CHAPTER III

SEMIFIELDS

In this chapter, we shall classify all 0-skew semifields up to isomorphism in Section 1. In Section 2 we shall give partial classifications of ∞-skew semifields.

Section 1. 0-Skew Semifields

<u>Definition 3.1.1.</u> A system $(K,+,\cdot,\leqslant)$ is called an <u>ordered 0-skew semifield</u> iff $(K,+,\cdot)$ is a 0-skew semifield and $\leqslant$ is an order on K satisfying the following properties:

(i) For any $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$,

(ii) For any $x, y \in K$, $x \leqslant y$ implies that $xz \leqslant yz$ and $zx \leqslant zy$ for all $z \geqslant 0$ in K and

(iii) $0 < 1$.

<u>Proposition 3.1.2</u> Let K be a 0-skew semifield. If there are $x, y \in K \smallsetminus \{0\}$ such that $x+y = 0$, then K is a skew field.

<u>Proof</u>: Assume that $x, y \in K \smallsetminus \{0\}$ are such that $x+y = 0$, let $z \in K$ be arbitrary. Since $x \in K \smallsetminus \{0\}$, x^{-1} exists. Therefore $zx^{-1}(x+y) = (zx^{-1})0 = 0$, it follows that $z+zx^{-1}y = 0$. Thus $zx^{-1}y$ is an additive inverse of z. Since $z \in K$ is arbitrary, b has an

additive inverse for every $b \varepsilon K$. Hence K is a skew field.#

Notation : Let K be an ordered 0-skew semifield. Then we will denote $D_K^+ = \{x \varepsilon K | x > 0\}$ and $D_K^- = \{x \varepsilon K | x < 0\}$. Note that $1 \varepsilon D_K^+$, so D_K^+ is never the empty set.

The following remarks follow immediately from Definition 3.1.1.

Remarks : 1) $x \varepsilon D_K^+$ and $y \varepsilon D_K^-$ imply that $xy \varepsilon D_K^-$,

2) $x \varepsilon D_K^-$ implies that $x^{-1} \varepsilon D_K^-$.

Proposition 3.1.3. Let K be a complete ordered 0-skew semifield which is not a skew field. Then D_K^+ is a complete ordered skew ratio semiring and D_K^- is a complete ordered semigroup with respect to addition if $D_K^- \neq \emptyset$.

Proof: First, we shall show that D_K^+ is a complete ordered skew ratio semiring. To prove this, let $x, y \varepsilon D_K^+$. So $x > 0$ and $y > 0$, which implies that $x+y \geqslant 0$. Suppose that $x+y = 0$, by Proposition 3.1.2, K is a skew field, contrary to the assumption. Thus $x+y \varepsilon D_K^+$. We see that $xy \geqslant x \cdot 0 = 0$. Suppose that $xy = 0$. Then $x^{-1}(xy) = x^{-1} \cdot 0$, therefore $y = 0$, a contradiction. So $xy \varepsilon D_K^+$. To show that $x^{-1} > 0$, suppose that $x^{-1} < 0$. By Definition 3.1.1, $x \cdot x^{-1} \leqslant x\ 0 = 0$, so $1 \leqslant 0$, a contradiction. Thus $x^{-1} > 0$. This shows that D_K^+ is an ordered skew ratio semiring.

Next, we shall show that D_K^+ is complete. Suppose that $A \subseteq D_K^+$ is a nonempty set having an upper bound in D_K^+, it follows that $A \subseteq K$. Since K is complete, A has a least upper bound in K. Let $z = \sup(A)$. So for every $a \in A$, $a \leqslant z$. Fix $b \in A$. We get that $0 < b \leqslant z$. Therefore $z \in D_K^+$. Thus D_K^+ is complete, as required.

Finally, suppose that $D_K^- \neq \emptyset$. To show that D^- is a complete ordered semigroup with respect to addition, let $x, y \in D_K^-$. Then $x < 0$ and $y < 0$. Thus $x+y \leqslant 0+y = y < 0$. Therefore $x+y \in D_K^-$. Thus D_K^- is an ordered semigroup. To show that D_K^- is complete, suppose that $A \subseteq D_K^-$ is a nonempty set having a lower bound in D_K^-, it follows that $A \subseteq K$. Since K is complete, A has a greatest lower bound in K. Let $t = \inf(A)$. Fix $d \in A$. We see that $t \leqslant d < 0$. Thus $t \in D_K^-$. Therefore we get that D_K^- is a complete. Thus the proposition is proved.#

<u>Proposition 3.1.4</u>. Let K be an ordered 0-skew semifield such that $1+1 \neq 1$. Then the prime 0-skew semifield of K is isomorphic to $\mathbb{Q}_o^+$ with the usual addition, multiplication and order.

<u>Proof</u>: Let $(P,+,\cdot,\leqslant)$ be the prime 0-skew semifield of $(K,+,\cdot,\leqslant)$. By Theorem 1.39, $(P,+,\cdot) \cong (\mathbb{Q}_o^+,+,\cdot)$ or $P \cong \mathbb{Z}_p$ where p is a prime number. Suppose that $P \cong \mathbb{Z}_p$ for some prime p. We shall denote an element in $\mathbb{Z}_p$ by $\bar{n}$ where $n = 1,2,3,\ldots,p$. Since $0 < 1$, by induction we get that $0 < 1 < 2 < \ldots < p$ in P. Thus $\bar{0} < \bar{1} < \bar{2}$ $\bar{0} < \bar{1} < \bar{2} < \ldots < \bar{p} = \bar{0}$, a contradiction. Therefore $P \ncong \mathbb{Z}_p$ for all prime p. This shows that $(P,+,\cdot) \cong (\mathbb{Q}_o^+,+,\cdot)$. Using the same arguement as before $(P,+,\cdot,\leqslant) \cong (\mathbb{Q}_o^+,+,\cdot,\leqslant)$.#

Theorem 3.1.5. Let $(K,+,\cdot,\leqslant)$ be a complete ordered 0-skew semifield such that $1+1 \neq 1$. Suppose that $(K,+,\cdot)$ is not a skew field. Then $(K,+,\cdot,\leqslant) \cong (\mathbb{R}_0^+,+,\cdot,\leqslant)$.

Proof: By Proposition 3.1.3 and Proposition 3.1.4, $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$. Now, we shall show that $D_K^- = \emptyset$. Suppose that $D_K^- \neq \emptyset$. Let $x \in D_K^-$ be arbitrary. Since $(K \setminus \{0\},\cdot)$ is a group, $x(K \setminus \{0\}) = K \setminus \{0\}$. Therefore $x(D_K^+ \cup D_K^-) = D_K^+ \cup D_K^-$, it follows that $xD_K^+ \cup xD_K^- = D_K^+ \cup D_K^-$. Since $xD_K^+ \subseteq D_K^-$, $xD_K^- \supseteq D_K^+$. So $xD_K^- \supseteq D_K^+$ for all $x \in D_K^-$.

Case 1: Suppose that $x+y < 0$ for all $x \in D_K^-$, for all $y \in K$. Let $y \in D_K^-$ and $z \in D_K^+$. By assumption, $y+z \in D_K^-$. Thus we get that $(y+z)D_K^- \supseteq D_K^+$, so there is a $t \in D_K^-$ such that $(y+z)t \in D_K^+$. Now, we have that $zt \in D_K^-$. By assumption again, $yt+zt \in D_K^-$. Therefore $(y+z)t \neq yt+zt$. This shows that K is not distributive, a contradiction, so this case can not occur.

Case 2: Suppose that $a+b \geqslant 0$ for some $a \in D_K^-$ and for some $b \in K$. If $a+b = 0$, then K is a skew field, a contradiction. Therefore $a+b > 0$. Let $C = \{c \in K | a + c > 0\}$. Clearly, $A \neq \emptyset$ since $b \in A$. Thus $0 < a+c \leqslant 0+c = c$ for all $c \in C$. Then 0 is a lower bound of C. Since K is complete, C has a greatest lower bound, say z^*. Therefore $z^* \geqslant 0$. We shall show that $a+z^* = 0$. To prove this, suppose that $a+z^* \neq 0$. Then either $a+z^* < 0$ or $a+z^* > 0$.

Subcase 2.1: $a+z^* > 0$. If $z^* = 0$, then $a+z^* = a+0 = a < 0$, a contradiction. Thus $z^* > 0$, it follows that $z^* \in D_K^+$. Since $(D_K^+,+,\cdot,\leqslant)$ ia isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$, $(D_K^+,\leqslant)$ is densely ordered, which implies that there exists an $r \in D_K^+$ such that $0 < r < a+z^*$. Let $0 < u < \min\{r,z^*\}$. Again, using the fact that $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(R^+,+,\cdot,\leqslant)$, there are $s,t,w \in D_K^+$ such that $r = u+w$, $a+z^* = r+s$ and $z^* = u+t$. Thus $a+z^* = r+s = u+w+s$. Therefore, substiting $u+t$ for z^* we have that $a+u+t = u+w+s$ which implies that $a+t = w+s > 0$. Then $t \in C$, hence $t \geqslant z^*$. Since $z^* = u+t$ where u, $t \in D_K^+$, $z^* > t$, a contradiction.

<u>Subcase 2.2</u>: $a+z^* < 0$.

<u>Step 1</u>. We shall show that $0 = \sup\{n^{-1}x | n \in \mathbb{Z}^+\}$ for all $x \in D_K^-$. Let $x \in D_K^-$ be arbitrary and let $B = \{n^{-1}x | n \in \mathbb{Z}^+\}$. Then B has 0 as an upper bound. Since $B \subseteq K$ and K is complete, B has a least upper bound. Let $y = \sup(B)$, so $y \leqslant 0$. Assume now that $y < 0$. Then $(n2)^{-1}x \leqslant y$ for all $n \in \mathbb{Z}^+$, it follows that $(2^{-1}n^{-1})x \leqslant y$ for all $n \in \mathbb{Z}^+$ which implies that $n^{-1}x \leqslant 2y$ for all $n \in \mathbb{Z}^+$. Therefore 2y is an upper bound of B, so $y \leqslant 2y$. But we have that $y < 0$, this implies that $2y = y+y \leqslant 0+y = y$. Thus $2y = y$, hence $2 = 1$, a contradiction. Then $0 = \sup(B)$.

<u>Step 2</u>. We shall show that $a+z > 0$ for all $z > z^*$. To prove this, let $z > z^*$ be arbitrary. Then there exists an $r \in C$ such that $z > r > z^*$. Thus $a+z \geqslant a+r > 0$.

<u>Step 3</u>. We shall show that there is $q > 0$ such that $(a+z^*)+q \leqslant 0$. To prove this, suppose not. Then $(a+z^*)+d > 0$ for all $d > 0$. We claim that $c+d > 0$ for all $c < 0$ and for all $d > 0$. To prove the claim, let $c < 0$ be arbitrary. If $c \geqslant a+z^*$, then $c+d \geqslant (a+z^*)+d > 0$ for all $d > 0$. If $c < a+z^*$, then by the fact that $0 = \sup\{n^{-1}c | n \in \mathbb{Z}^+\}$,

there exists an $n \in \mathbb{Z}^+$ such that $a+z^* < n^{-1}c$. Thus $n^{-1}c+d \geq (a+z^*)+d > 0$ for all $d > 0$. It follows that $n^{-1}(c+nd) > 0$ for all $d > 0$. Since $nD^+ = D^+$, $n^{-1}(c+d) > 0$ for all $d > 0$ which implies that $c+d > 0$ for all $d > 0$, so we have the claim. Let $t < 0$ and $s > 0$. By the claim $t+s > 0$. Since $t^{-1} < 0$, $(t+s)t^{-1} < 0$. But we have that $tt^{-1} = 1 > 0$ and $st^{-1} < 0$. Again, by the claim, $tt^{-1} + st^{-1} > 0$. Thus K is not distributive, a contradiction. This shows that there is $q > 0$ such that $(a+z^*)+q \leq 0$, as required.

By Step 3, there exists an $r > 0$ such that $(a+z^*)+r \leq 0$. But we have that $z^*+r > z^*$. By Step 2, $a+(z^*+r) > 0$. Thus $(a+z^*)+r > 0$, a contradiction.

Thus we have shown that $a+z^* = 0$. By Proposition 3.1.2, K is a skew field, a contradiction. Therefore $D_K^- = \emptyset$. This shows that $(K,+,\cdot,\leq) \cong (\mathbb{R}_0^+,+,\cdot,\leq)$.

Hence, the theorem is proved. #

<u>Notation</u>: Let K be an ordered 0-skew semifield and $z \in K$. Then we will denote $I_K(z) = \{y \in K \mid y+z = z\}$, $I_K^+(z) = I_K(z) \cap D_K^+$ and $I_K^-(z) = I_K(z) \cap D_K^-$.

Assume that $(K,+,\cdot,\leq)$ is a complete ordered 0-skew semifield which is not a skew field such that $1+1 = 1$. Then by Proposition 3.1.3, $(D_K^+,+,\cdot,\leq)$ is a complete ordered skew ratio semiring. By Theorem 2.5 and Theorem 2.6, $(D_K^+,+,\cdot,\leq)$ is isomorphic to exactly one of the following ratio semirings:

(1) $(\mathbb{R}^+, \min, \cdot, \leqslant)$.

(2) $(\{2^n | n \in \mathbb{Z}\}, \min, \cdot, \leqslant)$.

(3) $(\mathbb{R}^+, \max, \cdot, \leqslant)$.

(4) $(\{2^n | n \in \mathbb{Z}\}, \max, \cdot, \leqslant)$.

(5) $(\{1\}, +, \cdot, \leqslant)$.

Theorem 3.1.6. There does not exist an ordered 0-skew semifield $(K,+,\cdot,\leqslant)$ such that $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to either of the following two ordered ratio semirings:

(1) $(\mathbb{R}^+, \min, \cdot, \leqslant)$

(2) $(\{2^n | n \in \mathbb{Z}\}, \min, \cdot, \leqslant)$.

Proof: Assume that there exists an ordered 0-skew semifield $(K,+,\cdot,\leqslant)$ such that $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$ or $(\{2^n | n \in \mathbb{Z}\},\min,\cdot,\leqslant)$. Without loss of generality, suppose that $D_K^+ = \{2^n | n \in \mathbb{Z}\}$ or $\mathbb{R}^+$. Let x, y, z $\in$ K be such that $x = 0$, $y = 2^3$ and $z = 2^4$. Then $x < y$. By Definition 3.1.1, $x+z \leqslant y+z$. But we have that $x+z = 0+z = z = 2^4$ and $y+z = \min\{2^3, 2^4\} = 2^3$, this implies that $y+z < x+z$ which is a contradiction. #

Theorem 3.1.7. Let $(K,+,\cdot,\leqslant)$ be a complete ordered 0-skew semifield. If $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\max,\cdot,\leqslant)$, then $(K,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}_0^+,\max,\cdot,\leqslant)$.

Proof: Assume that $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\max,\cdot,\leqslant)$. Then for every a, b $\in D_K^+$, $a+b = \max\{a,b\}$. .... (i)

It is clear that $I_K^+(z) = \{t \in D_K^+ \mid t+z = z\} = \{t \in D_K^+ \mid t \leqslant z\}$ for all $z \in D_K^+$. .... (ii)

Now, we have that $K = D_K^- \cup \{0\} \cup D_K^+$. To show that $D_K^- = \emptyset$. To prove this, suppose not. Then $D_K^- \neq \emptyset$.

Step 1. We shall show that for every a, b ε D_K^+, a < b implies that zb < za and bz < az for all z ε D_K^-. To prove this, let a, b ε D_K^+ be arbitrary. Suppose that a < b. From (i), we have that a+b = b. Let z ε D_K^- be arbitrary. Then zb = z(a+b) = za+zb. Since zb < 0, za+zb $\leqslant$ za. Thus zb $\leqslant$ za. If za = zb, then $z^{-1}(za) = z^{-1}(zb)$ which implies that a = b, a contradiction. Therefore zb < za. Similarly, bz < az.

Step 2. We shall show that for every x, y ε D_K^-, x $\leqslant$ y iff x+y = x. To prove this, let x, y ε D_K^- be arbitrary. Suppose that x $\leqslant$ y. Then x = x+x $\leqslant$ x+y. Since y < 0, x+y $\leqslant$ x+0 = x. Thus x+y = x. On the other hand, suppose that x+y = x. If y < x, then by the first proof in this step shows that y+x = y, it follows that x+y $\neq$ y+x, a contradiction.

By Step 2 we see that addition in D_K^- is minimum, therefore it is clear that $I_K^-(w) = \{s \ \varepsilon \ D_K^- \mid s+w = w\} = \{s \ \varepsilon \ D_K^- \mid w \leqslant s\}$ for all w ε D_K^-. ....(iii)

Step 3. We shall show that for every x, y ε D_K^-, x < y iff $y^{-1} < x^{-1}$. To prove this, let x, y ε D_K^- be arbitrary. Suppose that x < y. By Step 2, x+y = x. Thus $xx^{-1} + yx^{-1} = xx^{-1}$, so $1 + yx^{-1} = 1$ which implies that $y^{-1} + x^{-1} = y^{-1}$. By Step 2, $y^{-1} \leqslant x^{-1}$. If $y^{-1} = x^{-1}$, then x = y, a contradiction. Therefore $y^{-1} < x^{-1}$. On the other hand, suppose that $y^{-1} < x^{-1}$. By the first proof in this step, x < y.

Step 4. We shall show that x+y = x or x+y = y for all x ε D_K^- and for all y ε D_K^+. To prove this, let x ε D^- and y ε D_K^+ be arbitrary.

Clearly, $x+y \neq 0$.

Case 1: $x+y < 0$. Since $x+y = (x+x)+y = x+(x+y)$, $x \in I_K^-(x+y)$. By (iii), $I_K^-(x+y) = \{s \in D_K^- \mid x+y \leqslant s\}$. Then $x+y \leqslant x$. But we have that $0 < y$, this implies that $x = x+0 \leqslant x+y$. Thus $x+y = x$.

Case 2: $x+y > 0$. Since $x+y = x+(y+y) = (x+y)+y$, $y \in I_K^+(x+y)$. By (ii), $I_K^+(x+y) = \{t \in D_K^+ \mid t \leqslant x+y\}$. Then $y \leqslant x+y$. Since $x < 0$, $x+y \leqslant 0+y = y$. Therefore $x+y = y$.

Step 5. We shall show that $I_K(x) = xI_K(1)$ for all $x \in K \setminus \{0\}$. To prove this, let $x \in K \setminus \{0\}$ be arbitrary. Suppose that $y \in I_K(x)$. Then $y+x = x$ which implies that $x^{-1}y+1 = 1$. Thus $x^{-1}y \in I_K(1)$, hence $y \in xI_K(1)$. Therefore $I_K(x) \subseteq xI_K(1)$. On the other hand, suppose that $b \in xI_K(1)$. Then $b = xt$ for some $t \in I_K(1)$, so $b+x = xt+x = x(t+1)$. Since $t \in I_K(1)$, $t+1 = 1$. Hence $b+x = x$. Thus $b \in I_K(x)$. Therefore $xI_K(1) \subseteq I_K(x)$. Hence $I_K(x) = xI_K(1)$.

Step 6. We shall show that for every $x \in K \setminus \{1\}$, $x^n \neq 1$ for all $n \in \mathbb{Z}^+$. To prove this, let $x \in K \setminus \{1\}$ be arbitrary. Suppose that $x^m = 1$ for some $m \in \mathbb{Z}^+$. Clearly $m > 1$, so $m-1 \in \mathbb{Z}^+$. Therefore $x(x^{m-1}+x^{m-2}+\ldots+1) = x^m+x^{m-1}+\ldots+x^2+x = 1+(x^{m-1}+x^{m-2}+\ldots+x)$ $= (x^{m-1}+x^{m-2}+\ldots+x)+1 = x^{m-1}+x^{m-2}+\ldots+x+1$. Then $x = 1$ which is a contradiction.

Step 7. We shall show that $I_K^-(1) \neq \emptyset$. To prove this, let $z,t \in K$ be such that $z < 0 < t$. By Step 4, $z+t = z$ or $z+t = t$.

Case 1: $z+t = z$. Then $z(1+z^{-1}t) = z$ which implies that $1+z^{-1}t = 1$. Since $z^{-1}t \in D_K^-$, $z^{-1}t \in I_K(1) \cap D_K^- = I_K^-(1)$. Thus $I_K^-(1) \neq \emptyset$.

<u>Case 2</u>: $z+t = t$. Then $t(t^{-1}z+1) = t$ which implies that $t^{-1}z+1 = 1$. Since $t^{-1}z \in D_K^-$, $t^{-1}z \in I_K(1) \cap D_K^- = I_K^-(1)$. Thus $I_K^-(1) \neq \emptyset$.

<u>Step 8</u>. We shall show that $I_K^-(1)$ has a greatest lower bound in D_K^-. It suffices to show that $I_K^-(1)$ has a lower bound in D_K^-. To prove this, suppose not. Then $I_K^-(1)$ has no lower bound in D_K^-. Let $d \in D_K^-$ be arbitrary, so d is not a lower bound of $I_K^-(1)$. Then there exists an $r \in I_K^-(1)$ such that $r < d$. Therefore $1 = 1+r \leqslant 1+d$. Since $d < 0$, $1+d \leqslant 1+0 = 1$. Thus $1+d = 1$, so $d \in I_K^-(1)$. Then $D_K^- \subseteq I_K^-(1)$. But we have that $I_K^-(1) \subseteq D_K^-$ this implies that $D_K^- = I_K^-(1)$. Therefore for every $a \in D_K^-$, $a+1 = 1$. Let $a_o \in D_K^-$, therefore $a_o+ 1 = 1$. Since $a_o^{-1} \in D_K^-$, $a_o^{-1}+ 1 = 1$ which implies that $1 + a_o = a_o$. Then $a_o = 1$ which is a contradiction since $a_o < 0$.

From now on, α will denote $\inf(I_K^-(1))$.

<u>Step 9</u>. We shall show that $\alpha \in I_K^-(1)$. To prove this, suppose not. Then $\alpha \notin I_K^-(1)$, so $\alpha+1 = \alpha$. Thus $1+\alpha^{-1} = 1$. Therefore $\alpha^{-1} \in I_K^-(1)$, if follows that $\alpha < \alpha^{-1}$. We claim that there exists an $s_o \in I_K^-(1)$ such that $s_o < \alpha^{-1}$. To prove this claim, suppose not. Then $\alpha^{-1} \leqslant s$ for all $s \in I_K^-(1)$. Thus α^{-1} is a lower bound of $I_K^-(1)$, so $\alpha^{-1} \leqslant \alpha$, a contradiction. Hence we have the claim. Let $s \in I_K^-(1)$ be such that $s < \alpha^{-1}$. By Step 3, $\alpha < s^{-1}$. Using a proof similar to the claim of this step, we can show that there exists an $r \in I_K^-(1)$ such that $r < s^{-1}$. Thus $1 = 1+r \leqslant 1+s^{-1}$. Since $s^{-1} < 0$, $1+s^{-1} \leqslant 0+1 = 1$. Then $s^{-1}+1 = 1$, so $1+s = s$. Since $s \in I_K^-(1)$, $s+1 = 1$. Thus $s = 1$, a contradiction.

<u>Step 10</u>. We shall show that $\alpha^{-1} < \alpha$. To prove this, suppose not. Then $\alpha \leqslant \alpha^{-1}$. By Step 9, $1 = 1+\alpha \leqslant 1+\alpha^{-1}$. Since $\alpha^{-1} < 0$,

$1+\alpha^{-1} \leqslant 1+0 = 1$. Thus $1+\alpha^{-1} = 1$ which implies that $\alpha+1 = \alpha$. Since $\alpha+1 = 1$, $\alpha = 1$, a contradiction.

Step 11. We shall show that there does not exist $y \varepsilon D_K^-$ such that $\alpha^{-1} < y < \alpha$. To prove this, suppose not. Let $y \varepsilon D_K^-$ be such that $\alpha^{-1} < y < \alpha$. Thus $y+1 = y$. By Step 3, $y^{-1} < (\alpha^{-1})^{-1} = \alpha$. Then $y^{-1}+1 = y^{-1}$ which implies that $1+y = 1$. Hence $y = 1$, a contradiction.

Step 12. We shall show that $\alpha^n+1 = 1$ for all $n \varepsilon \mathbb{Z}^+$. We shall prove this by using mathematical induction on $n \varepsilon \mathbb{Z}^+$. Let $n \varepsilon \mathbb{Z}^+$. If $n = 1$, then by Step 9, $\alpha+1 = 1$. Suppose that $\alpha^{n-1}+1 = 1$ for some $n-1 \geqslant 1$. Then $\alpha = \alpha(\alpha^{n-1}+1) = \alpha^n+\alpha$, it follows that $1 = \alpha+1 = (\alpha^n+\alpha)+1 = \alpha^n+(\alpha+1) = \alpha^n+1$. Hence $\alpha^n+1 = 1$ for all $n \varepsilon \mathbb{Z}^+$.

Step 13. We shall show that $\alpha^n < 0$ for all $n \varepsilon \mathbb{Z}^+$. We shall prove thia by using mathematical induction on $n \varepsilon \mathbb{Z}^+$. Let $n \varepsilon \mathbb{Z}^+$. If $n = 1$, then we are done. Suppose that $n = 2$. If $\alpha^2 > 0$, then $\alpha^2 = d$ for some $0 < d < 1$. Thus $\alpha = \alpha^{-1}d$. Let $0 < d < d' < 1$. By Step 1, $\alpha^{-1} < \alpha^{-1}d' < \alpha^{-1}d = \alpha$ which contradicts Step 11. Then $\alpha^2 < 0$. Let $n-1 \geqslant 2$. Suppose that $\alpha^k < 0$ for all $1 \leqslant k \leqslant n-1$. If $\alpha^n > 0$. then by Step 12 and (i), $\alpha^n \leqslant 1$. By Step 6. $0 < \alpha^n < 1$. Thus $\alpha^n = d$ for some $0 < d < 1$, so $\alpha^2 = \alpha^{-(n-2)}d < 0$. Let $0 < d < d_1 < 1$. By Step 1 and $\alpha^{-(n-2)} < 0$, $\alpha^{-(n-2)}d_1 < \alpha^{-(n-2)}d = \alpha^2$.

Case 1: $\alpha^{-(n-2)}d_1 = \alpha$. Then $0 < d_1 = \alpha^{n-2}\alpha = \alpha^{n-1}$ which is a contradiction.

Case 2: $\alpha < \alpha^{-(n-2)}d_1$. Then $\alpha < \alpha^{-(n-2)}d_1 < \alpha^2$. By Step 2, $\alpha+\alpha^{-(n-2)}d_1 = \alpha$. Thus $\alpha^{n-2}\alpha + d_1 = \alpha^{n-2}\alpha$, so $\alpha^{n-1}+ d_1 = \alpha^{n-1}$. ...(iv)
By Step 2 again, $\alpha^{-(n-2)}d_1+ \alpha^2 = \alpha^{-(n-2)}d_1$, so $\alpha^{-n+2}d_1+ \alpha^2= \alpha^{-n+2}d_1$.

Therefore $\alpha^{-1}(\alpha^{-n+2}d_1+\alpha^2) = \alpha^{-1}(\alpha^{-n+2}d_1)$. Thus $\alpha^{-n+1}d_1+\alpha = \alpha^{-n+1}d_1$, so $\alpha^{-(n-1)}d_1+\alpha = \alpha^{-(n-1)}d_1$. By Step 2, $\alpha^{-(n-1)}d_1 \leqslant \alpha$.

<u>Subcase 2.1</u>: $\alpha^{-(n-1)}d_1 = \alpha$. Then $d_1 = \alpha^{n-1}\alpha = \alpha^n = d$, a contradiction.

<u>Subcase 2.2</u>: $\alpha^{-(n-1)}d_1 < \alpha$. Then $\alpha^{-(n-1)}d_1+1 = \alpha^{-(n-1)}d_1$.

Thus $d_1+\alpha^{n-1} = d_1$. ....(v)

From (iv) and (v), we have that $\alpha^{n-1} = d_1 > 0$, a contradiction.

<u>Case 3</u>. $\alpha^{-(n-2)}d_1 < \alpha$. Then by Step 11, $\alpha^{-(n-2)}d_1 \leqslant \alpha^{-1}$. By Step 2, $\alpha^{-(n-2)}d_1+\alpha^{-1} = \alpha^{-(n-2)}d_1$. ....(▼)

We claim that $d_1+\alpha \neq d_1$. To prove this claim, suppose not. Then $d_1+\alpha = d_1$, so $1+\alpha d_1^{-1} = 1$. Thus $\alpha \leqslant \alpha d^{-1}$. Now, we have that $1 < d_1^{-1}$. By Step 1, $\alpha d_1^{-1} < \alpha$ which is a contradiction. Hence we have the claim. Now, we have that $n-2 \geqslant 1$. If $n-2 = 1$, then by (▼), $\alpha^{-1}d_1+\alpha^{-1} = \alpha^{-1}d_1$. By Step 2, $\alpha^{-1}d_1 \leqslant \alpha^{-1}$. Since $0 < d_1 < 1$ and $\alpha^{-1} < 0$, by Step 1, $\alpha^{-1} < \alpha^{-1}d_1$, a contradiction. Therefore $n-2 > 1$.

Hence $n-3 \geqslant 1$. From (▼), we have that $d_1+\alpha^{n-3} = d_1$. ....(vi)

If $n-3 = 1$, then by (vi), $d_1+\alpha = d_1$, a contradiction. Therefore $n-3 > 1$. Hence $n-4 \geqslant 1$ and $\alpha^{n-3} < 0$. From (vi), we have that $1+\alpha^{n-3}d_1^{-1} = 1$. Then $\alpha \leqslant \alpha^{n-3}d_1^{-1}$. By Step 2, $\alpha+\alpha^{n-3}d_1^{-1} = \alpha$ which implies that $d_1+\alpha^{n-4} = d_1$. ....(vii)

If $n-4 = 1$, then $d_1+\alpha = d_1$, a contradiction. Therefore $n-4 > 1$. Continue in this way. Since n is finite after a finite number of

Steps, $d_1 + \alpha = d_1$. This is a contradiction.

Hence $\alpha^n < 0$ for all $n \in \mathbb{Z}^+$.

<u>Step 14</u>. We shall show that $\alpha^n < \alpha^{n+1}$ for all $n \in \mathbb{Z}^+$. To prove this, Suppose not. Then $\alpha^{m+1} < \alpha^m$ for some $m \in \mathbb{Z}^+$. By Step 13 and Step 2, $\alpha^{m+1} + \alpha^m = \alpha^{m+1}$ which implies that $\alpha + 1 = \alpha$. Then $\alpha \notin I_K^-(1)$, a contradiction.

<u>Step 15</u>. We shall show that $I_K(\alpha) = \{y \in K | \alpha \leqslant y < 1\}$. To prove this, we first to prove that $I_K^+(\alpha) \neq \emptyset$. We claim that there exists a $u \in D_K^+$ such that $u + \alpha = \alpha$. Suppose not. Then by Step 4, $u + \alpha = u$ for all $u \in D_K^+$. Thus $1 + u^{-1}\alpha = 1$ for all $u \in D_K^+$, so $u^{-1}\alpha \in I_K^-(1)$ for all $u \in D_K^+$. Then $\alpha \leqslant u^{-1}\alpha$ for all $u \in D_K^+$. Let $L = \{u^{-1}\alpha | u \in D_K^+\}$, so α is a lower bound of L. Let $\beta = \inf(L)$. Then $\beta \leqslant u^{-1}\alpha$ for all $u \in D_K^+$. Let $1 < s$. Therefore $\beta \leqslant su^{-1}\alpha$ for all $u \in D_K^+$, so $s^{-1}\beta \leqslant u^{-1}\alpha$ for all $u \in D_K^+$. Thus $s^{-1}\beta$ is a lower bound of L, hence $s^{-1}\beta \leqslant \beta$. Since $s^{-1} < 1$, by Step 1, $\beta < s^{-1}\beta$, a contradiction. Hence we have the claim. By the claim, $I_K^+(\alpha) \neq \emptyset$.

Next, we shall show that $I_K^+(\alpha) = \{t \in D_K^+ | t < 1\}$. To prove this, let $a \in I_K^+(\alpha)$. Then $a + \alpha = \alpha$. If $1 \leqslant a$, then $1 = 1 + \alpha \leqslant a + \alpha = \alpha$, a contradiction. Thus $0 < a < 1$, so $a \in \{t \in D_K^+ | t < 1\}$. Hence $I_K^+(\alpha) \subseteq \{t \in D_K^+ | t < 1\}$. On the other hand, let $b \in \{t \in D_K^+ | t < 1\}$. If $b + \alpha = b$, then $1 + b^{-1}\alpha = 1$, so $b^{-1}\alpha \in I_K^-(1)$. Thus $\alpha \leqslant b^{-1}\alpha$. Since $1 < b^{-1}$, by Step 1, $\alpha b^{-1} < \alpha$, a contradiction. Then $b + \alpha \neq b$. By Step 4, $b + \alpha = \alpha$. Thus $b \in I_K^+(\alpha)$. Therefore $\{t \in D_K^+ | t < 1\} \subseteq I_K^+(\alpha)$. Hence $I_K^+(\alpha) = \{t \in D_K^+ | t < 1\}$.

Finally, by (iii), $I_K^-(\alpha) = \{s \in D_K^- | \alpha \leqslant s\}$. Then $I_K(\alpha) = I_K^-(\alpha) \cup \{0\} \cup I_K^+(\alpha) = \{s \in D_K^- | \alpha \leqslant s\} \cup \{0\} \cup \{t \in D_K^+ | t < 1\} = \{y \in K \mid \alpha \leqslant y < 1\}$.

Step 16. We shall show that there does not exist an $\ell \in D_K^-$ such that $\alpha^n < \ell < \alpha^{n+1}$ for all $n \in \mathbb{Z}^+$. To prove this, let $n \in \mathbb{Z}^+$. Now, we have that $I_K(\alpha^{n+1}) = \{y \in K \mid \alpha^{n+1} \leqslant y \leqslant d\}$ for some $0 < d < 1$. By Step 5,

$$\begin{aligned} I_K(\alpha^{n+1}) &= \alpha^{n+1} I_K(1) \\ &= \alpha^n(\alpha I_K(1)) \\ &= \alpha^n(I_K(\alpha)) \\ &= \alpha^n\{w \in K \mid \alpha \leqslant w < 1\} \quad \text{(by Step 15)} \\ &= \alpha^n(\{w \in K \mid \alpha \leqslant w \leqslant 1\} \smallsetminus \{1\}) \\ &= \alpha^n\{w \in K \mid \alpha \leqslant w \leqslant 1\} \smallsetminus \{\alpha^n\} \\ &= \alpha^n I_K(1) \smallsetminus \{\alpha^n\} \\ &= I_K(\alpha^n) \smallsetminus \{\alpha^n\} \\ &= \{v \in K \mid \alpha^n < v \leqslant d_1\} \quad \text{for some } 0 < d_1 < 1. \end{aligned}$$

Hence there does not exist an $\ell \in D_K^-$ such that $\alpha^n < \ell < \alpha^{n+1}$.

Step 17. We shall show that $0 = \sup\{\alpha^n \mid n \in \mathbb{Z}^+\}$. To prove this, let $T = \{\alpha^n \mid n \in \mathbb{Z}^+\}$. By Step 13, 0 is an upper bound of T. Let $\lambda = \sup\{\alpha^n \mid n \in \mathbb{Z}^+\}$. Then $\lambda \leqslant 0$. Suppose that $\lambda < 0$. Thus

$$\begin{aligned} \lambda \in \{z \in K \mid \alpha \leqslant z < 1\} &= I_K(\alpha) \quad \text{(by Step 16)} \\ &= \alpha I_K(1) \quad \text{(by Step 5)} \\ &= \alpha(I_K^+(1) \cup \{0\} \cup I_K^-(1)) \\ &= \alpha I_K^+(1) \cup \{0\} \cup \alpha I_K^-(1). \end{aligned}$$

Since $\lambda \neq 0$, $\lambda \in \alpha I_K^+(1)$ or $\lambda \in \alpha I_K^-(1)$.

Case 1: $\lambda \in \alpha I_K^+(1)$. Then $\lambda = \alpha d$ for some $0 < d \leq 1$. If $d = 1$, then $\lambda = \alpha < \alpha^2 \leq \lambda$, a contradiction. Thus $d \neq 1$. Let $0 < d < d' < 1$. By Step 1, $\alpha < \alpha d' < \alpha d = \lambda$. By Step 16, $\alpha d' = \alpha^m$ for some $m \in \mathbb{Z}^+ \setminus \{1\}$. Then $\alpha^{m-1} = d' > 0$ which contradicts Step 13.

Case 2: $\lambda \in \alpha I_K^-(1)$. Then $\lambda = \alpha u$ for some $u \in I_K^-(1)$.

Subcase 2.1: $\lambda = u$. Then $\alpha u = u$, so $\alpha = 1$ which is a contradiction.

Subcase 2.2: $\lambda < u$. Then $\alpha u < u < 0$. By Step 2, $\alpha u + u = \alpha u$. Thus $\alpha + 1 = \alpha$. By Step 9, $\alpha + 1 = 1$. Hence $\alpha = 1$, a contradiction.

Subcase 2.3: $u < \lambda$. Then $u = \alpha^m$ for some $m \in \mathbb{Z}^+$. Thus $\lambda = \alpha u = \alpha\alpha^m = \alpha^{m+1}$. By Step 14, $\lambda = \alpha^{m+1} < \alpha^{m+2} \leq \lambda$, a contradiction.

Hence $0 = \sup\{\alpha^n \mid n \in \mathbb{Z}^+\}$. From Step 14 and Step 16, we have that $I_K^-(1) = \{\alpha^n \mid n \in \mathbb{Z}^+\}$. Let $0 < r < 1$. By Step 1, $\alpha < \alpha r < 0$. Then $\alpha r = \alpha^m$ for some $m \in \mathbb{Z}^+ \setminus \{1\}$. Therefore $\alpha^{m-1} = r > 0$ which contradicts Step 14. Thus $D_K^- = \emptyset$

Hence, the theorem is proved. #

Remark 3.1.8. Let β be a symbol not representing any integer and let p be a positive integer greater than 1. Let $K_{(p)} = \{\beta^n \mid n \in \mathbb{Z}\} \cup \{0\}$. Define multiplication on $K_{(p)}$ by $\beta^n \cdot 0 = 0 \cdot \beta^n = 0 = 0 \cdot 0$ and $\beta^n\beta^m = \beta^{n+m}$ for all $m, n \in \mathbb{Z}$. Define addition on $K_{(p)}$ by $\beta^n + 0 = 0 + \beta^n = \beta^n$, $0 + 0 = 0$ and $\beta^n + \beta^m = \beta^{\min\{n,m\}}$ for all $m, n \in \mathbb{Z}$. Define order $\leq$ on $K_{(p)}$ by as follows: Let $m, n \in \mathbb{Z}$.

(1) If $m \equiv 0 \pmod{p}$, then $0 < \beta^m$.

(2) If $n \not\equiv 0 \pmod p$, then $\beta^n < 0$.

(3) If $n \equiv 0 \pmod p$ and $m \equiv 0 \pmod p$, then $\beta^m \leqslant \beta^n$ iff $n \leqslant m$.

(4) If $n \not\equiv 0 \pmod p$ and $m \not\equiv 0 \pmod p$, then $\beta^m \leqslant \beta^n$ iff $m \leqslant n$. Then $(K_{(p)},+,\cdot,\leqslant)$ is a complete ordered 0-skew semifield as is shown below.

Proof: Clearly, $(K_{(p)},+)$ is a commutative semigroup, $(K_{(p)},\cdot)$ is an abelian group with zero and the distributive law holds. Also, it is clear that $(K_{(p)},\leqslant)$ is an ordered set. We must first show that $(K_{(p)},+,\cdot,\leqslant)$ is an ordered 0-skew semifield.

We must show that for every $\beta^m, \beta^n \in K_{(p)}$, $\beta^m \leqslant \beta^n$ implies that $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$ for all $\beta^\ell \in K_{(p)}$. To prove this, let β^m, $\beta^n \in K_{(p)}$ be such that $\beta^m < \beta^n$. Let $\beta^\ell \in K_{(p)}$ be arbitrary.

Case 1: $m, n \equiv 0 \pmod p$. Then $n < m$.

Subcase 1.1: $\ell < n < m$. Then $\beta^m + \beta^\ell = \beta^\ell$ and $\beta^n + \beta^\ell = \beta^\ell$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 1.2: $n < \ell < m$. Then $\beta^m + \beta^\ell = \beta^\ell$ and $\beta^n + \beta^\ell = \beta^n$.

Subcase 1.2.1: $\ell \equiv 0 \pmod p$. Then $\beta^\ell < \beta^n$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 1.2.2: $\ell \not\equiv 0 \pmod p$. Then $\beta^\ell < \beta^n$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 1.3: $n < m < \ell$. Then $\beta^m + \beta^\ell = \beta^m$ and $\beta^n + \beta^\ell = \beta^n$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Case 2: $m, n \not\equiv 0 \pmod p$. Then $m < n$.

Subcase 2.1: $\ell < m < n$. Then $\beta^m + \beta^\ell = \beta^\ell$ and $\beta^n + \beta^\ell = \beta^\ell$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 2.2: $m < \ell < n$. Then $\beta^m + \beta^\ell = \beta^m$ and $\beta^n + \beta^\ell = \beta^\ell$

Subcase 2.2.1: $\ell \equiv 0 \pmod p$. Then $\beta^m < \beta^\ell$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 2.2.2: $\ell \not\equiv 0 \pmod p$. Then $\beta^m < \beta^\ell$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 2.3: $m < n < \ell$. Then $\beta^m + \beta^\ell = \beta^m$ and $\beta^n + \beta^\ell = \beta^n$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Case 3: $m \not\equiv 0 \pmod p$ and $n \equiv 0 \pmod p$.

Subcase 3.1: $m < n$

Subcase 3.1.1: $\ell < m < n$. Then $\beta^m + \beta^\ell = \beta^\ell$ and $\beta^n + \beta^\ell = \beta^\ell$. Then $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 3.1.2: $m < \ell < n$. Then $\beta^m + \beta^\ell = \beta^m$ and $\beta^n + \beta^\ell = \beta^\ell$

Subcase 3.1.2.1: $\ell \equiv 0 \pmod p$. Then $\beta^m < \beta^\ell$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 3.1.2.2: $\ell \not\equiv 0 \pmod p$. Then $\beta^m < \beta^\ell$. Then $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$.

Subcase 3.1.3: $n < m < \ell$. Then $\beta^m + \beta^\ell = \beta^m$ and $\beta^n + \beta^\ell = \beta^n$. Thus $\beta^m + \beta^\ell \leqslant \beta^n + \beta^\ell$

Subcase 3.2: $n < m$. The proof is similar to the proof of Subcase 3.1.

Next, we must show that for every $\beta^m, \beta^n \in K_{(p)}$, $\beta^m \leqslant \beta^n$ implies that $\beta^m\beta^\ell \leqslant \beta^n\beta^\ell$ for all $\beta^\ell > 0$. To prove this, let $\beta^m, \beta^n \in K_{(p)}$ be such that $\beta^m \leqslant \beta^n$. Let $\beta^\ell > 0$ be arbitrary.

Case 1: $n \equiv 0 \pmod p$ and $m \equiv 0 \pmod p$. Then $n \leqslant m$, so $\ell+n \leqslant \ell+m$. Now, we have that $\ell \equiv 0 \pmod p$. Then $n+\ell \equiv 0 \pmod p$ and $m+\ell \equiv 0 \pmod p$. Thus $\beta^{m+\ell} \leqslant \beta^{n+\ell}$. Therefore $\beta^m\beta^\ell \leqslant \beta^n\beta^\ell$.

Case 2: $m \not\equiv 0 \pmod p$ and $n \not\equiv 0 \pmod p$. Thus $m \leqslant n$. Now, we have that $\ell \equiv 0 \pmod p$. Thus $m+\ell \not\equiv 0 \pmod p$ and $n+\ell \not\equiv 0 \pmod p$ and $m+\ell \leqslant n+\ell$. Therefore $\beta^{m+\ell} \leqslant \beta^{n+\ell}$. Hence $\beta^m\beta^\ell \leqslant \beta^n\beta^\ell$.

Case 3: $m \not\equiv 0 \pmod p$ and $n \equiv 0 \pmod p$. Now, we have that $\ell \equiv 0 \pmod p$. Thus $m+\ell \not\equiv 0 \pmod p$ and $n+\ell \equiv 0 \pmod p$. Hence $\beta^{m+\ell} \leqslant \beta^{n+\ell}$. Thus $\beta^m\beta^\ell \leqslant \beta^n\beta^\ell$.

Lastly, we must show that $K_{(p)}$ is complete. To prove this, let $H \subseteq K_{(p)}$ be a nonempty set which has an upper bound. Let w be an upper bound of H.

Case 1: $w \leqslant 0$. Then $H \subseteq \{\beta^m \mid m \in \mathbb{Z} \setminus p\mathbb{Z}\}$. Since $(\{\beta^m \mid m \in \mathbb{Z} \setminus p\mathbb{Z}\}, \leqslant)$ is isomorphic to $(\mathbb{Z} \setminus p\mathbb{Z}, \leqslant)$ and $(\mathbb{Z} \setminus p\mathbb{Z}, \leqslant)$ is complete, H has a least upper bound.

Case 2: $0 < w$. Then $w = \beta^{n_1}$ for some $n_1 \in p\mathbb{Z}$. If w is a least upper bound then we are done. Suppose that w is not a least upper bound of H. Now, we have that $w = \beta^{kp}$ for some $k \in \mathbb{Z}$. If $\beta^{(k+1)p}$ is a least upper bound of H, then we are done. Suppose that $\beta^{(k+1)p}$ is not a least upper bound of H. Continue in this way.

Subcase 2.1: The process stops at $\beta^{(k+\ell)}$ for some $\ell \in \mathbb{Z}^+$. Then H has a least upper bound.

Subcase 2.2: The process does not stop. If 0 is a least upper bound, then we are done. Suppose that 0 is not a least upper bound of H. Then there exists an $r < 0$ which is an upper bound of H. Using the same proof as in the proof of Case 1, we obtain that H has a least upper bound in $K_{(p)}$.#

Remark 3.1.9. $K_{(2)}$ is isomorphic to the complete ordered 0-skew semifield $(\{-\sqrt{2}^m \mid m \in \mathbb{Z} \text{ is odd}\} \cup \{0\} \cup \{2^n \mid n \in \mathbb{Z}\}, \oplus, \cdot, \leqslant)$ where $\leqslant$, $\cdot$ are the usual order and multiplication and $x \oplus y = x$ iff $|x| \geqslant |y|$.

Let $A = \{-\sqrt{2}^m \mid m \in \mathbb{Z} \text{ is odd}\} \cup \{0\} \cup \{2^n \mid n \in \mathbb{Z}\}$. The isomorphism f from $K_{(2)}$ to A is given by $f(0) = 0$ and

$$f(\beta^n) = \begin{cases} -\sqrt{2}^{-n} & \text{if } n \not\equiv 0 \pmod 2, \\ 2^{-\frac{n}{2}} & \text{if } n \equiv 0 \pmod 2. \end{cases}$$ #

Theorem 3.1.10. Let $(K,+,\cdot,\leqslant)$ be a complete ordered 0-skew semifield. If $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\{2^n \mid n \in \mathbb{Z}\}, \max, \cdot, \leqslant)$. Then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following 0-semifields:

(1) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{0\}, \max, \cdot, \leqslant)$.

(2) $(K_{(p)},+,\cdot,\leqslant)$ for some $p > 1$ as in Remark 3.1.8.

Proof: Assume that $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\{2^n \mid n \in \mathbb{Z}\}, \max, \cdot, \leqslant)$. Now, we have that $K = D_K^- \cup \{0\} \cup D_K^+$. If $D_K^- = \emptyset$, then $(K,+,\cdot,\leqslant)$ is isomorphic to $(\{2^n | n \in \mathbb{Z}\} \cup \{0\}, \max,\cdot,\leqslant)$. Suppose that $D_K^- \neq \emptyset$. For simplicity, we shall assume that $D_K^+ = \{2^n | n \in \mathbb{Z}\}$. Step 1 to Step 12 of Theorem 3.1.7 hold with these hypotheses and the proofs are exactly the same. As in Theorem 3.1.7 α will denote $\inf(I_K^-(1))$.

Step 1. We shall show that $\alpha^m > 0$ for some $m \in \mathbb{Z}^+$. To prove this, suppose not. Then $\alpha^n < 0$ for all $n \in \mathbb{Z}^+$. Using the same proof as in the proof of Step 14 of Theorem 3.1.7 we get that $\alpha^n < \alpha^{n+1}$ for all $n \in \mathbb{Z} \setminus \{0\}$. ....(*)

Using the same proof given in Step 16 of Theorem 3.1.7 we get that there does not exist a $y \in D_K^-$ such that $\alpha^n < y < \alpha^{n+1}$ for all $n \in \mathbb{Z}^+$. .... (**)

We claim that $0 = \sup \{\alpha^n | n \in \mathbb{Z}^+\}$. To prove this claim, let $L = \{\alpha^n | n \in \mathbb{Z}^+\}$. By (*), 0 is an upper bound of L. Since $L \subseteq K$ and K is complete, L has a least upper bound. Let $\lambda = \sup(L)$. Then $\lambda \leqslant 0$. Suppose that $\lambda < 0$. Using the same argument as given in the proof of Step 15 in Theorem 3.1.7 we get that $I_K(\alpha) = \{y \in K | \alpha \leqslant y < 1\}$. ....(***)

Now, we have that $I_K(1) = \{z \in K \mid \alpha \leqslant z \leqslant 1\}$. By Step 5 of Theorem 3.1.7 and (***), $\{y \in K \mid \alpha \leqslant y < 1\} = I_K(\alpha) = \alpha I_K(1)$ $= \alpha\{z \in K \mid \alpha \leqslant z \leqslant 1\} = \alpha(\{s \in D_K^- | \alpha \leqslant s\} \cup \{0\} \cup \{t \in D_K^+ | t \leqslant 1\} =$ $\alpha\{s \in D_K^- | \alpha \leqslant s\} \cup \{0\} \cup \alpha\{t \in D_K^+ | t \leqslant 1\}$. ....(****)

Since $\alpha < \lambda < 0$, $\lambda \in \{y \in K \mid \alpha \leqslant y < 1\}$. From (****), we have that $\lambda \in \alpha\{s \in D_K^- | \alpha \leqslant s\} \cup \alpha\{t \in D_K^+ | t \leqslant 1\}$.

Case 1: $\lambda \in \alpha\{t \in D_K^+ | \ t \leqslant 1\}$. Then $\lambda = \alpha d$ for some $0 < d \leqslant 1$. Clearly, $d \neq 1$. Suppose that $d = \frac{1}{2}$.. Then $\lambda = \alpha(\frac{1}{2})$. Now, we have that $0 < \frac{1}{2} < 1$ and $\alpha^{-1} < 0$. By Step 1, $\alpha^{-1} < \alpha^{-1}(\frac{1}{2})$

Subcase 1.1: $\alpha^{-1}(\frac{1}{2}) < \alpha$. Then $\alpha^{-1} < \alpha^{-1}(\frac{1}{2}) < \alpha$, a contradiction.

Subcase 1.2: $\alpha \leqslant \alpha^{-1}(\frac{1}{2})$. If $\alpha = \alpha^{-1}(\frac{1}{2})$, then $\alpha^2 = \frac{1}{2}$, a contradiction. Thus $\alpha < \alpha^{-1}(\frac{1}{2})$.

Subcase 1.2.1: $\alpha^{-1}(\frac{1}{2}) < \lambda$. Then by (**), $\alpha^{-1}(\frac{1}{2}) = \alpha^m$ for some $m \in \mathbb{Z}^+ \setminus \{1\}$. Thus $\alpha^{m+1} = \frac{1}{2}$, a contradiction.

Subcase 1.2.2: $\lambda \leqslant \alpha^{-1}(\frac{1}{2})$. Then $\alpha(\frac{1}{2}) \leqslant \alpha^{-1}(\frac{1}{2})$, so $\alpha < \alpha^{-1}$, a contradiction.

Therefore $d \neq \frac{1}{2}$, so $\lambda = \alpha d$ for some $d < \frac{1}{2}$. Let $0 < d < d' < 1$. Then by Step 1, $\alpha < \alpha d' < \alpha d = \lambda$. From (**), $\alpha d' = \alpha^m$ for some $m \in \mathbb{Z}^+ \setminus \{1\}$. Hence $\alpha^{m-1} = d' > 0$, a contradiction.

Case 2: $\lambda \in \alpha\{s \in D_K^- | \ \alpha \leqslant s\}$. Then $\lambda = \alpha u$ for some $u \in \{s \in D_K^- | \alpha \leqslant s\}$.

Subcase 2.1: $\lambda = u$. Then $\alpha u = u$, so $\alpha = 1$ which is a contradiction.

Subcase 2.2: $\lambda < u$. Then $\alpha u < u < 0$. By Step 2 of Theorem 3.1.7, $\alpha u + u = \alpha u$ which implies that $\alpha + 1 = \alpha$. Since $\alpha + 1 = 1$, $\alpha = 1$, a contradiction.

Subcase 2.3: $u < \lambda$. Then $u = \alpha^m$ for some $m \in \mathbb{Z}^+$. Thus $\lambda = \alpha u = \alpha\alpha^m = \alpha^{m+1} < \alpha^{m+2} \leqslant \lambda$, a contradiction.

Hence $\lambda = 0$, so we have the claim.

Let $0 < s < 1$. By Step 1 of Theorem 3.1.7, $\alpha < \alpha s < 0$.

From (**), we have that $\alpha s = \alpha^{\ell}$ for some $\ell \in \mathbb{Z}^+ \setminus \{1\}$. Therefore $\alpha^{\ell-1} = s > 0$, a contradiction.

This shows that $\alpha^m > 0$ for some $m \in \mathbb{Z}^+$.

Let $B = \{n \in \mathbb{Z}^+ | \alpha^n > 0\}$. By Step 1, $B \neq \emptyset$. Let $p = \min(B)$. Then $p > 1$.

Step 2. We shall show that $\alpha^p = \frac{1}{2}$. To prove this, suppose not. Then $\alpha^p < \frac{1}{2}$. Now, we have that $I_K(\alpha) = I_K^-(\alpha) \cup \{0\} \cup I_K^+(\alpha)$. By Step 5 of Theorem 3.1.7, $I_K(\alpha) = \alpha I_K(1) = \alpha(I_K^-(1) \cup \{0\} \cup I_K^+(1))$ $= \alpha I_K^-(1) \cup \{0\} \cup \alpha I_K^+(1)$ which implies that $I_K^-(\alpha) \cup I_K^+(\alpha) = \alpha I_K^-(1) \cup \alpha I_K^+(1)$. Since $\alpha I_K^+(1) \subseteq I_K^-(\alpha)$, $I_K^+(\alpha) \subseteq \alpha I_K^-(1)$. Hence $\{t \in D^+ | t \leqslant \frac{1}{2}\} \subseteq \alpha\{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{p-1}, \alpha^{p+1}, \ldots, \alpha^{2p-1}, \alpha^{2p+3}, \ldots\}$. We see that $\alpha(\alpha^{p-1}) = \alpha^p < \frac{1}{2}$

$$\alpha(\alpha^{2p-1}) = \alpha^{2p} = (\alpha^p)^2 < (\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$$\vdots$$

$$\alpha(\alpha^{np-1}) = \alpha^{np} = (\alpha^p)^n < (\tfrac{1}{2})^n = \tfrac{1}{2^n} \quad \text{for all } n \in \mathbb{Z}^+.$$

Hence $\frac{1}{2} \notin \alpha I_K^-(1)$, a contradiction.

Step 3. For every $n \in \mathbb{Z}^+$, $n \not\equiv 0 \pmod p$ and $n+1 \not\equiv 0 \pmod p$ implies that there does not exist a $y \in D_K^-$ such that $\alpha^n < y < \alpha^{n+1}$. This proof is the same as the proof of Step 16 in Theorem 3.1.7.

Step 4. We shall show that for every $n \in \mathbb{Z}^+$, $n \not\equiv 0 \pmod p$ and $n+2 \not\equiv 0 \pmod p$ and $n+1 \equiv 0 \pmod p$ implies that there does not exist $y \in D_K^-$ such that $\alpha^n < y < \alpha^{n+2}$. To prove this, let $n \in \mathbb{Z}^+$ be such that $n \equiv 0 \pmod p$ and $n+2 \not\equiv 0 \pmod p$ and $n+1 \equiv 0 \pmod p$. We first claim that $\alpha^2 I_K(1) = \{u \in K \mid \alpha < u < 1\}$. ....(iii)

Using the same proof given in Step 15 of Theorem 3.1.7 we get that

$I_K(\alpha) = \{u \in K \mid \alpha \leqslant u < 1\}$. ....(iv)

We see that
$$\begin{aligned}
\alpha^2 I_K(1) &= \alpha(\alpha I_K(1)) \\
&= \alpha I_K(\alpha) \quad \text{(by Step 5 of Theorem 3.1.7)} \\
&= \alpha\{u \in K \mid \alpha \leqslant u < 1\} \quad \text{(by (iv))} \\
&= \alpha(\{u \in K \mid \alpha \leqslant u < 1\} \setminus \{1\}) \\
&= \alpha\{u \in K \mid \alpha \leqslant u < 1\} \setminus \{\alpha\}\} \\
&= \alpha I_K(1) \setminus \{\alpha\} \\
&= I_K(\alpha) \setminus \{\alpha\} \quad \text{(by Step 5 of Theorem 3.1.7)} \\
&= \{u \in K \mid \alpha < u < 1\} \quad \text{(by (iv))}.
\end{aligned}$$

Hence, we have the claim. Now, we have that

$I_K(\alpha^{n+2}) = \{y \in K \mid \alpha^{n+2} \leqslant y \leqslant \alpha^r\}$ for some $r \in p\mathbb{Z}_0^+$. ....(v)

By Step 5 of Theorem 3.1.7,
$$\begin{aligned}
I_K(\alpha^{n+2}) &= \alpha^{n+2} I(1) \\
&= \alpha^n(\alpha^2 I_K(1)) \\
&= \alpha^n\{u \in K \mid \alpha < u < 1\} \\
&= \alpha^n(\{u \in K \mid \alpha \leqslant u \leqslant 1\} \setminus \{\alpha, 1\}) \\
&= \alpha^n\{u \in K \mid \alpha \leqslant u \leqslant 1\} \setminus \{\alpha^{n+1}, \alpha^n\} \\
&= \alpha^n I_K(1) \setminus \{\alpha^{n+1}, \alpha^n\} \\
&= I_K(\alpha^n) \setminus \{\alpha^{n+1}, \alpha^n\} \\
&= \{v \in K \mid \alpha^n < v \leqslant \alpha^s\} \setminus \{\alpha^{n+1}\}
\end{aligned}$$

for some $s \in p\mathbb{Z}_0^+$. ....(vi)

By (v) and (vi), $\{y \in K \mid \alpha^{n+2} \leqslant y \leqslant \alpha^r\} = \{v \in K \mid \alpha^n < v \leqslant \alpha^s\} \setminus \{\alpha^{n+1}\}$ for some $r, s \in p\mathbb{Z}_0^+$. Hence there does not exist a $y \in D_K^-$ such that $\alpha^n < y < \alpha^{n+2}$.

Step 5. We shall show that for every $n \in \mathbb{Z}^+$, $n-1 \not\equiv 0 \pmod{p}$ and $(n+1) \not\equiv 0 \pmod{p}$ and $n \equiv 0 \pmod{p}$ implies that there does not exist a $y \in D_K^-$ such that $\alpha^{n-1} < y < \alpha^{n+1}$. To prove this, let $n \in \mathbb{Z}^+$ be such that $n-1 \not\equiv 0 \pmod{p}$ and $n+1 \equiv 0 \pmod{p}$ and $n \equiv 0 \pmod{p}$. Now, we have that

$I_K(\alpha^{n+1}) = \{y \in K \mid \alpha^{n+1} \leqslant y \leqslant \alpha^r\}$ for some $r \in p\mathbb{Z}_o^+$. ....(vii)

By Step 6 of Theorem 3.1.7,
$$\begin{aligned}
I_K(\alpha^{n+1}) &= \alpha^{n+1} I_K(1) \\
&= \alpha^{n-1}(\alpha^2 I_K(1)) \\
&= \alpha^{n-1}\{u \in K \mid \alpha < u < 1\} \quad \text{(by (iii))} \\
&= \alpha^{n-1}(\{u \in K \mid \alpha \leqslant u \leqslant 1\} \setminus \{\alpha, 1\}) \\
&= \alpha^{n-1} I_K(1) \setminus \{\alpha^n, \alpha^{n-1}\} \\
&= I_K(\alpha^{n-1}) \setminus \{\alpha^n, \alpha^{n-1}\} \\
&= \{v \in K \mid \alpha^{n-1} < v \leqslant \alpha^s\} \setminus \{\alpha^n\}
\end{aligned}$$

for some $s \in p\mathbb{Z}_o^+$. ....(viii)

By (vii) and (viii), $\{y \in K \mid \alpha^{n+1} \leqslant y \leqslant \alpha^r\} = \{v \in K \mid \alpha^{n-1} < v \leqslant \alpha^s\} \setminus \{\alpha^n\}$ for some $r, s \in p\mathbb{Z}_o^+$. Therefore there does not exist $y \in D_K^-$ such that $\alpha^{n-1} < y < \alpha^{n+1}$.

Step 6. We shall show that $0 = \sup \{\alpha^n \mid n \in \mathbb{Z}^+ \setminus p\mathbb{Z}^+\}$. To prove this, let $T = \{\alpha^n \mid n \in \mathbb{Z}^+ \setminus p\mathbb{Z}^+\}$. Clearly $T \neq \emptyset$ since $\alpha \in T$. Since $\alpha^n < 0$ for all $n \in \mathbb{Z} \setminus p\mathbb{Z}$, 0 is an upper bound of T. Since $T \subseteq K$ and K is complete, T has a least upper bound. Let $\lambda = \sup(T)$. Then $\lambda \leqslant 0$. Suppose that $\lambda < 0$. We claim that $\lambda \in I_K(\alpha^p)$. To prove the claim, suppose not. Then $\lambda \notin I_K(\alpha^p)$. Therefore $\lambda + \alpha^p = \lambda$. Thus

$1+\lambda^{-1}\alpha^{p} = 1$, it follows that $\alpha \leqslant \lambda^{-1}\alpha^{p} < 0$. By Step 2 of Theorem 3.1.7, 3.1.7, $\alpha+\lambda^{-1}\alpha^{p} = \alpha$ which implies that $\lambda+\alpha^{p-1} = \lambda$. By Step 2 of Theorem 3.1.7, $\lambda \leqslant \alpha^{p-1}$, a contradiction. Hence we have the claim.

By the claim, $\lambda \in I_K(\alpha^p) = \alpha^p I_K(1) =$

$\alpha^p\{y \in K \mid \alpha \leqslant y \leqslant 1\} = \alpha^p(\{s \in D_K^- \mid \alpha \leqslant s\} \cup \{0\} \cup \{t \in D_K^+ \mid t \leqslant 1\}$

$= \alpha^p\{s \in D_K^- \mid \alpha \leqslant s\} \cup \{0\} \cup \alpha^p\{t \in D_K^+ \mid t \leqslant 1\}$ which implies that

$\lambda \in \alpha^p\{s \in D^- \mid \alpha < s\}$. Therefore $\lambda = \alpha^p u$ for some $u \in \{s \in D_K^- \mid \alpha \leqslant s\}$.

Case 1: $\lambda = u$. Then $u = \alpha^p u$, so $\alpha^p = 1$, a contradiction.

Case 2: $\lambda < u$. Then $\alpha^p u < u$. Since $0 < \alpha^p < 1$ and $u < 0$, by Step 1, $u < \alpha^p u$, a contradiction.

Case 3: $u < \lambda$. Then $u = \alpha^r$ for some $r \in \mathbb{Z}^+ \setminus p\mathbb{Z}^+$. Thus $\lambda = \alpha^p u = \alpha^p\alpha^r = \alpha^{p+r} < \alpha^{2p+r} \leqslant \lambda$, a contradiction.

This shows that $0 = \sup\{\alpha^n \mid n \in \mathbb{Z}^+ \setminus p\mathbb{Z}^+\}$.

Hence $D_K^- = \{\alpha^n \mid n \in \mathbb{Z} \setminus p\mathbb{Z}\}$. Let K(p) be the complete ordered 0-semifield given in Remark 3.1.8. Define $f: (K,+,\cdot,\leqslant) \rightarrow (K_{(p)},+,\cdot,\leqslant)$ in the following way: Define $f(0) = 0$. Let $x \in K \setminus \{0\}$. If $x \in D_K^-$, then $x = \alpha^m$ for some $m \in \mathbb{Z} \setminus p\mathbb{Z}$. Define $f(x) = \beta^m$. If $x \in D_K^+$, then $x = 2^k$ for some $k \in \mathbb{Z}$. Define $f(x) = \beta^{-pk}$. Clearly, f is well-defined and f is a bijection.

We shall first show that for every $x, y \in K$, $x \leqslant y$ implies that $f(x) \leqslant f(y)$. To prove this, let $x, y \in K$ be such that $x \leqslant y$.

Case 1: $x \leqslant 0 \leqslant y$. This case is clear.

Case 2: $x \leqslant y < 0$. Then $x = \alpha^n$ for some $n \in \mathbb{Z} \setminus p\mathbb{Z}$ and $y = \alpha^m$ for some $m \in \mathbb{Z} \setminus p\mathbb{Z}$. Suppose that $m < n$. Then $n-m \in \mathbb{Z}^+$. By assumption and Step 2 of Theorem 3.1.7, $x+y = x$. Then $\alpha^n + \alpha^m = \alpha^n$ which implies that $\alpha^{n-m} + 1 = \alpha^{n-m}$. By Step 12 of Theorem 3.1.7 and the fact that $n-m \in \mathbb{Z}^+$, $\alpha^{n-m} + 1 = 1$. Thus $\alpha^{n-m} = 1$ which contradicts Step 6 of Theorem 3.1.7. Therefore $n \leqslant m$. Hence $\beta^n \leqslant \beta^m$. Thus $f(x) \leqslant f(y)$.

<u>Case 3</u>: $0 < x < y$: Then $x = 2^k$ for some $k \in \mathbb{Z}$ and $y = 2^\ell$ for some $\ell \in Z$. Since $2^k < 2^\ell$, $k < \ell$. It follows that $-p\ell < -pk$. Thus $\beta^{-pk} \leqslant \beta^{-p\ell}$. Hence $f(x) \leqslant f(y)$.

Next we shall show that for every $x, y \in K$, $f(x+y) = f(x)+f(y)$. To prove this, let $x, y \in K$ be arbitrary. If either $x = 0$ or $y = 0$, then the result is clear. Suppose that $x, y \in K \setminus \{0\}$.

<u>Case 1</u>: $x \in D_K^+$ and $y \in D_K^+$. Then $x = 2^k$ for some $k \in \mathbb{Z}$ and $y = 2^\ell$ for some $\ell \in \mathbb{Z}$. Without loss of generality, suppose that $x \leqslant y$. Then $x+y = y$, it follows that $f(x+y) = f(y)$. Now, we have that $k \leqslant \ell$. Then $-p\ell \leqslant -pk$. Therefore $f(x) + f(y) = \beta^{-pk} + \beta^{-p\ell} = \beta^{-p\ell} = f(y)$. Hence $f(x+y) = f(x) + f(y)$.

<u>Case 2</u>: $x \in D_K^-$ and $y \in D_K^-$. Then $x = \alpha^n$ for some $n \in \mathbb{Z} \setminus p\mathbb{Z}$ and $y = \alpha^m$ for some $m \in \mathbb{Z} \setminus p\mathbb{Z}$. Without loss of generality, suppose that $x \leqslant y$. By Step 2 of Theorem 3.1.7, $x+y = x$, therefore $f(x+y) = f(x)$. Since $\alpha^n + \alpha^m = \alpha^n$, $\alpha^{m-n} + 1 = 1$ which implies that $m-n \geqslant 0$. Therefore $n \leqslant m$. Thus $f(x) + f(y) = \beta^n + \beta^m = \beta^n = f(x)$. Hence $f(x+y) = f(x)+f(y)$. $f(x+y) = f(x) + f(y)$.

<u>Case 3</u>: $x \in D_K^-$ and $y \in D_K^+$. Then $x = \alpha^n$ for some $n \in \mathbb{Z} \setminus p\mathbb{Z}$ and $y = 2^k$ for some $k \in \mathbb{Z}$. By assumption and Step 4 of Theorem 3.1.7, $x+y = x$ or $x+y = y$.

Subcase 3.1: $x+y = x$. Then $f(x+y) = f(x)$. By Step 2, $y = 2^k = (2^{-1})^{-k} = (\alpha^p)^{-k} = \alpha^{-pk}$. Thus $\alpha^n + \alpha^{-pk} = \alpha^n$, it follows that $\alpha^{-pk-n} + 1 = 1$. Then $-pk - n > 0$, so $n < -pk$. Thus $f(x) + f(y) = \beta^n + \beta^{-pk} = \beta^n = f(x)$. Hence $f(x+y) = f(x) + f(y)$.

Subcase 3.2: $x+y = y$. Then $f(x+y) = f(y)$. By Step 2, $y = 2^k = (2^{-1})^{-k} = (\alpha^p)^{-k} = \alpha^{-pk}$. Thus $\alpha^n + \alpha^{-pk} = \alpha^{-pk}$, it follows that $\alpha^{n+pk} + 1 = 1$. Then $n+pk > 0$, so $-pk < n$. Thus $f(x) + f(y) = \beta^n + \beta^{-pk} = \beta^{-pk} = f(y)$. Hence $f(x+y) = f(x) + f(y)$.

Case 4: $x \in D_K^+$ and $y \in D_K^-$. The proof is similar to the proof of Case 3.

Lastly, we must show that for every $x, y \in K$, $f(xy) = f(x)f(y)$. To prove this, let $x, y \in K$ be arbitrary. If either $x = 0$ or $y = 0$, then the result is clear. Suppose that $x, y \in K \setminus \{0\}$.

Case 1: $x \in D_K^+$ and $y \in D_K^+$. Then $x = 2^k$ for some $k \in \mathbb{Z}$ and $y = 2^\ell$ for some $\ell \in \mathbb{Z}$. Thus $xy = 2^{k+\ell}$. Then $f(xy) = \beta^{-p(k+\ell)}$. Now, we have that $f(x) = \beta^{-pk}$ and $f(y) = \beta^{-p\ell}$. Therefore we get that $f(x)f(y) = (\beta^{-pk})(\beta^{-p\ell}) = \beta^{-p(k+\ell)}$. Hence $f(xy) = f(x)f(y)$.

Case 2: $x \in D_K^-$ and $y \in D_K^-$. Then $x = \alpha^n$ for some $n \in \mathbb{Z} \setminus p\mathbb{Z}$ and $y = \alpha^m$ for some $m \in \mathbb{Z} \setminus p\mathbb{Z}$. Thus $xy = \alpha^{n+m}$

Subcase 2.1: $n+m \in \mathbb{Z} \setminus p\mathbb{Z}$. Then $f(xy) = \beta^{n+m} = \beta^n\beta^m = f(x)f(y)$ $= f(x)f(y)$

Subcase 2.2: $n+m \in p\mathbb{Z}$. Then $n+m = p\ell$ for some $\ell \in \mathbb{Z}$. Thus $xy = \alpha^{n+m} = \alpha^{p\ell} = (\alpha^p)^\ell = (2^{-1})^\ell = 2^{-\ell}$. Therefore $f(xy) = \beta^{-p(-\ell)} = \beta^{p\ell} = \beta^{n+m} = \beta^n\beta^m = f(x)f(y)$.

Case 3: $x \in D_K^-$ and $y \in D_K^+$. Then $x = \alpha^m$ for some $m \in \mathbb{Z} \setminus p\mathbb{Z}$ and $y = 2^k$ for some $k \in \mathbb{Z}$. Thus $y = (2^{-1})^{-k} = (\alpha^p)^{-k} = \alpha^{-pk}$. Then $xy = \alpha^{m-pk}$ and $m-pk \in \mathbb{Z} \setminus p\mathbb{Z}$. Therefore we get that $f(xy) = \beta^{m-pk} = \beta^m\beta^{-pk} = f(x)f(y)$.

Case 4: $x \in D_K^+$ and $y \in D_K^-$. The proof is similar to the proof of Case 3.

This shows that f is an isomorphism.

To finish the proof we must show that if $p, q > 1$ are distinct, then $K_{(p)}$ is not isomorphic to $K_{(q)}$. Let $p, q \in \mathbb{Z}^+ \setminus \{1\}$ be such that $p \neq q$. Without loss of generality, suppose that $p < q$. Then $K_{(p)} \setminus \{0\}$ and $K_{(q)} \setminus \{0\}$ are infinite cyclic group. Let β_1 and β_2 be generators of $K_{(p)} \setminus \{0\}$ and $K_{(q)} \setminus \{0\}$, respectively. Suppose that there is an isomorphism $F: K_{(p)} \setminus \{0\} \rightarrow K_{(q)} \setminus \{0\}$. Then $F(\beta_1) = \beta_2$ or $F(\beta_1) = \beta_2^{-1}$. Now, we have that $\beta_1^p > 0$, $\beta_2^p < 0$ and $\beta_2^{-p} < 0$. Then $0 < F(\beta_1^p) = (F(\beta_1))^p = \beta_2^p$ or β_2^{-p}, a contradiction.

Hence $K_{(p)}$ is not isomorphic to $K_{(q)}$. #

Remark 3.1.11. Let $C = \{-(2^n) \mid n \in \mathbb{Z} \setminus \{0\}\} \cup \{0,1\}$. Let the order on C be the usual order. Define addition and multiplication on C by

$$x + y = x \quad \text{if } |x| \geq |y|$$

$$x \cdot y = \begin{cases} 1 & \text{if } x = y^{-1} \\ -|xy| & \text{if } x \neq y^{-1} \end{cases}$$

where $|x|$ is the absolute value of x. Then $(C,+,\cdot,\leq)$ is a complete ordered 0-semifield as is shown below.

Proof: Clearly, C is closed under +, $\cdot$ and $(C,\leqslant)$ is an ordered set.

To show that + is associative, let $x, y, z \in C$.

Case 1: $|x| < |y| < |z|$. Then $x+(y+z) = x+z = z$ and $(x+y)+z = y+z = z$. Thus $x+(y+z) = (x+y)+z$.

Case 2: $|x| < |z| < |y|$. Then $x+(y+z) = x+y = y$ and $(x+y)+z = y+z = y$. Thus $x+(y+z) = (x+y)+z$.

Case 3: $|y| < |x| < |z|$. Then $x+(y+z) = x+z = z$ and $(x+y)+z = x+z = z$. Thus $x+(y+z) = (x+y)+z$.

Case 4: $|y| < |z| < |x|$. Then $x+(y+z) = x+z = x$ and $(x+y)+z = x+z = x$. Thus $x+(y+z) = (x+y)+z$.

Case 5: $|z| < |x| < |y|$. Then $x+(y+z) = x+y = y$ and $(x+y)+z = y+z = y$. Thus $x+(y+z) = (x+y)+z$.

Case 6: $|z| < |y| < |x|$. Then $x+(y+z) = x+y = x$ and $(x+y)+z = x+z = x$. Thus $x+(y+z) = (x+y)+z$.

Case 7: $|x| = |y|$.

Subcase 7.1: $|y| < |z|$. Then $x+(y+z) = x+z = z$ and $(x+y)+z = x+z = z$. Thus $x+(y+z) = (x+y)+z$.

Subcase 7.2: $|z| < |y|$. Then $x+(y+z) = x+y = x$ and $(x+y)+z = x+z = x$. Thus $x+(y+z) = (x+y)+z$.

Case 8: $|y| = |z|$

Subcase 8.1: $|x| < |y|$. Then $x+(y+z) = x+y = y$ and $(x+y)+z = y+z = y$. Thus $x+(y+z) = (x+y)+z$.

Subcase 8.2: $|y| < |x|$. Then x+(y+z) = x+y = x and x+(y+z) = x+y = x. Thus x+(y+z) = (x+y)+z.

Case 9: $|x| = |z|$.

Subcase 9.1: $|x| < |y|$. Then x+(y+z) = x+y = y and (x+y)+z = y+z = y. Thus x+(y+z) = (x+y)+z.

Subcase 9.2: $|y| < |x|$. Then x+(y+z) = x+z = x and (x+y)+z = x+z = x. Thus x+(y+z) = (x+y)+z.

This shows that + is an associative.

To show that $(C \setminus \{0\},\cdot)$ is an abelian group, let $x \in C \setminus \{0\}$. We shall show that $x^{-1} \in C \setminus \{0\}$.

Case 1 x = 1. Then we are done.

Case 2 $x \in \{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\}$. Then $x = -(2^m)$ for some $m \in \mathbb{Z} \setminus \{0\}$. Now, we have that $-m \in \mathbb{Z} \setminus \{0\}$, so $x^{-1} = (-(2^m))^{-1} = -(2^{-m}) \in C \setminus \{0\}$.

Clearly, multiplication is commutative and associative and x1 = 1x = x for all $x \in C \setminus \{0\}$. Therefore $(C \setminus \{0\},\cdot)$ is an abelian group.

To show that $(C,+,\cdot,\leqslant)$ satisfies the distributive law, let $x, y, z \in C$.

Case 1: $|y| > |z|$. Then x(y+z) = xy and $|x||y| > |x||z|$. Thus $|xy| > |xz|$, so xy+xz = xy. Thus x(y+z) = xy+xz. Similarly, (y+z)x = yx+zx.

Case 2: $|y| < |z|$. This proof is similar to the proof of Case 1.

Case 3: $|y| = |z|$. Then $|x||y| = |x||z|$, so $|xy| = |xz|$.

Therefore $x(y+z) = xy+xz$. Similarly, $(y+z)x = yx+zx$.

We shall show that for every $x, y \in C$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in C$. To prove this, let x, y be such that $x \leqslant y$. Let $z \in C$ be arbitrary. If $z \in \{0,1\}$, then we are done. Suppose that $z \in \{-(2^n) \mid n \in \mathbb{Z} \setminus \{0\}\}$.

<u>Case 1</u>: $x, y \in \{0,1\}$. Then we are done.

<u>Case 2</u>: $x \in \{-(2^n) \mid n \in \mathbb{Z} \setminus \{0\}\}$ and $y \in \{0,1\}$.

<u>Subcase 2.1</u> $x \leqslant z < 0 \leqslant y$. Then $|z| \leqslant |x|$, so $x+z = x$. Thus $x+y \leqslant y+z$.

<u>Subcase 2.2</u> $z \leqslant x < 0 \leqslant y$. Then $|x| \leqslant |z|$, so $x+z = z$. Thus $x+y \leqslant y+z$.

<u>Case 3</u>: $x, y \in \{-(2^n) \mid n \in \mathbb{Z} \setminus \{0\}\}$.

<u>Subcase 3.1</u> $x \leqslant z \leqslant y < 0$. Then $|z| \leqslant |x|$ and $|y| \leqslant |z|$, so $x+z = x$ and $y+z = z$. Thus $x+z < y+z$

<u>Subcase 3.2</u> $z \leqslant x \leqslant y < 0$. Then $|y| \leqslant |x| \leqslant |z|$, so $x+z = z$ and $y+z = z$. Thus $x+z = y+z$.

<u>Subcase 3.3</u> $x \leqslant y \leqslant z < 0$. Then $|z| \leqslant |y| \leqslant |x|$, so $x+z = x$ and $y+z = y$. Thus $x+z \leqslant y+z$.

To show that for every $x, y \in C$, $x \leqslant y$ implies that $xz \leqslant yz$ for all $z \neq x^{-1}$, let $x, y \in C$ be such that $x \leqslant y$. Let $z \in C \setminus \{x^{-1}\}$. If $z \in \{0,1\}$, then we are done. Suppose that $z \in \{-(2^n) \mid n \in \mathbb{Z} \setminus \{0\}\}$.

<u>Case 1</u>: $x = 0$ and $y = 0$. This case is clear.

Case 2: $x = 0$ and $y = 1$. This case is clear.

Case 3: $x = 1$ and $y = 1$. This case is clear.

Case 4: $x \leqslant y < 0$. Then $x = -(2^n)$ and $y = -(2^m)$ and $z = -(2^\ell)$ for some $m, n, \ell \in \mathbb{Z} \setminus \{0\}$. Therefore we get that $xz = -|-(2^n)(-(2^\ell))| = -|2^{n+\ell}| = -(2^{n+\ell})$ and $yz = -|-(2^m)(-(2^\ell))| = -|2^{m+\ell}| = -(2^{m+\ell})$. Now, we have that $m \leqslant n$, so $m+\ell \leqslant n+\ell$. Thus $2^{m+\ell} \leqslant 2^{n+\ell}$, it follows that $-(2^{n+\ell}) \leqslant -(2^{m+\ell})$ Therefore $xz \leqslant yz$.

Lastly, to show that $(C,\leqslant)$ is complete. Now, we assume that $H \subseteq C$ be a nonempty set which has an upper bound. Let w be an upper bound of H. Now, we shall show that H has a least upper bound in C.

Case 1: $w \geqslant 0$. Then 0 or 1 is a least upper bound of H. Then we are done.

Case 2: $w < 0$. Then $a \leqslant w < 0$ for all $a \in H$. Thus $H \subseteq \{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\}$. We claim that $(\mathbb{Z} \setminus \{0\}, \leqslant)$ is isomorphic to $(\{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\}, \leqslant)$. To prove the claim, define $f: \{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\} \to \mathbb{Z} \setminus \{0\}$ in the following way: Let $x \in \{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\}$. Then $x = -(2^n)$ for some $n \in \mathbb{Z} \setminus \{0\}$. Define $f(x) = -n$. Clearly, f is well-defined and f is a bijection. To show that f is an order map, let $x, y \in \{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\}$ be such that $x \leqslant y$. Then $x = -(2^n)$ and $y = -(2^m)$ for some $n, m \in \mathbb{Z} \setminus \{0\}$. Therefore $m \leqslant n$, so $-n \leqslant -m$. Thus $f(x) = -n \leqslant -m = f(y)$, so we have the claim. By the claim $(\{-(2^n) | n \in \mathbb{Z} \setminus \{0\}\}, \leqslant)$ is complete. Hence H has a least upper bound in C. #

Theorem 3.1.12. Let $(K,+,\cdot,\leqslant)$ be a complete ordered 0-skew semifield. If $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\{1\},+,\cdot,\leqslant)$, then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following 0-semifield:

(1) $(\{0,1\},+,\cdot,\leqslant)$ as in **Boolean semifield**.

(2) $(C,+,\cdot,\leqslant)$ as in Remark 3.1.11.

Proof: Assume that $(D_K^+,+,\cdot,\leqslant)$ is isomorphic to $(\{1\},+,\cdot,\leqslant)$. Now, we have that $K = D_K^- \cup \{0\} \cup D_K^+$. If $D_K^- = \emptyset$, then $(K,+,\cdot,\leqslant)$ is isomorphic to (1). Suppose that $D_K^- \neq \emptyset$. Step 2 to Step 12 of Theorem 3.1.7 hold with these hypotheses and the proofs are exactly the same. As in Theorem 3.1.7 α will denote $\inf(I_K^-(1))$.

Step 1. We shall show that $\alpha^n < 0$ for all $n \in \mathbb{Z}\setminus\{0\}$. It suffices to show that $\alpha^n < 0$ for all $n \in \mathbb{Z}^+$. To prove this, suppose not. Then $0 < \alpha^m$ for some $m \in \mathbb{Z}^+$. If $\alpha^m = 0$, then $\alpha = \alpha^{-(m-1)}0 = 0$, a contradiction. Thus $0 < \alpha^m$, so $\alpha^m = 1$ which contradicts Step 6 of Theorem 3.1.7.

Step 2. For every $n \in \mathbb{Z}\setminus\{0\}$, $\alpha^n < \alpha^{n+1}$. This proof is the same as the proof of Step 14 in Theorem 3.1.7.

Step 3. We shall show that for every $x, y, z \in D_K^-$, $x < y$ and $z \neq x^{-1}$ implies that $zx < zy$ and $xz < yz$. To prove this, let $x, y, z \in D_K^-$ be such that $x < y$ and $z \neq x^{-1}$. By Step 2 of Theorem 3.1.7, $x+y = x$. Therefore $zx = z(x+y) = zx+zy$. ....(*)

Case 1: $zy \in D_K^+$. Then $zx+zy \leqslant zy$. From (*), we have that $zx \leqslant zy$. If $zx = zy$, then $x = y$ which is a contradiction. Therefore $zx < zy$.

Case 2: $zy \in D_K^-$. From (*) and Step 2 of Theorem 3.1.7, we have that $zx \leqslant zy$. If $zx = zy$, then $x = y$, a contradiction. Therefore $zx < zy$.

Hence $zx < zy$. Similarly, $xz < yz$.

Step 4. We shall show that for every $n \in \mathbb{Z} \setminus \{0\}$ there does not exist $y \in D_K^-$ such that $\alpha^n < y < \alpha^{n+1}$. To prove this, suppose not. Then $\alpha^m < y < \alpha^{m+1}$ for some $m \in \mathbb{Z} \setminus \{0\}$ and for some $y \in D_K^-$. By Step 2, $\alpha^m < y < \alpha^{m+2}$. ....(I)

Since $\alpha^{-m} = (\alpha^m)^{-1}$, $\alpha^{-m-1} \neq (\alpha^m)^{-1}$. From (I) and Step 3, we have that $(\alpha^{-m-1})\alpha^m < \alpha^{-m-1}y$. Then $\alpha^{-1} < \alpha^{-m-1}y$. ....(II)

If $\alpha^{-m-1} = y^{-1}$, then $\alpha^{m+1} = y$, a contradiction. Thus $\alpha^{-m-1} \neq y^{-1}$. From (I) and Step 3, $\alpha^{-m-1}y < (\alpha^{-m-1})\alpha^{m+2}$. Then $\alpha^{-m-1}y < \alpha$. ....(III)

From (II) and (III), we have that, $\alpha^{-1} < \alpha^{-m-1}y < \alpha$ which contradicts Step 11 of Theorem 3.1.7.

Step 5. We shall show that $0 = \sup\{\alpha^n | n \in \mathbb{Z}^+\}$. To prove this, let $L = \{\alpha^n | n \in \mathbb{Z}^+\}$. By Step 1, 0 is an upper bound of L. Since $L \subseteq K$ and K is complete, L has a least upper bound. Let $\lambda = \sup(L)$. Then $\lambda \leqslant 0$. Suppose that $\lambda < 0$. We claim that λ is lower discrete. To prove this claim, if $\lambda = \alpha^n$ for some $n \in \mathbb{Z} \setminus \{0\}$, then by Step 4, we are done. Suppose that $\lambda \neq \alpha^n$ for all $n \in \mathbb{Z} \setminus \{0\}$. Now, we have that $\alpha^{-1} < \alpha < 0$. ....(i)

If $\alpha^{-1}\lambda = (\alpha^{-1})^{-1}$, then $\alpha^{-1}\lambda = \alpha$. It follows that $\lambda = \alpha^2$, a contradiction. Thus $\alpha^{-1}\lambda \neq (\alpha^{-1})^{-1}$. From (i) and Step 3, we have that $\alpha^{-1}(\alpha^{-1}\lambda) < \alpha(\alpha^{-1}\lambda)$. Then $\alpha^{-2}\lambda < \lambda$. We shall show that there does not exist $y \in D_K^-$ such that $\alpha^{-2}\lambda < y < \lambda$. To prove this, suppose not. Then there exists a $y \in D_K^-$ such that $\alpha^{-2}\lambda < y < \lambda$. ....(ii)

If $\lambda^{-1}\alpha = (\alpha^{-2}\lambda)^{-1}$, then $\lambda^{-1}\alpha = \lambda^{-1}\alpha^2$. Thus $\alpha = \alpha^2$ which implies that $\alpha = 1$, a contradiction. Therefore $\lambda^{-1}\alpha \neq (\alpha^2\lambda)^{-1}$. From (ii) and Step 3, we have that $\alpha^{-2}\lambda(\lambda^{-1}\alpha) < y(\lambda^{-1}\alpha)$, hence

$\alpha^{-1} < y(\lambda^{-1}\alpha)$. ....(iii)

If $\lambda^{-1}\alpha = y^{-1}$, then $\alpha = \lambda y^{-1}$. Since $y < \lambda$, $y < \alpha^m < \lambda$ for some $m \in \mathbb{Z} \setminus \{0\}$. Thus $\alpha^m y^{-1} < \lambda y^{-1} = \alpha$, it follows that $\alpha^m y^{-1} + 1 = \alpha^m y^{-1}$. Then $\alpha^m + y = \alpha^m$. By Step 2 of Theorem 3.1.7, $\alpha^m \leqslant y$, a contradiction. Therefore $\lambda^{-1}\alpha \neq y^{-1}$. From (ii) and Step 3, we have that

$y(\lambda^{-1}\alpha) < \lambda(\lambda^{-1}\alpha) = \alpha$. ....(iv)

By (iii) and (iv), $\alpha^{-1} < y(\lambda^{-1}\alpha) < \alpha$ which contradicts Step 11 of Theorem 3.1.7. Then there does not exist $y \in D_K^-$ such that $\alpha^{-2}\lambda < y < \lambda$. Therefore λ is lower discrete, so we have the claim. Since $\lambda^- < \lambda$, $\lambda^- \leqslant \alpha^\ell < \lambda$ for some $\ell \in \mathbb{Z} \setminus \{0\}$. If $\lambda^- = \alpha^\ell$, then $\lambda^- < \alpha^{\ell+1} < \lambda$, a contradiction. Thus $\lambda^- < \alpha^\ell < \lambda$, a contradiction. Hence $\lambda = 0$.

<u>Step 6</u>. We shall show that $\{\alpha^{-n} | n \in \mathbb{Z}^+\}$ has no lower bound in D_K^-. To prove this, suppose not. Then $\{\alpha^{-n} | n \in \mathbb{Z}^+\}$ has lower bound in D_K^-. Thus $\{\alpha^{-n} | n \in \mathbb{Z}^+\}$ has a greatest lower bound. Let $w = \inf\{\alpha^{-n} | n \in \mathbb{Z}^+\}$, so $w \leqslant \alpha^{-n}$ for all $n \in \mathbb{Z}^+$. Suppose that $w = \alpha^{-m}$ for some $m \in \mathbb{Z}^+$. Since $\alpha^m < \alpha^{m+1}$, by Step 2, $(\alpha^{m+1})^{-1} < (\alpha^m)^{-1}$. Thus $\alpha^{-(m+1)} < \alpha^{-m} = w$, a contradiction. Therefore $w < \alpha^{-n}$ for all $n \in \mathbb{Z}^+$. By Step 2 again $(\alpha^{-n})^{-1} < w^{-1}$ for all $n \in \mathbb{Z}^+$, so $\alpha^n < w^{-1}$ for all $n \in \mathbb{Z}^+$. Since $0 = \sup\{\alpha^n | n \in \mathbb{Z}^+\}$, $0 \leqslant w^{-1}$ which is a contradiction.

From Step 4, Step 5 and Step 6, we have that $D_K^- = \{\alpha^n | n \in \mathbb{Z} \setminus \{0\}\}$. Let C be the complete ordered 0-semifield given in Remark 3.1.11. Define $f: (K,+,\cdot,\leqslant) \to (C,+,\cdot,\leqslant)$ in the following way: $f(0) = 0$

and $f(1) = 1$. Let $x \in D^-$ be arbitrary. Then $x = \alpha^n$ for some a unique $n \in \mathbb{Z} \setminus \{0\}$. Let $f(x) = -2^{-n}$. Clearly, f is well-defined and f is a bijection.

To show that f is an order map, let x, y $\in$ K be such that $x \leqslant y$. We must show that $f(x) \leqslant f(y)$

Case 1: $y \in \{0,1\}$. Then we are done.

Case 2: $x \leqslant y < 0$. Then $x = \alpha^n$ and $y = \alpha^m$ for some $m, n \in \mathbb{Z} \setminus \{0\}$. Therefore $\alpha^n \leqslant \alpha^m$, so $n \leqslant m$ which implies that $2^{-m} \leqslant 2^{-n}$. Thus $-(2^{-n}) \leqslant -(2^{-m})$. Hence $f(x) \leqslant f(y)$.

To show that $f(x+y) = f(x)+f(y)$ for all x, y $\in$ K. To prove this, let x, y $\in$ K be arbitrary.

Case 1: $x = 0$ and $y = 0$. Then we are done

Case 2: $x = 0$ and $y \in K$. Then $f(x+y) = f(0+y) = f(y) = 0+f(y) = f(0)+f(y) = f(x)+f(y)$.

Case 3: $y = 0$ and $x \in K$. This proof is similar to the proof of Case 2.

Case 4: $x = 1$ and $y = 0$. Then we are done.

Case 5: $x = 1$ and $y = 1$. Then we are done.

Case 6: $x = 1$ and $y \in D_K^-$. Then $y+1 = y$ or $y+1 = 1$.

Subcase 6.1: $y+1 = y$. Then $f(y+1) = f(y)$. ....(*)

Since $y \in D_K^-$ and $y+1 = y$, $y < \alpha$. Thus $y = \alpha^{-n}$ for some $n \in \mathbb{Z}^+$. Therefpre $f(y) = -(2^{-(-n)}) = -(2^n)$, so $|f(y)| = |-(2^n)| = 2^n > 1 = |f(1)|$. Then $f(y)+f(1) = f(y)$. ....(**)

From (*) and (**), $f(y+1) = f(y)+f(1)$.

Subcase 6.2: $y+1 = 1$. Then $f(y+1) = f(1)$. ....(I)

Since $y \in D_K^-$ and $y+1 = 1$, $\alpha \leqslant y$. Thus $y = \alpha^n$ for some $n \in \mathbb{Z}^+$.

Therefore $f(y) = -(2^{-n})$, so $|f(y)| = |-(2^{-n})| = 2^{-n} < 1 = |f(1)|$.

Then $f(y)+f(1) = f(1)$ ....(II)

From (I) and (II), $f(y+1) = f(y)+f(1)$.

Case 7: $x \in D_K^-$ and $y = 1$. This proof is similar to the proof of Case 6.

Case 8: $x \in D_K^-$ and $y \in D_K^-$. Then either $x \leqslant y$ or $y \leqslant x$.

Subcase 8.1: $x \leqslant y$. By Step 2 of Theorem 3.1.7, $x+y = x$. Then $f(x+y) = f(x)$. Since $x \leqslant y < 0$ and f is an order map, $f(x) \leqslant f(y) < 0$ which implies that $0 < |f(y)| < |f(x)|$. Thus $f(x)+f(y) = f(x)$. Hence $f(x+y) = f(x)+f(y)$.

Subcase 8.2: $y \leqslant x$. By Step 2 of Theorem 3.1.7, $y+x = y$. Then $f(y+x) = f(y)$. Since $y \leqslant x < 0$ and f is an order map, $f(y) \leqslant f(x) < 0$ which implies that $0 < |f(x)| \leqslant |f(y)|$. Thus $f(x)+f(y) = f(y)$. Hence $f(x+y) = f(x)+f(y)$.

Lastly, we must show that $f(xy) = f(x)f(y)$ for all $x, y \in K$. To prove this, let $x, y \in K$ be arbitrary.

Case 1: $x = 0$, $y \in K$. This case is clear.

Case 2: $y = 0$, $x \in K$. This case is clear.

Case 3: $x = 1$, $y \in K$. This case is clear.

Case 4: $y = 1$, $x \in K$. This case is clear.

Case 5: $x \in D_K^-$ and $y \in D_K^-$. Then $x = \alpha^n$ and $y = \alpha^m$ for some n, $m \in \mathbb{Z} \setminus \{0\}$. Thus $xy = \alpha^n\alpha^m = \alpha^{n+m}$. Therefore we get that $f(xy) = -(2^{-n-m})$. Now, we have that $f(x) = -(2^{-n})$ and $f(y) = -(2^{-m})$. Then $f(x)f(y) = -|-(2^{-n})(-(2^{-m}))| = -|2^{-n-m}| = -(2^{-n-m})$. Therefore $f(xy) = f(x)f(y)$.

Hence f is an isomorphism. #

Section 2. ∞-Skew Semifields

Definition 3.2.1. A system $(K,+,\cdot,\leqslant)$ is called an ordered ∞-skew semifield iff $(K,+,\cdot)$ is an ∞-skew semifield and $\leqslant$ is an order on K satisfying the following properties:

(i) For any $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \leqslant \infty$ in K.

(ii) For any $x, y \in K$, $x \leqslant y$ implies that $xz \leqslant yz$ and $zx \leqslant zy$ for all $z \leqslant \infty$ in K.

(iii) $1 < \infty$.

Notation: Let K be an ∞-skew semifield. Then we will denote $Cor_K(x) = \{y \in K \mid x+y = \infty\}$ for all $x \in K$.

Proposition 3.2.2. Let K be an ∞-skew semifield. Then the following properties hold:

(1) For every $x \in K$, $\infty \in Cor_K(x)$.

(2) $Cor_K(\infty) = K$.

(3) For every $x \in K \setminus \{\infty\}$, $Cor_K(x) = x\, Cor_K(1)$.

(4) If $Cor_K(1) = \{\infty\}$, then $x+y \neq \infty$ for all $x, y \in K \setminus \{\infty\}$.

(5) If $x \in Cor_K(1)$ and $x \neq \infty$, then $x^{-1} \in Cor_K(1)$.

Proof: The proof of 1) and 2) are obvious.

To show 3), let $x \in K \setminus \{\infty\}$ be arbitrary. We must show that $Cor_K(x) = x\, Cor_K(1)$. To prove this, let $y \in Cor_K(x)$ be arbitrary. Then $y+x = \infty$ which implies that $x^{-1}y + 1 = \infty$. Therefore $x^{-1}y \in Cor_K(1)$. Thus $y \in x\, Cor_K(1)$. Hence $Cor_K(x) \subseteq x\, Cor_K(1)$. On the other hand, let $z \in x\, Cor_K(1)$ be arbitrary. Then $x^{-1}z + 1 = \infty$ which implies that $z+x = \infty$. Therefore $z \in Cor_K(x)$. Hence $x\, Cor_K(1) \subseteq Cor_K(x)$. Thus $Cor_K(x) = x\, Cor_K(1)$.

To show 4), suppose that $Cor_K(1) = \{\infty\}$. Let $x, y \in K \setminus \{\infty\}$ be arbitrary. If $x+y = \infty$, then $y \in Cor_K(x)$. By 3), $y \in x\, Cor_K(1)$ which implies that $x^{-1}y \in Cor_K(1) = \{\infty\}$. Thus $x^{-1}y = \infty$. Then $\infty = x\infty = x(x^{-1}y) = (xx^{-1})y = y$, a contradiction. Hence $x+y \neq \infty$.

To show 5), let $x \in Cor_K(1) \setminus \{\infty\}$. Then $x+1 = \infty$ which implies that $1+x^{-1} = \infty$. Hence $x^{-1} \in Cor_K(1)$. #

Notation: Let K be an ordered ∞-skew semifield. Then we will denote $D_K^f = \{x \in K \mid x < \infty\}$ and $D_K^i = \{x \in K \mid x > \infty\}$. Note that $1 \in D_K^f$, so D_K^f is never the empty set.

Proposition 3.2.3. Let $(K,+,\cdot,\leqslant)$ be an ordered ∞-skew semifield. Then the following properties hold:

(1) For every $x, y \in K \setminus \{\infty\}$, $xy \in K \setminus \{\infty\}$.

(2) For every $x, y \in K$, $x < y$ implies that $xz < yz$ and $zx < zy$ for all $z \in D_K^f$.

(3) For every $x, y \in D_K^f$, $x^{-1} \in D_K^f$ and $xy \in D_K^f$ and $x+y \leqslant \infty$.

(4) For every $x \in D_K^f$ and for every $y \in D_K^i$, $xy \in D_K^i$.

(5) For every $x, y \in D_K^i$, $x+y < \infty$ implies that $xD_K^f \cap yD_K^i = \emptyset$.

(6) Suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Then for every $x, y \in D_K^f$, $x+y \in D_K^f$ and $xy^{-1} \in D_K^f$.

(7) Suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Then for every $x \in D_K^f$ and for every $y \in D_K^i$, $x+y = \infty$.

(8) Suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Then for every $x, y \in D_K^i$,

$$x+y = \begin{cases} \infty & \text{if } y \notin xD_K^f, \\ > \infty & \text{if } y \in xD_K^f. \end{cases}$$

(9) Suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Then for every $x, y \in D_K^i$,

$$xy = \begin{cases} < \infty & \text{if } y \in x^{-1}D_K^f, \\ > \infty & \text{if } y \notin x^{-1}D_K^f. \end{cases}$$

(10) Suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Then D_K^f is a normal subgroup of $K \setminus \{\infty\}$.

(11) Suppose that $Cor_K(1) = K$. Then $x+y = \infty$ for all $x \in D_K^f$ and for all $y \in D_K^i$.

<u>Proof:</u> The proof of (1) is obvious.

To show (2), let $x, y \in K$ be such that $x < y$. Let $z \in D_K^f$ be arbitrary. Then $z < \infty$. By Definition 3.2.1 (ii), $xz \leqslant yz$ and $zx \leqslant zy$. If $zx = zy$, then $z^{-1}(zx) = z^{-1}(zy)$ which implies that $x = y$, a contradiction. Thus $xz < yz$. Similarly, $zx < zy$.

To show (3), let $x, y \in D_K^f$ be arbitrary. Then $x < \infty$ and $y < \infty$. If $\infty \leqslant x^{-1}$, then by Definition 3.2.1 (ii), $\infty = \infty x \leqslant x^{-1}x = 1$, a contradiction. Therefore $x^{-1} < \infty$. Thus $x^{-1} \varepsilon D_K^f$. Now, we shall show that $xy < \infty$. Suppose that $\infty \leqslant xy$. By Definition 3.2.1 (i), $\infty = x^{-1}\infty \leqslant x^{-1}(xy) = (x^{-1}x)y = y$, a contradiction. Therefore $xy < \infty$. Hence $xy \varepsilon D_K^f$. It is clear that $x+y \leqslant \infty$.

To show (4), let $x \varepsilon D_K^f$ and $y \varepsilon D_K^i$ be arbitrary. Then $x < \infty$ and $y > \infty$. By Definition 3.2.1 (ii), $\infty \leqslant xy$. If $xy = \infty$ then $x^{-1}(xy) = x^{-1}\infty = \infty$ which implies that $y = \infty$, a contradiction. Hence $\infty < xy$. Therefore $xy \varepsilon D_K^i$.

To show (5), let $x, y \varepsilon D_K^i$ be such that $x+y < \infty$. We shall show that $xD_K^f \cap yD_K^f = \emptyset$. To prove this, suppose not. Then $xD_K^f \cap yD_K^f \neq \emptyset$. Thus $xD_K^f = yD_K^f$, so $y \varepsilon xD_K^f$. Then $y = xd$ for some $d \varepsilon D_K^f$. Therefore $x+y = x+xd = x(1+d)$. By (4), $x+y > \infty$, a contradiction. Hence $xD_K^f \cap yD_K^f = \emptyset$.

To show (6), suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Let $x, y \varepsilon D_K^f$ be arbitrary. By (3), $xy^{-1} \varepsilon D_K^f$ and $x+y \leqslant \infty$. If $x+y = \infty$ then $1 + xy^{-1} = \infty$. Thus $xy^{-1} \varepsilon Cor_K(1)$, so $xy^{-1} \varepsilon \{\infty\} \cup D_K^i$, a contradiction. Therefore $x+y < \infty$. Hence $x+y \varepsilon D_K^f$.

To show (7), suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Let $x \varepsilon D_K^f$ and $y \varepsilon D_K^i$ be arbitrary. By (3), $x^{-1} \varepsilon D_K^f$. By (4), $x^{-1}y \varepsilon D_K^i$. Then $x^{-1}y \varepsilon Cor_K(1)$, it follows that $x^{-1}y + 1 = \infty$. Therefore $x+y = \infty$.

To show (8), suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Let $x, y \varepsilon D_K^i$ be arbitrary. Now, we have that $K = D_K^f \cup Cor_K(1)$. Therefore

$K = xK = x(D_K^f \cup Cor_K(1)) = xD_K^f \cup xCor_K(1)$. ....(*)

By (*) and Proposition 3.2.2 (3), $K = xD_K^f \cup Cor_K(x)$. ....(**)

If $y \in xD_K^f$, then $y = xd$ for some $d \in D_K^f$. Thus $x+y = x+xd = x(1+d) > \infty$ by (4). If $y \notin xD_K^f$, then by (**), $y \in Cor_K(x)$. Thus $x+y = \infty$.

To show (9), suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. Let $x, y \in D_K^i$ be arbitrary. If $y \in x^{-1}D_K^f$, then $y = x^{-1}d$ for some $d \in D_K^f$. Thus $xy = x(x^{-1}d) = (xx^{-1})d = d$. Therefore $xy < \infty$. Suppose that $y \notin x^{-1}D_K^f$. To show that $xy > \infty$, suppose not. Then $xy \leqslant \infty$. If $xy = \infty$ then $x^{-1}(xy) = x^{-1}\infty = \infty$. Thus $y = \infty$, a contradiction. If $xy < \infty$, then $xy = s$ for some $s \in D_K^f$. Thus $y = x^{-1}s \in x^{-1}D_K^f$, a contradiction. Therefore $xy > \infty$.

To show (10), suppose that $Cor_K(1) = \{\infty\} \cup D_K^i$. We shall show that D_K^f is a normal subgroup of $K \setminus \{\infty\}$. Let $x \in K \setminus \{\infty\}$ and $d \in D_K^f$ be arbitrary. Now, we have that $D_K^f = (Cor_K(1))^c$. We must show that $xdx^{-1} \in D_K^f$. If $x \in D_K^f$, then we are done. Suppose that $x \in D_K^i$. If $xdx^{-1} \notin D_K^f$, then $xdx^{-1} \in Cor_K(1)$. Thus $xdx^{-1} + 1 = \infty$, so $dx^{-1} + x^{-1} = \infty$ which implies that $d+1 = \infty$, a contradiction. Therefore $xdx^{-1} \in D_K^f$. Hence D_K^f is a normal subgroup of $K \setminus \{\infty\}$.

To show (11), suppose that $Cor_K(1) = K$. Let $x, y \in K$ be arbitrary. If $x = \infty$ or $y = \infty$, then we are done. Suppose that $x \neq \infty$ and $y \neq \infty$. Then $x^{-1}y \in K$, so $x^{-1}y \in Cor_K(1)$. Therefore $x^{-1}y + 1 = \infty$. Hence $y+x = \infty$. #

<u>Proposition 3.2.4</u>. Let $(K,+,\cdot,\leqslant)$ be a complete ordered ∞-skew semifield. Then the following properties hold:

(1) $(D_K^f,\leqslant)$ is a complete ordered set.

(2) If $D_K^f \neq \{1\}$, then for every $x \in D_K^i$, xD_K^f has no upper bound and has no lower bound in D_K^i.

(3) If $(H,\cdot)$ is a subgroup of $(D_K^f,\cdot)$ and $H \neq \{1\}$, then H has neither a lower bound nor an upper bound in D_K^f.

(4) If $(Cor_K(1))^c \neq \emptyset$, then $D_K^f \subseteq (Cor_K(1))^c$.

Proof: To show (1), let $A \subseteq D_K^f$ be a nonempty set having a lower bound in D_K^f. Then $A \subseteq K$. Since K is complete, A has a greatest lower bound in K. Let $w = \inf(A)$. Fix $a \in A$. Then $w \leqslant a < \infty$. Therefore $w \in D_K^f$. Hence $(D_K^f,\leqslant)$ is a complete ordered set.

To show (2), suppose that $D_K^f \neq \{1\}$. Let $x \in D_K^i$ be arbitrary. Without loss of generality, suppose that xD_K^f has an upper bound. Let $z = \sup(xD_K^f)$. Then $xd \leqslant z$ for all $d \in xD_K^f$. Let $s \in D_K^f \setminus \{1\}$. Thus $xds \leqslant z$ for all $d \in xD_K^f$, it follows that $xd \leqslant zs^{-1}$ for all $d \in D_K^f$. Therefore zs^{-1} is an upper bound of xD_K^f. Thus $z \leqslant zs^{-1}$, so $zs \leqslant z$.

.............(*)

Similarly, $xds^{-1} \leqslant z$ for all $d \in xD_K^f$ which implies that zs is an upper bound of xD_K^f. Then $z \leqslant zs$. ....(**)

From (*) and (**), we have that $z = zs$ which implies that $1 = s$, a contradiction.

To show (3), by Proposition 3.2.3 (3), $(D_K^f,\cdot)$ is a group. Suppose that $(H,\cdot)$ is a subgroup of $(D_K^f,\cdot)$ and $H \neq \{1\}$. By (1) and Proposition 3.2.3 (2), $(D_K^f,\cdot,\leqslant)$ is a complete ordered group. By Proposition 1.25, Theorem 1.26 and Theorem 1.28, $(D_K^f,\cdot,\leqslant)$ is either isomorphic to $(\{2^n \mid n \in \mathbb{Z}\},\cdot,\leqslant)$ or $(\mathbb{R}^+,\cdot,\leqslant)$. Let $x \in H \setminus \{1\}$ be arbitrary. Then either $x < 1$ or $1 < x$. Without loss of generality, suppose that $x < 1$. Then x^n has the property that for every $r \in D_K^f$ there exists an $N \in \mathbb{Z}^+$ such that $n \geqslant N$ implies that $x^n < r$. Therefore H has no lower bound in D_K^f. Now, we have that $1 < x^{-1}$. Then $(x^{-1})^m$ has the property that for every $s \in D_K^f$ there exists an $M \in \mathbb{Z}^+$ such that $m > M$ implies that $(x^{-1})^m > s$. Therefore H has no upper bound in D_K^f.

To show (4), suppose that $(Cor_K(1))^c \neq \emptyset$. First, we shall show that $((Cor_K(1))^c,\cdot)$ is a group. To prove this, let $x, y \in (Cor_K(1))^c$ be arbitrary. Then $x+1 \neq \infty$ and $y+1 \neq \infty$. It follows that $1+x^{-1} \neq \infty$. Thus $x^{-1} \in (Cor_K(1))^c$. If $xy+1 = \infty$, then $xy + x + y + 1 = \infty$. Thus $x(y+1) + y + 1 = \infty$ which implies that $(x+1)(y+1) = \infty$. Since $x+1 \neq \infty$, $y+1 = \infty$, a contradiction. Therefore $xy+1 \neq \infty$. Thus $xy \in (Cor_K(1))^c$, it follows that $1 \in (Cor_K(1))^c$. Hence $((Cor_K(1))^c,\cdot)$ is a group.

Clearly, if $D_K^f = \{1\}$, then (4) is true. Suppose that $D_K^f \neq \{1\}$. We claim that $D_K^f \cap (Cor_K(1))^c \neq \{1\}$. To prove this claim, let $x \in D_K^f$ be such that $x < 1$. Then $x+1 \leqslant 1+1 < \infty$. Thus $x \in D_K^f \cap (Cor_K(1))^c$. Hence $D_K^f \cap (Cor_K(1))^c \neq \{1\}$, so we have the claim.

Let $y \in D_K^f$ be arbitrary. Now, we have that $(D_K^f \cap (Cor_K(1))^c,\cdot)$ is a subgroup of $(D_K^f,\cdot)$. By the claim and (3), x is not an upper bound of $D_K^f \cap (Cor_K(1))^c$. Then there exists a $t \in D_K^f \cap (Cor_K(1))^c$ such that $y < t$. Therefore $y+1 \leqslant t+1 < \infty$. Thus $y \in (Cor_K(1))^c$. Hence $D_K^f \subseteq (Cor_K(1))^c$. #

Theorem 3.2.5 Let K be a complete ordered ∞-skew semifield. Then either $Cor_K(1) = \{\infty\}$ or $Cor_K(1) = \{\infty\} \cup D^i_K$ or $Cor_K(1) = K$.

Proof: Assume that $Cor_K(1) \neq \{\infty\}$ and $Cor_K(1) \neq K$. We must show that $Cor_K(1) = \{\infty\} \cup D^i_K$. To prove this, suppose not. Then $Cor_K(1) \neq \{\infty\} \cup D^i_K$. Therefore $(Cor_K(1))^c \neq D^f_K$. By Proposition 3.2.4, $D^f_K \subset (Cor_K(1))^c$. Hence $(Cor_K(1))^c \cap D^i_K \neq \emptyset$.

Case 1: $(Cor_K(1))^c \cap D^i_K$ has no lower bound in D^i_K. Let $y \in D^i_K$ be arbitrary. Then there exists a $z \in (Cor_K(1))^c \cap D^i_K$ such that $z < y$. Thus $\infty < 1+z \leqslant 1+y$. Then $y \in (Cor_K(1))^c$. Therefore $D^i_K \subseteq (Cor_K(1))^c$, so $D^f_K \cup D^i_K \subseteq D^f_K \cup (Cor_K(1))^c = (Cor_K(1))^c \subseteq K \setminus \{\infty\}$. Thus $K \setminus \{\infty\} = (Cor_K(1))^c$. Hence $Cor_K(1) = \{\infty\}$, a contradiction.

Case 2: $(Cor_K(1))^c \cap D^i_K$ has a lower bound in D^i_K. Since $(Cor_K(1))^c \cap D^i_K \subseteq K$ and K is complete, $(Cor_K(1))^c \cap D^i_K$ has a greatest lower bound. Let $\alpha = \inf((Cor_K(1))^c \cap D^i_K)$. Then $\alpha > \infty$. We claim that α is an upper bound of $Cor_K(1)$. To prove this claim, suppose not. Then there exists a $v \in Cor_K(1)$ such that $\alpha < v$. If $\alpha \in ((Cor_K(1))^c$, then $\infty < \alpha+1 \leqslant v+1$, a contradiction. Thus $\alpha \notin (Cor_K(1))^c$. Since $\alpha = \inf(Cor_K(1))^c \cap D^i_K$, there exists a $u \in (Cor_K(1))^c \cap D^i_K$ such that $\alpha < u < v$. Then $\infty < 1+\alpha \leqslant u+1 \leqslant v+1 = \infty$, a contradiction. Hence we have the claim. Since $Cor_K(1) \subseteq K$ and K is complete, $Cor_K(1)$ has a least upper bound. Let $\beta = \sup(Cor_K(1))$ Then $\beta \leqslant \alpha$. If $\beta \leqslant \infty$, then $Cor_K(1) \subseteq D^f_K \subseteq (Cor_K(1))^c$, a contradiction. Therefore $\infty < \beta \leqslant \alpha$

Subcase 2.1: $\beta < \alpha$. Then $\infty < \beta < \alpha$. If there exists a $t \in K$ such that $\beta < t < \alpha$. Then $\infty < t$ and $t+1 \neq \infty$, so $t \in (Cor_K(1))^c \cap D_K^i$. Thus $\alpha \leqslant t$, a contradiction. Therefore there does not exist a $t \in K$ such that $\beta < t < \alpha$. ....(*)

Let $s \in D_K^f \setminus \{1\}$. Then $\alpha s \neq \alpha$.

Subcase 2.1.1: $\alpha s < \alpha$. By (*), $\alpha s \leqslant \beta < \alpha$. Therefore $\alpha \leqslant \beta s^{-1} < \alpha s^{-1}$, so $\beta < \alpha \leqslant \beta s^{-1}$. Thus $\beta s^{-1} + 1 \neq \infty$. Then $\beta + s \neq \infty$ which implies that $1 + \beta^{-1} s \neq \infty$. Thus $\alpha \leqslant \beta^{-1} s$, so $\alpha s^{-1} \leqslant \beta^{-1}$. If $\beta + 1 \neq \infty$, then $\beta \in (Cor_K(1))^c \cap D_K^i$. Therefore $\alpha \leqslant \beta$,,a contradiction. Thus $\beta + 1 = \infty$, so $1 + \beta^{-1} = \infty$. Then $\beta^{-1} < \beta$. Therefore $\alpha s^{-1} \leqslant \beta^{-1} \leqslant \beta < \alpha$. Hence $\alpha < \alpha s$, a contradiction.

Subcase 2.1.2: $\alpha < \alpha s$. Then $\alpha s^{-1} < \alpha$ and use the same proof as in Subcase 2.1.1.

Subcase 2.2: $\beta = \alpha$.

Subcase 2.2.1: $\alpha \in (Cor_K(1))^c \cap D_K^i$. Then $\beta = \alpha \notin Cor_K(1)$. Let $t \in D_K^f \setminus \{1\}$. Then $\beta t \neq \beta$. Suppose that $\beta t < \beta$. Then there exists a $u \in Cor_K(1)$ such that $\beta t < u < \beta$. Thus $\beta < ut^{-1}$, so $ut^{-1} + 1 \neq \infty$ which implies that $1 + u^{-1}t \neq \infty$. Therefore $\beta \leqslant u^{-1}t$, so $\beta t^{-1} \leqslant u^{-1}$. But we have that $u+1 = \infty$, this implies that $1+u^{-1} = \infty$. Thus $u^{-1} < \beta$. Therefore $\beta t^{-1} < \beta$. Then $\beta < \beta t$, a contradiction. Therefore $\beta < \beta t$. Then $\beta t^{-1} < \beta$ and use the same proof as the one just given to get a contradiction.

Subcase 2.2.2: $\alpha \notin (Cor_K(1))^c \cap D_K^i$. Let $w \in D_K^f \setminus \{1\}$. Then $\alpha w \neq \alpha$. Suppose $\alpha < \alpha w$. Then there exists a $v \in (Cor_K(1))^c \cap D_K^i$ such that $\alpha < v < \alpha w$. Thus $\alpha w^{-1} < vw^{-1} < \alpha$, so $vw^{-1} + 1 = \infty$ which implies that $v^{-1}w+1 = \infty$. Therefore $v^{-1}w \leqslant \alpha$. Then $v^{-1} \leqslant \alpha w^{-1}$. But we have that $v+1 \neq \infty$, this implies that $1+v^{-1} \neq \infty$. Thus $\alpha < v^{-1}$, it follows that $\alpha < v^{-1} \leqslant \alpha w^{-1}$. Hence $\alpha w < \alpha$, a contradiction. Therefore $\alpha w < \alpha$. Then $\alpha < \alpha w^{-1}$ and use the same proof as the one just given to get a contradiction.

Hence $Cor_K(1) = \{\infty\} \cup D_K^i$. #

From Theorem 3.2.5 we see that there are three type of complete ordered ∞-skew semifields:

(1) complete ordered ∞-skew semifield K with $Cor_K(1) = \{\infty\}$,

(2) complete ordered ∞-skew semifield K with $Cor_K(1) = \{\infty\}\ \ D_K^i$,

(3) complete ordered ∞-skew semifield K with $Cor_K(1) = K$.

If a complete ordered ∞-skew semifield K satisfies (1) then K is called a <u>type I ∞-skew semifield</u>, if K satisfies (2) then K is called a <u>type II ∞-skew semifield</u> and if K satisfies (3) then K is called a <u>type III ∞-skew semifield</u>.

<u>Proposition 3.2.6.</u> Let $(K,+,\cdot,\leqslant)$ be a type I ∞-skew semifield. Then the following properties hold:

(1) $(D_K^f,+,\cdot,\leqslant)$ is a complete ordered skew ratio semiring.

(2) Suppose that $1+1 \neq 1$. Then for every x, y ε D_K^i, $x+y < \infty$ implies that y is the unique element of the coset yD_K^f such that $x+y < \infty$.

(3) Suppose that $1+1 \neq 1$. Then for every y ε D_K^i and for every c, d ε D_K^f, $y+c = y+d$ implies that $c = d$.

(4) Suppose that $1+1 \neq 1$. Then for every c, d ε D_K^f and

for every $y \in D_K^i$, $c \neq d$ implies that $(y+c)D_K^f \cap (y+d)D_K^f = \emptyset$.

Proof: To show (1), by Proposition 3.2.4, $(D_K^f, \leqslant)$ is a complete ordered set. By Proposition 3.2.2 (4) and Proposition 3.2.3 (3), $(D_K^f,+,\cdot)$ is a skew ratio semiring. Hence $(D_K^f,+,\cdot,\leqslant)$ is a complete ordered skew ratio semiring.

To show (2), suppose that $1+1 \neq 1$. By (1) and Theorem 2.18, $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$ or $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$. Let $x,y \in D_K^i$ be such that $x+y < \infty$. We shall show that y is the unique element of yD_K^f such that $x+y < \infty$. Suppose not. Then there exists a $z \in (yD_K^f) \setminus \{y\}$ such that $x+z < \infty$. Thus $z = yd$ for some $d \in D_K^f \setminus \{1\}$.

Case 1: $1 < d$. Then $d = 1+c$ for some $c \in D_K^f$. Therefore $x+z$ $= x+yd = x+y(1+c) = x+y+yc$. ....(*)

Since $x+y \in D_K^f$ and $yc \in D_K^i$, $x+y+yc \geqslant \infty$. Thus $x+z \geqslant \infty$, a contradiction.

Case 2: $d < 1$. Then $1 = d+a$ for some $a \in D_K^f$. Thus $x+z = x(d+a)+yd$ $= xd+xa+yd = (x+y)d+xa$. ....(**)

Since $(x+y)d \in D_K^f$ and $xa \in D_K^i$, $(x+y)d+xa \geqslant \infty$. Thus $x+z \geqslant \infty$, a contradiction.

Hence there does not exist $z \in (yD_K^f) \setminus \{y\}$ such that $x+z < \infty$.

To show (3), suppose that $1+1 \neq 1$. By (1) and Theorem 2.18, $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$ or $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$. Let $y \in D_K^i$ and $c, d \in D_K^f$ be such that $y+c = y+d$. To show that $c = d$, suppose not. Then $c \neq d$. Without loss of generality, suppose that $c < d$. Then $d = c+a$ for some $a \in D_K^f$. Thus $y+c = y+d = y+(c+a) = (y+c)+a$.

.........................(i)

By $Cor_K(1) = \{\infty\}$ and Proposition 3.2.2 (4), $y+c \neq \infty$. From (i), we have that $(y+c)^{-1}(y+c) = (y+c)^{-1}(y+c)+(y+c)^{-1}a$ which implies that

$1 = 1 + (y+c)^{-1}a$. ....(ii)

Since $(y+c)^{-1} > \infty$ and $a < \infty$, $(y+c)^{-1}a > \infty$. From (ii), we have that $1 > \infty$, a contradiction. Hence $c = d$.

To show (4), suppose that $1+1 \neq 1$. By (1) and Theorem 2.18, $(D_K^f,+,\cdot,\leqslant)$ is isomorphic $(\mathbb{R}^+,+,\cdot,\leqslant)$ or $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$. Let $c, d \in D_K^f$ be such that $c \neq d$. Let $y \in D_K^i$ be arbitrary. To show that $(y+c)D_K^f \cap (y+d)D_K^f = \emptyset$, suppose not. Then $(y+c)D_K^f \cap (y+d)D_K^f \neq \emptyset$. Thus $(y+c)D_K^f = (y+d)D_K^f$. Therefore $y+d = (y+c)a$ for some $a \in D_K^f$.

<u>Case 1</u>: $c < d$. Then $d = c+b$ for some $b \in D_K^f$. Thus

$(y+c)a = y+d = y+(c+b) = (y+c)+b$. ....(iii)

By $Cor_K(1) = \{\infty\}$ and Proposition 3.2.2 (4), $y+c \neq \infty$. From (iii), we have that $(y+c)^{-1}(y+c)a = (y+c)^{-1}(y+c)+(y+c)^{-1}b$ which implies that

$a = 1 + (y+c)^{-1}b$. ....(iv)

Since $(y+c)^{-1} > \infty$ and $b < \infty$, $(y+c)^{-1}b > \infty$. From (iv), we have that $a > \infty$, a contradiction.

<u>Case 2</u>: $d < c$. This proof is similar to the proof of Case 1.

Therefore $(y+c)D_K^f \cap (y+d)D_K^f = \emptyset$. #

<u>Theorem 3.2.7</u>. Let $(K,+,\cdot,\leqslant)$ be a type I ∞-skew semifield such that $1+1 = 1$. Then $D_K^i = \emptyset$.

<u>Proof</u>: Assume that $D_K^i \neq \emptyset$. Let $x \in D_K^i$ be arbitrary. Since $(K \setminus \{\infty\},\cdot)$ is a group, $x(K \setminus \{\infty\}) = K \setminus \{\infty\}$. Thus

$D_K^i \cup D_K^f = x(D_K^f \cup D_K^i) = xD_K^f \cup xD_K^i$ . ...(*)

By Proposition 3.2.3 (4), $xD_K^f \subseteq D_K^i$. By (*), $D_K^f \subseteq xD_K^i$. ....(**)

<u>Case 1</u>: For every $a \in D_K^i$ and for every $b \in K \setminus \{\infty\}$, $a+b \in D_K^i$. Let $y \in D_K^i$ and $z \in D_K^f$. Then $\infty < y$ and $z < \infty$. Thus $\infty \leqslant y+z$. By $Cor_K(1) = \{\infty\}$ and Proposition 3.2.2 (4), $\infty < y+z$. Then $y+z \in D_K^i$. From (**), we have that $D_K^f \subsetneq (y+z)D_K^i$. Then there exists a $w \in D_K^i$ such that $(y+z)w \in D_K^f$. ....(***)

By Proposition 3.2.2 (4), $zw \in D_K^i$. By assumption, we have that $yw+zw \in D_K^i$. Then $(y+z)w \neq yw+zw$ which is a contradiction.

<u>Case 2</u>: There exist $x \in D_K^i$ and $y \in K \setminus \{\infty\}$ such that $x+y \notin D_K^i$. Let $a \in D_K^i$ and $b \in K \setminus \{\infty\}$ be such that $a+b \notin D_K^i$. Then $a+b \leqslant \infty$. By Proposition 3.2.2 (4), $a+b < \infty$.

<u>Subcase 2.1</u>: $b < \infty$. Then $\infty = \infty+b \leqslant a+b$, a contradiction.

<u>Subcase 2.2</u>: $\infty < b$. Then $\infty = \infty+(a+b) \leqslant (a+b)+b = a+(b+b) = a+b$, a contradiction.

Therefore we get that $D_K^i = \emptyset$.

<u>Theorem 3.2.8</u>. Let $(K,+,\cdot,\leqslant)$ be a type I ∞-skew semifield such that $1+1 = 1$. Then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) ∞-skew semifield with the almost trivial addition of order 2.

(2) $(\mathbb{R}_\infty^+, \oplus, \cdot, \leqslant)$ where $\cdot$ and $\leqslant$ are the usual multiplication

and order and

$$x \oplus y = \begin{cases} \min\{x,y\} & \text{if } x \neq \infty \text{ and } y \neq \infty, \\ \infty & \text{if } x = \infty \text{ or } y = \infty. \end{cases}$$

(3) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{\infty\}, *, \cdot, \leqslant)$ where $\cdot$ and $\leqslant$ are the usual multiplication and order and

$$x * y = \begin{cases} \min\{x,y\} & \text{if } x \neq \infty \text{ and } y \neq \infty, \\ \infty & \text{if } x = \infty \text{ or } y = \infty. \end{cases}$$

(4) $(\mathbb{R}^+_\infty, \max, \cdot, \leqslant)$.

(5) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{\infty\}, \max, \cdot, \leqslant)$.

Proof: The proof of theorem follows from Proposition 3.2.6, Theorem 3.2.7, Theorem 2.5 and Theorem 2.6. #

Theorem 3.2.9. Let $(K,+,\cdot,\leqslant)$ be a type I ∞-skew semifield such that $1+1 \neq 1$. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$. Then $(K,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+_\infty,+,\cdot,\leqslant)$ or $(\mathbb{R}^+_\infty,+,\cdot,\leqslant^*)$ where $\leqslant^* = \leqslant_{opp}$ on R^+ and $x <^* \infty$ for all $x \in \mathbb{R}^+$.

Proof: We have that $K = D^f_K \cup \{\infty\} \cup D^i_K$. Now, we shall show that $D^i_K = \emptyset$. To prove this, suppose not. Then $D^i_K \neq \emptyset$. Let $x \in D^f_K$ and $y \in D^i_K$. Then $x < \infty$ and $y > \infty$. Thus $\infty = \infty + x \leqslant x+y \leqslant \infty + y = \infty$. Therefore $x+y = \infty$. By Proposition 3.2.2.(4), $x+y \neq \infty$, a contradiction. Hence $D^i_K = \emptyset$. By Proposition 3.2.6 (1) and Theorem 2.18, $(K,+,\cdot,\leqslant)$ is is isomorphic to $(\mathbb{R}_\infty,+,\cdot,\leqslant)$ or $(\mathbb{R}^+_\infty,+,\cdot,\leqslant^*)$ where $\leqslant^* = \leqslant_{opp}$ on $\mathbb{R}^+$ and $x <^* \infty$ for all $x \in \mathbb{R}^+$. #

Remark 3.2.10. Let $A = \{1,\infty,t\}$. Define $\leqslant$ on A by $1 < \infty < t$ and define multiplication $\cdot$ on A by

$\cdot$	1	∞	t
1	1	∞	t
∞	∞	∞	∞
t	t	∞	1

There are two possible commutative binary operation + on A such that A is an ∞-semifield.

+	1	∞	t
1	∞	∞	∞
∞	∞	∞	∞
t	∞	∞	∞

+	1	∞	t
1	1	∞	∞
∞	∞	∞	∞
t	∞	∞	t

Note that table (1) make $\{1,\infty,t\}$ into an ∞-skew semifield with the trivial addition, table (2) make $\{1,\infty,t\}$ into an ∞-skew semifield with the almost trivial addition.

Remark 3.2.11. Let $B_1 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(2^m,1) \mid m \in \mathbb{Z}\}$.

Define + and $\cdot$ on B_1 by as follows:

$$z + \infty = \infty = z\infty = \infty z \quad \text{for all } z \in B_1.$$ Let

$x, y \in \{2^n \mid n \in \mathbb{Z}\}$ be arbitrary. Define

$$(x,0) + (y,0) = (\min\{x,y\},0),$$

$$(x,1) + (y,1) = (\min\{x,y\},1),$$

$$(x,0) + (y,1) = \infty ,$$

$$(x,0)(y,1) = (xy,1),$$

$$(y,1)(x,0) = (yx,1),$$

$$(x,1)(y,1) = (xy,1),$$

$$(x,0)(y,0) = (xy,0).$$

Define $\leqslant$ on B_1 by as follows: Let $x, y \in \{2^n | n \in \mathbb{Z}\}$

$(x,0) < \infty < (y,1)$,

$(x,0) \leqslant (y,0)$ iff $x \leqslant y$,

$(x,1) \leqslant (y,1)$ iff $x \leqslant y$.

Then $(B_1,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield as is shown below.

Proof: Clearly, + and $\cdot$ are closed.

We shall show that + is associative. To prove this, let $(x,c_1),(y,c_2),(z,c_3) \in B_1$ where $x, y, z \in \{2^n | n \in \mathbb{Z}\}$ and $c_1,c_2,c_3 \in \{0,1\}$. We must show that $(x,c_1)+[(y,c_2)+(z,c_3)] = [(x,c_1)+(y,c_2)]+(z,c_3)$.

Case 1: $c_1= c_2= c_3= 0$. Then $(x,0)+[(y,0)+(z,0)] = (x,0)+(\min\{y,z\},0)$ $= (\min\{x,y,z\},0)$ and $(x,0)+(y,0)]+(z,0) = (\min\{x,y\},0)+(z,0)$ $= (\min\{x,y,z\},0)$. Thus $(x,0)+[(y,0)+(z,0)] = [(x,0)+(y,0)]+(z,0)$.

Case 2: $c_1= c_2= 0$ and $c_3= 1$. Then $(x,0)+[(y,0)+(z,1)] = (x,1)+\infty$ $= \infty$ and $[(x,0)+(y,0)]+(z,1) = (\min\{x,y\},0)+(z,1) = \infty$ so $(x,0)+[(y,0)+(z,1)] = [(x,1)+(y,0)]+(z,1)$.

Case 3: $c_1= c_3= 0$ and $c_2= 1$. Then $(x,0)+[(y,1)+(z,0)] = (x,0)+\infty$ $= \infty$ and $[(x,0)+(y,1)]+(z,0) = \infty+(z,0) = \infty$. Thus $(x,0)+[(y,1)+(z,0)] = [(x,0)+(y,1)]+(z,0)$.

Case 4: $c_1= 0$ and $c_2= c_3= 1$. Then $(x,0)+[(y,1)+(z,1)] =$ $(x,0)+(\min\{y,z\},1) = \infty$ and $[(x,0)+(y,1)]+(z,1) = \infty+(z,1) = \infty$. Thus $(x,0)+[(y,1)+(z,1)] = [(x,0)+(y,1)]+(z,1)$.

Case 5: $c_1= c_2= c_3= 1$. Then $(x,1)+[(y,1)+(z,1)] = (x,1)+(\min\{y,z\},1)$

$= (\min\{x,y,z\},1)$ and $[(x,1)+(y,1)]+(z,1) = (\min\{x,y\},1)+(z,1)$ $= (\min\{x,y,z\},1)$. Thus $(x,1)+[(y,1)+(z,1)] = [(x,1)+(y,1)]+(z,1)$.

Case 6: $c_1 = 1$ and $c_2 = c_3 = 0$. Then $(x,1)+[(y,0)+(z,0)] = (x,1)+(\min\{y,z\},0) = \infty$ and $[(x,1)+(y,0)]+(z,0) = \infty+(z,0) = \infty$. Thus $(x,1)+[(y,0)+(z,0)] = [(x,1)+(y,0)]+(z,0)$.

Case 7: $c_1 = c_2 = 1$ and $c_3 = 0$. Then $(x,1)+[(y,1)+(z,0)] = (x,1)+\infty = \infty$ and $[(x,1)+(y,1)]+(z,0) = (\min\{x,y\},1)+(z,0) = \infty$. Thus $(x,1)+[(y,1)+(z,0)] = [(x,1)+(y,1)]+(z,0)$.

Case 8: $c_1 = c_3 = 1$ and $c_2 = 0$. Then $(x,1)+[(y,0)+(z,1)] = (x,1)+\infty = \infty$ and $[(x,1)+(y,0)]+(z,1) = \infty+(z,1) = \infty$. Thus $(x,1)+[(y,0)+(z,1)] = [(x,1)+(y,0)]+(z,1)$.

We shall show that $\cdot$ is associative. To prove this, let $(x,c_1),(x,c_2),(x,c_3) \in B_1 \setminus \{\infty\}$. Then $x,y,z \in \{2^n \mid n \in \mathbb{Z}\}$ and $c_1,c_2,c_3 \in \{0,1\}$.

Case 1: $c_1 = c_2 = c_3 = 0$. Then $(x,0)[(y,0)(z,0)] = (x,0)(yz,0) = (xyz,0)$ and $[(x,0)(y,0)](z,0) = (xy,0)(z,0) = (xyz,0)$. Thus $(x,0)[(y,0)(z,0)] = [(x,0)(y,0)](z,0)$.

Case 2: $c_1 = c_2 = 0$ and $c_3 = 1$. Then $(x,0)[(y,0)(z,1)] = (x,0)(yz,1) = (xyz,1)$ and $[(x,0)(y,0)](z,1) = (xy,0)(z,1) = (xyz,1)$. Thus $(x,0)[(y,0)(z,1)] = [(x,0)(y,0)](z,1)$.

Case 3: $c_1 = c_3 = 0$ and $c_2 = 1$. Then $(x,0)[(y,1)(z,0)] = (x,0)(yz,1) = (xyz,1)$ and $[(x,0)(y,1)](z,0) = (xy,1)(z,0) = (xyz,1)$. Thus $(x,0)[(y,1)(z,0)] = [(x,0)(y,1)](z,0)$.

Case 4: $c_1 = 0$ and $c_2 = c_3 = 1$. Then $(x,0)[(y,1)(z,1)] = (z,0)(yz,0)$ $= (xyz,0)$ and $[(x,0)(y,1)](z,1) = (xy,1)(z,1) = (xyz,0)$. Thus $(x,0)[(y,1)(z,1)] = [(x,0)(y,1)](z,1)$.

Case 5: $c_1 = 1$ and $c_2 = c_3 = 0$. Then $(x,1)[(y,0)(z,0)] = (x,1)(yz,0)$ $= (xyz,1)$ and $[(x,1)(y,0)](z,0) = (xy,1)(z,0) = (xyz,1)$. Thus $(x,1)[(y,0)(z,0)] = [(x,1)(y,0)](z,0)$.

Case 6: $c_1 = c_3 = 1$ and $c_2 = 0$. Then $(x,1)[(y,0)(z,1)] = (x,1)(yz,1)$ $= (xyz,0)$ and $[(x,1)(y,0)](z,1) = (xy,1)(z,1) = (xyz,0)$. Thus $(x,1)[(y,0)(z,1)] = [(x,1)(y,0)](z,1)$.

Case 7: $c_1 = c_2 = 1$ and $c_3 = 0$. Then $(x,1)[(y,1)(z,0)] = (x,1)(yz,1)$ $= (xyz,0)$ and $[(x,1)(y,1)](z,0) = (xy,0)(z,0) = (xyz,0)$. Thus $(x,1)[(y,1)(z,0)] = [(x,1)(y,1)](z,0)$.

Case 8: $c_1 = c_2 = c_3 = 1$. Then $(x,1)[(y,1)(z,1)] = (x,1)(yz,0)$ $= (xyz,1)$ and $[(x,1)(y,1)](z,1) = (xy,0)(z,1) = (xyz,1)$. Thus $(x,1)[(y,1)(z,1)] = [(x,1)(y,1)](z,1)$.

We shall show that $B_1 \setminus \{\infty\}$ is distributive. To prove this, let $(x,c_1),(y,c_2),(z,c_3) \in B_1 \setminus \{\infty\}$ be arbitrary. Then $x,y,z \in \{2^n | n \in \mathbb{Z}\}$ and $c_1,c_2,c_3 \in \{0,1\}$.

Case 1: $c_1 = c_2 = c_3 = 0$. Then $(x,0)[(y,0)+(z,0)] = (x,0)(\min\{y,z\},0)$ $= (\min\{xy,xz\},0)$ and $(x,0)(y,0)+(x,0)(z,0) = (xy,0)+(xz,0)$ $= (\min\{xy,xz\},0)$. Thus $(x,0)[(y,0)+(z,0)] = (x,0)(y,0)+(x,0)(z,0)$.

Case 2: $c_1 = c_2 = 0$ and $c_3 = 1$. Then $(x,0)[(y,0)+(z,1)] = (x,0)\infty = \infty$ and $(x,0)(y,0)+(x,0)(z,1) = (xy,0)+(xz,1) = \infty$. Thus $(x,0)[(y,0)+(z,1)] = (x,0)(y,0)+(x,0)(z,1)$.

<u>Case 3</u>: $c_1 = c_3 = 0$ and $c_2 = 1$. Then $(x,0)[(y,1)+(z,0)] = (x,0)\infty = \infty$ and $(x,0)(y,1)+(x,0)(z,0) = (xy,1)+(xz,0) = \infty$. Thus $(x,0)[(y,0)+(z,1)] = (x,0)(y,0)+(x,0)(z,1)$.

<u>Case 4</u>: $c_1 = 0$ and $c_2 = c_3 = 1$. Then $(x,0)[(y,1)+(z,1)] = (x,0)(\min\{y,z\},1) = (\min\{xy,xz\},1)$ and $(x,0)(y,1)+(x,0)(z,1) = (xy,1)+(xz,1) = (\min\{xy,xz\},1)$. Thus $(x,0)[(y,1)+(z,1)] = (x,0)(y,1)+(x,0)(z,1)$.

<u>Case 5</u>: $c_1 = 1$ and $c_2 = c_3 = 0$. Then $(x,1)[(y,0)+(z,0)] = (x,1)(\min\{y,z\},0) = (\min\{xy,xz\},1)$ and $(x,1)(y,0)+(x,1)(z,0) = (xy,1)+(xz,1) = (\min\{xy,xz\},1)$. Thus $(x,1)[(y,0)+(z,0)] = (x,1)(y,0)+(x,1)(z,0)$.

<u>Case 6</u>: $c_1 = c_3 = 1$ and $c_2 = 0$. Then $(x,1)[(y,0)+(z,1)] = (x,1)\infty = \infty$ and $(x,1)(y,0)+(x,1)(z,1) = (xy,1)+(xz,0) = \infty$. Thus $(x,1)[(y,0)+(z,1)] = (x,1)(y,0)+(x,1)(z,1)$.

<u>Case 7</u>: $c_1 = c_2 = 1$ and $c_3 = 0$. Then $(x,1)[(y,1)+(z,0)] = (x,1)\infty = \infty$ and $(x,1)(y,1)+(x,1)(z,0) = (xy,0)+(xz,1) = \infty$. Thus $(x,1)[(y,1)+(z,0)] = (x,1)(y,1)+(x,1)(z,0)$.

<u>Case 8</u>: $c_1 = c_2 = c_3 = 1$. Then $(x,1)[(y,1)+(z,1)] = (x,1)(\min\{y,z\},1) = (\min\{xy,xz\},0)$ and $(x,1)(y,1)+(x,1)(z,1) = (xy,0)+(xz,0) = (\min\{xy,xz\},0)$. Thus $(x,1)[(y,1)+(z,1)] = (x,1)(y,1)+(x,1)(z,1)$

Hence $B_1 \setminus \{\infty\}$ is distributive.

We shall show that $(B_1 \setminus \{\infty\},\cdot)$ is a group. To prove this, let $(x,c) \in B_1 \setminus \{\infty\}$ be arbitrary. Now, we have that $(1,0),(x^{-1},c) \in B_1 \setminus \{\infty\}$.

Case 1: $c = 0$. Then $(x,0)(1,0) = (x,0) = (1,0)(x,0)$ and $(x,0)(x^{-1},0) = (1,0) = (x^{-1},0)(x,0)$.

Case 2: $c = 1$. Then $(x,1)(1,0) = (x,1) = (1,0)(x,1)$ and $(x,1)(x^{-1},1) = (1,0) = (x^{-1},1)(x,1)$.

Thus $(1,0)$ is the identity of $B_1 \setminus \{\infty\}$ and (x^{-1},c) is an inverse of (x,c). Since $\cdot$ is associative, $(B_1 \setminus \{\infty\},\cdot)$ is a group.

We shall show that for every $(x,c),(y,d) \in B_1$, $(x,c) \leqslant (y,d)$ implies that $(x,c)+(z,b) \leqslant (y,d)+(z,b)$ and $(x,c)(z,b) \leqslant (y,d)(z,b)$ for all $(z,b) \leqslant \infty$. To prove this, let $(x,c),(y,d) \in B_1$ be such that $(x,c) \leqslant (y,d)$. Let $(z,b) \leqslant \infty$. If $(z,b) = \infty$, then we are done. Suppose that $(z,b) < \infty$. Then $b = 0$. If $(x,c) = \infty$ or $(y,d) = \infty$, then we are done. Suppose that $(x,c),(y,d) \in B_1 \setminus \{\infty\}$.

Case 1: $c = 0$. Then $d = 0$ or $d = 1$

Subcase 1.1: $d = 0$. Then $x \leqslant y$.

Subcase 1.1.1: $x \leqslant y \leqslant z$. Then $(x,0)+(z,0) = (\min\{x,z\},0) = (x,0)$ and $(y,0)+(z,0) = (\min\{y,z\},0) = (y,0)$. Thus $(x,0)+(z,0) \leqslant (y,0)+(z,0)$. Now, we have that $(x,0)(z,0) = (xz,0)$ and $(y,0)(z,0) = (yz,0)$. It follows from $xz \leqslant yz$ that $(x,0)(z,0) \leqslant (y,0)(z,0)$.

Subcase 1.1.2: $x \leqslant z \leqslant y$. Then $(x,0)+(z,0) = (\min\{x,z\},0) = (x,0)$ and $(y,0)+(z,0) = (\min\{y,z\},0) = (z,0)$. Thus $(x,0)+(z,0) \leqslant (y,0)+(z,0)$. The proof that $(x,0)(z,0) \leqslant (y,0)(z,0)$ is similar to the one given in Subcase 1.1.1.

Subcase 1.1.3: $z \leqslant x \leqslant y$. This proof is similar to the proof of Subcase 1.1.1.

Subcase 1.2: $d = 1$. Then $(x,0)+(z,0) = (\min\{x,z\},0)$ and $(y,1)+(z,0) = \infty$. Thus $(x,0)+(z,0) \leqslant (y,1)+(z,0)$. Now, we have that $(x,0)(z,0) = (xz,0)$ and $(y,1)(z,0) = (yz,1)$. Thus $(x,0)(z,0) \leqslant (y,1)(z,0)$.

Case 2: $c = 1$. Then $d = 1$. Therefore $x \leqslant y$. Then $(x,1)+(z,0) = = (y,1)+(z,0)$. Thus $(x,1)+(z,0) \leqslant (y,1)+(z,0)$. Now, we have that $(x,1)(z,0) = (xz,1)$ and $(y,1)(z,0) = (yz,1)$. It follows from $xz \leqslant yz$ that $(x,1)(z,0) \leqslant (y,1)(z,0)$.

Lastly, a proof semilar to the one given in Remark 3.1.8 shows that B_1 is a complete. #

The proof of the following remarks is similar to the proof of Remark 3.2.11.

Remark 3.2.12. Let $B_2 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(2^m,1) \mid m \in \mathbb{Z}\}$. Define $+$ and $\cdot$ on B_2 as in Remark 3.2.11 and define $\leqslant$ on B_2 by as follows: Let $x, y \in \{2^n \mid n \in \mathbb{Z}\}$. Define

$$(x,0) < \infty < (y,1),$$

$$(x,0) \leqslant (y,0) \text{ iff } x \leqslant y,$$

$$(x,1) \leqslant (y,1) \text{ iff } y \leqslant x.$$

Then $(B_2,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.13. Let $B_3 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(2^m,1) \mid m \in \mathbb{Z}\}$. Define $\cdot$ and $\leqslant$ as in Remark 3.2.11. Let $x, y \in \{2^n \mid n \in \mathbb{Z}\}$. Define $+$ on B_3 by follows:

$$(x,0)+(y,0) = (\max\{x,y\},0),$$

$$(x,1)+(y,1) = (\max\{x,y\},1),$$

$$(x,0)+(y,1) = \infty.$$

Then $(B_3,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

<u>Remark 3.2.14</u>. Let $B_4 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(2^m,1) \mid m \in \mathbb{Z}\}$. Define + and $\cdot$ on B_4 as om Remark 3.2.13 and define $\leqslant$ on B_4 as in Remark 3.2.12. Then $(B_4,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

<u>Remark 3.2.15</u>. Let $C_1 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(\sqrt{2^m},1) \mid m \in \mathbb{Z} \text{ is odd}\}$. Let x_1, $x_2 \in \{2^n \mid n \in \mathbb{Z}\}$ and let y_1, $y_2 \in \{\sqrt{2^m} \mid m \in \mathbb{Z} \text{ is odd}\}$. Define + and $\cdot$ on C_1 as follows:

$$(x_1,0)+(x_2,0) = (\min\{x,y\},0),$$

$$(y_1,0)+(y_2,0) = (\min\{x,y\},0),$$

$$(x_1,0)+(y_1,0) = \infty,$$

$$(x_1,0)(x_2,0) = (x_1x_2,0),$$

$$(x_1,0)(y_1,1) = (x_1y_1,1),$$

$$(y_1,1)(x_1,0) = (y_1x_1,1),$$

$$(y_1,1)(y_2,1) = (y_1y_2,0) \text{ and}$$

$$z+\infty = \infty = z\infty = \infty z \quad \text{for all } z \in C_1.$$

Define $\leqslant$ on C_1 by as follows:

$$(x_1,0) < \infty < (y_1,1),$$

$$(x_1,0) \leqslant (x_2,0) \text{ iff } x_1 \leqslant x_2,$$

$$(y_1,1) \leqslant (y_2,1) \text{ iff } y_1 \leqslant y_2.$$

Then $(C_1,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifeild.

Remark 3.2.16. Let $C_2 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(\sqrt{2^m},1) \mid m \in \mathbb{Z} \text{ is odd}\}$. Define + and $\cdot$ on C_2 as in Remark 3.2.15. Let $x_1, x_2 \in \{2^n \mid n \in \mathbb{Z}\}$ and let $y_1, y_2 \in \{\sqrt{2^m} \mid m \in \mathbb{Z} \text{ is odd}\}$. Define $\leqslant$ on C_2 as follows:

$$(x_1,0) < \infty < (y_1,1),$$

$$(x_1,0) \leqslant (x_2,0) \text{ iff } x_1 \leqslant x_2,$$

$$(y_1,1) \leqslant (y_2,1) \text{ iff } y_2 \leqslant y_1.$$

Then $(C_2,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.17. Let $C_3 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(\sqrt{2^m},1) \mid m \in \mathbb{Z} \text{ is odd}\}$. Define $\cdot$ and $\leqslant$ on C_3 as in Remark 3.2.15. Let x_1, $x \in \{2^n \mid n \in \mathbb{Z}\}$ and let $y_1, y_2 \in \{\sqrt{2^m} \mid m \in \mathbb{Z} \text{ is odd}\}$. Define + on C_3 as follows:

$$(x_1,0)+(x_2,0) = (\max\{x_1,x_2\},0),$$

$$(y_1,1)+(y_2,1) = (\max\{y_1,y_2\},1),$$

$$(x_1,0)+(y_1,1) = \infty \text{ and}$$

$$z+\infty = \infty \quad \text{for all } z \in C_3.$$

Then $(C_3,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.18. Let $C_4 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(\sqrt{2^m},1) \mid m \in \mathbb{Z} \text{ is odd}\}$. Define + and $\cdot$ on C_4 as in Remark 3.2.17 and define $\leqslant$ on C_4 as in Remark 3.2.16. Then $(C_4,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.19. Let $E_1 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Let $x, y \in \mathbb{R}^+$. Define + and $\cdot$ by :

$$(x,0)+(y,0) = (x+y,0),$$

$$(x,0)+(y,1) = \infty,$$

$$(x,1)+(y,1) = (x+y,1),$$

$$(x,0)(y,0) = (xy,0),$$

$$(x,0)(y,1) = (xy,1),$$

$$(y,1)(x,0) = (yx,1),$$

$$(x,1)(y,1) = (xy,0)$$

and $z+\infty = \infty = z\infty = \infty z$ for all $z \in E_1$.

Define $\leqslant$ on E_1 as follows:

$$(x,0) < \infty < (y,1),$$

$$(x,0) \leqslant (y,0) \text{ iff } x \leqslant y,$$

$$(x,1) \leqslant (y,1) \text{ iff } x \leqslant y.$$

Then $(E_1,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

<u>Remark 3.2.20</u>. Let $E_2 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define + and $\cdot$ on E_2 as in Remark 3.2.19. Let $x, y \in \mathbb{R}^+$. Define $\leqslant$ on E_2 as follows:

$$(x,0) < \infty < (y,1)$$

$$(x,0) \leqslant (y,0) \text{ iff } x \leqslant y,$$

$$(x,1) \leqslant (y,1) \text{ iff } y \leqslant x.$$

Then $(E_2,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

<u>Remark 3.2.21</u>. Let $E_3 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define + and $\cdot$ on E_3 as in Remark 3.2.19. Let $x, y \in \mathbb{R}^+$. Define $\leqslant$ on E_3 as follows:

$$(x,0) < \infty < (y,1),$$

$$(x,0) \leqslant (y,0) \text{ iff } y \leqslant x,$$

$$(x,1) \leqslant (y,1) \text{ iff } x \leqslant y.$$

Then $(E_3,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.22. Let $E_4 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define $+$ and $\cdot$ on E_4 as in Remark 3.2.19. Let $x, y \in \mathbb{R}^+$. Define $\leqslant$ on E_4 as follows:

$$(x,0) < \infty < (x,1),$$

$$(x,0) \leqslant (y,0) \text{ iff } y \leqslant x,$$

$$(x,1) \leqslant (y,1) \text{ iff } y \leqslant x.$$

Then $(E_4,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.23. Let $E_5 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define $\cdot$ and $\leqslant$ on E_5 as in Remark 3.2.19. Let $x, y \in \mathbb{R}^+$. Define $+$ on E_5 as follows:

$$(x,0)+(y,0) = (\max\{x,y\},0),$$

$$(x,1)+(y,1) = (\max\{x,y\},1),$$

$$(x,0)+(y,1) = \infty$$

and $z+\infty = \infty$ for all $z \in E_5$.

Then $(E_5,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.24. Let $E_6 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define $+$ and $\cdot$ on E_6 as in Remark 3.2.23 and define $\leqslant$ on E_6 as in Remark 3.2.20. Then $(E_6,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.25. Let $E_7 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define $\cdot$ and on E_7 as in Remark 3.2.19. Let $x, y \in \mathbb{R}^+$. Define $+$ on E_7 as follows:

$$(x,0)+(y,0) = (\min\{x,y\},0),$$

$$(x,1)+(y,1) = (\min\{x,y\},1),$$

$$(x,0)+(y,1) = \infty$$

and $z+\infty = \infty$ for all $z \in E_7$.

Then $(E_7,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Remark 3.2.26. Let $E_8 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$. Define $+$ and $\cdot$ on E_8 as in Remark 3.2.25 and define $\leqslant$ on E_8 as given in Remark 3.2.20. Then $(E_8,+,\cdot,\leqslant)$ is a complete ordered ∞-skew semifield.

Proposition 3.2.27. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Then $(D_K^f,+,\cdot,\leqslant)$ is a complete ordered skew ratio semiring.

Proof: Assume that $(K,+,\cdot,\leqslant)$ is a type II ∞-skew semifield. Then $\mathrm{Cor}_K(1) = \{\infty\} \cup D_K^i$. By Proposition 3.2.3 (6), $(D_K^f,+,\cdot,\leqslant)$ is an ordered skew ratio semiring. By Proposition 3.2.4 (1), $(D_K^f,\leqslant)$ is a complete. Hence $(D_K^f,+,\cdot,\leqslant)$ is a complete ordered skew ratio semiring. #

Theorem 3.2.28. Let K be a type II ∞-skew semifield. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$. If $D_K^i \neq \emptyset$, then there exists a $t \in D_K^i$ such that $K = D_K^f \cup \{\infty\} \cup tD_K^f$ and $t^2 \in D_K^f$.

Proof: Assume that $D_K^i \neq \emptyset$. Now, we have that $\mathrm{Cor}_K(1) = \{\infty\} \cup D_K^i$. Then $K = D_K^f \cup \{\infty\} \cup D_K^i = D_K^f \cup \mathrm{Cor}_K(1)$. ....(1)

Let $x \in D_K^i$ be arbitrary. From (1), we have that $K = xK = x(D_K^f \cup \mathrm{Cor}_K(1))$ $= xD_K^f \cup x\,\mathrm{Cor}_K(1)$. ....(2)

By Proposition 3.2.2 (3) and (2), $K = xD_K^f \cup \mathrm{Cor}_K(x)$. ....(3)

Since $D_K^f \cap \mathrm{Cor}_K(1) = \emptyset$, $xD_K^f \cap \mathrm{Cor}_K(x) = \emptyset$. ....(4)

Now, we have that $K\setminus\{\infty\} = D_K^f \cup D_K^i = D_K^f \cup (\bigcup_{x\in D_K^i} xD_K^f)$. Since $D_K^f \cap D_K^i = \emptyset$ and $D_K^f \cap (\bigcup_{x\in D_K^i} xD_K^f) = \emptyset$ and $\bigcup_{x\in D_K^i} xD_K^f \neq \emptyset$, $\bigcup_{x\in D_K^i} xD_K^f = D_K^i$. Suppose that there are two disjoint cosets contained in D_K^i. Let vD_K^f and wD_K^f be distinct cosets of D_K^i. Then $v < w$ or $w < v$. Without loss of generality, suppose that $w < v$. Let $a \in D_K^f$ and let $u = wa$. Then $u \in wD_K^f$. Now, we have that $1+a \in D_K^f$ and $w \in D_K^i$. By Proposition 3.2.3, $w(1+a) \in D_K^i$. Thus $\infty < w(1+a) = w+wa = w+u \leqslant v+u$. ....(5)

Since $u \in wD_K^f$, $u \notin vD_K^f$. By (4), $u \in Cor_K(v)$. Therefore $u+v = \infty$ which contradicts (5).

Hence the number of cosets in D_K^i is 0 or 1. If the number of coset in D_K^i is 0, then $D_K^i = \emptyset$, a contradiction. Therefore the number of coset in D_K^i is 1. Thus $K = D_K^f \cup \{\infty\} \cup tD_K^f$ for some $t \in D_K^i$. Then $K = tK = tD_K^f \cup \{\infty\} \cup t^2D_K^f$ which implies that $t^2D_K^f = D_K^f$. Therefore $t^2 \in D_K^f$. #

<u>Theorem 3.2.29</u>. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\{2^n \mid n \in \mathbb{Z}\},\min,\cdot,\leqslant)$, then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{\infty\},*,\cdot,\leqslant)$ as in (3) of Theorem 3.2.8.

(2) $(B_1,+,\cdot,\leqslant)$ as in Remark 3.2.11.

(3) $(B_2,+,\cdot,\leqslant)$ as in Remark 3.2.12.

(4) $(C_1,+,\cdot,\leqslant)$ as in Remark 3.2.15.

(5) $(C_2,+,\cdot,\leqslant)$ as in Remark 3.2.16.

<u>Proof</u>: Assume that $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\{2^n|n \in Z\},\min,\cdot,\leqslant)$. Now, we have that $K = D_K^f \cup \{\infty\} \cup D_K^i$. If $D_K^i = \emptyset$, then $(K,+,\cdot,\leqslant)$ is isomorphic to (1). Suppose that $D_K^i \neq \emptyset$. By Theorem 3.2.28, there exists a $z \in D_K^i$ such that $K = D_K^f \cup \{\infty\} \cup zD_K^f$ and $z^2 \in D_K^f$. Thus $D_K^i = zD_K^f$. For simplicity, we shall assume that $D_K^f = \{2^n|n \in \mathbb{Z}\}$. Now, we shall show that $zr = rz$ for all $r \in D_K^f$. Let $r \in D_K^f$ be arbitrary. We shall first show that $z^2r^2 \in D_K^f$ and $(zr)^2 \in D_K^f$. Clearly, $z^2r^2 \in D_K^f$. We see that $(zr)^2 = (zr)(zr) = z(rz)z$.

.................(*)

Since $rz \in D_K^i = zD_K^f$, $rz = zs$ for some $s \in D_K^f$. From (*), we have that $(zr)^2 = z(zs)r = z^2sr \in D_K^f$.

Next, we shall show that $(zr)^2 = z^2r^2$. To prove this, suppose that $(zr)^2 \neq z^2r^2$. If $(zr)^2 < z^2r^2$, then $zrzr < z^2r^2$ which implies that $zrz < z^2r$. ....(**)

Since $rz \in D_K^i = zD_K^f$, $rz = zs$ for some $s \in D_K^f$. From (**), we have that $z(zs) < z^2r$. Thus $z^2s < z^2r$, it follows that $s < r$. Therefore $s+r = s$. Then $zs+zr = zs$. Therefore we get that $rz+zr = rz$. Thus $rz(zr) + zr(zr) = rz(zr)$ which implies that $z^2r^2+(zr)^2 = z^2r^2$. Since z^2r^2, $(zr)^2 \in D_K^f$, $z^2r^2 \leqslant (zr)^2$, a contradiction. If $z^2r^2 < (zr)^2$, then $z^2r^2 < zrzr$ which implies that $z^2r < zrz$. ....(***)

Since $rz \in D_K^i = zD_K^f$, $rz = zw$ for some $w \in D_K^f$. From (***), we have that $z^2r < z(zw) = (zz)w = z^2w$, it follows that $r < w$. Therefore $r+w = r$, so $zr+zw = zr$. Then $zr+rz = zr$. Thus $rz(zr)+rz(zr) = zr(zr)$

which implies that $(zr)^2 + z^2r^2 = (zr)^2$. Since $z^2r^2, (zr)^2 \varepsilon D_K^f)$, $(zr)^2 \leqslant z^2r^2$, a contradiction. Hence $(zr)^2 = z^2r^2$. ....(****)

Finally, from (****), we have that zrzr = zzrr. Then $z^{-1}(zrzr) = z^{-1}(zzrr)$, hence rzr = zrr. Therefore $(rzr)r^{-1} = (zrr)r^{-1}$. Thus rz = zr. Therefore we get that zr = rz for all $r \varepsilon D_K^f$.

..............(i)

<u>Case 1</u>: $z^2 = 2^{2m}$ for some $m \varepsilon \mathbb{Z}$. Thus $z^2(2^{2m})^{-1} = 1$. By (i), $(z2^{-m})^2 = 1$. Let $t = z2^{-m}$. Then $t \varepsilon D_K^i$ and $t^2 = 1$ and $tD_K^f = (z2^{-m})D_K^f = z(2^{-m}D_K^f) = zD_K^f = D_K^i$. ....(ii)

Using a proof similar to the proof of (i) we can show that ts = st for all $s \varepsilon D_K^f$. ....(iii)

Now, we have that $t2 \neq t$. Then $t2 > t$ or $t2 < t$.

<u>Subcase 1.1</u>: **t2 > t**. Now, we shall show that the following properties hold:

(a) $t2^{n+1} > t2^n$ for all $n \varepsilon \mathbb{Z}^+$,

(b) For every $m, n \varepsilon \mathbb{Z}^+$, $m < n$ implies that $t2^m < t2^n$,

(c) For every $m, n \varepsilon \mathbb{Z}^-$, $m < n$ implies that $t2^m < t2^n$.

To show (a), let $n \varepsilon \mathbb{Z}^+$ be arbitrary. We shall prove this by using induction on $n \varepsilon \mathbb{Z}^+$. If n = 1, then we are done. Suppose that (a) is true for some $n-1 \geqslant 1$. Then $t2^n > t2^{n-1}$. Thus $(t2^n)2 > (t2^{n-1})2$, so $t2^{n+1} > t2^n$. Hence $t2^{n+1} > t2^n$ for all $n \varepsilon \mathbb{Z}^+$.

To show (b), let $m, n \varepsilon \mathbb{Z}^+$ be such that $m < n$. Then there exists an $\ell \varepsilon \mathbb{Z}^+$ such that $m+\ell = n$. It follows from (a) that $t2^m < t2^{m+1} <\ldots< t2^{m+\ell} = t2^n$.

To show (c), let $m, n \in \mathbb{Z}^-$ be such that $m < n$. Now, we have that $t2^{-1} < t$. A proof similar to the proof of (a) shows that $t2^n < t2^{n+1}$ for all $n \in \mathbb{Z}^-$. Since $m < n$, $m+\ell = n$ for some $\ell \in \mathbb{Z}^+$. Thus $t2^m < t2^{m+1} < \ldots < t2^{m+\ell} = t2^n$. From (b) and (c), we have that for every $m, n \in \mathbb{Z}$, $m < n$ implies that $t2^m < t2^n$. ....(iv)

Let $B_1 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(2^m,1) \mid m \in \mathbb{Z}\}$. Define +, $\cdot$ and $\leqslant$ as are given in Remark 3.2.11. Define $f: K \to B_1$ in the following way: $f(\infty) = \infty$. Let $x \in K \setminus \{\infty\}$. If $x \in D_K^f$, then $x = 2^n$ for some $n \in \mathbb{Z}$. Define $f(x) = (x,0)$. If $x \in D_K^i$, then by (i), $x = tr$ for some $r \in D_K^f$. Define $f(x) = (r,1)$. Clearly, f is well-defined and f is a bijection.

(I) To show that for every $x, y \in K$, $x \leqslant y$ implies that $f(x) \leqslant f(y)$, let $x, y \in K$ be such that $x \leqslant y$.

Case I.1: $x \leqslant y < \infty$. This case is clear.

Case I.2: $x < \infty < y$. Then $y = tr$ for some $r \in D_K^f$. Now, we have that $f(x) = (x,0)$ and $f(y) = (r,1)$. Therefore $f(x) = (x,0) < (r,1) = f(y)$.

Case I.3: $\infty < x \leqslant y$. Then by (i), $x = tr$ for some $r \in D_K^f$ and $y = ts$ for some $s \in D_K^f$. Thus $f(x) = (r,1)$ and $f(y) = (s,1)$. Then there are $n, m \in Z$ such that $r = 2^n$ and $s = 2^m$. Therefore $t2^n = x \leqslant y = t2^m$. If $m < n$, then by (iv), $t2^m < t2^n$, a contradiction. Thus $n \leqslant m$. Therefore $r = 2^n \leqslant 2^m = s$, hence $(r,1) \leqslant (s,1)$. Therefore $f(x) \leqslant f(y)$.

(II) To show that $f(x+y) = f(x)+f(y)$ for all $x, y \in K$, let $x, y \in K$ be arbitrary. If $x = \infty$ or $y = \infty$, then we are done. Suppose that $x, y \in K \setminus \{\infty\}$.

<u>Case II.1</u>: $x \leqslant y < \infty$. This case is clear.

<u>Case II.2</u>: $x < \infty < y$. Then $y = tr$ for some $r \in D_K^f$. Thus $f(x) = (x,0)$ and $f(y) = (r,1)$. Therefore $f(x)+f(y) = (x,0)+(r,1) = \infty$. Since $x < \infty < y$, $\infty \leqslant x+y \leqslant \infty$. Then $x+y = \infty$. Thus $f(x+y) = f(\infty) = \infty$. Hence $f(x+y) = f(x)+f(y)$.

<u>Case II.3</u>: $\infty < x \leqslant y$. Then $x = tr$ for some $r \in D_K^f$ and $y = ts$ for some $s \in D_K^f$. Thus $x+y = tr+ts = t(r+s) = t(\min\{r,s\})$. Thus $f(x+y) = (\min\{r,s\},1)$. Now, we have that $f(x) = (r,1)$ and $f(y) = (s,1)$. Then $f(x)+f(y) = (r,1)+(s,1) = (\min\{r,s\},1)$. Hence $f(x+y) = f(x)+f(y)$.

(III) To show that $f(xy) = f(x)f(y)$ for all $x, y \in K$, let $x, y \in K$ be arbitrary. If $x = \infty$ or $y = \infty$, then we are done. Suppose that $x, y \in K \setminus \{\infty\}$.

<u>Case III.1</u>: $x \leqslant y < \infty$. This case is clear.

<u>Case III.2</u>: $x < \infty < y$. Then $y = tr$ for some $r \in D_K^f$. By Proposition 3.2.3 (4), $xy \in D_K^i$. Thus $xy = tr_1$ for some $r_1 \in D_K^f$. By (iii), $xr = r_1$. Then $f(xy) = (r_1,1) = (xr,1) = (x,0)(r,1) = f(x)f(y)$.

<u>Case III.3</u>: $y < \infty < x$. This proof is similar to the proof of Case III.2.

<u>Case III.4</u>: $\infty < x \leqslant y$. Then $x = tr$ for some $r \in D_K^f$ and $y = ts$ for some $s \in D_K^f$. By (iii), $xy = t^2rs = rs$. Therefore $f(xy) = (rs,0) = (r,1)(s,1) = f(x)f(y)$.

Therefore f is an isomorphism. Hence $(K,+,\cdot,\leqslant)$ is isomorphic to (2).

<u>Subcase 1.2</u>: $t2 < t$. Now, we shall show that the following

properties hold:

(d) $t2^{n+1} < t2^n$ for all $n \varepsilon \mathbb{Z}^+$,

(e) For every $m, n \varepsilon \mathbb{Z}^+$, $m < n$ implies that $t2^n < t2^m$,

(f) For every $m, n \varepsilon \mathbb{Z}^-$, $m < n$ implies that $t2^n < t2^m$.

To show (d), let $n \varepsilon \mathbb{Z}^+$ be arbitrary. We shall prove this by using induction on $n \varepsilon \mathbb{Z}^+$. If $n = 1$, then we are done. Suppose that (d) is true for some $n-1 \geqslant 1$. Then $t2^n < t2^{n-1}$. Thus $(t2^n)2 < (t2^{n-1})2$, so $t2^{n+1} < t2^n$. Hence $t2^{n+1} < t2^n$ for all $n \varepsilon \mathbb{Z}^+$.

To show (e), let $m, n \varepsilon \mathbb{Z}^+$ be such that $m < n$. Then there exists an $\ell \varepsilon \mathbb{Z}^+$ such that $m+\ell = n$. From (d), we have that $t2^n = t2^{m+} < \ldots < t2^{m+1} < t2^m$.

To show (f), let $m, n \varepsilon \mathbb{Z}^-$ be such that $m < n$. Now, we have that $t < t2^{-1}$. A proof is similar to the proof of (d) shows that $t2^{n+1} < t2^n$ for all $n \varepsilon \mathbb{Z}^-$. Since $m < n$, $m+\ell = n$ for some $\ell \varepsilon \mathbb{Z}^+$. Thus $t2^n = t2^{m+\ell} < \ldots < t2^{m+2} < t2^{m+1} < t2^m$. From (e) and (f), we have that for every $m, n \varepsilon \mathbb{Z}$, $m < n$ implies that $t2^n < t2^m$. ......(v)

Let $B_2 = \{(2^n,0) | n \varepsilon \mathbb{Z}\} \cup \{\infty\} \cup \{(2^m,1) | m \varepsilon \mathbb{Z}\}$. Define $+$, $\cdot$ and $\leqslant$ as are given in Remark 3.2.12. Define $F: K \to B_2$ in the following way: $F(\infty) = \infty$. Let $x \varepsilon K \setminus \{\infty\}$. If $x \varepsilon D_K^f$, then $x = 2^n$ for some $n \varepsilon Z$. Define $F(x) = (x,0)$. If $x \varepsilon D_K^i$, then $x = tr$ for some $r \varepsilon D_K^f$. Define $F(x) = (r,1)$. Clearly, F is well-defined and F is a bijection.

To show that for every $x, y \varepsilon K$, $x \leqslant y$ implies that $F(x) \leqslant F(y)$, let $x, y \varepsilon K$ be such that $x \leqslant y$. If $x, y \varepsilon D_K^f$, then we are done. If $x < \infty < y$, then $F(x) < \infty < F(y)$. Suppose that $\infty < x \leqslant y$. Then by (i), $x = tr$ for some $r \varepsilon D_K^f$ and $y = ts$ for some $s \varepsilon D_K^f$. Thus $F(x) = (r,1)$ and $F(y) = (s,1)$. Then there are $n, m \varepsilon \mathbb{Z}$ such that $r = 2^n$ and $s = 2^m$. Therefore $t2^n = x < y = t2^m$. If $n < m$, then by (v), $t2^m < t2^n$, a contradiction. Therefore $m \leqslant n$. Thus $s = 2^m < 2^n = r$. Then $(r,1) \leqslant (s,1)$. Hence $F(x) \leqslant F(y)$.

Using a proof similar to the one used in Subcase 1.1 we can show that F is a homomorphism. Therefore we get that F is an isomorphism. Hence $(K,+,\cdot,\leqslant)$ is isomorphic to (3).

Case 2: $z^2 = 2^N$ for some $N \in \mathbb{Z}_{odd}$. Then $z^2 = 2^{2m-1}$ for some $m \in \mathbb{Z}$. Thus $z^2(2^{2m})^{-1} = 2^{-1}$. By (i), $(z2^{-m})^2 = 2^{-1}$. Let $w = z2^{-m}$. Then $w \in D_K^i$ and $w^2 = 2^{-1}$ and $wD_K^f = z2^{-m}D_K^f = zD_K^f = D_K^i$. Now, we have that $w2 \neq w$. Then $w2 > w$ or $w < w2$.

Subcase 2.1: $w2 > w$. Using a proof similar to the proof of (iv) in Subcase 1.1 we get that for every $m, n \in \mathbb{Z}$, $m < n$ implies that $w2^m < w2^n$. ....(vi)

Let $C_1 = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(\sqrt{2^m},1) \mid m \in \mathbb{Z} \text{ is odd}\}$. Define $+$, $\cdot$ and $\leqslant$ as are given in Remark 3.2.15. Define $g: K \rightarrow C_1$ in the following way: $g(\infty) = \infty$. Let $x \in K \setminus \{\infty\}$. If $x \in D_K^f$, then $x = 2^n$ for some $n \in \mathbb{Z}$. Define $g(x) = (x,0)$. If $x \in D_K^i$, then $x = wr$ for some $r \in D_K^f$. Then $r = 2^m$ for some $m \in \mathbb{Z}$. Define $g(x) = (\sqrt{2^{2m-1}},1)$. Clearly, g is well-defined and g is a bijection.

(I) To show that for every $x, y \in K$, $x \leqslant y$ implies that $g(x) \leqslant g(y)$, let $x, y \in K$ be such that $x \leqslant y$.

Case I.1: $x \leqslant y \leqslant \infty$. This case is clear.

Case I.2: $x < \infty < y$. Then $y = wr$ for some $r \in D_K^f$. Thus $r = 2^m$ for some $m \in \mathbb{Z}$. Therefore $g(y) = (\sqrt{2^{2m-1}},1)$. Now, we have that $x = 2^n$ for some $n \in \mathbb{Z}$. Thus $g(x) = (2^n,0)$. Hence $g(x) = (2^n,0) < \infty < (\sqrt{2^{2m-1}},1) = g(y)$.

<u>Case I.3</u>: $\infty < x \leqslant y$. Then $x = wr$ for some $r \in D_K^f$ and $y = ws$ for some $s \in D_K^f$. Thus $r = 2^m$ for some $s \in D_K^f$. Thus $r = 2^m$ for some $m \in \mathbb{Z}$ and $s = 2^n$ for some $n \in \mathbb{Z}$. Therefore $g(x) = (\sqrt{2^{2m-1}},1)$ and $g(y) = (\sqrt{2^{2n-1}},1)$. Now, we have that $w2^m < w2^n$. If $n < m$, then by (vi), $w2^n < w2^n$, a contradiction. Therefore $m \leqslant n$, it follows that $\sqrt{2^{2m-1}} \leqslant \sqrt{2^{2n-1}}$. Hence $g(x) = (\sqrt{2^{2m-1}},1) \leqslant (\sqrt{2^{2n-1}},1) = g(y)$.

(II) To show that $g(x+y) = g(x)+g(y)$ for all $x, y \in K$, let $x, y \in K$. If $x = \infty$ or $y = \infty$, then we are done. Suppose that $x, y \in K \setminus \{\infty\}$.

<u>Case II.1</u>: $x, y \in D_K^f$. This case is clear.

<u>Case II.2</u>: $x \in D_K^f$ and $y \in D_K^i$. Then by Theorem 3.2.25, $x+y = \infty$. Now, we have that $x = 2^m$ for some $m \in \mathbb{Z}$ and $y = wr$ for some $r \in D_K^f$. Then $r = 2^n$ for some $n \in \mathbb{Z}$. Thus $g(x) = (2^m,0)$ and $g(y) = (\sqrt{2^{2n-1}},1)$. Thus $g(x)+g(y) = (2^m,0)+(\sqrt{2^{2n-1}},1) = \infty$. Hence $g(x+y) = g(\infty) = \infty = \infty$ $g(x)+g(y)$.

<u>Case II.3</u>: $x \in D_K^i$ and $y \in D_K^f$. This proof is similar to the proof of Case II.2.

<u>Case II.4</u>: $x \in D_K^i$ and $y \in D_K^i$. Then $x = wr$ for some $r \in D_K^f$ and $y = ws$ for some $s \in D_K^f$. Thus $r = 2^m$ for some $m \in \mathbb{Z}$ and $s = 2^n$ for some $n \in \mathbb{Z}$. Without loss of generality, suppose that $m \leqslant n$. Then $x+y = z2^m + z2^n = z(2^m+2^n) = z2^m$. Therefore $g(x+y) = (\sqrt{2^{2m-1}},1)$. Now, we have that $g(x) = (\sqrt{2^{2m-1}},1)$ and $g(y) = (\sqrt{2^{2n-1}},1)$. Then $g(x)+g(y) = (\sqrt{2^{2m-1}},1)+(\sqrt{2^{2n-1}},1) = (\min\{\sqrt{2^{2m-1}},\sqrt{2^{2n-1}}\},1) = (\sqrt{2^{2m-1}},1)$

Hence $g(x+y) = g(x)+g(y)$.

(III) To show that $g(xy) = g(x)g(y)$ for all $x, y \in K$, for $x, y \in K$ be arbitrary. If $x = \infty$ or $y = \infty$, then we are done. Suppose that $x, y \in K \setminus \{\infty\}$.

Case III.1: $x, y \in D_K^f$. This case is clear.

Case III.2: $x \in D_K^f$ and $y \in D_K^i$. Then $x = 2^m$ for some $m \in \mathbb{Z}$ and $y = w2^n$ for some $n \in \mathbb{Z}$. Using a proof similar to the proof of (i) we can show that $wd = dw$ for all $d \in D_K^f$. Thus $xy = w2^{m+n}$. Thus $g(xy) = (\sqrt{2^{2(m+n)-1}},1)$. Now, we have that $g(x) = (2^m,0)$ and $g(y) = (\sqrt{2^{2n-1}},1)$. Therefore we get that $g(x)g(y) = (2^m,0)(\sqrt{2^{2n-1}},1)$ $= (2^m\sqrt{2^{2n-1}},1) = (\sqrt{2^{2m+2n-1}},1)$. Hence $g(xy) = g(x)g(y)$.

Case III.3: $x \in D_K^i$ and $y \in D_K^f$. This proof is similar to Case III.2.

Case III.4: $x \in D_K^i$ and $y \in D_K^i$. Then $x = w2^m$ for some $m \in \mathbb{Z}$ and $y = w2^n$ for some $n \in \mathbb{Z}$. Using a proof similar to the proof of (i) we can show that $wb = bw$ for all $b \in D_K^f$. Then $xy = w^2 2^{m+n} = 2^{m+n-1}$. Thus $g(xy) = (2^{m+n-1},0)$. Now, we have that $g(x) = (\sqrt{2^{2m-1}},1)$ and $g(y) = (\sqrt{2^{2n-1}},1)$. Then $g(x)g(y) = (\sqrt{2^{2m-1}},1)(\sqrt{2^{2n-1}},1) = (\sqrt{2^{2m+2n-2}},0)$ $= (\sqrt{2^{2(m+n-1)}},0) = (2^{m+n-1},0)$. Hence $g(xy) = g(x)g(y)$.

Thus g is an isomorphism. Hence $(K,+,\cdot,\leqslant)$ is isomorphic to (4).

Subcase 2.2: $w2 < w$. Using a proof similar to the proof of Subcase 2.1 we get that $(K,+,\cdot,\leqslant)$ is isomorphic to (5). #

Theorem 3.2.30. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield.

Suppose that for every x, y ε K, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all z ε K. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\{2^n | n \varepsilon \mathbb{Z}\}, \max,\cdot,\leqslant)$. Then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) $(\{2^n | n \varepsilon \mathbb{Z}\} \cup \{\infty\}, \max,\cdot,\leqslant)$.

(2) $(B_3,+,\cdot,\leqslant)$ as in Remark 3.2.13.

(3) $(B_4,+,\cdot,\leqslant)$ as in Remark 3.2.14.

(4) $(C_3,+,\cdot,\leqslant)$ as in Remark 3.2.15.

(5) $(C_4,+,\cdot,\leqslant)$ as in Remark 3.2.16.

The proof of Theorem 3.2.30 is similar to the proof of Theorem 3.2.29.

Theorem 3.2.31. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that for every x, y ε K, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all z ε K. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\{1\},+,\cdot,\leqslant)$. Then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) ∞-skew semifield with the almost trivial addition of order 2.

(2) ∞-skew semifield with the almost trivial addition of order 3.

Proof: Assume that $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\{1\},+,\cdot,\leqslant)$. Now, we have that $K = D_K^f \cup \{\infty\} \cup D_K^i$. If $D_K^i = \emptyset$, then $(K,+,\cdot,\leqslant)$ is isomorphic to (1). Suppose that $D_K^i \neq \emptyset$. By Theorem 3.2.26, there exists a $t \varepsilon D_K^i$ such that $K = D_K^f \cup \{\infty\} \cup tD_K^f$ and $t^2 \varepsilon D_K^f$. Thus $D_K^i = tD_K^f$. Therefore we get that $(K,+,\cdot,\leqslant)$ is isomorphic to (2). #

Proposition 3.2.32. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$ and suppose that $D_K^i \neq \emptyset$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$, then for every $a, b \in D_K^i$, $a < a+b$ or for every $c, d \in D_K^i$, $c+d < c$.

Proof: Assume that $D_K^i \neq \emptyset$. By Theorem 3.2.28, $D_K^i = tD_K^f$ and $t^2 \in D_K^f$ for some $t \in D_K^i$. Suppose that $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$. For simplicity, we shall assume that $D_K^f = \mathbb{R}^+$. Now, we have that $2t \neq t$. Then either $2t < t$ or $t < 2t$.

Case 1: $t < 2t$. We shall show that for every $a, b \in D_K^i$, $a < a+b$.

Step 1.1. We shall show that for every $m, n \in \mathbb{Z}^+$, $m < n$ implies that $mt < nt$. We claim that for every $n \in \mathbb{Z}^+$, $nt < (n+1)t$. Let $n \in \mathbb{Z}^+$. We shall prove this by using induction on $n \in \mathbb{Z}^+$. If $n = 1$, then we are done. Suppose that the claim is true for some $n-1 \geqslant 1$. Then $(n-1)t < nt$. Therefore $(n-1)t+t \leqslant nt+t$, it follows that $nt \leqslant (n+1)t$. If $nt = (n+1)t$, then $n = n+1$, a contradiction. Thus $nt < (n+1)t$, so we have the claim.

Suppose that $m, n \in \mathbb{Z}^+$ are such that $m < n$. Then $m+\ell = n$ for some $\ell \in \mathbb{Z}^+$. Therefore $mt < (m+1)t < \ldots < (m+n)t = nt$.

Step 1.2. We shall show that for every $r, s \in \mathbb{Q}^+$, $r < s$ implies that $rt < st$. To prove this, let $r, s \in \mathbb{Q}^+$ be such that $r < s$. Then $r = \frac{m}{n}$ and $s = \frac{p}{q}$ for some $m, n, p, q \in \mathbb{Z}^+$. Thus $\frac{m}{n} < \frac{p}{q}$, it follows that $qm < np$. By Step 1.1, $qmt < npt$. Therefore $\frac{m}{n} t < \frac{p}{q} t$. Hence $rt < st$.

Step 1.3. We shall show that $t\mathbb{Q}^+$ has no lower bound in D_K^i. To prove this, suppose not. Then $t\mathbb{Q}^+$ has a lower bound in D_K^i. Since K is complete and $D_K^i \subseteq K$, $t\mathbb{Q}^+$ has an infimum in D_K^i. Let $z = \inf(t\mathbb{Q}^+)$. Then $\infty < z$. Therefore $z \leqslant rt$ for all $r \in \mathbb{Q}^+$. Thus $z \leqslant \frac{r}{2}t$ for all $r \in \mathbb{Q}^+$, so $2z \leqslant rt$ for all $r \in \mathbb{Q}^+$. Then $2z$ is a lower bound of $t\mathbb{Q}^+$. Thus $2z \leqslant z$. ....(1)

Similarly, $z \leqslant 2rt$ for all $r \in \mathbb{Q}^+$. It follows that $2^{-1}z \leqslant z$. Thus $z \leqslant 2z$. From (1), we have that $z = 2z$ which implies that $1 = 2$, a contradiction. Hence $t\mathbb{Q}^+$ has no lower bound in D_K^i.

Step 1.4. We shall show that $t < t+dt$ for all $d \in D_K^f$. Let $r \in \mathbb{Q}^+$ be arbitrary. Then $1 < 1+r$. By Step 1.2, $t < (1+r)t = t+rt$. Hence $t < t+rt$ for all $r \in \mathbb{Q}^+$. ....(2)

Suppose that $d \in D_K^f$ is arbitrary. Then $dt \in D_K^i$. By Step 1.3, dt is not a lower bound of $t\mathbb{Q}^+$. Then there exists an $r_d \in \mathbb{Q}^+$ such that $r_d t < dt$. Thus $t+r_d t \leqslant t+dt$. From (2) we have that $t < t+dt$.

Now, we shall show that for every $x, y \in D_K^i$, $x < x+y$. Let $x, y \in D_K^i$ be arbitrary. Then $x = ct$ and $y = dt$ for some $c, d \in D_K^f$. Thus $c^{-1}d \in D_K^f$. By Step 1.4, $t < t+c^{-1}dt$ which implies that $ct < ct+dt$. Hence $x < x+y$.

Case 2: $2t < t$. Using a proof similar to the one used in Case 1 we can show that $x+y < x$ for all $x, y \in D_K^i$.

Hence, the theorem is proved. #

Theorem 3.2.33. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield.

Suppose that for every x, y ε K, $x \leqslant y$ implies that $x+z \leqslant y+z$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$, then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) $(\mathbb{R}^+,+,\cdot,\leqslant)$.

(2) $(E_1,+,\cdot,\leqslant)$ as in Remark 3.2.19.

(3) $(E_2,+,\cdot,\leqslant)$ as in Remark 3.2.20.

Proof: Assume that $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$. Now, we have that $K = D_K^f \cup \{\infty\} \cup D_K^i$. If $D_K^i = \emptyset$, then $(K,+,\cdot,\leqslant)$ is isomorphic to (1).

Suppose that $D_K^i \neq \emptyset$. By Theorem 3.2.28, there exists a $z \varepsilon D^i$ such that $K = D_K^f \cup \{\infty\} \cup zD_K^f$ and $z^2 \varepsilon D_K^f$. For simplicity, we shall assume that $D_K^f = \mathbb{R}^+$. Then $z^2 = a$ for some $a \varepsilon \mathbb{R}^+$. Thus $a = b^2$ for some $b \varepsilon \mathbb{R}^+$. Therefore $z^2 = b^2$. Using a proof similar to the proof of (i) in Theorem 3.2.29 we can show that $(zb^{-1})^2 = 1$. Let $t = zb^{-1}$. Then $t \varepsilon D_K^i$ and $t^2 = 1$ and $tD_K^f = (zb^{-1})D_K^f = z(b^{-1}D_K^f)$
$= zD_K^f = D_K^i$. ....(*)

Using a proof similar to the proof of (i) in Theorem 3.2.29 we can show that $tr = rt$ for all $r \varepsilon D_K^f$. ....(**)

Now, we have that $2t \neq t$. Then either $t < 2t$ or $2t < t$.

Case 1: $t < 2t$. Then by the proof of Proposition 3.2.32, $a < a+b$ for all $a, b \varepsilon D_K^i$. Let $E_1 = ((\mathbb{R}^+ \times \{1\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\}),+,\cdot,\leqslant)$ be given as in Remark 3.2.19. Define $F: K \to E_1$ in the following way: $F(\infty) = \infty$. Define $F(x) = (x,0)$ for all $x \varepsilon D_K^f$. Let $y \varepsilon D_K^i$. By (*), $y = ts$ for some $s \varepsilon D_K^f$. Define $F(y) = (s,1)$. Clearly, F is well-defined and F is a bijection.

(1.1) To show that for every x, y ε K, $x \leqslant y$ implies that $F(x) \leqslant F(y)$, let x, y ε K be such that $x \leqslant y$. If $x = y$, then we are done. Suppose that $x < y$.

Subcase 1.1.1: $x < y \leqslant \infty$. This case is clear.

Subcase 1.1.2: $x \leqslant \infty < y$. This case is clear.

Subcase 1.1.3: $\infty < x < y$. Then $x = tr$ for some $r \varepsilon D_K^f$ and $y = ts$ for some $s \varepsilon D_K^f$. Thus $tr < ts$. If $s < r$, then there exists a $u \varepsilon D_K^f$ such that $s+u = r$. Therefore $ts+tu = tr$. Then $ts+tu < ts$ and $ts, tu \varepsilon D_K^i$, a contradiction. Hence $r \leqslant s$. Therefore $(r,1) \leqslant (s,1)$. Then $F(x) \leqslant F(y)$.

This shows that for every x, y ε K, $x \leqslant y$ implies that $F(x) \leqslant F(y)$.

(1.2) To show that for every x, y ε K, $F(x+y) = F(x)+F(y)$ and $F(xy) = F(x)F(y)$, let x, y ε K be arbitrary. If $x = \infty$ or $y = \infty$, then we are done. Suppose that x, y ε $K \setminus \{\infty\}$.

Subcase 1.2.1: $x \varepsilon D_K^f$ and $y \varepsilon D_K^f$. This case is clear.

Subcase 1.2.2: $x \varepsilon D_K^f$ and $y \varepsilon D_K^i$. Then by (*), $y = tr$ for some $r \varepsilon D_K^f$. By Proposition 3.2.3(7), $x+y = \infty$. Thus $F(x+y) = F(\infty) = \infty$ and $F(x)+F(y) = (x,0)+(r,1) = \infty$. Hence $F(x+y) = F(x)+F(y)$. From (**), we have that $xy = txr$. By Proposition 3.2.3 (4), $xy \varepsilon D_K^i$. By (*), $xy = tr_1$. Hence $xr = r_1$. Therefore we get that $F(xy) = (r_1,1) = (xr,1) = (x,0)(r,1) = F(x)F(y)$.

Subcase 1.2.3: $x \varepsilon D_K^i$ and $y \varepsilon D_K^f$. This proof is similar to the proof of Subcase 1.2.2.

Subcase 1.2.4: $x \varepsilon D_K^i$ and $y \varepsilon D_K^i$. Then $x = tr$ for some $r \varepsilon D_K^f$ and $y = ts$ for some $s \varepsilon D_K^f$. Then $x+y = t(r+s)$. Then

$F(x+y) = (r+s,1) = (r,1)+(s,1) = F(x)+F(y)$. By (**), $xy = t^2rs = rs$

Therefore $F(xy) = (rs,0) = (r,1)(s,1) = F(x)F(y)$.

Therefore F is an isomorphism. Hence $(K,+,\cdot,\leqslant)$ is isomorphic to (2).

Case 2: $2t < t$. Then by the proof of Proposition 3.2.32, $a+b < a$ for all $a, b \in D_K^i$. Let $E_2 = ((\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\}),+,\cdot,\leqslant)$ be given as in Remark 3.2.20. Using a proof similar to the proof of Case 1 we can show that $(K,+,\cdot,\leqslant)$ is isomorphic to (3).

Hence, the theorem is proved. #

Theorem 3.2.34. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant_{opp})$, then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) $(\mathbb{R}^+_\infty,+,\cdot,\leqslant^*)$ where + and $\cdot$ are the usual addition and multiplication, respectively and $\leqslant^* = \leqslant_{opp}$ on $\mathbb{R}^+$ and $x < \infty$ for all $x \in \mathbb{R}^+$.

(2) $(E_3,+,\cdot,\leqslant)$ as in Remark 3.2.21.

(3) $(E_4,+,\cdot,\leqslant)$ as in Remark 3.2.22.

The proof of Theorem 3.2.34 is similar to the proof of Theorem 3.2.33.

Proposition 3.2.35. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$ and suppose that $D_K^i \neq \emptyset$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(R^+,\max,\cdot,\leqslant)$, then for every $a, b \in D_K^i$, $a \leqslant a+b$ or for every $c, d \in D_K^i$, $c+d \leqslant d$.

<u>Proof</u>: Assume that $D_K^i \neq \emptyset$ and suppose that $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\max,\cdot,\leqslant)$. By Theorem 3.2.28, $D_K^i = tD_K^f$ and $t^2 = 1$ for some $t \in D_K^i$. For simplicity, we shall assume that $D_K^f = \mathbb{R}^+$. Now, we have that $2t \neq t$. Then either $2t < t$ or $t < 2t$.

<u>Case 1</u>: $t < 2t$. We shall show that for every $a, b \in D_K^i$, $a \leqslant a+b$. First, we shall show that $t \leqslant t+dt$ for all $d \in D_K^f$. Let $d \in D_K^f$ be arbitrary.

<u>Subcase 1.1</u>: $d \leqslant 1$. Then $t = (1+d)t = t+dt$.

<u>Subcase 1.2</u>: $1 < d$. Then there exists an $m \in \mathbb{Z}^+$ be such that $2^m \leqslant d < 2^{m+1}$. Since $2^m t < 2^{m+1}t$, $2^m t+dt \leqslant 2^{m+1}t+dt$. Then $(2^m+d)t \leqslant (2^{m+1}+d)t$. Thus $dt \leqslant 2^{m+1}t$. ....(1)

Let $2^{-(m+1)} \leqslant b \leqslant 2^{-m}$ be arbitrary. Then $2^m \leqslant 2^{2m+1}b \leqslant 2^{m+1}$. From (1), we have that $2^{2m+1}bt \leqslant 2^{m+1}t$, it follows that $bt \leqslant 2^{-m}t$.

......................(2)

Now, we have that $2^{-(m+1)} < d^{-1} < 2^{-m}$. By (2), $d^{-1}t \leqslant 2^{-m}t$, so $d^{-1}t < t$. Therefore $t < dt$. Hence $t = t+t \leqslant t+dt$.

Therefore we get that $t \leqslant t+dt$ for all $d \in D_K^f$. Now, we shall show that $a \leqslant a+b$ for all $a, b \in D_K^i$. Let $a, b \in D_K^i$ be arbitrary. Then $a = rt$ for some $r \in D_K^f$ and $b = st$ for some $s \in D_K^f$. Since $r^{-1}s \in D_K^f$, $t \leqslant t+r^{-1}st$. Then $rt \leqslant rt+st$. Hence $a \leqslant a+b$.

<u>Case 2</u>: $2t < t$. We shall show that for every $c, d \in D_K^i$, $c+d \leqslant d$.

We shall first show that $t+ut \leqslant ut$ for all $u \in D_K^f$. Let $u \in D_K^f$ be arbitrary.

<u>Subcase 2.1</u>: $1 \leqslant u$. Then $t+ut = (1+u)t = ut$.

<u>Subcase 2.2</u>: $u < 1$. Then there exists an $n \in \mathbb{Z}^+$ such that $2^{-n-1} \leqslant u < 2^{-n}$. Since $2^{-n}t < 2^{-(n+1)}t$, $2^{-n}t+ut \leqslant 2^{-(n+1)}t+ut$. Thus $(2^{-n}+u)t \leqslant (2^{-(n+1)}+u)t$. Therefore $2^{-n}t \leqslant ut$ which implies that $t < 2^{-n}t \leqslant ut$. Hence $t+ut \leqslant ut$.

Therefore we get that $t+ut \leqslant ut$ for all $u \in D_K^f$. Now, we shall show that $c+d \leqslant d$ for all $c, d \in D_K^i$. Let $c, d \in D_K^i$ be arbitrary. Then $c = rt$ for some $r \in D_K^f$ and $d = st$ for some $s \in D_K^f$. Since $r^{-1}s \in D_K^f$, $t+r^{-1}st \leqslant r^{-1}st$. Then $rt+st \leqslant st$. Hence $c+d \leqslant d$. #

<u>Proposition 3.2.36</u>. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that for every $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$ and suppose that $D_K^i \neq \emptyset$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\min,\cdot,\leqslant)$, then for every $a, b \in D_K^i$, $a+b \leqslant a$ or for every $c, d \in D_K^i$, $c \leqslant c+d$.

The proof of Proposition 3.2.36 is similar to the proof of Proposition 3.2.35.

<u>Theorem 3.2.37</u>. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\max,\cdot,\leqslant)$, then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) $(\mathbb{R}^+,\max,\cdot,\leqslant)$.

(2) $(E_5,+,\cdot,\leqslant)$ as in Remark 3.2.23.

(3) $(E_6,+,\cdot,\leqslant)$ as in Remark 3.2.24.

Proof: Assume that $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\min,\cdot,\leqslant)$. Now, we have that $K = D_K^f \cup \{\infty\} \cup D_K^i$. If $D_K^i = \emptyset$, then $(K,+,\cdot,\leqslant)$ is isomorphic to (1).

Suppose that $D_K^i \neq \emptyset$. For simplicity, we shall assume that $D_K^f = \mathbb{R}^+$. By Theorem 3.2.28, there exists a $t \in D_K^i$ such that $D_K^f = tD_K^f$ and $t^2 = 1$ for some $t \in D_K^i$. Using a proof similar to the proof of (i) in Theorem 3.2.29 we can show that $tr = rt$ for all $r \in D_K^f$. ...(*)

Now, we have that $2t \neq t$. Then either $t < 2t$ or $2t < t$.

Case 1: $t < 2t$. Then by the proof of Proposition 3.2.30, $a \leqslant a+b$ for all $a, b \in D_K^i$. ....(**)

Let $E_5 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$ be given as in Remark 3.2.23. Define $F: K \to E_5$ in the following way: $F(\infty) = \infty$. $F(x) = (x,0)$ for all $x \in D_K^f$. Let $y \in D_K^i$. Then $y = tr$ for some $r \in D_K^f$. Define $F(y) = (r,1)$. Clearly. F is well-defined and F is a bijection.

(1.1) To show that for every $x, y \in K$, $F(x+y) = F(x)+F(y)$ and $F(xy) = F(x)F(y)$, let $x, y \in K$ be arbitrary. If $x = \infty$ or $y = \infty$, then we are done. Suppose that $x, y \in K \setminus \{\infty\}$.

Subcase 1.1.1: $x \in D_K^f$ and $y \in D_K^f$. This case is clear.

Subcase 1.1.2: $x \in D_K^f$ and $y \in D_K^i$. Then $y = tr$ for some $r \in D_K^f$. By Proposition 3.2.3(7), $x+y = \infty$. Thus $F(x+y) = \infty$ and $F(x)+F(y) = (x,0)+(r,1) = \infty$. Hence $F(x+y) = F(x)+F(y)$. From (*), we have that $xy = txr$. By Proposition 3.2.3 (4), $xy \in D_K^i$. By (*),

$xy = tr_1$. Hence $xr = r_1$. Therefore we get that $F(xy) = (r_1,1) = (xr,1) = (x,0)(r,1) = F(x)F(y)$.

<u>Subcase 1.1.3</u>: $x \in D_K^i$ and $y \in D_K^f$. This proof is similar to the proof of Subcase 1.1.2.

<u>Subcase 1.1.4</u>: $x \in D_K^i$ and $y \in D_K^i$. Then $x = tr$ for some $r \in D_K^f$ and $y = ts$ for some $s \in D_K^f$. Then $x+y = t(r+s)$. Without loss of generality, suppose that $r \leqslant s$. Thus $x+y = ts$. Then $F(x+y) = (s,1) = (\max\{r,s\},1) = (r,1)+(s,1) = F(x)+F(y)$. By (*), $xy = t^2rs = rs$. Therefore $F(xy) = (rs,0) = (r,1)(s,1) = F(x)F(y)$.

(1.2) To show that for every $x, y \in K$, $x \leqslant y$ implies that $F(x) < F(y)$, let $x, y \in K$ be such that $x < y$. If $x = y$, then we are done. Suppose that $x < y$.

<u>Subcase 1.2.1</u>: $x < y \leqslant \infty$. This case is clear.

<u>Subcase 1.2.2</u>: $x \leqslant \infty < y$. This case is clear.

<u>Subcase 1.2.3</u>: $\infty \leqslant x < y$. If $x = \infty$, then by Subcase 1.2.2 we are done. Suppose that $\infty < x < y$. Then $x = tr$ for some $r \in D_K^f$ and $y = ts$ for some $s \in D_K^f$. Thus $tr < ts$. If $s < r$, then $s+r = r$. Thus $ts+tr = tr$. Therefore $ts+tr < tr$ and $tr, ts \in D_K^i$ which contradicts (**). Then $r \leqslant s$, so $(r,1) \leqslant (s,1)$. Hence $F(x) \leqslant F(y)$.

Therefore we get that F is an isomorphism. Hence $(K,+,\cdot,\leqslant)$ is isomorphic to (2).

<u>Case 2</u>: $2t < t$. Using a proof similar to the proof of Case 1 we can show that $(K,+,\cdot,\leqslant)$ is isomorphic to (3). #

Theorem 3.2.38. Let $(K,+,\cdot,\leqslant)$ be a type II ∞-skew semifield. Suppose that $x, y \in K$, $x \leqslant y$ implies that $x+z \leqslant y+z$ for all $z \in K$. If $(D_K^f,+,\cdot,\leqslant)$ is isomorphic to $(\mathbb{R}^+,\min,\cdot,\leqslant)$, then $(K,+,\cdot,\leqslant)$ is isomorphic to exactly one of the following ∞-skew semifields:

(1) $(\mathbb{R}^+_\infty, +, \cdot,\leqslant)$ as in (2) of Theorem 3.2.8.

(2) $(E_7,+,\cdot,\leqslant)$ as in Remark 3.2.25.

(3) $(E_8,+,\cdot,\leqslant)$ as in Remark 3.2.26.

The proof of Theorem 3.2.38 is similar to the proof of Theorem 3.2.37.

We cannot classify type III ∞-skew semifields. We close this section by giving some examples.

Example 3.2.39. Let $H_1 = (\{2^n | n \in \mathbb{Z}\}\times\{0\}) \cup \{\infty\}$. Define $+$, $\cdot$ and $\leqslant$ as follows: Let $x, y \in \{2^n | n \in \mathbb{Z}\}$ be arbitrary. Define

$$(x,0)+(y,0) = \infty,$$
$$(x,0)+\infty = \infty.$$

Define
$$(x,0)(y,0) = \infty,$$
$$(x,0)\infty = \infty = \infty(x,0).$$

Define
$$(x,0) < \infty,$$
$$(x,0) \leqslant (y,0) \text{ iff } x \leqslant y.$$

Then $(H_1,+,\cdot,\leqslant)$ is a type III ∞-skew semifield.

Example 3.2.40. Let $H_2 = (\mathbb{R}^+\times\{0\}) \cup \{\infty\}$. Define $+$, $\cdot$ and $\leqslant$ on H_2 is as in Example 3.2.24.

Then $(H_2,+,\cdot,\leqslant)$ is a type III ∞-skew semifield.

<u>Example 3.2.41.</u> Let $H_3 = (\{2^n | n \in \mathbb{Z}\} \times \{0\}) \cup \{\infty\} \cup (\{2^n | n \in \mathbb{Z}\} \times \{1\})$. Define $\cdot$ and $\leqslant$ on H_3 as are given in Remark 3.2.11. Define + on H_3 as follows:

Let $x, y \in \{2^n | n \in \mathbb{Z}\}$ be arbitrary. Define

$$(x,0)+(y,0) = \infty,$$

$$(x,0)+(y,1) = \infty,$$

$$(x,1)+(y,1) = \infty$$

and $x+\infty = \infty$ for all $z \in H_3$.

Then $(H_3,+,\cdot,\leqslant)$ is a type III ∞-skew semifield.

<u>Example 3.2.42.</u> Let $H_4 = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{\infty\})$. Define +, $\cdot$ and $<$ on H_4 as in Example 3.2.26.

Then $(H_4,+,\cdot,\leqslant)$ is a type III ∞-skew semifield.

<u>Example 3.2.43.</u> An ordered ∞-skew semifield with the trivial addition of order 2 and 3 are type III ∞-skew semifield.