\mu, & \text{if } \mu \in \Pi(\lambda), \\ 0, & \text{if } \mu \notin \Pi(\lambda). \end{cases}$$

For convenience in calculation, we find the multiplicity of weight μ in $V(\lambda)$ in the language of roots and weights as in Sections 1.1–1.2.

Definition 1.5.4. [4] Let λ be a dominant weight. Define the *formal character* $\mathrm{ch}_{V(\lambda)}$ of $V(\lambda)$ as follows:

$$\mathrm{ch}_{V(\lambda)} = \sum_{\mu\in\Pi(\lambda)} m_\lambda(\mu)e^\mu.$$

Since $m_\lambda(\mu) = 0$ whenever $\mu \notin \Pi(\lambda)$, we can extend the summation to all integral $\mu \in \Lambda$.

Theorem 1.5.5. *Let λ be a dominant weight. Then*

$$\mathrm{ch}_{V(\lambda)} = S(e^\lambda) + \sum_{\substack{\mu\in\Lambda^+ \\ \mu\prec\lambda}} m_\lambda(\mu)S(e^\mu).$$

Proof. Note that $\mathrm{ch}_{V(\lambda)} = \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) e^\mu$ and $\Pi(\lambda) = \bigcup_{\substack{\mu \in \Lambda^+ \\ \mu \preceq \lambda}} \mathcal{W}\mu$ from Theorem 1.2.14.

Then

$$\mathrm{ch}_{V(\lambda)} = \sum_{\substack{\mu \in \Lambda^+ \\ \mu \preceq \lambda}} \sum_{\kappa \in \mathcal{W}\mu} m_\lambda(\kappa) e^\kappa.$$

Since weights which are conjugate under $\mathcal{W}$ have the same multiplicity from Theorem 1.4.11, for each dominant weight μ such that $\mu \preceq \lambda$,

$$\sum_{\kappa \in \mathcal{W}\mu} m_\lambda(\kappa) e^\kappa = m_\lambda(\mu) \sum_{\kappa \in \mathcal{W}\mu} = m_\lambda(\mu) S(e^\mu).$$

Note that $m_\lambda(\lambda) = 1$. Therefore, we obtain that

$$\mathrm{ch}_{V(\lambda)} = S(e^\lambda) + \sum_{\substack{\mu \in \Lambda^+ \\ \mu \prec \lambda}} m_\lambda(\mu) S(e^\mu).$$

□

Theorem 1.5.6. [**Weyl Character Formula**, [6]] *Let Φ be a root system and Δ a fixed base. Let λ be dominant weight. Then*

$$\mathrm{ch}_{V(\lambda)} = \frac{A(\lambda + \delta)}{A(\delta)}.$$

As a result, we obtain the following magnificant theorem.

Theorem 1.5.7. *For the fundamental weight λ, we obtain that*

$$\chi_\lambda = S(e^\lambda) + \sum_{\substack{\mu \in \Lambda^+ \\ \mu \prec \lambda}} m_\lambda(\mu) S(e^\mu).$$

Proof. This follows from the fact that $\mathrm{ch}_{V(\lambda)} = \chi_\lambda$. □

We can see from Theorem 1.5.7 that in order to find the desired relation, it is enough to focus on finding multiplicities $m_\lambda(\mu)$ where $\mu \in \Lambda^+$ and $\mu \prec \lambda$. Although, there are many methods to find multiplicities, we apply Fruthenthal's recursive formula (1.5.2). Actually, the right hand side of (1.5.2) is a finite sum. We can see that the calculation involves positive roots and weights. Moreover, we know that positive roots and weights are vectors in a real Euclidean vector space. As a result, we make use of the standard basis, i.e., we write each vector in a linear combination of standard basis elements and apply the fact that the multiplicity of μ with highest weight λ is zero if μ is not a weight. Finally, we obtain the general formula of the multiplicities of weights μ with highest weight λ being fundamental weights, so we know the particular formula in Theorem 1.5.7 of each root system.

In this literature, results of root systems whose Dynkin diagrams are $\mathsf{A}_n, \mathsf{B}_n, \mathsf{C}_n$ and D_n are presented in the same manner in Chapters II, III, IV and V, respectively. They are outline of the root system, example ($\mathsf{A}_2, \mathsf{B}_2, \mathsf{C}_3$ and D_4, respectively) and the relation between $S(e^{\lambda_m})$ and χ_{λ_m}. We give the outline of and the relation between $S(e^{\lambda_m})$ and χ_{λ_m} in the root system whose Dynkin diagram is G_2 in Chapter VI. Since we use the same technique to prove results in Chapters III–V, we provide in details for those in chapter III only and we omit the proofs of some properties in Chapters IV and V.

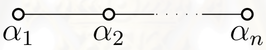
CHAPTER II

ROOT SYSTEM A_n

We consider $\mathbb{R}^{n+1}$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\epsilon_1, \ldots, \epsilon_{n+1}$ be the standard basis vectors of $\mathbb{R}^{n+1}$. The $\mathbb{Z}$-span of this basis is a lattice, denoted by I. Let V be the n-dimensional subspace of $\mathbb{R}^{n+1}$ orthogonal to the vector $\epsilon_1 + \cdots + \epsilon_{n+1}$ and let $I' = I \cap V$.

2.1. Outline of the Root System A_n

Let $\Phi = \{\alpha \in I' : (\alpha, \alpha) = 2\}$. Note that all elements of Φ have the same length.

- Φ is a root system in V of rank n.
- Dynkin diagram is 

- Roots are $\epsilon_r - \epsilon_s$ for all $r \neq s$ and $1 \leq r, s \leq n+1$.
- Simple roots are

$$\alpha_r = \epsilon_r - \epsilon_{r+1} = (0, \ldots, 0, \overset{r^{\text{th}}}{1}, -1, 0, \ldots, 0) \quad \text{for all } 1 \leq r \leq n.$$

- Positive roots are

$$\begin{aligned} \epsilon_r - \epsilon_s &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1} \\ &= (0, \ldots, 0, \overset{r^{\text{th}}}{1}, 0, \ldots, 0, \overset{s^{\text{th}}}{-1}, 0, \ldots, 0) \quad \text{for all } 1 \leq r < s \leq n+1. \end{aligned}$$

- Fundamental weights are

$$\begin{aligned} \lambda_r &= \frac{n-r+1}{n+1} \sum_{j=1}^{r} \epsilon_j - \frac{r}{n+1} \sum_{j=r+1}^{n+1} \epsilon_j \\ &= \frac{n-r+1}{n+1} \sum_{j=1}^{r} j\alpha_j + \frac{r}{n+1} \sum_{j=r+1}^{n} (n-j+1)\alpha_j \quad \text{for all } 1 \leq r \leq n. \end{aligned}$$

2.2. Root system A_2

We give an example of a root system whose Dynkin diagram is A_2. Let V be the subspace of dimension 2 of $\mathbb{R}^3$ orthogonal to the vector (1,1,1), i.e.,

$$V = \big\langle (1,-1,0), (0,1,-1) \big\rangle.$$

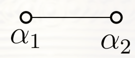
Let $\alpha_1 = (1,-1,0)$ and $\alpha_2 = (0,1,-1)$. Then

$$\Phi = \big\{ \pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2) \big\} = \big\{ \pm(1,-1,0), \pm(0,1,-1), \pm(1,0,-1) \big\}$$

is a root system in V and

$$\Delta = \big\{ \alpha_1, \alpha_2 \big\} = \big\{ (1,-1,0), (0,1,-1) \big\}$$

is a base of Φ. The Dynkin diagram is



We can see that positive roots are

$$\alpha_1 = (1,-1,0), \qquad \alpha_2 = (0,1,-1) \qquad \text{and} \qquad \alpha_1 + \alpha_2 = (1,0,-1),$$

so $\delta = (1,0,-1)$. The fundamental weights are

$$\lambda_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right) \qquad \text{and} \qquad \lambda_2 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right).$$

We know that

$$\sigma_{\alpha_1} : \begin{array}{l} \alpha_1 \mapsto -\alpha_1, \\ \\ \alpha_2 \mapsto \alpha_1 + \alpha_2, \end{array} \qquad \text{and} \qquad \sigma_{\alpha_2} : \begin{array}{l} \alpha_1 \mapsto \alpha_1 + \alpha_2, \\ \\ \alpha_2 \mapsto -\alpha_2. \end{array}$$

Moreover,

$$\sigma_{\alpha_1} : \begin{array}{l} \lambda_1 \mapsto -\lambda_1 + \lambda_2, \\ \\ \lambda_2 \mapsto \lambda_2, \end{array} \qquad \text{and} \qquad \sigma_{\alpha_2} : \begin{array}{l} \lambda_1 \mapsto \lambda_1, \\ \\ \lambda_2 \mapsto \lambda_1 - \lambda_2. \end{array}$$

The Weyl group $\mathcal{W}$ of Φ is $\langle \sigma_{\alpha_1}, \sigma_{\alpha_2} \rangle$, i.e.,

$$\mathcal{W} = \big\{ i_V, \sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_1}\sigma_{\alpha_2}, \sigma_{\alpha_2}\sigma_{\alpha_1}, \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_1} \big\}.$$

Next, for each fundamental weight λ, we consider the elementary symmetric sum $S(e^{\lambda})$ and the character χ_λ with highest weight λ (see also Definition 1.2.20). We see that

$$\mathcal{W}\lambda_1 = \big\{ \lambda_1, -\lambda_2, \lambda_2 - \lambda_1 \big\} \qquad \text{and} \qquad \mathcal{W}\lambda_2 = \big\{ \lambda_2, -\lambda_1, \lambda_1 - \lambda_2 \big\},$$

so

$$S(e^{\lambda_1}) = e^{\lambda_1} + e^{-\lambda_2} + e^{\lambda_2 - \lambda_1} \qquad \text{and} \qquad S(e^{\lambda_2}) = e^{\lambda_2} + e^{-\lambda_1} + e^{\lambda_1 - \lambda_2}.$$

In order to determine χ_{λ_1} and χ_{λ_2}, we need the followings.

$$A(e^{\lambda_1+\delta}) = e^{2\lambda_1+\lambda_2} - e^{-2\lambda_1+3\lambda_2} - e^{3\lambda_1-\lambda_2} + e^{-3\lambda_1+2\lambda_2} + e^{\lambda_1-3\lambda_2} - e^{-\lambda_1-2\lambda_2},$$

$$A(e^{\lambda_2+\delta}) = e^{\lambda_1+2\lambda_2} - e^{-\lambda_1+3\lambda_2} - e^{3\lambda_1-2\lambda_2} + e^{-3\lambda_1+\lambda_2} + e^{2\lambda_1-3\lambda_2} - e^{-2\lambda_1-\lambda_2},$$

and

$$A(e^{\delta}) = e^{\lambda_1+\lambda_2} - e^{\lambda_2} - e^{\lambda_1} + e^{-\lambda_1} + e^{-\lambda_2} - e^{-\lambda_1-\lambda_2}.$$

Thus,

$$\chi_{\lambda_1} = \frac{e^{2\lambda_1+\lambda_2} - e^{-2\lambda_1+3\lambda_2} - e^{3\lambda_1-\lambda_2} + e^{-3\lambda_1+2\lambda_2} + e^{\lambda_1-3\lambda_2} - e^{-\lambda_1-2\lambda_2}}{e^{\lambda_1+\lambda_2} - e^{\lambda_2} - e^{\lambda_1} + e^{-\lambda_1} + e^{-\lambda_2} - e^{-\lambda_1-\lambda_2}}$$
$$= e^{\lambda_1} + e^{-\lambda_2} + e^{\lambda_2-\lambda_1}$$

and

$$\chi_{\lambda_2} = \frac{e^{\lambda_1+2\lambda_2} - e^{-\lambda_1+3\lambda_2} - e^{3\lambda_1-2\lambda_2} + e^{-3\lambda_1+\lambda_2} + e^{2\lambda_1-3\lambda_2} - e^{-2\lambda_1-\lambda_2}}{e^{\lambda_1+\lambda_2} - e^{\lambda_2} - e^{\lambda_1} + e^{-\lambda_1} + e^{-\lambda_2} - e^{-\lambda_1-\lambda_2}}$$
$$= e^{\lambda_2} + e^{-\lambda_1} + e^{\lambda_1-\lambda_2}.$$

We observe that

$$\chi_{\lambda_1} = S(e^{\lambda_1}) \qquad \text{and} \qquad \chi_{\lambda_2} = S(e^{\lambda_2}).$$

On the other hand, we remind from Theorem 1.5.7 that for $i = 1$ and 2

$$\chi_{\lambda_i} = S(e^{\lambda_i}) + \sum_{\substack{\mu\in\Lambda^+ \\ \mu\prec\lambda_i}} m_{\lambda_i}(\mu) S(e^{\mu}).$$

Moreover, we find out that λ_1 and λ_2 are minimal so that

$$\chi_{\lambda_1} = S(e^{\lambda_1}) \qquad \text{and} \qquad \chi_{\lambda_2} = S(e^{\lambda_2}).$$

2.3. The Relation between $S(e^{\lambda_m})$ and χ_{λ_m}

Now, we let Φ be the root system whose Dynkin diagram is A_n, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the base of Φ, and $\lambda_1, \ldots, \lambda_n$ the fundamental weights described in Section 2.1. Recall from Theorem 1.5.7 that for each fundamental weight λ_m,

$$\chi_{\lambda_m} = S(e^{\lambda_m}) + \sum_{\substack{\mu\in\Lambda^+ \\ \mu\prec\lambda_m}} m_{\lambda_m}(\mu)S(e^{\mu}).$$

We discover that all the fundamental weights are minimal. Remind that $\lambda \in \Lambda^+$ is minimal if and only if $\mu \in \Lambda^+$, $\mu \preceq \lambda$ implies $\mu = \lambda$.

Proposition 2.3.1. *All fundamental weights are nonzero minimal.*

Proof. Let λ_m be a fundamental weight. Then λ_m is nonzero. Thank to Theorem 1.2.17, it suffices to show that λ_m is minuscule. Let $\beta \in \Phi$. Then $\beta = \pm\alpha_i$ for some $1 \le i \le n$ or $\beta = \pm(\alpha_i + \cdots + \alpha_j)$ for some $1 \le i < j \le n$. By the definition of λ_m, we have $(\lambda_m, \alpha_k^\vee) = \delta_{mk}$ for all $1 \le k \le n$. If $\beta = \pm\alpha_i$, then it is obvious that $(\lambda_m, \pm\alpha_i^\vee) \in \{0, \pm 1\}$. Since $(\alpha_i, \alpha_i) = 2$ for all $1 \le i \le n$, we have $(\lambda_m, \alpha_i) = (\lambda_m, \alpha_i^\vee) = \delta_{mk}$ for all $1 \le k \le n$. Let $\beta = \pm(\alpha_i + \cdots + \alpha_j)$. Then we know that $(\beta, \beta) = 2$ and

$$\begin{aligned}\big(\lambda_m, \pm(\alpha_i + \cdots + \alpha_j)^\vee\big) &= 2\frac{\big(\lambda_m, \pm(\alpha_i + \cdots + \alpha_j)\big)}{\big(\pm(\alpha_i + \cdots + \alpha_j), \pm(\alpha_i + \cdots + \alpha_j)\big)} \\ &= \big(\lambda_m, \pm(\alpha_i + \cdots + \alpha_j)\big) \in \{0, \pm 1\}.\end{aligned}$$

We conclude that λ_m is minuscule, so λ_m is nonzero minimal. □

Note 2.3.2. Note that when a dominant weight λ is minimal, there is no dominant weights μ such that $\mu \prec \lambda$.

Theorem 2.3.3. *Let m be a positive integer such that $1 \le m \le n$. Then*

$$\chi_{\lambda_m} = S(e^{\lambda_m}).$$

Proof. This follows from Theorem 1.5.7 and Proposition 2.3.1. □

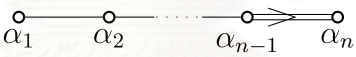
CHAPTER III

ROOT SYSTEM B_n

In this chapter, we fix $n \in \mathbb{N} \setminus \{1\}$. We consider $\mathbb{R}^n$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis vectors of $\mathbb{R}^n$. The $\mathbb{Z}$-span of this basis is a lattice, denoted by I.

3.1. Outline of the Root System B_n

Let $\Phi = \{\alpha \in I : (\alpha, \alpha) = 1 \text{ or } 2\}$. Note that the squared length of an element of Φ is 1 or 2.

- Φ is a root system in $\mathbb{R}^n$ of rank n.
- Dynkin diagram is 

- Short roots are

$$\pm\epsilon_r \quad \text{(of squared length 1)} \quad \text{for all } 1 \le r \le n.$$

Long roots are

$$\pm(\epsilon_r \pm \epsilon_s) \quad \text{(of squared length 2)} \quad \text{for all } 1 \le r \neq s \le n.$$

- Simple roots are

$$\alpha_r = \epsilon_r - \epsilon_{r+1} = (0, \ldots, 0, \overset{r\text{th}}{1}, -1, 0, \ldots, 0) \qquad \text{for all } 1 \le r \le n-1 \quad \text{and}$$

$$\alpha_n = \epsilon_n = (0, \ldots, 0, 1).$$

- Positive roots are

$$
\begin{aligned}
\epsilon_r &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_n && \text{for all } 1 \le r \le n, \\
\epsilon_r - \epsilon_s &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1} && \text{for all } 1 \le r < s \le n, \text{ and} \\
\epsilon_r + \epsilon_s &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1} \\
&\qquad + 2\alpha_s + 2\alpha_{s+1} + \cdots + 2\alpha_n && \text{for all } 1 \le r < s \le n.
\end{aligned}
$$

- Fundamental weights are

$$
\begin{aligned}
\lambda_r &= \epsilon_1 + \epsilon_2 + \cdots + \epsilon_r \\
&= \alpha_1 + 2\alpha_2 + \cdots + (r-1)\alpha_{r-1} + r(\alpha_r + \alpha_{r+1} + \cdots + \alpha_n) \\
&= (\underbrace{1, \ldots, 1}_{r \text{ terms}}, 0, \ldots, 0) \qquad \text{for all } 1 \le r \le n-1, \text{ and} \\
\lambda_n &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) \\
&= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n) \\
&= \left(\frac{1}{2}, \ldots, \frac{1}{2}\right).
\end{aligned}
$$

- Let $\lambda_0 = 0$. Then $\lambda_0 \prec \lambda_1 \prec \cdots \prec \lambda_{n-1}$.

3.2. Root System B_2

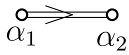
We give an example of a root system whose Dynkin diagram is B_2. We consider $\mathbb{R}^2$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\alpha_1 = (1, -1)$ and $\alpha_2 = (0, 1)$. Then

$$\Phi = \big\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\big\} = \big\{\pm(1,-1), \pm(0,1), \pm(1,0), \pm(1,1)\big\}$$

is a root system in $\mathbb{R}^2$ and

$$\Delta = \big\{\alpha_1, \alpha_2\big\} = \big\{(1,-1), (0,1)\big\}$$

is a base of Φ. The Dynkin diagram is



We can see that positive roots are

$$\alpha_1 = (1,-1),\ \alpha_2 = (0,1),\ \alpha_1+\alpha_2 = (1,0)\ \text{ and }\ \alpha_1+2\alpha_2 = (1,1),$$

so $\delta = \left(\frac{3}{2}, \frac{1}{2}\right)$. The fundamental weights are

$$\lambda_1 = (1,0) \qquad \text{and} \qquad \lambda_2 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

We know that

$$\sigma_{\alpha_1} : \begin{array}{l} \alpha_1 \mapsto -\alpha_1, \\ \\ \alpha_2 \mapsto 3\alpha_1 + \alpha_2, \end{array} \qquad \text{and} \qquad \sigma_{\alpha_2} : \begin{array}{l} \alpha_1 \mapsto \alpha_1 + \alpha_2, \\ \\ \alpha_2 \mapsto -\alpha_2. \end{array}$$

Moreover,

$$\sigma_{\alpha_1} : \begin{array}{l} \lambda_1 \mapsto -\lambda_1 + 2\lambda_2, \\ \\ \lambda_2 \mapsto \lambda_2, \end{array} \qquad \text{and} \qquad \sigma_{\alpha_2} : \begin{array}{l} \lambda_1 \mapsto \lambda_1, \\ \\ \lambda_2 \mapsto \lambda_1 - \lambda_2. \end{array}$$

The Weyl group $\mathcal{W}$ of Φ is $\big\langle \sigma_{\alpha_1}, \sigma_{\alpha_2} \big\rangle$, i.e.,

$$\mathcal{W} = \big\{i_V, \sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_1}\sigma_{\alpha_2}, \sigma_{\alpha_2}\sigma_{\alpha_1}, \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_1}, \sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_2}, \sigma_{\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_2}\big\}.$$

Next, for each fundamental weight λ, we consider the elementary symmetric sum $S(e^\lambda)$ and the character χ_λ with highest weight λ (see also Definition 1.2.20). We see that

$$\mathcal{W}\lambda_1 = \{\lambda_1, -\lambda_1, \lambda_1 - 2\lambda_2, -\lambda_1 + 2\lambda_2\}$$

and

$$\mathcal{W}\lambda_2 = \{\lambda_2, -\lambda_1 + \lambda_2, \lambda_1 - \lambda_2, -\lambda_2\}.$$

Then

$$S(e^{\lambda_1}) = e^{\lambda_1} + e^{-\lambda_1} + e^{\lambda_1 - 2\lambda_2} + e^{-\lambda_1 + 2\lambda_2}$$

and

$$S(e^{\lambda_2}) = e^{\lambda_2} + e^{-\lambda_1 + \lambda_2} + e^{\lambda_1 - \lambda_2} + e^{-\lambda_2}.$$

We need the followings in order to calculate χ_{λ_1} and χ_{λ_2}.

$$\begin{aligned} A(e^{\lambda_1+\delta}) &= e^{2\lambda_1+\lambda_2} - e^{-3\lambda_1+\lambda_2} + e^{-2\lambda_1-\lambda_2} + e^{3\lambda_1-5\lambda_2} - e^{-2\lambda_1+5\lambda_2} \\ &\quad - e^{2\lambda_1-5\lambda_2} + e^{-3\lambda_1+5\lambda_2} - e^{3\lambda_1-\lambda_2}, \\ A(e^{\lambda_2+\delta}) &= e^{\lambda_1+2\lambda_2} - e^{-3\lambda_1+2\lambda_2} + e^{-\lambda_1-2\lambda_2} + e^{3\lambda_1-4\lambda_2} - e^{-\lambda_1+4\lambda_2} \\ &\quad - e^{\lambda_1-4\lambda_2} + e^{-3\lambda_1+4\lambda_2} - e^{3\lambda_1-2\lambda_2} \qquad \text{and} \\ A(e^{\delta}) &= e^{\lambda_1+\lambda_2} - e^{-2\lambda_1+\lambda_2} + e^{-\lambda_1-\lambda_2} + e^{2\lambda_1-3\lambda_2} - e^{-\lambda_1+3\lambda_2} \\ &\quad + e^{-2\lambda_1+3\lambda_2} - e^{2\lambda_1-\lambda_2}. \end{aligned}$$

Thus,

$$\chi_{\lambda_1} = \frac{A(e^{\lambda_1+\delta})}{A(\delta)} = 1 + e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-2\lambda_2} + e^{-\lambda_1+2\lambda_2}$$

and

$$\chi_{\lambda_2} = \frac{A(e^{\lambda_2+\delta})}{A(\delta)} = e^{-\lambda_2} + e^{\lambda_1-\lambda_2} + e^{\lambda_2} + e^{-\lambda_1+\lambda_2}.$$

We observe that

$$\chi_{\lambda_1} = S(e^{\lambda_1}) + 1 \quad \text{and} \quad \chi_{\lambda_2} = S(e^{\lambda_2}).$$

On the other hand, we remind from Theorem 1.5.7 that for $i = 1$ and 2

$$\chi_{\lambda_i} = S(e^{\lambda_i}) + \sum_{\substack{\mu\in\Lambda^+ \\ \mu\prec\lambda_i}} m_{\lambda_i}(\mu)S(e^{\mu}).$$

We discover that a dominant weight μ such that $\mu \prec \lambda_1$ is only 0 and λ_2 is minimal. Moreover, if we calculate by Fruthenthal's recursive formula, then we obtain that $m_{\lambda_1}(0) = 1$. Thus

$$\chi_{\lambda_1} = S(e^{\lambda_1}) + 1 \quad \text{and} \quad \chi_{\lambda_2} = S(e^{\lambda_2}).$$

3.3. The Relation between $S(e^{\lambda_m})$ and χ_{λ_m}

Let Φ be the root system whose Dynkin diagram is B_n, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a base, $\lambda_1, \ldots, \lambda_n$ the fundamental weights described in Section 3.1 and $\mathcal{W}$ the Weyl group of Φ.

For each fundamental weight λ, we want to find the multiplicities of dominant weights μ such that $\mu \prec \lambda$. For each $m < n$, if μ is a dominant weight such that $\mu \prec \lambda_m$, then μ is only $\lambda_0, \lambda_1, \ldots, \lambda_{m-2}$ or λ_{m-1}, so it restricts us to find only $m_{\lambda_m}(\lambda_k)$ where $0 \le k < m$.

From the Fruthenthal's recursive formula (1.5.2) for weight μ with highest weight λ, we, first, consider the multiplier of $m_{\lambda_m}(\lambda_k)$ $(0 \le k < m)$ in its left hand side. Note that $m_{\lambda_m}(\lambda_m) = 1$

Proposition 3.3.1. *Let m and k be nonnegative integers such that $0 \leq k < m < n$. Then*

$$(\lambda_m + \delta, \lambda_m + \delta) - (\lambda_k + \delta, \lambda_k + \delta) = (m-k)(2n - m - k + 1).$$

Proof. We can see that $\lambda_m - \lambda_k = (0, \ldots, 0, \overset{k+1^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0)$ and

$$\delta = \left(n - \frac{1}{2}, n - \frac{3}{2}, n - \frac{2i-1}{2}, \ldots, \frac{1}{2}\right) \quad \text{where } 1 \leq i \leq n.$$

Then

$$\begin{aligned}
(\lambda_m - \lambda_k, \delta) &= \sum_{j=k+1}^{m} \left(n + \frac{1}{2} - j\right) \\
&= (m-k)\left(n + \frac{1}{2}\right) - \left(\frac{m(m+1)}{2} - \frac{k(k+1)}{2}\right) \\
&= (m-k)\left(\frac{2n - m - k}{2}\right).
\end{aligned}$$

Now, since $(\lambda_m, \lambda_m) = m$ and $(\lambda_k, \lambda_k) = k$, it follows that

$$\begin{aligned}
(\lambda_m + \delta, \lambda_m + \delta) - (\lambda_k + \delta, \lambda_k + \delta) &= (\lambda_m, \lambda_m) - (\lambda_k, \lambda_k) + 2(\lambda_m - \lambda_k, \delta) \\
&= (m-k) + 2(m-k)\left(\frac{2n-m-k}{2}\right) \\
&= (m-k)(2n - m - k + 1).
\end{aligned}$$

□

Next, in order to know $m_{\lambda_m}(\lambda_k)$ $(0 \leq k < m)$, we require values of $m_{\lambda_m}(\lambda_k + t\alpha)$ where $\alpha \in \Phi^+$ and $t \in \mathbb{N}$. We use two main techniques to find these. First, we write positive roots in terms of standard basis elements, i.e., either ϵ_r (for some $1 \leq r \leq n$) or $\epsilon_r \pm \epsilon_s$ (for some $1 \leq r < s \leq n$). Second, we apply the fact that $m_{\lambda_m}(\lambda_k + t\alpha) = 0$

when $\lambda_k + t\alpha$ is not a weight in $\Pi(\lambda_k)$. There are two main cases to calculate these $m_{\lambda_m}(\lambda_k + t\alpha)$ as follows:

Case 1 $\alpha = \epsilon_r$ where $1 \leq r \leq n$

(1.1) $r \leq k$ and $t \in \mathbb{N}$

(1.2) $r > k$ and $t \in \mathbb{N} \setminus \{1\}$

(1.3) $r > k$ and $t = 1$

Case 2 $\alpha = \epsilon_r \pm \epsilon_s$ where $1 \leq r < s \leq n$

(2.1) $r \leq k$ and $t \in \mathbb{N}$

(2.2) $r > k$ and $t \in \mathbb{N} \setminus \{1\}$

(2.3) $r > k$ and $t = 1$

Proposition 3.3.2. *Let k, m and r be nonnegative integers such that $1 \leq r \leq k < m < n$. Then $m_{\lambda_m}(\lambda_k + t\epsilon_r) = 0$ for all $t \in \mathbb{N}$.*

Proof. Let $t \in \mathbb{N}$. We claim that $\lambda_k + t\epsilon_r \npreceq \lambda_m$. We consider

$$\lambda_m - (\lambda_k + t\epsilon_r) = (0, \ldots, 0, \overset{r^{\text{th}}}{-t}, 0, \ldots, 0, \overset{(k+1)^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0).$$

Suppose that $\lambda_k + t\epsilon_r \preceq \lambda_m$. Then there exist $a_1, \ldots, a_n \in \mathbb{Z}_0^+$ such that

$$\begin{aligned}\lambda_m - (\lambda_k + t\epsilon_r) &= \sum_{i=1}^{n} a_i\alpha_i = \sum_{i=1}^{n-1} a_i(\epsilon_i - \epsilon_{i+1}) + a_n\epsilon_n \\ &= (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, \ldots, a_{n-1} - a_{n-2}, a_n - a_{n-1}).\end{aligned}$$

That is,

$$\begin{aligned}&(0, \ldots, 0, \overset{r^{\text{th}}}{-t}, 0, \ldots, 0, \overset{(k+1)^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0) \\ &\qquad = (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, \ldots, a_{n-1} - a_{n-2}, a_n - a_{n-1}).\end{aligned}$$

Then, $a_1 = a_2 = \cdots = a_{r-1} = 0$ so that $a_r = -t$ which is a contradiction since $a_r \in \mathbb{Z}_0^+$ but $t \in \mathbb{N}$. Thus $\lambda_k + t\epsilon_r \not\leq \lambda_m$. By Theorem 1.4.7, we conclude that $m_{\lambda_m}(\lambda_k + t\epsilon_r) = 0$. □

Proposition 3.3.3. *Let $k, m, r, s \in \mathbb{Z}_0^+$ such that $1 \leq r \leq k < m < n$ and $r < s$. Then $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$ for all $t \in \mathbb{N}$.*

Proof. Let $t \in \mathbb{N}$.

Case 1 $r < s \leq k$

We know that

$$\lambda_k + t(\epsilon_r \pm \epsilon_s) = \big(1, \ldots, 1, \overset{r^{\text{th}}}{(1+t)}, 1, \ldots, 1, \overset{s^{\text{th}}}{(1 \pm t)}, 1, \ldots, \overset{k^{\text{th}}}{1}, 0, \ldots, 0\big).$$

Therefore,

$$\begin{aligned}\lambda_m - \big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) \\ &= (0, \ldots, 0, \overset{r^{\text{th}}}{-t}, 0, \ldots, 0, \overset{s^{\text{th}}}{\mp t}, 0, \ldots, 0, \overset{(k+1)^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0).\end{aligned}$$

By the same reason as in the proof of Proposition 3.3.2, we assure that

$$m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0.$$

Case 2 $r \leq k < s$

We use the same method as in Case 1 and obtain that

$$m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0.$$

□

Recall from Theorem 1.4.11 that weights which are conjugate under the Weyl group have the same multiplicity. Then we apply this fact to obtain two following propositions.

Proposition 3.3.4. *Let b, k and m be nonnegative integers such that $1 < b \leq n-k$, and $0 \leq k < m < n$. Then $m_{\lambda_m}(\lambda_{k+1}) = m_{\lambda_m}(\lambda_k + \epsilon_{k+b})$.*

Proof. Since ϵ_{k+1} and ϵ_{k+b} are positive roots of the same length, they are conjugate under $\mathcal{W}$. Furthermore, $(\epsilon_{k+1}, \epsilon_{k+b}) = 0$, so $\sigma_{\epsilon_{k+1}-\epsilon_{k+b}}(\epsilon_{k+1}) = \epsilon_{k+b}$ by Proposition 1.1.10. Because of the orthogonality between $\epsilon_{k+1} - \epsilon_{k+b}$ and λ_k, then $\sigma_{\epsilon_{k+1}-\epsilon_{k+b}}(\lambda_k) = \lambda_k$. Thus $\sigma_{\epsilon_{k+1}-\epsilon_{k+b}}(\lambda_k + \epsilon_{k+1}) = \lambda_k + \epsilon_{k+b}$, i.e., $\lambda_k + \epsilon_{k+1}$ and $\lambda_k + \epsilon_{k+b}$ are conjugate under $\mathcal{W}$. In conclusion, from Theorem 1.4.11, we have

$$m_{\lambda_m}(\lambda_{k+1}) = m_{\lambda_m}(\lambda_k + \epsilon_{k+1}) = m_{\lambda_m}(\lambda_k + \epsilon_{k+b}).$$

$\square$

Proposition 3.3.5. *Let b, c, k and m be nonnegative integers such that $1 \leq b < c \leq n-k$ and $0 \leq k < m < n$. Then $m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+b} \pm \epsilon_{k+c})\big)$.*

Proof. Since $\epsilon_{k+1} + \epsilon_{k+2}$ and $\epsilon_{k+b} \pm \epsilon_{k+c}$ are positive roots of the same length, they are conjugate under $\mathcal{W}$. Let $\alpha = \epsilon_{k+1} + \epsilon_{k+2}$ and $\beta = \epsilon_{k+b} \pm \epsilon_{k+c}$. Also, one of $\sigma_{\alpha-\beta}(\beta) = \alpha$, $\sigma_\alpha\sigma_\beta\sigma_\alpha(\beta) = \alpha$, or $\sigma_\alpha\sigma_\beta\sigma_\beta\sigma_\alpha(\beta) = \alpha$ holds. For each case, λ_k is fixed, so that $\lambda_k + \epsilon_{k+1} + \epsilon_{k+2}$ is mapped to $\lambda_k + (\epsilon_{k+b} \pm \epsilon_{k+c})$, i.e., they are conjugate under $\mathcal{W}$. By Theorem 1.4.11, we conclude that

$$m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+1} + \epsilon_{k+2})\big) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+b} \pm \epsilon_{k+c})\big).$$

$\square$

From the proof of Propositions 3.3.4 and 3.3.5, we obtain two corollaries.

Corollary 3.3.6. *Let $t \in \mathbb{N}$, then $t\epsilon_{k+b}$ conjugates to $t\epsilon_{k+1}$ and $\lambda_k + t\epsilon_{k+b}$ conjugates to $\lambda_k + t\epsilon_{k+1}$ for all $1 \leq b \leq n-k$.*

Corollary 3.3.7. *Let $t \in \mathbb{N}$, then $t(\epsilon_{k+b} \pm \epsilon_{k+c})$ conjugates to $t(\epsilon_{k+1} \pm \epsilon_{k+2})$ and $\lambda_k + t(\epsilon_{k+b} \pm \epsilon_{k+c})$ conjugates to $\lambda_k + t(\epsilon_{k+1} \pm \epsilon_{k+2})$ for $1 \leq b < c \leq n-k$.*

Proposition 3.3.8. *Let k, m and r be nonnegative integers such that $0 \leq k < r \leq n$, and $t \in \mathbb{N} \setminus \{1\}$. Then $m_{\lambda_m}(\lambda_k + t\epsilon_r) = 0$.*

Proof. We know from Corollary 3.3.6 that $\lambda_k + t\epsilon_r$ is conjugate to $\lambda_k + t\epsilon_{k+1}$. Then it suffices to show that $m_{\lambda_m}(\lambda_k + t\epsilon_{k+1}) = 0$. Of course,

$$\lambda_k + t\epsilon_{k+1} = (1, \ldots, \overset{k^{\text{th}}}{1}, t, 0, \ldots, 0).$$

Therefore,

$$\lambda_m - (\lambda_k + t\epsilon_{k+1}) = (0, \ldots, \overset{k^{\text{th}}}{0}, 1-t, 1, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0).$$

Since $t \geq 2$, we attain $\lambda_k + t\epsilon_{k+1} \npreceq \lambda_m$ so that $m_{\lambda_m}(\lambda_k + t\epsilon_{k+1}) = 0$. $\square$

Proposition 3.3.9. *Let k, m, r and s be nonnegative integers such that $0 \leq k < r < s \leq n$ and $t \in \mathbb{N} \setminus \{1\}$. Then $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$.*

Proof. This is similar to the proof of Proposition 3.3.8. $\square$

Consequently, we can reduce the finite sum in the right hand side of (1.5.2) to the simpler form.

Theorem 3.3.10. *Let k and m be nonnegative integers such that $0 \le k < m < n$. Then*

$$\sum_{\alpha\in\Phi^+}\sum_{t=1}^{\infty} m_{\lambda_m}(\lambda_k+t\alpha)(\lambda_k+t\alpha,\alpha)$$

$$= (n-k)m_{\lambda_m}(\lambda_{k+1}) + 2(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}).$$

Proof. Let α be a positive root and $t \in \mathbb{N}$. Then $\alpha = \epsilon_r$ for some $1 \le r \le n$ or $\alpha = \epsilon_r \pm \epsilon_s$ for some $1 \le r < s \le n$.

Case 1 $\alpha = \epsilon_r$

(1.1) $r \le k$ and $t \in \mathbb{N}$

It follows from Proposition 3.3.2 that $m_{\lambda_m}(\lambda_k + t\epsilon_r) = 0$.

(1.2) $r > k$ and $t \in \mathbb{N}\setminus\{1\}$

It follows from Proposition 3.3.8 that $m_{\lambda_m}(\lambda_k + t\epsilon_r) = 0$.

(1.3) $r > k$ and $t = 1$

By Proposition 3.3.4, we acquire the fact that

$$m_{\lambda_m}(\lambda_{k+1}) = m_{\lambda_m}(\lambda_k + \epsilon_{k+1}) = m_{\lambda_m}(\lambda_k + \epsilon_r).$$

Case 2 $\alpha = \epsilon_r \pm \epsilon_s$

(2.1) $r \le k$ and $t \in \mathbb{N}$

It follows from Proposition 3.3.3 that $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$.

(2.2) $r > k$ and $t \in \mathbb{N}\setminus\{1\}$

It follows from Proposition 3.3.9 that $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$.

(2.3) $r > k$ and $t = 1$

By Proposition 3.3.5, we reach the fact that

$$m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+1} + \epsilon_{k+2})\big) = m_{\lambda_m}\big(\lambda_k + (\epsilon_r \pm \epsilon_s)\big).$$

We conclude from all the cases that

$$\begin{aligned}
&\sum_{\alpha\in\Phi^+}\sum_{t=1}^{\infty} m_{\lambda_m}(\lambda_k+t\alpha)(\lambda_k+t\alpha,\alpha)\\
&=\sum_{r=k+1}^{n} m_{\lambda_m}(\lambda_k+\epsilon_r)(\lambda_k+\epsilon_r,\epsilon_r)+\sum_{r=k+1}^{n-1}\sum_{s=r+1}^{n} m_{\lambda_m}(\lambda_k+\epsilon_r\pm\epsilon_s)(\lambda_k+\epsilon_r\pm\epsilon_s,\epsilon_r\pm\epsilon_s)\\
&=(n-k)m_{\lambda_m}(\lambda_{k+1})+4\binom{n-k}{2}m_{\lambda_m}(\lambda_{k+2})\\
&=(n-k)m_{\lambda_m}(\lambda_{k+1})+2(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}).
\end{aligned}$$

□

We are ready to provide the particular form of Fruthenthal's multiplicity recursive formula for fundamental weights λ_k with highest weight λ_m where $1 \le k < m < n$. By Proposition 3.3.1, Theorem 3.3.10 and the formula (1.5.2), we obtain that

$$\begin{aligned}
&(m-k)(2n-m-k+1)m_{\lambda_m}(\lambda_k)\\
&\qquad\qquad =2(n-k)m_{\lambda_m}(\lambda_{k+1})+4(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}). \quad (3.3.11)
\end{aligned}$$

Next, we present the general formula for the multiplicities of λ_k with highest weight λ_m where $0 \le k < m < n$.

Proposition 3.3.12. *Let $m \in \mathbb{N}$ be such that $1 \le m < n$. Then*

$$m_{\lambda_m}(\lambda_{m-i}) = \binom{n-m+i}{\lfloor \frac{i}{2} \rfloor}$$

for all $1 \le i \le m$.

Proof. We prove by using the strong induction.

Basis step

We replace k by $m-1, m-2$ and $m-3$ in the formula (3.3.11), respectively. Then we obtain that

$$m_{\lambda_m}(\lambda_{m-1}) = 1 = \binom{n-m+1}{0}$$

$$\begin{aligned}
m_{\lambda_m}(\lambda_{m-2}) &= \frac{2(n-m+2)m_{\lambda_m}(\lambda_{m-1}) + 4(n-m+2)(n-m+1)m_{\lambda_m}(\lambda_m)}{2(2n-2m+2+1)} \\
&= \frac{(n-m+2)\big(1+2(n-m+1)\big)}{(2n-2m+2+1)} \\
&= n-m+2 \\
&= \binom{n-m+2}{\lfloor \frac{2}{2} \rfloor} \\
m_{\lambda_m}(\lambda_{m-3}) &= \frac{2(n-m+3)m_{\lambda_m}(\lambda_{m-2}) + 4(n-m+3)(n-m+2)m_{\lambda_m}(\lambda_{m-1})}{3(2n-2m+3+1)} \\
&= \frac{(n-m+3)(n-m+2)}{(n-m+2)} \\
&= n-m+3 \\
&= \binom{n-m+3}{\lfloor \frac{3}{2} \rfloor}
\end{aligned}$$

Induction step

Suppose that the statement is true for $1, \ldots, i-1$.

Case 1 i is even.

Then $i = 2b$ for some $b \in \mathbb{N}$, so

$$\begin{aligned}
&m_{\lambda_m}(\lambda_{m-2b}) \\
&= \frac{2(n-m+2b)m_{\lambda_m}(\lambda_{m-2b+1}) + 4(n-m+2b)(n-m+2b-1)m_{\lambda_m}(\lambda_{m-2b+2})}{2b(2n-2m+2b+1)}
\end{aligned}$$

$$= \frac{2(n-m+2b)\binom{n-m+2b-1}{b-1} + 4(n-m+2b)(n-m+2b-1)\binom{n-m+2b-2}{b-1}}{2b(2n-2m+2b+1)}$$

$$= \frac{(n-m+2b)! + (n-m+2b)!(2n-2m+2b)}{(n-m+b)!b!(2n-2m+2b+1)}$$

$$= \frac{(n-m+2b)!}{(n+m+b)!b!} = \binom{n-m+2b}{b} = \binom{n-m+2b}{\lfloor \frac{2b}{2} \rfloor}.$$

Case 2 i is odd.

Then $i = 2b+1$ for some $b \in \mathbb{Z}_0^+$, so

$$m_{\lambda_m}(\lambda_{m-(2b+1)})$$

$$= \frac{2(n-m+2b+1)m_{\lambda_m}(\lambda_{m-2b}) + 4(n-m+2b+1)(n-m+2b)m_{\lambda_m}(\lambda_{m-2b+1})}{(2b+1)(2n-2m+2b+1+1)}$$

$$= \frac{2(n-m+2b+1)\binom{n-m+2b}{b} + 4(n-m+2b+1)(n-m+2b)\binom{n-m+2b-1}{b-1}}{(2b+1)(2n-2m+2b+1+1)}$$

$$= \frac{(n-m+2b+1)! + 2b(n-m+b+1)!}{(2b+1)(n-m+2b+1)!b!}$$

$$= \frac{(n-m+2b+1)!}{(n-m+b+1)!b!} = \binom{n-m+2b+1}{b} = \binom{n-m+2b+1}{\lfloor \frac{2b+1}{2} \rfloor}.$$

Thus, for each $1 \leq i \leq m$, we conclude that

$$m_{\lambda_m}(\lambda_{m-i}) = \binom{n-m+i}{\lfloor \frac{i}{2} \rfloor}.$$

□

Proposition 3.3.13. *The fundamental weight λ_n is nonzero minimal.*

Proof. Remember from Section 3.1 that $\lambda_n = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ so that λ_n is nonzero.

We show that λ_n is minuscule. Let $\alpha \in \Phi$. Then $\alpha = \pm\epsilon_r$ or $\pm(\epsilon_r \pm \epsilon_s)$ where $1 \leq r < s \leq n$.

Case 1 $\alpha = \pm\epsilon_r$

We consider that

$$\left(\left(\frac{1}{2},\dots,\frac{1}{2}\right), \pm\epsilon_r^\vee\right) = 2\left(\left(\frac{1}{2},\dots,\frac{1}{2}\right), \pm\epsilon_r\right) \in \{0, \pm 1\}.$$

Case 2 $\alpha = \pm(\epsilon_r \pm \epsilon_s)$

We consider that

$$\left(\left(\frac{1}{2},\dots,\frac{1}{2}\right), \pm(\epsilon_r \pm \epsilon_s)^\vee\right) = \left(\left(\frac{1}{2},\dots,\frac{1}{2}\right), \pm(\epsilon_r \pm \epsilon_s)\right) \in \{0, \pm 1\}.$$

From these two cases, we obtain that $(\lambda_n, \alpha^\vee) \in \{0, \pm 1\}$. Then λ_n is minuscule, so λ_n is minimal from Theorem 1.2.14. $\square$

Recall that $\lambda_0 = 0$.

Theorem 3.3.14. *Let m be a positive integer such that $m \leq n$. Then*

$$\chi_{\lambda_m} = \begin{cases} S(e^{\lambda_m}), & \textit{if } m = n, \\ \displaystyle\sum_{i=0}^{m} \binom{n-m+i}{\lfloor \frac{i}{2} \rfloor} S(e^{\lambda_{m-i}}), & \textit{if } m < n. \end{cases}$$

Proof. Recall from Theorem 1.5.7 that

$$\chi_{\lambda_m} = S(e^{\lambda_m}) + \sum_{\substack{\mu \in \Lambda^+ \\ \mu \prec \lambda_m}} m_{\lambda_m}(\mu) S(e^{\mu}).$$

Proposition 3.3.13 shows that λ_n is minimal in its subposet of $(\Lambda^+, \prec)$ so that $\chi_{\lambda_n} = S(e^{\lambda_n})$.

Next, let $m < n$. Then the dominant weights which are less than λ_m are $\lambda_0, \lambda_1, \dots, \lambda_{m-1}$ since $\lambda_0 \prec \lambda_1 \prec \cdots \prec \lambda_{m-1} \prec \lambda_m$. Moreover, Proposition 3.3.12

assures us that

$$\begin{aligned}
\chi_{\lambda_m} &= S(e^{\lambda_m}) + \sum_{i=1}^{m} \binom{n-m+i}{\lfloor \frac{i}{2} \rfloor} S(e^{\lambda_{m-i}}) \\
&= \sum_{i=0}^{m} \binom{n-m+i}{\lfloor \frac{i}{2} \rfloor} S(e^{\lambda_{m-i}}).
\end{aligned}$$

□

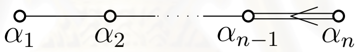
CHAPTER IV

ROOT SYSTEM C_n

In this chapter, we fix $n \in \mathbb{N} \setminus \{1, 2\}$. We consider $\mathbb{R}^n$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis vectors of $\mathbb{R}^n$. The $\mathbb{Z}$-span of this basis is a lattice, denoted by I. Let $\Phi' = \big\{\alpha \in I : (\alpha, \alpha) = 1 \text{ or } 2\big\}$.

4.1. Outline of the Root System C_n

Let $\Phi = \big\{\alpha^\vee : \alpha \in \Phi'\big\}$. Note that the squared length of an element of Φ is 2 or 4.

- Φ is root system in $\mathbb{R}^n$ of rank n.
- Dynkin diagram is 

- Short roots are

$$\pm(\epsilon_r \pm \epsilon_s) \text{ (of squared length 2)} \quad \text{for all } 1 \le r < s \le n.$$

Long roots are

$$\pm 2\epsilon_r \text{ (of squared length 4)} \quad \text{for all } 1 \le r \le n.$$

- Simple roots are

$$\alpha_r = \epsilon_r - \epsilon_{r+1} = (0, \ldots, 0, \overset{r\text{th}}{1}, -1, 0, \ldots, 0) \quad \text{for all } 1 \le r \le n-1 \quad \text{and}$$

$$\alpha_n = 2\epsilon_n = (0, \ldots, 0, 2).$$

- Positive roots are

$$\epsilon_r - \epsilon_s = \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1}, \qquad \text{for all } 1 \le r < s \le n,$$

$$\epsilon_r + \epsilon_s = \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1}$$

$$+ 2\alpha_s + 2\alpha_{s+1} + \cdots + 2\alpha_{n-1} + 2\alpha_n, \qquad \text{for all } 1 \le r < s \le n, \text{ and}$$

$$2\epsilon_r = 2\alpha_r + 2\alpha_{r+1} + \cdots + 2\alpha_{n-1} + 2\alpha_n, \qquad \text{for all } 1 \le r \le n.$$

- Fundamental weights are

$$\begin{aligned}\lambda_r &= \epsilon_1 + \epsilon_2 + \cdots + \epsilon_r \\ &= \alpha_1 + 2\alpha_2 + \cdots + (r-1)\alpha_{r-1} + r\left(\alpha_r + \alpha_{r+1} + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n\right) \\ &= (\underbrace{1, \ldots, 1}_{r \text{ terms}}, 0, \ldots, 0) \qquad \text{for all } 1 \le r \le n.\end{aligned}$$

- Let $\lambda_0 = 0$. Then

$$\lambda_0 \prec \lambda_2 \prec \cdots \prec \lambda_{2b} \qquad \text{for all } 2b \le n$$

and

$$\lambda_1 \prec \lambda_3 \prec \cdots \prec \lambda_{2b+1} \qquad \text{for all } 2b+1 \le n.$$

Moreover, if i is odd (even) and j is even (odd) then $\lambda_i \not\prec \lambda_j$ and $\lambda_j \not\prec \lambda_i$.

4.2. Root System C_3

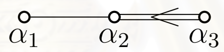
We give the example of a root system whose Dynkin diagram is C_3. We consider $\mathbb{R}^3$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$ and $\alpha_3 = (0, 0, 2)$. Then

$$\Phi = \big\{\pm(2\alpha_1 + 2\alpha_2 + \alpha_3), \pm(\alpha_1 + 2\alpha_2 + \alpha_3), \pm(\alpha_1 + \alpha_2 + \alpha_3), \pm(\alpha_1 + \alpha_2), \pm\alpha_1,$$
$$\pm (2\alpha_2 + \alpha_3), \pm(\alpha_2 + \alpha_3), \pm\alpha_2, \pm\alpha_3\big\}$$
$$= \big\{\pm(2,0,0), \pm(1,1,0), \pm(1,0,1), \pm(1,0,-1), \pm(1,-1,0), \pm(0,2,0), \pm(0,1,1),$$
$$\pm (0,1,-1), \pm(0,0,2)\big\}$$

is a root system in $\mathbb{R}^3$ and

$$\Delta = \big\{\alpha_1, \alpha_2, \alpha_3\big\} = \big\{(1,-1,0), (0,1,-1), (0,0,2)\big\}$$

is a base of Φ. The Dynkin diagram is



We can see that positive roots are

$$\alpha_1 = (1,-1,0) \qquad \alpha_1 + \alpha_2 = (1,0,-1) \qquad \alpha_1 + \alpha_2 + \alpha_3 = (1,0,1)$$
$$\alpha_2 = (0,1,-1) \qquad \alpha_2 + \alpha_3 = (0,1,1) \qquad \alpha_1 + 2\alpha_2 + \alpha_3 = (1,1,0)$$
$$\alpha_3 = (0,0,2) \qquad 2\alpha_2 + \alpha_3 = (0,2,0) \qquad 2\alpha_1 + 2\alpha_2 + \alpha_3 = (2,0,0),$$

so $\delta = (3,2,1)$. The fundamental weights are

$$\lambda_1 = (1,0,0)\,, \qquad \lambda_2 = (1,1,0) \qquad \text{and} \qquad \lambda_3 = (1,1,1)\,.$$

We know that

$$\sigma_{\alpha_1}: \begin{array}{l} \alpha_1 \mapsto -\alpha_1, \\ \alpha_2 \mapsto \alpha_1+\alpha_2, \\ \alpha_3 \mapsto \alpha_3, \end{array} \qquad \sigma_{\alpha_2}: \begin{array}{l} \alpha_1 \mapsto \alpha_1+\alpha_2, \\ \alpha_2 \mapsto -\alpha_2, \\ \alpha_3 \mapsto 2\alpha_2+\alpha_3, \end{array} \qquad \sigma_{\alpha_3}: \begin{array}{l} \alpha_1 \mapsto \alpha_1, \\ \alpha_2 \mapsto \alpha_2+\alpha_3, \\ \alpha_3 \mapsto -\alpha_3. \end{array}$$

Moreover,

$$\sigma_{\alpha_1}: \begin{array}{l} \lambda_1 \mapsto -\lambda_1+\lambda_2, \\ \lambda_2 \mapsto \lambda_2, \\ \lambda_3 \mapsto \lambda_3, \end{array} \qquad \sigma_{\alpha_2}: \begin{array}{l} \lambda_1 \mapsto \lambda_1, \\ \lambda_2 \mapsto \lambda_1-\lambda_2+\lambda_3, \\ \lambda_3 \mapsto \lambda_3, \end{array} \qquad \sigma_{\alpha_3}: \begin{array}{l} \lambda_1 \mapsto \lambda_1, \\ \lambda_2 \mapsto \lambda_2, \\ \lambda_3 \mapsto 2\lambda_2-\lambda_3. \end{array}$$

The Weyl group $\mathcal{W}$ of Φ is $\langle \sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_3} \rangle$ consisting of 48 elements.

Next, for each fundamental weight λ, we consider the elementary symmetric sum $S(e^{\lambda})$ and the character χ_λ with highest weight λ (see also Definition 1.2.20). We see that

$$\begin{aligned} \mathcal{W}\lambda_1 &= \{-\lambda_1, \lambda_1, \lambda_1-\lambda_2, \lambda_2-\lambda_1, \lambda_2-\lambda_3, \lambda_3-\lambda_2\}, \\ \mathcal{W}\lambda_2 &= \{-\lambda_2, 2\lambda_1-\lambda_2, \lambda_2, -2\lambda_1+\lambda_2, \lambda_1-\lambda_3, -\lambda_1+\lambda_2-\lambda_3, \lambda_1+\lambda_2-\lambda_3, \\ &\qquad -\lambda_1+2\lambda_2-\lambda_3, -\lambda_1+\lambda_3, \lambda_1-2\lambda_2+\lambda_3, -\lambda_1-\lambda_2+\lambda_3, \lambda_1-\lambda_2+\lambda_3\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}\lambda_3 = \{&-\lambda_3, 2\lambda_1-\lambda_3, 2\lambda_2-\lambda_3, -2\lambda_1+2\lambda_2-\lambda_3, \lambda_3, -2\lambda_1+\lambda_3, -2\lambda_2+\lambda_3, \\ &2\lambda_1-2\lambda_2+\lambda_3\}. \end{aligned}$$

Then

$$S(e^{\lambda_1}) = e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{-\lambda_1+\lambda_2} + e^{\lambda_2-\lambda_3} + e^{-\lambda_2+\lambda_3},$$

$$S(e^{\lambda_2}) = e^{-\lambda_2} + e^{2\lambda_1-\lambda_2} + e^{\lambda_2} + e^{-2\lambda_1+\lambda_2} + e^{\lambda_1-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} + e^{\lambda_1+\lambda_2-\lambda_3}$$
$$+ e^{-\lambda_1+2\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_3} + e^{\lambda_1-2\lambda_2+\lambda_3} + e^{-\lambda_1-\lambda_2+\lambda_3} + e^{\lambda_1-\lambda_2+\lambda_3},$$

and

$$S(e^{\lambda_3}) = e^{-\lambda_3} + e^{2\lambda_1-\lambda_3} + e^{2\lambda_2-\lambda_3} + e^{-2\lambda_1+2\lambda_2-\lambda_3} + e^{\lambda_3} + e^{-2\lambda_1+\lambda_3}$$
$$+ e^{-2\lambda_2+\lambda_3} + e^{2\lambda_1-2\lambda_2+\lambda_3}.$$

We calculate that

$$\chi_{\lambda_1} = e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{-\lambda_1+\lambda_2} + e^{\lambda_2-\lambda_3} + e^{-\lambda_2+\lambda_3},$$

$$\chi_{\lambda_2} = 2 + e^{-\lambda_2} + e^{2\lambda_1-\lambda_2} + e^{\lambda_2} + e^{-2\lambda_1+\lambda_2} + e^{\lambda_1-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} + e^{\lambda_1+\lambda_2-\lambda_3}$$
$$+ e^{-\lambda_1+2\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_3} + e^{\lambda_1-2\lambda_2+\lambda_3} + e^{-\lambda_1-\lambda_2+\lambda_3} + e^{\lambda_1-\lambda_2+\lambda_3},$$

and

$$\chi_{\lambda_3} = e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{-\lambda_1+\lambda_2} + e^{-\lambda_3} + e^{2\lambda_2-\lambda_3} + e^{\lambda_2-\lambda_3} + e^{2\lambda_1-\lambda_3}$$
$$+ e^{-2\lambda_1+2\lambda_2-\lambda_3} + e^{\lambda_3} + e^{-2\lambda_1+\lambda_3} + e^{-2\lambda_2+\lambda_3} + e^{2\lambda_1-2\lambda_2+\lambda_3} + e^{-\lambda_2+\lambda_3}.$$

We observe that

$$\chi_{\lambda_1} = S(e^{\lambda_1}), \quad \chi_{\lambda_2} = S(e^{\lambda_2}) + 2 \quad \text{and} \quad \chi_{\lambda_3} = S(e^{\lambda_3}) + S(e^{\lambda_1}).$$

On the other hand, we remind from Theorem 1.5.7 that for $i = 1, 2$ and 3

$$\chi_{\lambda_i} = S(e^{\lambda_i}) + \sum_{\substack{\mu \in \Lambda^+ \\ \mu \prec \lambda_i}} m_{\lambda_i}(\mu) S(e^{\mu}).$$

We discover that a dominant weight μ such that $\mu \prec \lambda_2$ is only 0, a dominant weight μ such that $\mu \prec \lambda_3$ is only λ_1 and λ_1 is minimal. Moreover, if we calculate by Fruthenthal's recursive formula, then we obtain that $m_{\lambda_2}(0) = 2$ and $m_{\lambda_3}(\lambda_1) = 1$. Hence

$$\chi_{\lambda_1} = S(e^{\lambda_1}), \quad \chi_{\lambda_2} = S(e^{\lambda_2}) + 2 \quad \text{and} \quad \chi_{\lambda_3} = S(e^{\lambda_3}) + S(e^{\lambda_1}).$$

4.3. The Relation between $S(e^{\lambda_m})$ and χ_{λ_m}

Let Φ be the root system whose Dynkin diagram is C_n, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a base, $\lambda_1, \ldots, \lambda_n$ the fundamental weights described in Section 4.1 and $\mathcal{W}$ the Weyl group of Φ. We use the same technique used in the root system having Dynkin diagram C_n to find the desired relation. We can check that λ_n is not minimal. Hence we consider the case where $1 \leq m \leq n$.

Proposition 4.3.1. *Let m and k be nonnegative integers such that $0 \leq k < m \leq n$. Then*

$$(\lambda_m + \delta, \lambda_m + \delta) - (\lambda_k + \delta, \lambda_k + \delta) = (m - k)(2n - m - k + 2).$$

Proof. We see that $\lambda_m - \lambda_k = (0, \ldots, \overset{k+1^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0)$ and

$$\delta = (n, n-1, n-2, \ldots, n-i, \ldots, 2, 1) \qquad \text{where } 0 \leq i \leq n.$$

Then

$$
\begin{aligned}
(\lambda_m - \lambda_k, \delta) &= \sum_{i=k+1}^{m} (n+1-i) \\
&= (m-k)(n+1) - \left(\frac{m(m+1) - k(k+1)}{2}\right) \\
&= (m-k)\left(\frac{2n-m-k+1}{2}\right).
\end{aligned}
$$

Now, since $(\lambda_m, \lambda_m) = m$ and $(\lambda_k, \lambda_k) = k$, it follows that

$$
\begin{aligned}
(\lambda_m + \delta, \lambda_m + \delta) - (\lambda_k + \delta, \lambda_k + \delta) &= (\lambda_m, \lambda_m) - (\lambda_k, \lambda_k) + 2(\lambda_m - \lambda_k, \delta) \\
&= (m-k) + 2(m-k)\left(\frac{2n-m-k+1}{2}\right) \\
&= (m-k)(2n-m-k+2).
\end{aligned}
$$

$\square$

Note also that all positive roots are $\pm 2\epsilon_r$ where $1 \le r \le n$ or $\epsilon_r \pm \epsilon_s$ where $1 \le r < s \le n$.

Proposition 4.3.2. *Let k, m and r be nonnegative integers such that $1 \le r \le k < m \le n$ and $r < s$. Then $m_{\lambda_m}(\lambda_k + 2t\epsilon_r) = 0$ for all $t \in \mathbb{N}$.*

Proof. Let $t \in \mathbb{N}$. We know that

$$
\lambda_m - (\lambda_k + 2t\epsilon_r) = (0, \dots, 0, \overset{r^{\text{th}}}{1 - 2t}, 0, \dots, 0, \overset{(k+1)^{\text{th}}}{1}, 1, \dots, 1, \overset{m^{\text{th}}}{1}, 0, \dots, 0).
$$

Suppose that $\lambda_k + 2t\epsilon_r \preceq \lambda_m$. Then there exist $a_1, \dots, a_n \in \mathbb{N}_0$ such that

$$\lambda_m - (\lambda_k + 2t\epsilon_r) = \sum_{i=1}^{n} a_i\alpha_i = \sum_{i=1}^{n-1} a_i(\epsilon_i - \epsilon_{i+1}) + 2a_n\epsilon_n$$

$$= (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, \ldots, a_{n-1} - a_{n-2}, 2a_n - a_{n-1}).$$

That is

$$(0, \ldots, 0, \overset{r^{\text{th}}}{1 - 2t}, 0, \ldots, 0, \overset{(k+1)^{\text{th}}}{1}, 1, \ldots, 1, \overset{m^{\text{th}}}{1}, 0, \ldots, 0)$$

$$= (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, \ldots, a_{n-1} - a_{n-2}, 2a_n - a_{n-1}).$$

Then $a_1 = a_2 = \cdots = a_{r-1} = 0$ and $a_r = 1 - 2t$ which is a contradiction. Thus $\lambda_k + 2t\epsilon_r \not\preceq \lambda_m$. By Theorem 1.4.7, we conclude that $m_{\lambda_m}(\lambda_k + 2t\epsilon_r) = 0$.

$\square$

Proposition 4.3.3. *Let k, m, r and s be nonnegative integers such that $1 \leq r \leq k < m \leq n$ and $r < s$. Then $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$ for all $t \in \mathbb{N}$.*

Proof. Let $t \in \mathbb{N}$

Case 1 $r < s \leq k$

We know that

$$\lambda_k + t(\epsilon_r \pm \epsilon_s) = \big(1, \ldots, \overset{r^{\text{th}}}{(1 + t)}, 1, \ldots, \overset{s^{\text{th}}}{(1 \pm t)}, \overset{k^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0\big).$$

Therefore,

$$\lambda_m - (\lambda_k + t(\epsilon_r \pm \epsilon_s))$$

$$= (0, \ldots, 0, \overset{r^{\text{th}}}{-t}, 0, \ldots, 0, \overset{s^{\text{th}}}{\mp t}, 0, \ldots, 0, \overset{(k+1)^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, \ldots, 0).$$

By the similar argument to the proof of Proposition 4.3.2, we assert that

$$m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0.$$

Case 2 $r \leq k < s$

We use the same method as in Case 1 and obtain that

$$m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0.$$

□

Proposition 4.3.4. *Let b, c, k and m be nonnegative integers such that $1 \leq b < c \leq n-k$ and $m < n$. Then $m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+b} \pm \epsilon_{k+c})\big)$.*

Corollary 4.3.5. *We have that $t(\epsilon_{k+1} \pm \epsilon_{k+2})$ are conjugate to $t(\epsilon_{k+b} \pm \epsilon_{k+c})$ and $\lambda_k + t(\epsilon_{k+1} \pm \epsilon_{k+2})$ are conjugate to $\lambda_k + t(\epsilon_{k+b} \pm \epsilon_{k+c})$ for $1 \leq b < c \leq n-k$.*

Proposition 4.3.6. *Let k, m, r and s be nonnegative integers such that $0 \leq k < r < s \leq n$ and $t \in \mathbb{N}\backslash\{1\}$. Then $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$.*

Theorem 4.3.7. *Let k and m be nonnegative integers such that $0 \leq k < m \leq n$. Then*

$$\sum_{\alpha \in \Phi^+} \sum_{t=1}^{\infty} m_{\lambda_m}(\lambda_k + t\alpha)(\lambda_k + t\alpha, \alpha) = 2(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}).$$

Proof. Let α be a positive root and $t \in \mathbb{N}$. Then $\alpha = 2\epsilon_r$ for $1 \leq r \leq n$ or $\alpha = \epsilon_r \pm \epsilon_s$ for $1 \leq r < s \leq n$.

Case 1 $\alpha = 2\epsilon_r$ and $t \in \mathbb{N}$

It follows from Proposition 4.3.2 that $m_{\lambda_m}(\lambda_k + 2t\epsilon_r) = 0$.

Case 2 $\alpha = \epsilon_r \pm \epsilon_s$

(2.1) $r \leq k$ and $t \in \mathbb{N}$

It follows from Proposition 4.3.3 that $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s\big) = 0$.

(2.2) $r > k$ and $t \in \mathbb{N} \setminus \{1\}$

It follows from Proposition 4.3.6 that $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s\big) = 0$.

(2.3) $r > k$ and $t = 1$

By Proposition 4.3.4, we reach the fact that

$$m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+1} + \epsilon_{k+2})\big) = m_{\lambda_m}\big(\lambda_k + (\epsilon_r \pm \epsilon_s\big).$$

We conclude from all the cases that

$$\begin{aligned}\sum_{\alpha\in\Phi^+}\sum_{t=1}^{\infty} m_{\lambda_m}(\lambda_k + t\alpha)(\lambda_k + t\alpha, \alpha) &= \sum_{r=k+1}^{n-1}\sum_{s=r+1}^{n} m_{\lambda_m}(\lambda_k + \epsilon_r \pm \epsilon_s)(\lambda_k + \epsilon_r \pm \epsilon_s, \epsilon_r \pm \epsilon_s) \\ &= 4\binom{n-k}{2} m_{\lambda_m}(\lambda_{k+2}) \\ &= 2(n-k)(n-k-1) m_{\lambda_m}(\lambda_{k+2}).\end{aligned}$$

□

We are ready to provide the reduce form of Fruthenthal's multiplicity recursive formula for fundamental weights λ_k with highest weight λ_m where $1 \leq k < m < n$. By Proposition 4.3.1, Theorem 4.3.7 and the formula (1.5.2), we obtain that

$$(m-k)(2n-m-k+2) m_{\lambda_m}(\lambda_k) = 4(n-k)(n-k-1) m_{\lambda_m}(\lambda_{k+2}). \qquad (4.3.8)$$

Next, we present the general formula for multiplicity of λ_k, where $k = m - 2i$ for some i, with highest weight λ_m by using the formula (4.3.8).

Lemma 4.3.9. *Let i and $m \in \mathbb{N}$ be such that $1 \leq 2i \leq m < n-1$. Then*

$$m_{\lambda_m}(\lambda_{m-2i}) = \binom{n-m+2i}{i}\frac{n-m+1}{n-m+i+1}.$$

Proof. We prove by strong induction. Recall that $m_{\lambda_m}(\lambda_m) = 1$

Basis step

We replace k by $m-2$ in the formula (4.3.8), then

$$m_{\lambda_m}(\lambda_{m-2}) = \frac{4(n-m+2)(n-m+1)m_{\lambda_m}(\lambda_m)}{2(2n-2m+4)}$$

$$= n-m-1$$

$$= \binom{n-m+2}{1}\frac{n-m+1}{n-m+1+1}$$

Induction step

Suppose that the statement is true for $1, \ldots, i-1$. Then

$$\begin{aligned} m_{\lambda_m}(\lambda_{m-2i}) &= \frac{4(n-m+2i)(n-m+2i-1)m_{\lambda_m}(\lambda_{m-2i+2})}{2i(2n-2m+2i+2)} \\ &= \frac{(n-m+2i)(n-m+2i-1)\binom{n-m+2i-2}{i-1}(n-m+1)}{i(n-m+i+1)(n-m+i)} \\ &= \frac{(n-m+2i)!(n-m+1)}{i!(n-m+i)!(n-m+i+1)} \\ &= \binom{n-m+2i}{i}\frac{n-m+1}{n-m+i+1} \end{aligned}$$

□

Theorem 4.3.10. *Let m be a positive integer such that $m \leq n$. Then*

$$\chi_{\lambda_m} = \sum_{0 \leq 2i \leq m} \binom{n-m+2i}{i}\frac{n-m+1}{n-m+i+1} S(e^{\lambda_{m-2i}}).$$

Proof. Recall from Theorem 1.5.7 that

$$\chi_{\lambda_m} = S(e^{\lambda_m}) + \sum_{\substack{\mu \in \Lambda^+ \\ \mu \prec \lambda_m}} m_{\lambda_m}(\mu) S(e^{\mu}).$$

We know that the dominant weights which are less than λ_m are λ_{m-2i} for some $i \in \mathbb{N}$ since

$$\lambda_0 \prec \lambda_2 \prec \cdots \prec \lambda_{2b} \qquad\qquad \text{for all } 2b \leq m$$

or

$$\lambda_1 \prec \lambda_3 \prec \cdots \prec \lambda_{2b+1} \qquad\qquad \text{for all } 2b+1 \leq m.$$

Moreover, Proposition 4.3.9 assures us that

$$\begin{aligned}\chi_{\lambda_m} &= S(e^{\lambda_m}) + \sum_{2\leq 2i\leq m} \binom{n-m+2i}{i} \frac{n-m+1}{n-m+i+1} S(e^{\lambda_{m-2i}}) \\ &= \sum_{0\leq 2i\leq m} \binom{n-m+2i}{i} \frac{n-m+1}{n-m+i+1} S(e^{\lambda_{m-2i}}).\end{aligned}$$

$\square$

CHAPTER V

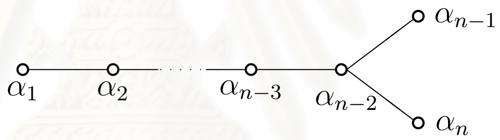
ROOT SYSTEM D_n

In this chapter, we fix $n \in \mathbb{N}\backslash\{1, 2, 3\}$. We consider $\mathbb{R}^n$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis vectors of $\mathbb{R}^n$. The $\mathbb{Z}$-span of this basis is a lattice, denoted by I.

5.1. Outline of the Root System D_n

Let $\Phi = \{\alpha \in I : (\alpha, \alpha) = 2\}$. Note that all elements of Φ have the same length.

- Φ is a root system in $\mathbb{R}^n$ of rank n.
- Dynkin diagram is



- Roots are

$$\pm(\epsilon_r \pm \epsilon_s) \quad \text{for all } 1 \leq r < s \leq n.$$

- Simple roots are

$$\alpha_r = \epsilon_r - \epsilon_{r+1} = (0, \ldots, 0, \overset{r\text{th}}{1}, -1, 0, \ldots, 0) \text{ for all } 1 \leq r \leq n-1 \text{ and}$$

$$\alpha_n = \epsilon_{n-1} + \epsilon_n = (0, \ldots, 0, 1, 1).$$

- Positive roots are

$$\begin{aligned}
\epsilon_r - \epsilon_s &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1}, && \text{for all } 1 \leq r < s \leq n, \\
\epsilon_r + \epsilon_n &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_n && \text{for all } 1 \leq r \leq n-1, \\
\epsilon_r + \epsilon_s &= \alpha_r + \alpha_{r+1} + \cdots + \alpha_{s-1} + 2\alpha_s \\
&\qquad + 2\alpha_{s+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n && \text{for all } 1 \leq r < s \leq n-1.
\end{aligned}$$

- Fundamental weights are

$$\begin{aligned}
\lambda_r &= \epsilon_1 + \epsilon_2 + \cdots + \epsilon_r \\
&= \alpha_1 + 2\alpha_2 + \cdots + (r-1)\alpha_{r-1} + r(\alpha_r + \alpha_{r+1} + \cdots + \alpha_{n-2}) \\
&\qquad + \frac{r}{2}(\alpha_{n-1} + \alpha_n) \\
&= (\underbrace{1, \ldots, 1}_{r \text{ terms}}, 0, \ldots, 0) \quad \text{for all } 1 \leq r \leq n-2,
\end{aligned}$$

$$\begin{aligned}
\lambda_{n-1} &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} - \epsilon_n) \\
&= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n\right) \\
&= \left(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}\right), \text{ and} \\
\lambda_n &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + \epsilon_n) \\
&= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n\right) \\
&= \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\right).
\end{aligned}$$

- Let $\lambda_0 = 0$. Then

$$\lambda_0 \prec \lambda_2 \prec \cdots \prec \lambda_{2b} \qquad\qquad \text{for all } 2b \leq n-2$$

and

$$\lambda_1 \prec \lambda_3 \prec \cdots \prec \lambda_{2b+1} \qquad\qquad \text{for all } 2b+1 \leq n-2.$$

Moreover, if i is odd (even) and j is even (odd) then $\lambda_i \not\prec \lambda_j$ and $\lambda_j \not\prec \lambda_i$.

5.2. Root System D_4

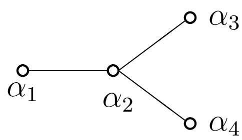
We give an example of a root system whose Dynkin diagram is D_4. We consider $\mathbb{R}^4$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $\alpha_1 = (1,-1,0,0)$, $\alpha_2 = (0,1,-1,0)$, $\alpha_3 = (0,0,1,-1)$ and $\alpha_4 = (0,0,1,1)$. Then

$$\begin{aligned}\Phi = \big\{&\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4), \pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \pm(\alpha_1+\alpha_2+\alpha_4), \pm(\alpha_1+\alpha_2+\alpha_3)\\ &\pm(\alpha_1+\alpha_2), \pm\alpha_1, \pm(\alpha_2+\alpha_3+\alpha_4), \pm(\alpha_2+\alpha_4), \pm(\alpha_2+\alpha_3), \pm\alpha_2, \pm\alpha_4, \pm\alpha_3\big\}\\ = \big\{&\pm(1,1,0,0), \pm(1,0,1,0), \pm(1,0,0,1), \pm(1,0,0,-1) \pm(1,0,-1,0), \pm(1,-1,0,0),\\ &\pm(0,1,1,0), \pm(0,1,0,1), \pm(0,1,0,-1), \pm(0,1,-1,0), \pm(0,0,1,1), \pm(0,0,1,-1)\big\}.\end{aligned}$$

is a root system in $\mathbb{R}^4$ and

$$\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{(1,-1,0,0), (0,1,-1,0), (0,0,1,-1), (0,0,1,1)\}$$

is a base of Φ. The Dynkin diagram is



We can see that positive roots are

$$\begin{array}{lll}
\alpha_1 = (1,-1,0,0) & \alpha_1+\alpha_2 = (1,0,-1,0) & \alpha_1+\alpha_2+\alpha_3 = (1,0,0,-1) \\
\alpha_2 = (0,1,-1,0) & \alpha_2+\alpha_3 = (0,1,0,-1) & \alpha_2+\alpha_3+\alpha_4 = (0,1,1,0) \\
\alpha_3 = (0,0,1,-1) & \alpha_2+\alpha_4 = (0,1,0,1) & \alpha_1+\alpha_2+\alpha_4 = (1,0,0,1) \\
\alpha_4 = (0,0,1,1) & &
\end{array}$$

$$\alpha_1+2\alpha_2+\alpha_3+\alpha_4 = (1,1,0,0) \qquad \text{and} \qquad \alpha_1+\alpha_2+\alpha_3+\alpha_4 = (1,0,1,0).$$

We obtain that $\delta = (3,2,1,0)$. The fundamental weights are

$$\lambda_1 = (1,0,0,0)\,,\ \lambda_2 = (1,1,0,0)\,,\ \lambda_3 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),\ \text{and } \lambda_4 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

We know that

$$\sigma_{\alpha_1}: \begin{cases} \alpha_1 & \mapsto -\alpha_1 \\ \alpha_2 & \mapsto \alpha_1+\alpha_2 \\ \alpha_3 & \mapsto \alpha_3 \\ \alpha_4 & \mapsto \alpha_4 \end{cases} \qquad \sigma_{\alpha_2}: \begin{cases} \alpha_1 & \mapsto \alpha_1+\alpha_2 \\ \alpha_2 & \mapsto -\alpha_2 \\ \alpha_3 & \mapsto \alpha_2+\alpha_3 \\ \alpha_4 & \mapsto \alpha_2+\alpha_4 \end{cases}$$

$$\sigma_{\alpha_3}: \begin{cases} \alpha_1 & \mapsto \alpha_1 \\ \alpha_2 & \mapsto \alpha_2+\alpha_3 \\ \alpha_3 & \mapsto -\alpha_3 \\ \alpha_4 & \mapsto \alpha_4 \end{cases} \qquad \sigma_{\alpha_4}: \begin{cases} \alpha_1 & \mapsto \alpha_1 \\ \alpha_2 & \mapsto \alpha_2+\alpha_4 \\ \alpha_3 & \mapsto \alpha_3 \\ \alpha_4 & \mapsto -\alpha_4 \end{cases}$$

Moreover,

$$\sigma_{\alpha_1}:\begin{cases}\lambda_1 & \mapsto -\lambda_1+\lambda_2\\ \lambda_2 & \mapsto \lambda_2\\ \lambda_3 & \mapsto \lambda_3\\ \lambda_4 & \mapsto \lambda_4\end{cases}\qquad \sigma_{\alpha_2}:\begin{cases}\lambda_1 & \mapsto \lambda_1\\ \lambda_2 & \mapsto \lambda_1-\lambda_2+\lambda_3+\lambda_4\\ \lambda_3 & \mapsto \lambda_3\\ \lambda_4 & \mapsto \lambda_4\end{cases}$$

$$\sigma_{\alpha_3}:\begin{cases}\lambda_1 & \mapsto \lambda_1\\ \lambda_2 & \mapsto \lambda_2\\ \lambda_3 & \mapsto \lambda_2-\lambda_3\\ \lambda_4 & \mapsto \lambda_4\end{cases}\qquad \sigma_{\alpha_4}:\begin{cases}\lambda_1 & \mapsto \lambda_1\\ \lambda_2 & \mapsto \lambda_2\\ \lambda_3 & \mapsto \lambda_3\\ \lambda_4 & \mapsto \lambda_2-\lambda_4\end{cases}$$

The Weyl group $\mathcal{W}$ of Φ is $\langle\sigma_{\alpha_1},\sigma_{\alpha_2},\sigma_{\alpha_3},\sigma_{\alpha_4}\rangle$ consisting of 192 elements.

Next, for each fundamental weight λ, we consider the elementary symmetric sum $S(e^{\lambda})$ and the character χ_λ with highest weight λ (see also Definition 1.2.20). We calculate that

$$S(e^{\lambda_1}) = e^{-\lambda_1}+e^{\lambda_1}+e^{\lambda_1-\lambda_2}+e^{-\lambda_1+\lambda_2}+e^{\lambda_2-\lambda_3-\lambda_4}+e^{\lambda_3-\lambda_4}+e^{-\lambda_3+\lambda_4}+e^{-\lambda_2+\lambda_3+\lambda_4},$$

$$\begin{aligned}S(e^{\lambda_2}) = {}& e^{-\lambda_2}+e^{2\lambda_1+\lambda_2}+e^{\lambda_2}+e^{-2\lambda_1+\lambda_2}+e^{\lambda_2-2\lambda_3}+e^{-\lambda_2+2\lambda_3}+e^{\lambda_2-2\lambda_4}+e^{\lambda_1-\lambda_3-\lambda_4}\\ &+e^{-\lambda_1+\lambda_2-\lambda_3-\lambda_4}+e^{\lambda_1+\lambda_2-\lambda_3-\lambda_4}+e^{-\lambda_1+2\lambda_2-\lambda_3-\lambda_4}+e^{-\lambda_1+\lambda_3-\lambda_4}+e^{\lambda_1+\lambda_3-\lambda_4}\\ &+e^{\lambda_1-\lambda_2+\lambda_3-\lambda_4}+e^{-\lambda_1+\lambda_2+\lambda_3-\lambda_4}+e^{-\lambda_1-\lambda_3+\lambda_4}+e^{\lambda_1-\lambda_3+\lambda_4}+e^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}\\ &+e^{-\lambda_1+\lambda_2-\lambda_3+\lambda_4}+e^{-\lambda_1+\lambda_3+\lambda_4}+e^{\lambda_1-2\lambda_2+\lambda_3+\lambda_4}+e^{-\lambda_1-\lambda_2+\lambda_3+\lambda_4}\\ &+e^{\lambda_1-\lambda_2+\lambda_3+\lambda_4}+e^{-\lambda_2+2\lambda_4}\end{aligned}$$

$$S(e^{\lambda_3}) = e^{-\lambda_3}+e^{\lambda_2-\lambda_3}+e^{\lambda_3}+e^{-\lambda_2+\lambda_3}+e^{\lambda_1-\lambda_4}+e^{-\lambda_1+\lambda_2-\lambda_4}+e^{-\lambda_1+\lambda_4}+e^{\lambda_1-\lambda_2+\lambda_4},$$

and

$$S(e^{\lambda_4}) = e^{\lambda_1-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_3} + e^{\lambda_1-\lambda_2+\lambda_3} + e^{-\lambda_4} + e^{\lambda_2-\lambda_4} + e^{\lambda_4} + e^{-\lambda_2+\lambda_4}.$$

Also,

$$\begin{aligned}
\chi_{\lambda_1} &= e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{-\lambda_1+\lambda_2} + e^{\lambda_2-\lambda_3-\lambda_4} + e^{\lambda_3-\lambda_4} + e^{-\lambda_3+\lambda_4} + e^{-\lambda_2+\lambda_3+\lambda_4},\\
\chi_{\lambda_2} &= e^{-\lambda_2} + e^{2\lambda_1+\lambda_2} + e^{\lambda_2} + e^{-2\lambda_1+\lambda_2} + e^{\lambda_2-2\lambda_3} + e^{-\lambda_2+2\lambda_3} + e^{\lambda_2-2\lambda_4} + e^{\lambda_1-\lambda_3-\lambda_4}\\
&\quad + e^{-\lambda_1+\lambda_2-\lambda_3-\lambda_4} + e^{\lambda_1+\lambda_2-\lambda_3-\lambda_4} + e^{-\lambda_1+2\lambda_2-\lambda_3-\lambda_4} + e^{-\lambda_1+\lambda_3-\lambda_4} + e^{\lambda_1+\lambda_3-\lambda_4}\\
&\quad + e^{\lambda_1-\lambda_2+\lambda_3-\lambda_4} + e^{-\lambda_1+\lambda_2+\lambda_3-\lambda_4} + e^{-\lambda_1-\lambda_3+\lambda_4} + e^{\lambda_1-\lambda_3+\lambda_4} + e^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}\\
&\quad + e^{-\lambda_1+\lambda_2-\lambda_3+\lambda_4} + e^{-\lambda_1+\lambda_3+\lambda_4} + e^{\lambda_1-2\lambda_2+\lambda_3+\lambda_4} + e^{-\lambda_1-\lambda_2+\lambda_3+\lambda_4}\\
&\quad + e^{\lambda_1-\lambda_2+\lambda_3+\lambda_4} + e^{-\lambda_2+2\lambda_4} + e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{-\lambda_1+\lambda_2} + e^{\lambda_2-\lambda_3-\lambda_4}\\
&\quad + e^{\lambda_3-\lambda_4} + e^{-\lambda_3+\lambda_4} + e^{-\lambda_2+\lambda_3+\lambda_4} + 4,\\
\chi_{\lambda_3} &= e^{-\lambda_3} + e^{\lambda_2-\lambda_3} + e^{\lambda_3} + e^{-\lambda_2+\lambda_3} + e^{\lambda_1-\lambda_4} + e^{-\lambda_1+\lambda_2-\lambda_4} + e^{-\lambda_1+\lambda_4} + e^{\lambda_1-\lambda_2+\lambda_4}
\end{aligned}$$

and

$$\chi_{\lambda_4} = e^{\lambda_1-\lambda_3} + e^{-\lambda_1+\lambda_2-\lambda_3} + e^{-\lambda_1+\lambda_3} + e^{\lambda_1-\lambda_2+\lambda_3} + e^{-\lambda_4} + e^{\lambda_2-\lambda_4} + e^{\lambda_4} + e^{-\lambda_2+\lambda_4}.$$

We observe that

$$\chi_{\lambda_1} = S(e^{\lambda_1}),\ \chi_{\lambda_2} = S(e^{\lambda_2}) + S(e^{\lambda_1}) + 4,\ \chi_{\lambda_3} = S(e^{\lambda_3}) \text{ and } \chi_{\lambda_4} = S(e^{\lambda_4}).$$

On the other hand, we remind from Theorem 1.5.7 that for $i = 1, 2, 3$ and 4

$$\chi_{\lambda_i} = S(e^{\lambda_i}) + \sum_{\substack{\mu\in\Lambda^+\\ \mu\prec\lambda_i}} m_{\lambda_i}(\mu) S(e^{\mu}).$$

We discover that a dominant weight μ such that $\mu \prec \lambda_2$ is only 0 and λ_1, λ_3 and λ_4 are minimal. Moreover, if we calculate by Fruthenthal's recursive formula, then we obtain that $m_{\lambda_2}(0) = 4$. Thus

$$\chi_{\lambda_1} = S(e^{\lambda_1}),\ \chi_{\lambda_2} = S(e^{\lambda_2}) + S(e^{\lambda_1}) + 4,\ \chi_{\lambda_3} = S(e^{\lambda_3}) \text{ and } \chi_{\lambda_4} = S(e^{\lambda_4}).$$

5.3. The Relation between $S(e^{\lambda_m})$ and χ_{λ_m}

Now, we let Φ be the root system whose Dynkin diagram is D_n, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a base, $\lambda_1, \ldots, \lambda_n$ the fundamental weights described in Section 5.1 and $\mathcal{W}$ the Weyl group of Φ. First, we consider the case where $m \leq n-2$.

Proposition 5.3.1. *Let m and k be nonnegative integers such that $0 \leq k < m \leq n-2$. Then*

$$(\lambda_m + \delta, \lambda_m + \delta) - (\lambda_k + \delta, \lambda_k + \delta) = (m-k)(2n-m-k).$$

Proof. We see that $\lambda_m - \lambda_k = (0, \ldots, \overset{k+1^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0)$ and,

$$\delta = (n-1, n-2, n-3, \ldots, n-i, \ldots, 1, 0) \quad \text{where } 4 \leq i \leq n.$$

Thus,

$$\begin{aligned}
(\lambda_m - \lambda_k, \delta) &= \sum_{i=k+1}^{m} (n-i) \\
&= (m-k)n - \left(\frac{m(m+1) - k(k+1)}{2}\right) \\
&= (m-k)\left(\frac{2n-m-k-1}{2}\right).
\end{aligned}$$

Now, since $(\lambda_m, \lambda_m) = m$ and $(\lambda_k, \lambda_k) = k$, it follows that

$$(\lambda_m+\delta,\lambda_m+\delta)-(\lambda_k+\delta,\lambda_k+\delta) = (\lambda_m,\lambda_m)-(\lambda_k,\lambda_k)+2(\lambda_m-\lambda_k,\delta)$$
$$= (m-k)+2(m-k)\left(\frac{2n-m-k-1}{2}\right)$$
$$= (m-k)(2n-m-k).$$

□

Proposition 5.3.2. *Let k, m, r and s be nonnegative integers such that $1 \le r \le k < m \le n-2$ and $r < s$. Then $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$ for all $t \in \mathbb{N}$.*

Proof. Let $t \in \mathbb{N}$.

Case 1 $r < s \le k$

We know that

$$\lambda_k + t(\epsilon_r \pm \epsilon_s) = \big(1, \ldots, \overset{r^{\text{th}}}{(1+t)}, 1, \ldots, \overset{s^{\text{th}}}{(1 \pm t)}, \overset{k^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, 0, \ldots, 0\big).$$

Therefore,

$$\lambda_m - (\lambda_k + t(\epsilon_r \pm \epsilon_s))$$
$$= (0, \ldots, 0, \overset{r^{\text{th}}}{-t}, 0, \ldots, 0, \overset{s^{\text{th}}}{\mp t}, 0, \ldots, 0, \overset{(k+1)^{\text{th}}}{1}, \ldots, \overset{m^{\text{th}}}{1}, \ldots, 0).$$

Suppose that $\lambda_k + t(\epsilon_r \pm \epsilon_s) \preceq \lambda_m$. Then there exist $a_1, \ldots, a_n \in \mathbb{N}_0$ such that

$$\lambda_m - (\lambda_k + t(\epsilon_r \pm \epsilon_s)) = \sum_{i=1}^{n} a_i \alpha_n = \sum_{i=1}^{n-1} a_i(\epsilon_i - \epsilon_{i+1}) + a_n(\epsilon_{n-1} + \epsilon_n)$$
$$= (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, \ldots, 2a_{n-1} - a_{n-2}, a_n - a_{n-1}).$$

That is

$$(0,\ldots,0,\overset{r^{\text{th}}}{-t},0,\ldots,0,\overset{s^{\text{th}}}{\mp t},0,\ldots,0,\overset{(k+1)^{\text{th}}}{1},\ldots,\overset{m^{\text{th}}}{1},0,\ldots,0)$$

$$= (a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, \ldots, 2a_{n-1} + a_{n-2}, a_n - a_{n-1}).$$

Then, $a_1 = a_2 = \cdots = a_{r-1} = 0$ so that $a_r = -t$ which is a contradiction. Thus $\lambda_k + t(\epsilon_r \pm \epsilon_r) \not\preceq \lambda_m$. By Theorem 1.4.7, we conclude that

$$m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0.$$

Case 2 $r \leq k < s$

We use the same method as in Case 1 and obtain that

$$m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0.$$

□

Proposition 5.3.3. *Let b, c, k and m be nonnegative integers such that $1 \leq b < c \leq n-k$ and $k < m \leq n-2$. Then $m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+b} \pm \epsilon_{k+c})\big)$.*

Corollary 5.3.4. *We have that $t(\epsilon_{k+1} \pm \epsilon_{k+2})$ are conjugate to $t(\epsilon_{k+b} \pm \epsilon_{k+c})$ and $\lambda_k + t(\epsilon_{k+1} \pm \epsilon_{k+2})$ are conjugate to $\lambda_k + t(\epsilon_{k+b} \pm \epsilon_{k+c})$ for $1 \leq b < c \leq n-k$ and $k < m \leq n-2$.*

Proposition 5.3.5. *Let k, m, r and s be nonnegative integers such that $0 \leq k < r < s \leq n$ and $k < m \leq n-2$ for all $t \in \mathbb{N}\backslash\{1\}$. Then $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s)\big) = 0$.*

Theorem 5.3.6. *Let k and m be nonnegative integers such that $0 \leq k < m \leq n-2$. Then*

$$\sum_{\alpha\in\Phi^+}\sum_{t=1}^{\infty} m_{\lambda_m}(\lambda_k + t\alpha)(\lambda_k + t\alpha, \alpha) = 2(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}).$$

Proof. Let α be a positive root and $t \in \mathbb{N}$. Then $\alpha = \epsilon_r \pm \epsilon_s$ for $1 \leq r < s \leq n$.

Case 1 $r \leq k$ and $t \in \mathbb{N}$

It follows from Proposition 5.3.2 that $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s\big) = 0$.

Case 2 $r > k$ and $t \in \mathbb{N} \setminus \{1\}$

It follows from Proposition 5.3.5 that $m_{\lambda_m}\big(\lambda_k + t(\epsilon_r \pm \epsilon_s\big) = 0$.

Case 3 $r > k$ and $t = 1$

By Proposition 5.3.3, we reach the fact that

$$m_{\lambda_m}(\lambda_{k+2}) = m_{\lambda_m}\big(\lambda_k + (\epsilon_{k+1} + \epsilon_{k+2})\big) = m_{\lambda_m}\big(\lambda_k + (\epsilon_r \pm \epsilon_s)\big).$$

We conclude from all the cases that

$$\begin{aligned}\sum_{\alpha\in\Phi^+}\sum_{t=1}^{\infty} m_{\lambda_m}(\lambda_k + t\alpha)(\lambda_k + t\alpha, \alpha) &= \sum_{r=k+1}^{n-1}\sum_{s=r+1}^{n} m_{\lambda_m}(\lambda_k + \epsilon_r \pm \epsilon_s)(\lambda_k + \epsilon_r \pm \epsilon_s, \epsilon_r \pm \epsilon_s) \\ &= 4\binom{n-k}{2} m_{\lambda_m}(\lambda_{k+2}) \\ &= 2(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}).\end{aligned}$$

□

We are ready to provide the reduce form of Fruthenthal's multiplicity recursive formula for fundamental weights λ_k with highest weight λ_m where $1 \leq k < m \leq n-2$. By Proposition 5.3.1, Theorem 5.3.6 and the formula (1.5.2), we obtain that

$$(m-k)(2n-m-k)m_{\lambda_m}(\lambda_k) = 4(n-k)(n-k-1)m_{\lambda_m}(\lambda_{k+2}). \tag{5.3.7}$$

Next, we present the general formula for multiplicity of λ_k, where $k = m - 2i$ for some i, with highest weight λ_m by using the formula (5.3.7).

Lemma 5.3.8. *Let i and $m \in \mathbb{N}$ be such that $1 \leq 2i \leq m \leq n-2$. Then*

$$m_{\lambda_m}(\lambda_m - 2i) = \binom{n-m+2i}{i}.$$

Proof. We prove by strong induction. Recall that $m_{\lambda_m}(\lambda_m) = 1$.

Basis step

We replace k by $m-2$ in the formula (5.3.7), then

$$\begin{aligned} m_{\lambda_m}(\lambda_{m-2}) &= \frac{4(n-m+2)(n-m+1)m_{\lambda_m}(\lambda_m)}{2(2n-2m+2)} \\ &= n-m+2 = \binom{n-m+2}{1}. \end{aligned}$$

Induction step

Suppose that the statement is true for $1, \ldots, i-1$. Then

$$\begin{aligned} m_{\lambda_m}(\lambda_{m-2i}) &= \frac{4(n-m+2i)(n-m+2i-1)m_{\lambda_m}(\lambda_{m-2i+2})}{2i(2n-2m+2i)} \\ &= \frac{(n-m+2i)(n-m+2i-1)\binom{n-m+2i-2}{i-1}}{i(n-m+i)} \\ &= \frac{(n-m+2i)!}{(n-m+i)!i!} = \binom{n-m+2i}{i}. \end{aligned}$$

□

Lemma 5.3.9. *The fundamental weights λ_{n-1} and λ_n are minimal.*

Proof. Remember from Section 5.1 that $\lambda_{n-1} = \left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right)$ and $\lambda_n = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. We show that λ_{n-1} and λ_{n-1} are minuscule. Let $\alpha \in \Phi$. Then $\alpha = \pm(\epsilon_r \pm \epsilon_s)$ where $r \neq s$.

$$\left(\left(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2}\right),\pm(\epsilon_r\pm\epsilon_s)^\vee\right)=\left(\left(\frac{1}{2},\ldots,-\frac{1}{2}\right),\pm(\epsilon_r\pm\epsilon_s)\right)\in\{0,\pm1\}.$$

Hence, λ_{n-1} is minuscule.

$$\left(\left(\frac{1}{2},\ldots,\frac{1}{2}\right),\pm(\epsilon_r\pm\epsilon_s)^\vee\right)=\left(\left(\frac{1}{2},\ldots,\frac{1}{2}\right),\pm(\epsilon_r\pm\epsilon_s)\right)\in\{0,\pm1\}.$$

Therefore, λ_n is minuscule. We obtain that λ_{n-1} and λ_n are minimal from Theorem 1.2.14. □

Theorem 5.3.10. *Let m be a positive integer such that such that $m \leq n$. Then*

$$\chi_{\lambda_m}=\begin{cases} S(e^{\lambda_m}), & \text{if } m=n,n-1,\\ \displaystyle\sum_{0\leq 2i\leq m}\binom{n-m+2i}{i}S(e^{\lambda_{m-2i}}), & \text{if } m\leq n-2.\end{cases}$$

Proof. Recall from Theorem 1.5.7 that

$$\chi_{\lambda_m}=S(e^{\lambda_m})+\sum_{\substack{\mu\in\Lambda^+\\ \mu\prec\lambda_m}}m_{\lambda_m}(\mu)S(e^{\mu}).$$

Lemma 5.3.9 shows that λ_n and λ_{n-1} are minimal in its subposet of $(\Lambda^+,\prec)$, then $\chi_{\lambda_n}=S(e^{\lambda_n})$ and $\chi_{\lambda_{n-1}}=S(e^{\lambda_{n-1}})$. Next, let $m\leq n-2$. We know that the dominant weights which are less than λ_m are λ_{m-2i} for some $i\in\mathbb{N}$ since

$$\lambda_0\prec\lambda_2\prec\cdots\prec\lambda_{2b} \qquad \text{for all } 2b\leq m$$

or

$$\lambda_1\prec\lambda_3\prec\cdots\prec\lambda_{2b+1} \qquad \text{for all } 2b+1\leq m.$$

Moreover, Proposition 5.3.8 assures us that

$$\begin{aligned}\chi_{\lambda_m} &= S(e^{\lambda_m}) + \sum_{2\leq 2i\leq m}\binom{n-m+2i}{i}S(e^{\lambda_{m-2i}})\\ &= \sum_{0\leq 2i\leq m}\binom{n-m+2i}{i}S(e^{\lambda_{m-2i}}).\end{aligned}$$

We conclude that

$$\chi_{\lambda_m} = \begin{cases} S(e^{\lambda_m}), & \text{if } m = n, n-1, \\ \displaystyle\sum_{0\leq 2i\leq m}\binom{n-m+2i}{i}S(e^{\lambda_{m-2i}}), & \text{if } m \leq n-2. \end{cases}$$

□

CHAPTER VI

ROOT SYSTEM G_2

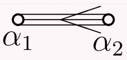
We consider $\mathbb{R}^3$ as the vector space over $\mathbb{R}$ with the usual inner product. Let $(1,0,0),(0,1,0)$ and $(0,0,1)$ be the standard basis vectors of $\mathbb{R}^3$. The $\mathbb{Z}$-span of this basis is a lattice, denoted by I. Let V be the 2-dimensional subspace of $\mathbb{R}^3$ orthogonal to the vector $(1,1,1)$ and $I' = I \cap V$.

6.1. Outline of the Root System G_2

Let $\Phi = \{\alpha \in I' : (\alpha,\alpha) = 2 \text{ or } 6\}$, i.e.,

$$\Phi = \{\pm(1,0,-1), \pm(1,-1,0), \pm(0,1,-1), \pm(-1,2,-1), \pm(2,-1,-1), \pm(-1,-1,2)\}.$$

Note that the squared length of an element of Φ is 2 or 6.

- Φ is a root system in V of rank 2.
- Dynkin diagram is 

- Short roots are

$$\pm(1,-1,0), \pm(1,0,-1) \text{ and } \pm(0,1,-1) \text{ (of squared length 2).}$$

Long roots are

$$\pm(2,-1,-1), \pm(-1,2,-1) \text{ and } \pm(-1,-1,2) \text{ (of squared length 6).}$$

- Simple roots are

$$\alpha_1 = (1,-1,0) \quad \text{and} \quad \alpha_2 = (-2,1,1).$$

- Positive roots are

$$\alpha_1 = (1,-1,0),\ \alpha_2 = (-2,1,1),\ \alpha_1+\alpha_2 = (-1,0,1),$$

$$2\alpha_1+\alpha_2 = (0,-1,1),\ 3\alpha_1+\alpha_2 = (1,-2,1) \text{ and } 3\alpha_1+2\alpha_2 = (-1,-1,2).$$

- Fundamental weights are

$$\lambda_1 = 2\alpha_1+\alpha_2 = (0,-1,1) \quad \text{and} \quad \lambda_2 = 3\alpha_1+2\alpha_2 = (-1,-1,2).$$

- Let $\lambda_0 = 0$. Then $\lambda_0 \prec \lambda_1 \prec \lambda_2$.

6.2. The Relation between $S(e^{\lambda_m})$ and χ_{λ_m}

In this section, we let Φ be the root system whose Dynkin diagram is G_2, $\Delta = \{\alpha_1,\alpha_2\}$ the base of Φ, and λ_1,λ_2 the fundamental weights described in Section 6.1. In addition, we know that $\delta = \lambda_1+\lambda_2 = (-1,-2,3)$. In order to determine $S(e^{\lambda_1})$ and $S(e^{\lambda_2})$, first, we find the Weyl group $\mathcal{W}$ of Φ. Remind that

$$\sigma_{\alpha_1} : \begin{array}{l} \alpha_1 \mapsto -\alpha_1, \\ \alpha_2 \mapsto 3\alpha_1+\alpha_2, \end{array} \qquad \text{and} \qquad \sigma_{\alpha_2} : \begin{array}{l} \alpha_1 \mapsto \alpha_1+\alpha_2, \\ \alpha_2 \mapsto -\alpha_2. \end{array}$$

Moreover,

$$\sigma_{\alpha_1} : \begin{array}{l} \lambda_1 \mapsto -\lambda_1+\lambda_2, \\ \lambda_2 \mapsto \lambda_1, \end{array} \qquad \text{and} \qquad \sigma_{\alpha_2} : \begin{array}{l} \lambda_1 \mapsto \lambda_1, \\ \lambda_2 \mapsto 3\lambda_1-\lambda_2. \end{array}$$

Then $\mathcal{W}$ consists of 12 elements as follows:

$$i_V, \sigma_\alpha, \sigma_\beta, \sigma_\alpha\sigma_\beta, \sigma_\beta\sigma_\alpha, \sigma_\alpha\sigma_\beta\sigma_\alpha, \sigma_\beta\sigma_\alpha\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta,$$

$$\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha, \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha, \sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta.$$

Next, for each fundamental weight λ, we consider the elementary symmetric sum $S(e^{\lambda})$ and the character χ_λ with highest weight λ (see also Definition 1.2.20). We see that

$$\mathcal{W}\lambda_1 = \{\lambda_1, \lambda_1 - \lambda_2, -2\lambda_1 + \lambda_2, -\lambda_1, 2\lambda_1 - \lambda_2, -\lambda_1 + \lambda_2\}$$

and

$$\mathcal{W}\lambda_2 = \{\lambda_2, 3\lambda_1 - 2\lambda_2, -3\lambda_1 + 2\lambda_2, -3\lambda_1 + \lambda_2, 3\lambda_1 - \lambda_2, -\lambda_2\}.$$

Then

$$S(e^{\lambda_1}) = e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{-2\lambda_1+\lambda_2} + e^{-\lambda_1} + e^{2\lambda_1-\lambda_2} + e^{-\lambda_1+\lambda_2}$$

and

$$S(e^{\lambda_2}) = e^{\lambda_2} + e^{3\lambda_1-2\lambda_2} + e^{-3\lambda_1+2\lambda_2} + e^{-3\lambda_1+\lambda_2} + e^{3\lambda_1-\lambda_2} + e^{-\lambda_2}.$$

We need the followings in order to calculate χ_{λ_1} and χ_{λ_2}.

$$\begin{aligned} A(e^{\lambda_1+\delta}) &= e^{5\lambda_1+4\lambda_2} - e^{7\lambda_1+4\lambda_2} - e^{2\lambda_1-3\lambda_2} + e^{7\lambda_1-3\lambda_2} + e^{-2\lambda_1-\lambda_2} - e^{5\lambda_1-\lambda_2} \\ &\quad - e^{-5\lambda_1+\lambda_2} + e^{2\lambda_1+\lambda_2} + e^{-7\lambda_1+3\lambda_2} - e^{-2\lambda_1+3\lambda_2} - e^{-7\lambda_1+4\lambda_2} + e^{-5\lambda_1+4\lambda_2}, \end{aligned}$$

$$\begin{aligned} A(e^{\lambda_2+\delta}) &= e^{7\lambda_1+5\lambda_2} - e^{8\lambda_1-5\lambda_2} - e^{\lambda_1-3\lambda_2} + e^{8\lambda_1-3\lambda_2} + e^{-\lambda_1-2\lambda_2} - e^{7\lambda_1-2\lambda_2} \\ &\quad - e^{-7\lambda_1+2\lambda_2} + e^{\lambda_1+2\lambda_2} + e^{-8\lambda_1+3\lambda_2} - e^{-\lambda_1+3\lambda_2} - e^{-8\lambda_1+5\lambda_2} + e^{-7\lambda_1+5\lambda_2}, \end{aligned}$$

and

$$\begin{aligned} A(e^{\delta}) &= e^{4\lambda_1+3\lambda_2} - e^{5\lambda_1-3\lambda_2} - e^{\lambda_1-2\lambda_2} + e^{5\lambda_1-2\lambda_2} + e^{-\lambda_1-2\lambda_2} - e^{4\lambda_1-\lambda_2} \\ &\quad - e^{-4\lambda_1+\lambda_2} + e^{\lambda_1+\lambda_2} + e^{-5\lambda_1+2\lambda_2} - e^{\lambda_1+2\lambda_2} - e^{-5\lambda_1+3\lambda_2} + e^{-4\lambda_1+3\lambda_2}. \end{aligned}$$

Thus,

$$\chi_{\lambda_1} = 1 + e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{2\lambda_1-\lambda_2} + e^{-2\lambda_1+\lambda_2} + e^{-\lambda_1+\lambda_2}$$

and

$$\chi_{\lambda_2} = 2 + e^{-\lambda_1} + e^{\lambda_1} + e^{\lambda_1-\lambda_2} + e^{2\lambda_1-\lambda_2} + e^{-2\lambda_1+\lambda_2} + e^{-\lambda_1+\lambda_2} + e^{\lambda_2}$$
$$+ e^{3\lambda_1-2\lambda_2} + e^{-3\lambda_1+2\lambda_2} + e^{-3\lambda_1+\lambda_2} + e^{3\lambda_1-\lambda_2} + e^{-\lambda_2}.$$

We conclude that

$$\chi_{\lambda_1} = S(e^{\lambda_1}) + 1 \qquad \text{and} \qquad \chi_{\lambda_2} = S(e^{\lambda_2}) + S(e^{\lambda_1}) + 2.$$

On the other hand, we remind from Theorem 1.5.7 that for $i = 1$ and 2

$$\chi_{\lambda_i} = S(e^{\lambda_i}) + \sum_{\substack{\mu\in\Lambda^+\\ \mu\prec\lambda_i}} m_{\lambda_i}(\mu) S(e^{\mu}).$$

We discover that a dominant weight μ such that $\mu \prec \lambda_1$ is only 0 and dominant weights μ such that $\mu \prec \lambda_2$ are $0, \lambda_1$, where $0 \prec \lambda_1$. Moreover, if we calculate by Fruthenthal's recursive formula, then we obtain that $m_{\lambda_1}(0) = 1, m_{\lambda_2}(\lambda_1) = 1$ and $m_{\lambda_2}(0) = 2$. Thus

$$\chi_{\lambda_1} = S(e^{\lambda_1}) + 1 \quad \text{and} \quad \chi_{\lambda_2} = S(e^{\lambda_2}) + S(e^{\lambda_1}) + 2.$$

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