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DARK ENERGY AND STABILIZATION OF THE UNIVERSE IN EXTRA DIMENSIONAL MODEL

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การขยายตัวของเอกภพด้วยอัตราเร่งในยุคปัจจุบันได้ถูกค้นพบจากการสังเกตการณ์ซุปเปอร์โนวาชนิด Ia เราเรียกพลังงานที่ทำให้เอกภพขยายตัวด้วยอัตราเร่งนี้ว่าพลังงานมืด ในอีกด้านหนึ่งทฤษฎีสตริงเชื่อว่าอวกาศสามารถมีมิติได้มากกว่าสามมิติ ถ้ากาลอวกาศมีมิติเพิ่มเติมนอกเหนือไปจากกาลอวกาศสี่มิติที่เราสัมผัสได้แสดงว่ามิติเพิ่มเติมเหล่านี้จะต้องขดม้วนอยู่และซ่อนตัวจากการสังเกตการณ์ของเรา นอกจากนี้จะต้องมีกระบวนการบางอย่างในการสร้างเสถียรภาพให้กับมิติเพิ่มเติมได้ไม่เช่นนั้นแล้วเราก็ควรจะรับรู้ถึงการมีอยู่ของมันได้จากการทดลอง ในปี 2007 ไบรอัน กรีน และ แจนนา เลวิน ได้เสนอว่าพลังงานคาสิเมียร์จากการรวมกันระหว่างสนามโบซอนกับสนามเฟอร์มิออนจะให้เสถียรภาพกับมิติเพิ่มเติมได้และในทางตรงกันข้ามพลังงานคาสิเมียร์นี้จะทำหน้าที่เป็นพลังงานสุญญากาศผลักอวกาศสามมิติให้ขยายตัวออกด้วยอัตราเร่งดังนั้นจึงมีความเป็นไปได้ที่พลังงานคาสิเมียร์จะเป็นพลังงานมืดได้ ในวิทยานิพนธ์ฉบับนี้เราได้ทำการศึกษาผลของสนามอีเธอร์ซึ่งเป็นสนามเวกเตอร์ที่ละเมิดสมมาตรลอเร็นซ์ต่อเสถียรภาพของมิติเพิ่มเติม ในแบบจำลองจะกำหนดให้ค่าเจาะจงของสนามอีเธอร์ไม่เท่ากับศูนย์ในทิศของมิติเพิ่มเติม ผลที่ได้แสดงให้เห็นว่าเฉพาะเงื่อนไขเริ่มต้นบางค่าสนามอีเธอร์จะไปหน่วงการสั่นของมิติเพิ่มเติมส่งผลให้มิติเพิ่มเติมมีเสถียรภาพได้แม้ว่าจะมีสสารกระจายตัวอยู่ในเอกภพมากที่สุดก็ตาม

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According to the observations of Type-Ia Supernovae, the expansion of the universe is accelerating in the late-time epoch. We call the energy form which causes the accelerated expansion of the universe a dark energy. On the other hand, there is motivation from the string theory that our spacetime has had some extra directions more than three spatial dimensions in space. If spacetime has some extra dimensions in addition to the 4-dimensions that we perceive, they should be compactified and hidden from us and there must be some mechanism to stabilize them, otherwise we would have observed them in the experiments. In 2007, Greene and Levin proposed that the Casimir energy of certain combinations of massless and massive bulk fields can give a stabilizing potential for the moduli fields while driving the accelerated expansion in the 3-large spatial direction. This is the hallmark behaviour of the vacuum dark energy. In this thesis we investigate the effect of a space-like Lorentz-violating aether field with a fixed expectation value along the extra dimensional directions on the moduli stabilization mechanism. We found that the aether field can slow down the oscillations of the shape moduli fields in the universe although the non-relativistic matter is dominating.

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Chapter I

Introduction

The discovery of the expansion of the universe by Edwin Hubble in 1929 suggested that the universe is not static. Recently the observations of Type Ia Supernovae by the High-z supernovae Search Team in 1998 [1], followed by the Supernovae Cosmology Project in 1999 [2] show that the universe is not only expanding, it is accelerating. The unknown component giving rise to the late-time cosmic acceleration is called the dark energy. The Supernovae Type Ia observations show that about 70% of the energy density today is in the form of the dark energy. The dark matter is about 25%, while the baryons is about 4% of the total energy density.

In fact, spacetime can have the extra dimensions in addition to left-right, forward-backward, up-down directions and time. If spacetime has the extra dimensions, they need to be somehow hidden from us. A way to compactify the extra dimension is the Kaluza-Klein scenario, to compactify the dimensions in the shape look like a doughnut. However, this proposal together with cosmological measurements have some problems such as we need a mechanism to stabilize the size of the extra dimensions remain unobservable while the three spatial dimensions grow large.

Recently, Brian Greene and Janna Levin suggest that Casimir energy of some combinations of massless and massive scalar and fermionic fields can be solved this problem [3]. By regularization of combining Casimir energy of the bulk scalar and fermionic field with different mass, the total Casimir energy is a function of the size of the extra dimension. The result shown that Casimir energy can stabilize the size of the extra dimension while it act as a cosmological constant to drive the accelerated expansion of the universe. This proposal is called Casimir dark energy.

However, the Casimir dark energy model fails to stabilize the extra dimension if we include the matter content. During the matter dominant epoch of the universe, the matter energy density was modify the Casimir energy potential in which there is no minimum. In this thesis we introduce a Lorentz violating vector aether field to solve this problem [4].

This thesis is organized as the following way. In chapter I, we review the action principle in the general relativity, the standard hot big bang model and the dark energy problem [5, 6, 7, 8, 9, 11, 12]. In chapter II, we consider the Casimir energy of certain combinations of massless and massive fields in the $M^{1+3} \times T^2$ spacetime that plays a crucial role in the accelerated expansion of the late time universe [13]. We start by discussing the cosmological dynamics on $M^{1+3} \times T^2$ spacetime and then calculate the Casimir energy in the spacetime with toroidally compactified extra dimensions. Then, we construct effective potential contributed by the Casimir energy. This effective potential gives a non-trivial minimum that can stabilize the size of the extra dimensions.

In chapter III, we review the extra dimensional model, the Kaluza-Klein theory and the ADD scenario in details [14, 15, 16, 17]. Then we review the Einstein-Æther theory, a theory of gravity with vector field that breaks Lorentz symmetry. Then, we investigate the aether compactification in 5-dimensions [18], and apply this model to the spacetime with toroidally compactified extra dimensions.

We study the effect of aether field in the stabilization mechanism in Chapter IV. In the first part of this chapter, we derive cosmological equation of motion in our $M^{1+3} \times T^2$ spacetime background with aether field. Then we calculate the Casimir energy in the case that a scalar field couple to an aether field with a coupling constant α_ϕ. Next, we consider the role of aether field in the stabilization mechanism of the extra dimensions both in the vacuum and the matter dominated universe. Finally we summarize our results in chapter V.

1.1 The Action Principle in General Relativity

In the framework of general relativity, gravity is not the force from some additional field propagating through spacetime. The dynamical field variable giving rise to the gravitation is the metric tensor describing the geometry of the spacetime itself. In other words, gravity is a manifestation of the curvature of our spacetime.

To obtain field equation governing the spacetime curvature in the presence of matter and energy, let us consider the Einstein-Hilbert action [5, 6]

$$S_{EH} = \int \mathrm{d}^4 x \sqrt{-g} R, \tag{1.1}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar and g is the determinant of the metric tensor. The Greek indices $\mu, \nu, ...$ run from 0 to 3 while the Latin indices $i, j, ...$ run from 1 to 3, the same convention applies to the whole thesis except when indicated otherwise. We use the Einstein's summation convention that when the term has the same upper and lower indices, we sum over all the indices $a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3$. Because $a_\mu b^\mu = a_\nu b^\nu$, the repeated indices are called the dummy indices. We consider a variation of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}. \tag{1.2}$$

Note that $g_{\mu\nu}$ satisfies the relation $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$ where δ^μ_ν is the Kronecker's delta, $\delta^\mu_\nu = 1$ for $\mu = \nu$ and otherwise $\delta^\mu_\nu = 0$, therefore $g^{\mu\nu}$ is the inverse metric tensor. From a variation of the metric in Eq.(1.2) we have variation of the Christoffel connection

$$\Gamma^\lambda_{\mu\nu} \to \Gamma^\lambda_{\mu\nu} + \delta\Gamma^\lambda_{\mu\nu}, \tag{1.3}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{1.4}$$

Hence

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} + \delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}(g^{\lambda\sigma} + \delta g^{\lambda\sigma})\left[\partial_\nu(g_{\mu\sigma} + \delta g_{\mu\sigma}) + \partial_\mu(g_{\nu\sigma} + \delta g_{\nu\sigma}) - \partial_\sigma(g_{\mu\nu} + \delta g_{\mu\nu})\right] \\ &= \Gamma^\lambda_{\mu\nu} + \frac{1}{2}\delta g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}) \\ &\quad + \frac{1}{2}g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}), \end{aligned} \tag{1.5}$$

keeping the first order terms of the metric. Since $g^{\lambda\sigma}g_{\sigma\rho} = \delta^\lambda_\rho$ it is straightforward to express relation between variations of the metric and the inverse metric

$$\delta g^{\lambda\sigma} = -g^{\sigma\rho}g^{\lambda\kappa}\delta g_{\kappa\rho}. \tag{1.6}$$

Substitute into Eq.(1.5), we obtain

$$\delta\Gamma^\lambda_{\mu\nu} = -g^{\lambda\kappa}\delta g_{\kappa\rho}\Gamma^\rho_{\mu\nu} + \frac{1}{2}g^{\lambda\sigma}(\partial_\nu \delta g_{\mu\sigma} + \partial_\mu \delta g_{\nu\sigma} - \partial_\sigma \delta g_{\mu\nu}). \tag{1.7}$$

From the covariant derivative of the variation of the metric tensor

$$\nabla_\nu \delta g_{\mu\sigma} = \partial_\nu \delta g_{\mu\sigma} - \Gamma^\rho_{\mu\nu}\delta g_{\rho\sigma} - \Gamma^\rho_{\sigma\nu}\delta g_{\mu\rho}, \tag{1.8}$$

hence

$$\partial_\nu \delta g_{\mu\sigma} + \partial_\mu \delta g_{\nu\sigma} - \partial_\sigma \delta g_{\mu\nu} = \nabla_\nu \delta g_{\mu\sigma} + \nabla_\mu \delta g_{\nu\sigma} - \nabla_\sigma \delta g_{\mu\nu} - \Gamma^\rho_{\sigma\nu}\delta g_{\mu\rho}. \tag{1.9}$$

Then a variation of the Christoffel connection in Eq.(1.7) can be written in the form

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\nabla_\nu\delta g_{\mu\sigma} + \nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\delta g_{\mu\nu}). \tag{1.10}$$

The variation $\delta\Gamma^{\lambda}_{\mu\nu}$ is the difference of two connections and it is a tensor. However, the connection $\Gamma^{\lambda}_{\mu\nu}$ itself is not a tensor because it is defined by partial derivatives. Therefore we can take the covariant derivative of $\delta\Gamma^{\lambda}_{\mu\nu}$

$$\nabla_\kappa\delta\Gamma^{\lambda}_{\mu\nu} = \partial_\kappa\delta\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\mu\kappa}\delta\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\rho}_{\nu\kappa}\delta\Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\rho\kappa}\delta\Gamma^{\rho}_{\mu\nu}, \tag{1.11}$$

then contracting the indices ν and λ

$$\nabla_\kappa\delta\Gamma^{\lambda}_{\mu\lambda} = \partial_\kappa\delta\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\rho}_{\mu\kappa}\delta\Gamma^{\lambda}_{\rho\lambda} - \Gamma^{\rho}_{\lambda\kappa}\delta\Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\rho\kappa}\delta\Gamma^{\rho}_{\mu\lambda}, \tag{1.12}$$

with a little algebra we can show that

$$\nabla_\lambda\delta\Gamma^{\lambda}_{\mu\kappa} - \nabla_\kappa\delta\Gamma^{\lambda}_{\mu\lambda} = \partial_\lambda\delta\Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa\delta\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\rho\lambda}\delta\Gamma^{\rho}_{\mu\kappa} + \Gamma^{\rho}_{\mu\kappa}\delta\Gamma^{\lambda}_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\rho\kappa} - \Gamma^{\lambda}_{\rho\kappa}\delta\Gamma^{\rho}_{\mu\lambda}. \tag{1.13}$$

Ricci tensor is defined by

$$R_{\mu\kappa} = \partial_\lambda\Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\rho}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\rho}, \tag{1.14}$$

then the first order in variation of the Ricci tensor is

$$\delta R_{\mu\kappa} = \partial_\lambda\delta\Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa\delta\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\rho\lambda}\delta\Gamma^{\rho}_{\mu\kappa} + \Gamma^{\rho}_{\mu\kappa}\delta\Gamma^{\lambda}_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\rho\kappa} - \Gamma^{\lambda}_{\rho\kappa}\delta\Gamma^{\rho}_{\mu\lambda}, \tag{1.15}$$

or

$$\delta R_{\mu\kappa} = \nabla_\lambda\delta\Gamma^{\lambda}_{\mu\kappa} - \nabla_\kappa\delta\Gamma^{\lambda}_{\mu\lambda}, \tag{1.16}$$

this relation is known as the Palatini identity.

We have

$$\delta S_{EH} = (\delta S)_1 + (\delta S)_2 + (\delta S)_3, \tag{1.17}$$

where

$$(\delta S)_1 = \int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}, \tag{1.18}$$

$$(\delta S)_2 = \int d^4x R\delta\sqrt{-g}, \tag{1.19}$$

$$(\delta S)_3 = \int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \tag{1.20}$$

From the Palatini identity we can show that the last equation is a total divergence,

$$\sqrt{-g}g^{\mu\kappa}\delta R_{\mu\kappa} = \sqrt{-g}(\nabla_\lambda\delta\Gamma^{\lambda}_{\mu\kappa} - \nabla_\kappa\delta\Gamma^{\lambda}_{\mu\lambda}) \tag{1.21}$$

$$= \sqrt{-g}\left[\nabla_\lambda(g^{\mu\kappa}\delta\Gamma^{\lambda}_{\mu\kappa}) - \nabla_\kappa(g^{\mu\kappa}\delta\Gamma^{\lambda}_{\mu\lambda})\right]. \tag{1.22}$$

The formula for the divergence of a vector can be written in the form

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\ V^\mu), \tag{1.23}$$

and $g^{\mu\kappa}\delta\Gamma^\lambda_{\mu\kappa}$ is a vector, hence

$$\begin{aligned}\sqrt{-g}g^{\mu\kappa}\delta R_{\mu\kappa} &= \partial_\lambda(\sqrt{-g}\ g^{\mu\kappa}\delta\Gamma^\lambda_{\mu\kappa}) - \partial_\kappa(\sqrt{-g}\ g^{\mu\kappa}\delta\Gamma^\lambda_{\mu\lambda}) \\ &= \partial_\lambda\left[\sqrt{-g}\left(g^{\mu\kappa}\delta\Gamma^\lambda_{\mu\kappa} - g^{\mu\lambda}\delta\Gamma^\kappa_{\mu\kappa}\right)\right] \\ &= \partial_\lambda W^\lambda. \end{aligned}\tag{1.24}$$

By the Stokes' theorem, an integral over volume element can be converted to an integral with respect to surface at the boundary. Because variation of the metric is zero on the boundary, then the surface integral does not contribute to the variation.

We can calculate $(\delta S)_2$ by using the following identity

$$\frac{1}{detM}\delta(detM) = Tr(M^{-1}\delta M). \tag{1.25}$$

Take the square matrix M to be the metric tensor $g_{\mu\nu}$ then M^{-1} is $g^{\mu\nu}$ and $\det M = g$, so

$$\frac{1}{g}\delta g = Tr(g^{\mu\nu}\delta g_{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu}. \tag{1.26}$$

Multiply Eq.(1.6) with $g_{\mu\nu}$, then contract an index λ and σ with μ and ν respectively,

$$\begin{aligned} g_{\mu\nu}\delta g^{\mu\nu} &= -g_{\mu\nu}\delta g^{\sigma\rho}g^{\lambda\kappa}\delta g^{\kappa\rho} \\ &= -\delta^\rho_\mu g^{\mu\kappa}\delta g^{\kappa\rho} \\ &= -g^{\mu\nu}\delta g_{\mu\nu}, \end{aligned}\tag{1.27}$$

because κ is dummy index and substitute it into Eq.(1.26), this gives a variation of g

$$\delta g = -gg^{\mu\nu}\delta g_{\mu\nu}. \tag{1.28}$$

For the linear approximation, $\delta\sqrt{-g} = \frac{\partial\sqrt{-g}}{\partial g}\delta g$ then

$$\frac{\delta\sqrt{-g}}{\delta g} = -\frac{1}{2\sqrt{-g}}, \tag{1.29}$$

from Eq.(1.26) and Eq.(1.29) and therefore

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{1.30}$$

Remember that $(\delta S)_3$ does not contribute to the action, therefore

$$\delta S_{EH} = \int \mathrm{d}^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{1.31}$$

From the functional derivative of the action

$$\delta S = \int \sum_i \left(\frac{\delta S}{\delta \Phi^i}\delta\Phi^i\right) d^n x, \tag{1.32}$$

where Φ^i is a complete set of field variable. The principle of least action leads to $\frac{\delta S}{\delta \Phi^i} = 0$, so

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{1.33}$$

this equation is called the Einstein's field equation in vacuum.

For the Einstein's field equation in the presence of matter and energy, the action with minimal coupling becomes

$$S = \frac{1}{16\pi G}S_{EH} + S_M, \tag{1.34}$$

where S_M is the action for the matter. Varying this action using the same procedure as above, we obtain

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{1.35}$$

We define

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{1.36}$$

to be the energy-momentum tensor, therefore

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.37}$$

is a complete Einstein's field equation where $G_{\mu\nu}$ is the Einstein tensor. The left-handed-side of the Einstein's field equation characterizes the geometry of space-time while the right-handed-side describes energy and momentum of matter. It is useful to rewrite this equation in a different form. By contracting Eq.(1.37)

$$R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu = 8\pi G T^\mu_\nu, \tag{1.38}$$

since $\delta^\mu_\mu = 4$, contraction gives $R = -8\pi G T$ where $R^\mu_\mu = R$ and $T^\mu_\mu = T$. Plugging this into Eq.(1.37), we obtain

$$R^\mu_\nu = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \tag{1.39}$$

1.2 The Standard Cosmological Model

At small scales the universe consists of very rich structure of stars, galaxies, local group of galaxies and cluster of galaxies. However, the large scale structure of the universe in order around 3,000 Mpc ($1Mpc \approx 3.26 \times 10^6$ *light years*) appears to be the same in every direction, the property called isotropy. In fact, it is not reasonable to assume that we are in special position in the universe, so the isotropic condition (or any physical conditions) is identical throughout the universe. We call this property a homogeneity. The isotropy and homogeneity of the universe on large scale structure are known as the"Cosmological Principle". The Einstein's field equations of general relativity explain how the energy density change with time. But without the Cosmological principle these equations are difficult to solve. There are many equations and each equations depend on the other. The Cosmological principle reduce the many equations that describe the entire universe to a single Friedman equation. Remarkably the cosmological principle allows us to understand the evolution of our spacetime background. The small scales structure is not satisfying the cosmological principle. However, it can be described through perturbations around the smooth background. The recent data from the observation such as the Cosmic Microwave Background Radiation (CMBR) reveal that the CMB photons coming from different places of the sky have almost the same temperature. This is the crucial evidence for the validity of the Cosmological Principle [7, 8, 9].

1.2.1 The Big Bang Model and Hubble's Law

In the hot big bang model it is believed that the universe began in a very hot and dense state called the Big Bang. At early times, the universe was filled with a hot plasma of elementary particles such as electrons, protons, and photons. At the temperature higher than the binding energy between nucleons in the nucleus and binding energy between proton and electron in the hydrogen atom, photons can scatter with nuclei and atom to knock them out of bound state, hence nuclei and atoms could not form. According to the modern cosmology it is strongly believed that the universe must have undergone a period of rapid expansion during the first moment after the big bang. Such a period of accelerating expansions in the very early universe is called the inflation. The idea of inflation was proposed in the 1980's to solve some cosmological problems such as the flatness problem, horizon promblem and the monopole problem [7]. After the end of inflation the universe cooled down, if it cools below the temperature characterized by the binding energy of hydrogen atom then an electron can combine with a proton to form a hydrogen atom. This era is known as the recombination (electron and proton recombined into a neutral hydrogen). In the recombination era the universe is sufficiently cold so that photons do not scatter much with elementary particles leading to the decoupling between radiation and matter. Consequently the universe became effectively transparent to photons. The photons are not colliding with other particles in the universe and they remain to the present day. The discovery of the Cosmic Microwave Background (CMB) in 1965 [10] confirmed the success of the standard hot big bang model.

In the expansion of the homogeneous and isotropic universe, the relative velocity and distance with respect to an observer obey the Hubble's law

$$\vec{v}\,(\vec{r},t) = H\,(t)\,\vec{r}, \tag{1.40}$$

where $H(t)$ is a parameter depending only on time called the Hubble parameter. Let us consider Figure 1

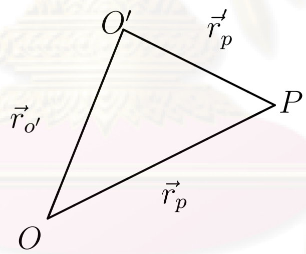

Figure1

where
O and $O^{'}$ are observers,
$\vec{r}_{o'}$ is a position vector of $O^{'}$ with respect to observer O,
$\vec{r}_p$ is a position vector of point P with respect to observer O,
$\vec{r}_p^{\,'}$ is a position vector of point P with respect to observer $O^{'}$,
we have

$$\vec{r}_p^{\,'} = \vec{r}_p - \vec{r}_{o'}, \tag{1.41}$$

then

$$\vec{v}^{\,'}\,(\vec{r}_p^{\,'}) = \vec{v}\,(\vec{r}_p) - \vec{v}\,(\vec{r}_{o'})\,, \tag{1.42}$$

denote that $\vec{v}^{\,'}(\vec{r}_p^{\,'})$ is the velocity of point P with respect to observer $O^{'}$, $\vec{v}\,(\vec{r}_p)$ and $\vec{v}\,(\vec{r}_{o'})$ is the velocity of point P and velocity of observer $O^{'}$ relative to observer O respectively. The cosmological principle states that no special position in homogeneous and isotropic universe therefore Eq.(1.40) should have the same functional form at any point which implies the Hubble's law. It can be verified by the following steps

$$\begin{aligned} \vec{v}'(\vec{r}_p') &= H(t)\vec{r}_p - H(t)\vec{r}_{o'} \\ &= H(t)(\vec{r}_p - \vec{r}_{o'}) \\ &= H(t)\vec{r}_p', \end{aligned} \tag{1.43}$$

you can see that the Hubble's law is the same functional form under the translation from one to another.

Let us rewrite the Hubble's law in a differential form

$$\vec{v} = \frac{d\vec{r}}{dt} = H(t)\vec{r}, \tag{1.44}$$

and integrate it we obtain

$$\vec{r} = \chi e^{\int_0^t H(t)dt}, \ \chi \equiv \vec{r}(t=0), \tag{1.45}$$

or

$$\vec{r} = a(t)\chi, \tag{1.46}$$

where $a(t) = e^{\int_0^t H(t)dt}$, $a(0) \equiv 1$ is called the scale factor with cosmic time t. The coordinate χ is known as the comoving coordinate. A freely falling particle is at rest in this coordinate. The Hubble's law or Eq.(1.44) tells us how distance between two points changes with time. So in the homogeneous and isotropic universe all objects move away (or contract) from the observer at any points. Differentiate Eq.(1.46)

$$\frac{d\vec{r}}{dt} = \vec{v} = \dot{a}(t)\chi = \frac{\dot{a}}{a}\vec{r} = H(t)\vec{r}, \tag{1.47}$$

we turn to the Hubble's law and hence the Hubble parameter is equal to

$$H(t) = \frac{\dot{a}}{a}. \tag{1.48}$$

1.2.2 Relativistic Cosmology

The assumption of homogeneity and isotropy force the line element into the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.49}$$

which is called the Friedmann-Robertson-Walker (FRW) metric. $a(t)$ is the scale factor and coordinate $(x^1, x^2, x^3) = (r, \theta, \phi)$ are the comoving coordinates. In

Eqs.(1.49) the Greek indices μ and ν run from 0 to 3. However this metric has isotropy in space but not in time direction. Constant k in the FRW metric describes the geometry of the spacetime where $k = 0, +1, -1$ corresponds to flat, sphere, and hyperbolic geometry respectively [5, 6, 7, 8, 9].

From the FRW metric, the non zero Christoffel symbols are given by

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2}\ ;\ \Gamma^1_{11} = \frac{kr}{1-kr^2};\\
\Gamma^0_{22} &= a\dot{a}r^2\ ; \qquad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta;\\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a};\\
\Gamma^1_{22} &= -r\left(1-kr^2\right);\ \Gamma^1_{33} = -r\left(1-kr^2\right)\sin^2\theta;\\
\Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r};\\
\Gamma^2_{33} &= -\sin\theta\cos\theta;\ \Gamma^3_{23} = \cot\theta.
\end{aligned} \tag{1.50}$$

The non zero component of the Ricci tensor are

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}\\
R_{11} &= \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}\\
R_{22} &= r^2\left(a\ddot{a}+2\dot{a}^2+2k\right)\\
R_{33} &= r^2\left(a\ddot{a}+2\dot{a}^2+2k\right)\sin^2\theta,
\end{aligned} \tag{1.51}$$

and the Ricci scalar is then

$$R = 6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}\right] \tag{1.52}$$

where a dot denotes a derivative with respect to the cosmic time t. Eq.(1.49) show that the scale factor $a(t)$ is the only single dynamical variable therefore the information about dynamics of the universe is contained in it. To derive the differential equation for the time evolution of the scale factor from the Einstein's field equation, we will choose a perfect fluid as a matter and energy filling the universe. The energy-momentum tensor then take the form

$$T^\mu_\nu = diag\left(-\rho, p, p, p\right), \tag{1.53}$$

where ρ is the energy density and p is the pressure. Note that the trace is given by

$$T = T^\mu_\mu = -\rho + 3p. \tag{1.54}$$

Plugging these objects into the Einstein's field equation in the form

$$R_{\mu\nu} = 8\pi G\left(T^\mu_\nu - \frac{1}{2}g_{\mu\nu}T\right). \tag{1.55}$$

The $\mu\nu = 00$ equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{1.56}$$

known as the acceleration equation and the $\mu\nu = ij$ equation is

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G\left(\rho + 3p\right), \tag{1.57}$$

Use the acceleration equation to eliminate second derivative in Eq.(1.57), we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.58}$$

this is known as the Friedmann equation.

The Einstein tensor satisfies the Bianchi identity

$$\nabla_\mu G^\mu_\nu = 0. \tag{1.59}$$

From the Einstein's field equation it follow that the energy-momentum tensor vanishes

$$\nabla_\mu T^\mu_\nu = 0, \tag{1.60}$$

which gives the energy conservation in zero component and the Euler equation familiar in fluid mechanics in the spatial components. Consider the zero component of Eq.(1.60)

$$\begin{aligned} 0 &= \nabla_\mu T^\mu_0 \\ &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\nu} T^\nu_0 - \Gamma^\nu_{\mu 0} T^\mu_\nu \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}\left(\rho + p\right), \\ \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) &= 0. \end{aligned} \tag{1.61}$$

This is the continuity equation.

Now, we have three equations that explain how the universe expand. there is the acceleration equation Eq.(1.56), the Friedmann equation Eq.(1.58), and the continuity equation Eq.(1.61). However, these three equations are not independent because the acceleration equation can be derived by differentiating the Friedmann equation with respect to cosmic time, t and use the continuity equation to eliminate $\dot{\rho}$ term. Thus, we have three unknowns, the scale factor $a\left(t\right)$, the energy density $\rho\left(t\right)$, and the pressure $p\left(t\right)$ with two independent equations. Therefore we need another condition. The additional equation called the equation of state is a relation between the pressure and energy density of the component in the universe. Fortunately, in cosmology we usually deal with low density fluid for which the equation of state can be written in a linear form

$$p = w\rho, \tag{1.62}$$

where w is a dimensionless number. We can rewrite the continuity equation in the following form

$$\frac{\dot{\rho}}{\rho} = -3\,(1+w)\,\frac{\dot{a}}{a}. \tag{1.63}$$

Now we have three unknowns and three independent equations so a system of equations can be solved. If w does not depend on time we can integrate Eq.(1.63) to obtain

$$\rho \propto a^{-3(1+w)}. \tag{1.64}$$

The recent observations have shown that the current universe is close to a spatially flat geometry, $k = 0$. Therefore we need to solve the Friedmann equation in flat geometry to obtain analytic solution for $a\,(t)$ that describe the time evolution of our flat universe

$$a \propto (t - t_0)^{\frac{2}{3(1+w)}}\,, \tag{1.65}$$

where t_0 is the initial time. The two well known cosmological fluids are matter or non-relativistic particles and radiation or relativistic particles. The pressure of non-relativistic particles is negligible so the equation of state is $w \approx 0$. Then the evolution of the universe during the matter dominated era is given by $\rho_{matter} \propto a^{-3}$ and $a \propto (t - t_0)^{\frac{2}{3}}$. For relativistic particles such as photons, the equation of state is $p = \frac{1}{3}\rho_{rad}$ or $w = \frac{1}{3}$. This relation can be derived from statistical mechanics and classical electrodynamics treatment. The evolution of the universe during the radiation dominated era is then given by $\rho_{rad} \propto a^{-4}$ and $a \propto (t - t_0)^{\frac{1}{2}}$.

1.2.3 Dark Energy Problem

From the previous sections we know that ordinary matter such as baryons and radiation has zero and positive pressure respectively. The pressure corresponds to these components will cause the expansion of the universe to slow down. On the other hand, the direct evidence of late-time cosmic acceleration was reported by Perlmutter and Riess from the Supernovae Type Ia in 1998 [1, 2]. The source for current acceleration of the universe was called dark energy. The word dark mean its still mystery for us to answer the question What is the physical cause of the dark energy?

The first candidate for dark energy is the cosmological constant Λ [11, 12]. Historically, it was introduced by Einstein to obtain a static universe. Let us consider in static universe, $\dot{a} = \ddot{a} = 0$ then the equation of state give $w = -\frac{1}{3}$ hence the universe was filled by a fluid with negative pressure. However, what physical matter has a negative pressure so the Einstein's field equation has no static solution. In order to solve this problem he introduced an extra term into the left hand side of his field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.66}$$

where the additional Λ term is the cosmological constant. In fact the Einstein's field equations allow addition of the $\Lambda g_{\mu\nu}$ term because the covariant derivative of the metric tensor, $g_{\mu\nu}$, is always zero. The Einstein tensor, $G^{\mu\nu}$ and the energy-momentum tensor $T^{\mu\nu}$ still satisfy the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ and energy conservation $\nabla_\mu T^{\mu\nu} = 0$. In the FRW background with equation of state $p = 0$ (matter dominate), Eq.(1.66) gives the differential equation for the scale factor

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_M + \frac{\Lambda}{3}. \tag{1.67}$$

It is shown that a positive Λ term represent a repulsive force to balance the attractive gravitational force from ordinary matter. However, Einstein abandoned the Λ term from his equation when Hubble's discovery of the expansion of the universe in 1929 is confirmed, meaning that the universe is not static. In addition, after the discovery of the accelerated expansion of the universe in 1998 the cosmological constant is back again as a candidate of dark energy.

What is the physical cause of the cosmological constant? In the quantum field theory context, it suggests that the empty space is not really empty. It permits particle-antiparticle pairs to spontaneously appear and then annihilated in a vacuum. This fact inspires some cosmologists to interpret the cosmological constant as a sort of vacuum energy density than the effect of spacetime geometry or gravity. Therefore we should be put the Λ term into the energy-momentum tensor part or into the right hand side of the Einstein's field equation. Let us consider the equation of state $p_{vac} = -\rho_{vac}$, $w = -1$ so ρ_{vac} is a constant in time from the Eq.(1.63). We can decompose the energy-momentum tensor into a matter piece $T^{(M)}_{\mu\nu}$ and a vacuum piece $T^{(vac)}_{\mu\nu} = -\rho g_{\mu\nu}$, the Einstein's field equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G(T^{(M)}_{\mu\nu} - \rho_{vac} g_{\mu\nu}). \tag{1.68}$$

Comparison with Eq.(1.66) we see that the cosmological constant is equivalent to the vacuum energy density

$$\rho_{vac} = \frac{\Lambda}{8\pi G}. \tag{1.69}$$

As a result vacuum energy has a behavior like cosmological constant representing a repulsive force to cancel the attractive gravitational force which is the hallmark behavior of the dark energy.

Quantum field theory is an important theoretical background for the cosmological constant. Cosmological constant corresponds to the vacuum energy of quantum field in the ground state. The vacuum energy or zero-point energy of a free quantum field given by

$$\rho_{vac} = \int_0^\infty \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}, \tag{1.70}$$

is ultraviolet divergent. However, we trust our theory is correct up to a certain momentum cut off k_{max} at the Plank scale $M_{Pl} = (8\pi G)^{-1/2} \sim 10^{18}$ GeV. We then obtain the vacuum energy density as [8, 9]

$$\rho_{vac} \sim k_{max}^4 \sim 10^{72}\mathrm{Ge}V^4. \tag{1.71}$$

According to the cosmic acceleration to day, we require the energy density of the cosmological constant ρ_Λ to be

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = \Lambda M_{Pl}^2 \sim 10^{-48}\mathrm{Ge}V^4, \tag{1.72}$$

much smaller than the ρ_{vac} in Eq.(1.71) which implies that $\frac{\rho_\Lambda}{\rho_{vac}} \sim 10^{-120}$ [8, 9]. Thus ρ_Λ needs to be fine tuned at the level of one per 10^{120} around the Plank epoch in order to satisfies the current cosmic acceleration expansion. An extreme fine tuning is unacceptable and the question what is the physical cause of the cosmological constant has no satisfactory answer.

If the source of dark energy is not the cosmological constant then we seek some alternative models to describe the late-time accelerated expansion of the universe. Recently, there are two approaches to construct alternative models of dark energy. The simplest way is to modify the energy-momentum tensor in the right hand side of the Einstein's field equation. This approach is called modified matter such as quintessence [19, 20], k-essence [21, 22], Phantom [23], and Chameleon scalar field [24]. Another approach is the modified gravity which modifies the left hand side of the Einstein's field equation. The models that an element to this class such as f(R) gravity [25], scalar-tensor theory [26], Gauss-Bonet dark energy [27] and DGP (Dvali Gabadadze Porrati) model [28].

Chapter II

Casimir Dark Energy Model

According to the previous chapter, physicists strongly believe that the universe consists of a sort of energy dubbed dark energy, which contributes to the accelerated expansion in spatial directions. However, the physics and dynamics of dark energy has not been discovered until now. Recently, it was found by Brian Greene and Janna Levin that the Casimir energy arise from fields fluctuations in space with extra dimensions could play a crucial role in the accelerated expansion of the late-time universe which is the hallmark behavior of dark energy [3]. In fact by certain combinations of fields with different masses and spins, the total Casimir energy from those can give a stabilizing potential that stabilizes the size of extra dimensions. If their size is constant or independent of time then the large directions feel the Casimir energy as a sort of vacuum energy to accelerate expansion of the spatial directions.

However, the shape moduli, τ_1, τ_2, of the extra dimensions were not considered in the previous work of Greene and Levin. Therefore this chapter is a review of some works of Burikham et.al [13], to include these moduli in the cosmological dynamics by assuming the extra dimensions are T^2. This review is organized as the following. In section 2.1, we review the moduli space of the torus. In section 2.2, we have constructed the cosmological dynamics on $M^{1+3} \times T^2$ spacetime. In section 2.3, we present the zeta function regularization to determine the Casimir energy of massive and massless real scalar fields in the spacetime with toroidally compactified extra dimensions. Then we go on to construct effective potential contributed by Casimir energy of massive and massless field in $M^{1+3} \times T^2$ in section 2.4.

2.1 The Moduli Space of the Torus

We can construct a two-dimensional torus by identifying a region in the complex plane as shown in Figure 2.

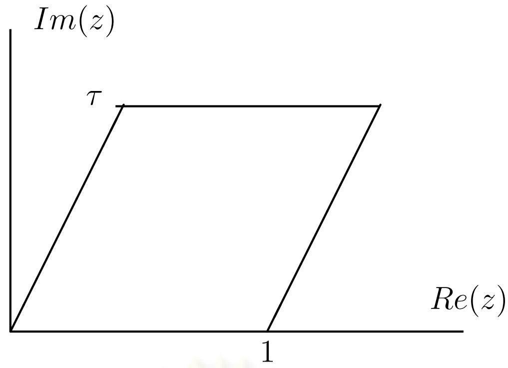

Figure 2

A two-dimensional torus can be described with a single complex parameter $\tau = \tau_1 + i\tau_2$ called the modulus. And the identifications is

$$z \equiv z + 1 \quad and \quad z \equiv z + \tau. \tag{2.1}$$

Note that τ is a parameter identifying how the torus is distorted. We can define a complex coordinate

$$z = y^1 + \tau y^2, \tag{2.2}$$

where $y^1, y^2 \in [0, 2\pi]$. Taking differential for z we obtain

$$dz = dy^1 + \tau dy^2, \tag{2.3}$$

because the volume is 1 if the determinant of the metric is 1. Therefore the interval on the torus can be written as

$$ds^2 = \frac{dz d\bar{z}}{\tau_2} = h_{ij} dy^i dy^j, \tag{2.4}$$

this implies the flat metric on the torus

$$(h_{ij}) = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \tag{2.5}$$

For $\tau_1 = 0$ and $\tau_2 = 1$ this would be the metric δ_{ab} [29]. However, there are some values of parameter τ that corresponds to the same torus. These transformation are S and T transformations where T : $\tau \to \tau + 1$ and S : $\tau \to -\frac{1}{\tau}$. Using S and T transformations we can identify the region in the complex τ-plane containing points that represent equivalent torus. It is called the fundamental domain for the moduli space of the torus:

$$-\frac{1}{2} \leq Re(\tau) \leq \frac{1}{2}, \quad Im(\tau) > 0, \quad |\tau| \geq 1. \tag{2.6}$$

2.2 Cosmological Dynamics in $M^{1+3} \times T^2$ Spacetime

Begin with the Einstein Hillbert action on the product space $M^{1+n} \times T^p$ between a (1+n)-large dimensional spacetime and a compactified p-dimensional toroidal space

$$S = \int d^{1+n}x d^p y \sqrt{-gh} \left[\frac{1}{16\pi G_{(1+n)}} R_{(1+n)} - \rho_{(1+n)} \right], \qquad (2.7)$$

where $\rho_{(1+n)}, G_{(1+n)}$ and $R_{(1+n)}$ are the Casimir energy density. It is a function of the metric h_{ij}, gravitational constant and the Ricci scalar in (1+n)-dimensional spacetime respectively. We assume the metric to be homogeneous but anisotropic

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + h_{ij} dy^i dy^j, \qquad (2.8)$$

where the four dimensional metric $g_{\mu\nu}$ with $\mu, \nu = 0, ..., n$ is the Friedmann-Robertson-Walker metric in flat universe ($k = 0$). The metric h_{ij} represents the p-dimensional compact space with compact coordinate $y^i \in [0, 2\pi]$. Note that $i, j = 1, ..., p$.

In this thesis, we investigate on the cosmological dynamics of a 4-dimensional flat spacetime with two extra dimensions ($n = 3, p = 2$). The metric h_{ij} on the torus is

$$(h_{ij}) = \frac{b^2}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \qquad (2.9)$$

where $\tau = \tau_1 + i\tau_2$ and b^2 is the volume moduli or the scale factor for the extra dimensions. To obtain the differential equation of the scale factors $a(t)$ and $b(t)$ that describe dynamic of the universe. First, we calculate the Einstein tensor and energy momentum tensor for our matter field and then plug it into the Einstein's field equation. Note that, we rewrite the interval in the form $ds^2 = g_{ab} dx^a dx^b$ where $a, b = 0, 1, 2, 3, 5, 6$ therefore $h_{11} = g_{55}, h_{12} = g_{56}, h_{21} = g_{65}$ and $h_{22} = g_{66}$. Note that from the above notation we rewrite a metric tensor for our spacetime background as

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{b^2}{\tau_2} & \frac{b^2\tau_1}{\tau_2} \\ 0 & 0 & 0 & 0 & \frac{b^2\tau_1}{\tau_2} & \frac{b^2|\tau|^2}{\tau_2} \end{pmatrix}, \qquad (2.10)$$

and its inverse is

$$(g^{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{|\tau|^2}{b^2\tau_2} & -\frac{\tau_1}{b^2\tau_2} \\ 0 & 0 & 0 & 0 & -\frac{\tau_1}{b^2\tau_2} & \frac{1}{b^2\tau_2} \end{pmatrix}. \tag{2.11}$$

The Christoffel connection is defined by

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}), \tag{2.12}$$

the non-zero components are

$$\begin{aligned}
\Gamma^0_{11} &= \Gamma^0_{22} = \Gamma^0_{33} = a\dot{a}, \\
\Gamma^0_{55} &= \frac{b\dot{b}}{\tau_2} - \frac{b^2\dot{\tau}_2}{2\tau_2^2}, \\
\Gamma^0_{66} &= \frac{b^2\dot{\tau}_1\tau_1}{\tau_2} + \frac{b^2\dot{\tau}_2}{2} + \frac{b\dot{b}\tau_1^2}{\tau_2} - \frac{b^2\dot{\tau}_2\tau_1^2}{2\tau_2^2} + b\dot{b}\tau_2, \\
\Gamma^0_{56} &= \frac{b^2\dot{\tau}_1}{2\tau_2} + \frac{b\dot{b}\tau_1}{\tau_2} - \frac{b^2\dot{\tau}_2\tau_1}{2\tau_2^2}, \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, \\
\Gamma^5_{05} &= \frac{\dot{b}}{b} - \frac{\dot{\tau}_2}{2\tau_2} - \frac{\tau_1\dot{\tau}_1}{2\tau_2^2}, \\
\Gamma^5_{06} &= \frac{\dot{\tau}_1}{2} - \frac{\tau_1\dot{\tau}_2}{\tau_2} - \frac{\tau_1^2\dot{\tau}_1}{2\tau^{2_2}}, \\
\Gamma^6_{05} &= \frac{\dot{\tau}_1}{2\tau_2^2}, \\
\Gamma^6_{06} &= \frac{\dot{b}}{b} + \frac{\dot{\tau}_2}{2\tau_2} + \frac{\tau_1\dot{\tau}_1}{2\tau_2^2}.
\end{aligned} \tag{2.13}$$

The Ricci tensor is defined by

$$R_{ab} = \partial_c\Gamma^c_{ab} - \partial_a\Gamma^c_{bc} + \Gamma^c_{dc}\Gamma^d_{ab} - \Gamma^c_{db}\Gamma^d_{ac}. \tag{2.14}$$

For our metric tensor, the non-zero components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= -\frac{1}{2}\frac{\dot{\tau}_2^2}{\tau_2^2} - \frac{1}{2}\frac{\dot{\tau}_1^2}{\tau_2^2} - 3\frac{\ddot{a}}{a} - 2\frac{\ddot{b}}{b}, \\
R_{11} &= R_{22} = R_{33} = 2\dot{a}^2 + 2\frac{a\dot{a}\dot{b}}{b} + a\ddot{a}, \\
R_{55} &= 3\frac{b\dot{a}\dot{b}}{a\tau_2} + \frac{\dot{b}^2}{\tau_2} - \frac{3}{2}\frac{b^2\dot{a}\dot{\tau}_2}{a\tau_2^2} - \frac{b\dot{b}\dot{\tau}_2}{\tau_2^2} + \frac{1}{2}\frac{b^2\dot{\tau}_2^2}{\tau_2^3} - \frac{1}{2}\frac{b^2\dot{\tau}_1^2}{\tau_2^3} + \frac{b\ddot{b}}{\tau_2} - \frac{1}{2}\frac{b^2\ddot{\tau}_2}{\tau_2^2}, \\
R_{56} &= 3\frac{b\tau_1\dot{a}\dot{b}}{a\tau_2} + \frac{\tau_1\dot{b}^2}{\tau_2} - \frac{3}{2}\frac{b^2\tau_1\dot{a}\dot{\tau}_2}{a\tau_2^2} - \frac{b\tau_1\dot{b}\dot{\tau}_2}{\tau_2^2} + \frac{1}{2}\frac{b^2\tau_1\dot{\tau}_2^2}{\tau_2^3} + \frac{3}{2}\frac{b^2\dot{a}\dot{\tau}_1}{a\tau_2} \\
&\quad + \frac{b\dot{b}\dot{\tau}_1}{\tau_2} - \frac{b^2\dot{\tau}_1\dot{\tau}_2}{\tau_2^2} - \frac{1}{2}\frac{b^2\tau_1\dot{\tau}_1^2}{\tau_2^3} + \frac{b\tau_1\ddot{b}}{\tau_2} - \frac{1}{2}\frac{b^2\tau_1\ddot{\tau}_2}{\tau_2^2} + \frac{1}{2}\frac{b^2\ddot{\tau}_1}{\tau_2}, \\
R_{65} &= 3\frac{b\tau_1\dot{a}\dot{b}}{a\tau_2} + \frac{\tau_1\dot{b}^2}{\tau_2} - \frac{3}{2}\frac{b^2\tau_1\dot{a}\dot{\tau}_2}{a\tau_2^2} - \frac{b\tau_1\dot{b}\dot{\tau}_2}{\tau_2^2} + \frac{1}{2}\frac{b^2\tau_1\dot{\tau}_2^2}{2\tau_2^3} + \frac{3}{2}\frac{b^2\dot{a}\dot{\tau}_1}{a\tau_2} \\
&\quad + \frac{b\dot{b}\dot{\tau}_1}{\tau_2} - \frac{b^2\dot{\tau}_1\dot{\tau}_2}{\tau_2^2} - \frac{1}{2}\frac{b^2\tau_1\dot{\tau}_1^2}{\tau_2^3} + \frac{b\tau_1\ddot{b}}{\tau_2} - \frac{1}{2}\frac{b^2\tau_1\ddot{\tau}_2}{\tau_2^2} + \frac{1}{2}\frac{b^2\ddot{\tau}_1}{\tau_2}, \\
R_{66} &= 3\frac{b\tau_2\dot{a}\dot{b}}{a} + 3\frac{b\tau_1^2\dot{a}\dot{b}}{a\tau_2} + \tau_2\dot{b}^2 + \frac{\tau_1^2\dot{b}^2}{\tau_2} + \frac{3}{2}\frac{b^2\dot{a}\dot{\tau}_2}{a} - \frac{3}{2}\frac{b^2\tau_1^2\dot{a}\dot{\tau}_2}{a\tau_2^2} + b\dot{b}\dot{\tau}_2 \\
&\quad - \frac{b\tau_1^2\dot{b}\dot{\tau}_2}{\tau_2^2} - \frac{1}{2}\frac{b^2\dot{\tau}_2^2}{\tau_2} + \frac{1}{2}\frac{b^2\tau_1^2\dot{\tau}_2^2}{\tau_2^3} + 3\frac{b^2\tau_1\dot{a}\dot{\tau}_1}{a\tau_2} + 2\frac{b\tau_1\dot{b}\dot{\tau}_1}{\tau_2} - 2\frac{b^2\tau_1\dot{\tau}_1\dot{\tau}_2}{\tau_2^2} \\
&\quad + \frac{1}{2}\frac{b^2\dot{\tau}_1^2}{\tau_2} - \frac{1}{2}\frac{b^2\tau_1^2\dot{\tau}_1^2}{\tau_2^3} + b\tau_2\ddot{b} + \frac{b\tau_1^2\ddot{b}}{\tau_2} + \frac{1}{2}b^2\ddot{\tau}_2 - \frac{1}{2}\frac{b^2\tau_1^2\ddot{\tau}_2}{\tau_2^2} + \frac{b^2\tau_1\ddot{\tau}_1}{\tau_2}.
\end{aligned}
$$

Next, the Ricci scalar can be calculated as the following

$$
\begin{aligned}
R &= g^{ab}R_{ab} \\
&= 6\frac{\dot{a}^2}{a^2} + 12\frac{\dot{a}\dot{b}}{ab} + 2\frac{\dot{b}^2}{b^2} + \frac{\dot{\tau}_2^2}{2\tau_2^2} + \frac{\dot{\tau}_1^2}{2\tau_2^2} + 6\frac{\ddot{a}}{a} + 4\frac{\ddot{b}}{b}.
\end{aligned} \tag{2.15}
$$

At this point, the Einstein tensor is defined by

$$
G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}. \tag{2.16}
$$

Note that we use the mixed form of Einstein's field equation to construct the equation of motion for the scale factors. In our spacetime background, using the above results we obtain the non-zero components of the Einstein tensor G^a_b as

$$
\begin{aligned}
G^0_0 &= -3\frac{\dot{a}^2}{a^2} - 6\frac{\dot{a}\dot{b}}{ab} - \frac{\dot{b}^2}{b^2} + \frac{1}{4}\frac{\dot{\tau}_2^2}{\tau_2^2} + \frac{1}{4}\frac{\dot{\tau}_1^2}{\tau_2^2}, \\
G^1_1 &= G^2_2 = G^3_3 = -\frac{\dot{a}^2}{a^2} - 4\frac{\dot{a}\dot{b}}{ab} - \frac{\dot{b}^2}{b^2} - \frac{1}{4}\frac{\dot{\tau}_2^2}{\tau_2^2} - \frac{1}{4}\frac{\dot{\tau}_1^2}{\tau_2^2} - 2\frac{\ddot{a}}{a} - 2\frac{\ddot{b}}{b}, \\
G^5_5 &= -3\frac{\dot{a}^2}{a^2} - 3\frac{\dot{a}\dot{b}}{ab} - \frac{3}{2}\frac{\dot{a}\dot{\tau}_2}{a\tau_2} - \frac{\dot{b}\dot{\tau}_2}{b\tau_2} + \frac{1}{4}\frac{\dot{\tau}_2^2}{\tau_2^2} - \frac{3}{2}\frac{\tau_1\dot{a}\dot{\tau}_1}{a\tau_2^2} - \frac{\tau_1\dot{b}\dot{\tau}_1}{b\tau_2^2} + \frac{\tau_1\dot{\tau}_1\dot{\tau}_2}{\tau_2^3} \\
&\quad -\frac{3}{4}\frac{\dot{\tau}_1^2}{\tau_2^2} - 3\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{1}{2}\frac{\ddot{\tau}_2}{\tau_2} - \frac{1}{2}\frac{\tau_1\ddot{\tau}_1}{\tau_2^2} \\
G^5_6 &= -3\frac{\tau_1\dot{a}\dot{\tau}_2}{a\tau_2} - 2\frac{\tau_1\dot{b}\dot{\tau}_2}{b\tau_2} + \frac{\tau_1\dot{\tau}_2^2}{\tau_2^2} + \frac{3}{2}\frac{\dot{a}\dot{\tau}_1}{a} - \frac{3}{2}\frac{\dot{a}\dot{\tau}_1}{a\tau_2^2} + \frac{\dot{b}\dot{\tau}_1}{b} - \frac{\tau_1^2\dot{b}\dot{\tau}_1}{b\tau_2^2} \\
&\quad -\frac{\dot{\tau}_1\dot{\tau}_2}{\tau_2} + \frac{\tau_1^2\dot{\tau}_1\dot{\tau}_2}{\tau_2^3} - \frac{\tau_1\dot{\tau}_1^2}{\tau_2^2} - \frac{\tau_1\ddot{\tau}_2}{\tau_2} + \frac{\ddot{\tau}_1}{2} - \frac{1}{2}\frac{\tau_1^2\ddot{\tau}_1}{\tau_2^2} \\
G^6_5 &= \frac{3}{2}\frac{\dot{a}\dot{\tau}_1}{a\tau_2^2} + \frac{\dot{b}\dot{\tau}_1}{b\tau_2^2} - \frac{\dot{\tau}_1\dot{\tau}_2}{\tau_2^3} + \frac{1}{2}\frac{\ddot{\tau}_1}{\tau_2^2} \\
G^6_6 &= -3\frac{\dot{a}^2}{a^2} - 3\frac{\dot{a}\dot{b}}{ab} + \frac{3}{2}\frac{\dot{a}\dot{\tau}_2}{a\tau_2} + \frac{\dot{b}\dot{\tau}_2}{b\tau_2} - \frac{3}{4}\frac{\dot{\tau}_2^2}{\tau_2^2} + \frac{3}{2}\frac{\tau_1\dot{a}\dot{\tau}_1}{a\tau_2^2} + \frac{\tau_1\dot{b}\dot{\tau}_1}{b\tau_2^2} - \frac{\tau_1\dot{\tau}_1\dot{\tau}_2}{\tau_2^3} \\
&\quad +\frac{1}{4}\frac{\dot{\tau}_1^2}{\tau_2^2} - 3\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{1}{2}\frac{\ddot{\tau}_2}{\tau_2} + \frac{1}{2}\frac{\tau_1\ddot{\tau}_1}{\tau_2^2}.
\end{aligned}
$$

The energy-momentum tensor can be calculated from the action for matter field

$$S_M = -\int d^n x\sqrt{-g}\rho_{(1+n)}(h^{ij}), \tag{2.17}$$

therefore

$$
\begin{aligned}
\delta S_M &= -\int d^n x\rho\delta\sqrt{-g} - \int d^n x\sqrt{-g}\delta\rho \\
&= \int d^n x\sqrt{-g}\left(\frac{1}{2}\rho g_{ab} - \frac{\partial\rho}{\partial g^{ab}}\right)\delta g^{ab},
\end{aligned}
$$

then

$$\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{ab}} = \frac{1}{2}\rho g_{ab} - \frac{\partial\rho}{\partial g^{ab}}. \tag{2.18}$$

From the definition of the energy-momentum tensor in Eq.(1.36) we obtain

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial\rho}{\partial g^{ab}}. \tag{2.19}$$

With one index raised and use the chain rule in the second term this takes the convenient form

$$
\begin{aligned}
T^a_b &= g^{ac}T_{cb} \\
&= -\rho\delta^a_b + 2g^{ac}\left(\frac{\partial b}{\partial g^{cb}}\partial_b\rho + \frac{\partial\tau_1}{\partial g^{cb}}\partial_{\tau_1}\rho + \frac{\partial\tau_2}{\partial g^{cb}}\partial_{\tau_2}\rho\right),
\end{aligned} \tag{2.20}
$$

where

$$
\begin{aligned}
b &= \left[g^{55}g^{66} - \left(g^{56}\right)^2\right]^{-\frac{1}{4}}, \\
\tau_1 &= -\frac{g^{56}}{g^{66}}, \\
\tau_2 &= \sqrt{\frac{g^{55}}{g^{66}} - \left(\frac{g^{56}}{g^{66}}\right)^2}.
\end{aligned}
$$

We obtain the non-zero components of the energy-momentum tensor

$$
\begin{aligned}
T^0_0 &= -\rho, \\
T^1_1 &= T^2_2 = T^3_3 = -\rho, \\
T^5_5 &= -\rho + \frac{b}{2}\left(\frac{\tau_1^2 - \tau_2^2}{\tau_2^2}\right)\partial_b\rho + 2\tau_1\partial_{\tau_1}\rho + \left(\frac{\tau_2^2 - \tau_1^2}{\tau_2}\right)\partial_{\tau_2}\rho \\
T^5_6 &= \left(\frac{b\tau_1^3}{2\tau_2^2} - \frac{b\tau_1}{2} + b\right)\partial_b\rho - 2\tau_2^2\partial_{\tau_1}\rho + \left(3\tau_1\tau_2 + \frac{\tau_1^3}{\tau_2}\right)\partial_{\tau_2}\rho \\
T^6_5 &= -\frac{b\tau_1}{2\tau_2^2}\partial_b\rho - 2\partial_{\tau_1}\rho + \frac{\tau_1}{\tau_2}\partial_{\tau_2}\rho \\
T^6_6 &= -\rho + \frac{b}{2}\left(\frac{\tau_1^2 - \tau_2^2}{\tau_2^2}\right)\partial_b\rho - \frac{\tau^2}{\tau_2}\partial_{\tau_2}\rho.
\end{aligned}
$$

Using the following combinations of the Einstein tensors

$$
\begin{aligned}
G^0_0 &= 8\pi G T^0_0, \\
G^1_1 - G^5_5 - G^6_6 - G^0_0 &= 8\pi G\left(T^1_1 - T^5_5 - T^6_6 - T^0_0\right), \\
3G^1_1 - G^5_5 - G^6_6 + G^0_0 &= 8\pi G\left(3T^1_1 - T^5_5 - T^6_6 + T^0_0\right), \\
G^6_5 &= 8\pi G T^6_5, \\
2G^5_6 - \tau_1 G^5_5 - \tau_1 G^6_6 - 2\tau_2^2 G^6_5 &= 8\pi G\left(2T^5_6 - \tau_1 T^5_5 - \tau_1 T^6_6 - 2\tau_2^2 T^6_5\right),
\end{aligned}
$$

we obtain the differential equations govern the cosmological dynamics

$$
3H_a^2 + H_b^2 + 6H_aH_b - \frac{1}{4\tau_2^2}\left(\dot{\tau}_1^2 + \dot{\tau}_2^2\right) = 8\pi G\rho_{6D},
$$

$$
\dot{H}_a + 3H_a^2 + 2H_aH_b = \frac{8\pi G}{4}\left\{2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2}\right)^2\right]b\partial_b\rho_{6D} - 2\tau_1\partial_{\tau_1}\rho_{6D} + \frac{2\tau_1^2}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\},
$$

$$
\dot{H}_b + 2H_b^2 + 3H_aH_b = -\frac{8\pi G}{4}\left\{-2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2}\right)^2\right]b\partial_b\rho_{6D} - 2\tau_1\partial_{\tau_1}\rho_{6D} + \frac{2\tau_1^2}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\},
$$

$$
\ddot{\tau}_1 + \left(3H_a + 2H_b - 2\frac{\dot{\tau}_2}{\tau_2}\right)\dot{\tau}_1 = -16\pi G\tau_2^2\left\{\frac{b\tau_1}{2\tau_2^2}\partial_b\rho_{6D} + 2\partial_{\tau_1}\rho_{6D} - \frac{\tau_1}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\},
$$

$$\frac{\ddot{\tau}_2}{\tau_2}+\frac{\dot{\tau}_1^2-\dot{\tau}_2^2}{\tau_2^2}+3H_a\frac{\dot{\tau}_2}{\tau_2}+2H_b\frac{\dot{\tau}_2}{\tau_2}=8\pi G\{\frac{b\tau_1^2}{\tau_2^2}\partial_b\rho_{6D}+2\tau_1\partial_{\tau_1}\rho_{6D}-2\tau_2\left[1+\left(\frac{\tau_1}{\tau_2}\right)^2\right]\partial_{\tau_2}\rho_{6D}\}, \tag{2.21}$$

where G is the $6-D$ gravitational constant. We have defined the Hubble's parameter $H_a=\frac{\dot{a}}{a}$ and $H_b=\frac{\dot{b}}{b}$, with a dotted sign represents the time derivative and ρ_{6D} is the Casimir energy density in six dimensional spacetime.

2.3 Casimir Energy Calculation

In this section, we will determine the Casimir energy associated with a scalar field of mass M in our spacetime background. The fermionic degrees of freedom will contribute to the Casimir energy with the same expression as the bosonic degree of freedom except for an extra minus sign. Let $V_n=L^n$ be the spatial volume of non-compact spacetime, and $V_p=l^n$ be the volume of compact space. If we assume $L\gg l$, the zero-point energy of scalar field can be evaluated by

$$\hat{E}_{cas}=\frac{1}{2}(\frac{L}{2\pi})^n\sum_{n_i,n_j}\int_{-\infty}^{\infty}d^nk\sqrt{\delta^{ab}k_ak_b+h^{ij}n_in_j+M^2}, \tag{2.22}$$

where $k_a; a=1,...,n$ is the momentum in each non-compact spatial direction, $n_i\in Z; i=1,...,p$ is the momentum number in each compact direction.
Using

$$\int d^dkf(k)=\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int dkk^{d-1}f(k), \tag{2.23}$$

then

$$\int_{-\infty}^{\infty}d^nk\sqrt{\delta^{ab}k_ak_b+h^{ij}n_in_j+M^2}=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^{\infty}dkk^{n-1}\sqrt{\delta^{ab}k_ak_b+h^{ij}n_in_j+M^2},$$

change variable $k=\sqrt{v(h^{ij}n_in_j+M^2}$ it is easy to show that

$$\hat{E}_{cas}=\frac{1}{2}(\frac{L}{2\pi})^n\sum_{n_in_j}(h^{ij}n_in_j+M^2)^{\frac{n+1}{2}}\int_0^{\infty}dvv^{\frac{n-2}{2}}\sqrt{1+v}. \tag{2.24}$$

We can evaluate the integral into the Gamma function by using the Beta function formula

$$B(1+r,-s-r-1)=\int_0^{\infty}t^r(1+t)^sdt=\frac{\Gamma(r+1)\Gamma(-s-r-1)}{\Gamma(-s)}. \tag{2.25}$$

Therefore

$$\int_0^{\infty}dvv^{\frac{n-2}{2}}\sqrt{1+v}=\frac{\Gamma(\frac{n}{2})\Gamma(\frac{-n-1}{2})}{\Gamma(-\frac{1}{2})},$$

and impose $\frac{n+1}{2} = -s$, we obtain

$$\hat{E}_{cas} = \frac{1}{2}(\frac{2\pi}{L})^{1+2s}\frac{\Gamma(s)}{\pi^{\frac{1+2s}{2}}\Gamma(-\frac{1}{2})}\sum_{n_i n_j}(h^{ij}n_i n_j + M^2)^{-s}. \quad (2.26)$$

Note that

$$\begin{aligned}\sum_{n_i n_j}(h^{ij}n_i n_j + M^2)^{-s} &= F(s;\frac{|\tau|^2}{b^2\tau_2},-\frac{2\tau_1}{b^2\tau_2},\frac{1}{b^2\tau_2};M^2) \\ &= \sum_{n_1,n_2}(\frac{|\tau|^2}{b^2\tau_2}n_1^2 - \frac{2\tau_1}{b^2\tau_2}n_1 n_2 + \frac{1}{b^2\tau_2}n_2^2 + M^2)^{-s}, \quad (2.27)\end{aligned}$$

which is known as extended Chowla-Selberg zeta function [30]. After a few steps of analytic manipulation by using Poisson resummation formula and property of the modified Bessel function see in appendix A, we obtain

$$\begin{aligned}F(s;\frac{|\tau|^2}{b^2\tau_2},-\frac{2\tau_1}{b^2\tau_2},\frac{1}{b^2\tau_2};M^2) &= b^{2s}\{2\tau_2^s\zeta_{EH}(s;\tau_2 b^2 M^2) \\ &+2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\tau_2^{1-s}\zeta_{EH}(s-\frac{1}{2};\frac{b^2M^2}{\tau_2}) \\ &+\sum_{m,k=1}^{\infty}\frac{8\pi^s}{\Gamma(s)}\sqrt{\tau_2}k^{s-\frac{1}{2}}\frac{\cos(2\pi\tau_1 mk)}{(m^2+\frac{b^2M^2}{\tau_2})^{\frac{s}{2}-\frac{1}{4}}} \\ &K_{s-\frac{1}{2}}(2\pi\tau_2 k\sqrt{m^2+\frac{b^2M^2}{\tau_2}})\}, \quad (2.28)\end{aligned}$$

where the Epstein-Hurwitz zeta function $\zeta_{EH}(s;q)$ is defined by

$$\begin{aligned}\zeta_{EH}(s;q) &= \frac{1}{2}\sum_{n\in z}\left(n^2+q\right)^{-s} \\ &= -\frac{q^{-s}}{2}+\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma\left(s\right)}q^{-s+\frac{1}{2}}+\sum_{n=1}^{\infty}\frac{2\pi^s q^{-\frac{s}{2}+\frac{1}{4}}}{\Gamma(s)}n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi n\sqrt{q}\right).\end{aligned}$$

Note that the first sum indicates that the term $n = 0$ is excluded. Insert Eq.(2.36) into Eq.(2.34) and eliminate the infinite terms due to the pole of $\Gamma(s = -2)$ and $\Gamma(s - 1 = -3)$. To eliminate the divergent terms corresponding to the constant energy density in the bulk. Both of them do not depend on any parameters of the shape moduli and hence can be safely dropped from the physically relevant Casimir effects by renormalization. The final regulated Casimir energy density of massive real scalar field $\rho_{4D}(h^{ij})$ in 4-dimensional spacetime is

$$\begin{aligned}
\rho_{4D}(b^2,\tau_1,\tau_2) &= \frac{\hat{E}_{cas}}{V_m} \\
&= -(4\pi^2 b^2)^s\{2\tau_2^s(\tau_2 b^2 M^2)^{-\frac{s}{2}+\frac{1}{4}}\sum_{k=1}^{\infty} k^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi k b M\sqrt{\tau_2}) \\
&\quad +2\tau_2^{1-s}(\frac{b^2M^2}{\tau_2})^{-\frac{s}{2}+\frac{1}{2}}\sum_{k=1}^{\infty} k^{s-1}K_{s-1}(\frac{2\pi k b M}{\sqrt{\tau_2}}) \\
&\quad +4\sqrt{\tau_2}\sum_{k,m=1}^{\infty} k^{s-\frac{1}{2}}\frac{\cos(2\pi\tau_1 m k)}{(m^2+\frac{b^2M^2}{\tau_2})^{\frac{s}{2}-\frac{1}{4}}}K_{s-\frac{1}{2}}(2\pi\tau_2 k\sqrt{m^2+\frac{b^2M^2}{\tau_2}})\}.
\end{aligned}$$

In the case of massless scalar fields ($M=0$), we use the formula

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(\frac{z}{2})^{-\nu},$$

to approximate the modified Bessel function if z approaches zero in the Casimir energy density. Such that

$$\begin{aligned}
K_{s-\frac{1}{2}}(2\pi k b M\sqrt{\tau_2}) &= K_{\frac{1}{2}-s}(2\pi k b M\sqrt{\tau_2}) \sim \frac{\Gamma(\frac{1}{2}-s)(2\pi k\sqrt{\tau_2}bM)^{s-\frac{1}{2}}}{2^{s+\frac{1}{2}}}, \\
K_{s-1}(\frac{2\pi k b M}{\sqrt{\tau_2}}) &= K_{1-s}(\frac{2\pi k b M}{\sqrt{\tau_2}}) \sim \frac{\Gamma(1-s)(2\pi k\frac{bM}{\sqrt{\tau_2}})^{s-1}}{2^s}, \\
K_{s-\frac{1}{2}}(2\pi\tau_2 k\sqrt{m^2+\frac{b^2M^2}{\tau_2}}) &= K_{s-\frac{1}{2}}(2\pi k\tau_2 m),
\end{aligned}$$

the Casimir energy density becomes

$$\begin{aligned}
\rho_{4D}(b^2,\tau_1,\tau_2) &= -(4\pi^2b^2)^s\{\tau_2^s\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-s)\zeta(1-2s)+\tau_2^{1-s}\pi^{s-1}\Gamma(1-s)\zeta(2-2s) \\
&\quad +4\sqrt{\tau_2}\sum_{m,k=1}^{\infty}(\frac{k}{m})^{s-\frac{1}{2}}\cos(2\pi m k\tau_1)K_{s-\frac{1}{2}}(2\pi m k\tau_2)\}, \qquad (2.29)
\end{aligned}$$

where $\zeta(s)$ is the familiar Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The Casimir energy density in (1+3+2) dimensions is given by $\rho_{6D} = \frac{\rho_{4D}}{(2\pi b)^2}$.

As it is pointed out in the work of Ponton and Poppitz [31]. Since the Casimir energy density is preserved under the symmetry $\tau \to -\frac{1}{\tau}$, and $\tau \to \tau+1$ of the torus. Therefore it is sufficient to consider only the fundamental region where $\tau \geq 1$, and $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$ of the shape moduli space. In the fundamental region, there are two minima and one saddle point of the Casimir energy density. The saddle point locate at $\tau_1 = 0$, $\tau_2 = 1$ and the two minima locate at $\tau_1 = \pm\frac{1}{2}$, $\tau_2 = \frac{\sqrt{3}}{2}$. This is shown in figured 2.1.

2.4 Particle Spectrum and Effective Potential for Moduli Fields

It is demonstrated in [3] that a careful mixing of massless and massive, bosonic and fermionic degree of freedom of the bulk fields can lead to a Casimir energy density with local minimum with respect to the scale factor b of the compact extra dimensions. In the torus case with the shape moduli τ_1, τ_2 it can be shown that the true minimum of the mixed Casimir energy density (and thus the potential) locates at $\tau_1 = \pm 1/2$, $\tau_2 = \sqrt{3}/2$, in contrast to the case of undistorted torus considered in the previous work where the shape moduli are set to $\tau_1 = 0$, $\tau_2 = 1$.

The simplest model of the bulk fields in our spacetime background consists of: (i) a massless boson, (ii) a massless fermion, (iii) a massive fermion of mass m_f and (iv) a massive boson with mass λm_f. It was found that for the range $0.4 < \lambda < 0.42$ and $m_f = 5$, the mixed Casimir energy density has local minimum. There is no particular reason for why the ratio of the masses of the massive boson and fermion took the specific value in this range. An issue in mixing bosonic and fermionic degree of freedom to obtain the Casimir energy density with a local minimum is the energy density must be positive.

As a result, the value of the total Casimir energy density at $\tau_1 = \pm 1/2$, $\tau_2 = \sqrt{3}/2$ is lower than the value of the saddle point $\tau_1 = 0$, $\tau_2 = 1$, for all range of λ. However, for $\lambda \leq 0.407$, the energy density become negative around the minimum and therefore violates the positive energy condition. A negative value of the energy density will not stabilize the dynamics and the size of the torus. Therefore we choose the value $\lambda = 0.408$ for our simulation of the cosmological dynamics.

2.5 Cosmological Dynamics and Evidence of Stability of The Moduli Space

By numerically solving the differential equations in section 2.2 , the accelerated expansion of the large three spatial directions and the stabilization of the extra dimensions can be demonstrated to occur at the minimum of the Casimir energy density. The saddle point $\tau_1 = 0$, $\tau_2 = 1$ is an unstable equilibrium of the dynamics.

When the initial conditions of the cosmological equations of motion take vanishes within a small vicinity of the saddle point, $\tau_1 = 0$, $\tau_2 = 1$, it will roll down to the true minimum at $\tau_1 = \pm 1/2$, $\tau_2 = \sqrt{3}/2$ even with small amount of perturbations. The rolling of the cosmological dynamics, $a(t), b(t), \tau_1(t), \tau_2(t), H_a(t)$, and $H_b(t)$ to the true minimum of the total Casimir energy density is shown in the figure 2.2.

The pressure in each dimensions defined as $p = -\frac{\partial(\rho V)}{\partial V}$ where ρ is the Casimir energy density and V is the volume. Now the pressure in the three large spatial directions is $p_a = -\frac{\partial(\rho V_a)}{\partial V_a}$ where $V_a \propto a^3$. Because of the Casimir energy density is independent of the scale factor, $a(t)$ in large directional dimensions so $p_a = -\rho$ or $w_a = -1$. The physical pressures in the direction of the two dimensional torus are defined by

$$p^*_K \equiv T^K_K, \tag{2.30}$$

where $K = 5, 6$. This definition leads to the equation of state parameter

$$w_K = \frac{p^*_K}{\rho}, \tag{2.31}$$

therefore

$$w_5 = -1 + \frac{b}{2\rho}(\frac{\tau_1^2 - \tau_2^2}{\tau_2^2})\partial_b\rho + \frac{2\tau_1}{\rho}\partial_b\rho + \frac{1}{\rho}(\frac{\tau_2^2 - \tau_1^2}{\tau_2})\partial_{\tau_2}\rho \tag{2.32}$$

$$w_6 = -1 + \frac{b}{2\rho}(\frac{\tau_1^2 - \tau_2^2}{\tau_2^2})\partial_b\rho - \frac{1}{\rho}(\frac{\tau^2}{\tau_2})\partial_{\tau_2}\rho \tag{2.33}$$

At the stabilized point we know that $\dot{H}_a = \dot{H}_b = H_b = \dot{\tau}_1 = \dot{\tau}_2 = 0$ it can be shown that $w_5 = w_6 = -2$ by solving the cosmological equation of motion in six dimensions.

The Casimir energy density in the off-diagonal components of the stress tensor induce the shear viscosity. We obtain cosmological dynamical equations with viscosities by introducing the energy-momentum tensor of a viscous fluid. Let $U^A = (1, 0, 0, 0, 0, 0)$ be the six-velocity of the fluid in the comoving coordinates. In terms of the projection tensor $h_{AB} = g_{AB} + U_A U_B$, the energy-momentum tensor of fluid with bulk viscosity ζ and shear viscosity η is given by:

$$T_{AB} = \rho U_A U_B + (p - \zeta\theta)h_{AB} - 2\eta\sigma_{AB}. \tag{2.34}$$

Here $\theta \equiv \nabla_A U^A$ is the scalar expansion and $\sigma_{AB} = h^C_A h^D_B \nabla_{(C} U_{D)} - 1/5 h_{AB}\theta$ is the shear tensor. The non-zero components of the energy-momentum tensor are

$$\begin{aligned}
T^0_0 &= -\rho, \\
T^1_1 &= T^2_2 = T^3_3 = p_a, \\
T^5_5 &= (p_b - \zeta_b\theta) - 2\eta_b[\frac{3}{5}(H_b - H_a) - \frac{\dot{\tau}_2}{2\tau_2} - \frac{\tau_1\dot{\tau}_1}{2\tau_2^2}], \\
T^6_6 &= (p_b - \zeta_b\theta) - 2\eta_b[\frac{3}{5}(H_b - H_a) + \frac{\dot{\tau}_2}{2\tau_2} - + \frac{\tau_1\dot{\tau}_1}{2\tau_2^2}], \\
T^5_6 &= 2\eta_b[\frac{\tau_1\dot{\tau}_2}{\tau_2} + (\tau_1^2 - \tau_2^2)\frac{\dot{\tau}_1}{2\tau_2^2}], \\
T^6_5 &= -\eta_b\frac{\dot{\tau}_1}{\tau_2^2}.
\end{aligned}$$

We assume there is no viscosity in the noncompact directions because of the Cosmological principle. Therefore Einstein's field equations in this case can be written as

$$\dot{H}_a + 3H_a^2 + 2H_aH_b = \frac{8\pi G}{4}[\rho_{6D} + p_a - 2(p_b - \zeta_b\theta) + \frac{12}{5}\eta_b(H_b - H_a)], \quad (2.35)$$

$$\dot{H}_b + 2H_b^2 + 3H_aH_b = \frac{8\pi G}{4}[\rho_{6D} - 3p_a + 2(p_b - \zeta_b\theta) - \frac{12}{5}\eta_b(H_b - H_a)], \quad (2.36)$$

$$\ddot{\tau}_1 + (3H_a + 2H_b - 2\frac{\dot{\tau}_2}{\tau_2})\dot{\tau}_1 = 16\pi G\eta_b\dot{\tau}_1, \quad (2.37)$$

$$\frac{\ddot{\tau}_2}{\tau_2} + \frac{\dot{\tau}_1^2 - \dot{\tau}_2^2}{\tau_2^2} + 3H_a\frac{\dot{b}}{\tau_2} + 2H_b\frac{\dot{\tau}_2}{\tau_2} = 48\pi G\eta_b\frac{\dot{\tau}_2}{\tau_2}. \quad (2.38)$$

The shear viscosity in compact dimensions at the stabilized point η_b^{stab} can be identified by Eq.(2.38) to be

$$\eta_b^{stab} = \frac{3H_{a,stab}}{16\pi G}, \quad (2.39)$$

where $H_{a,stab}$ is the Hubble parameter at the stabilized point of the compactified space.

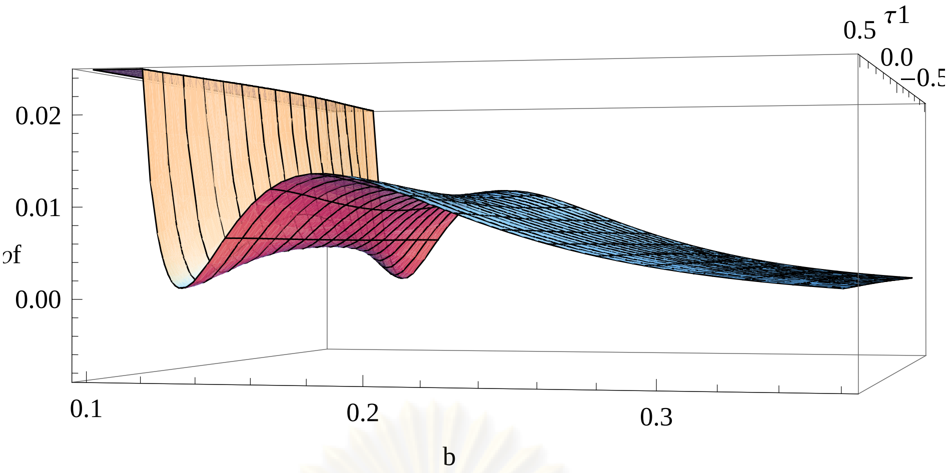

Figure 2.1: The total Casimir energy density in 6-dimensions for $M = 5, \lambda = 0.408$ and $\tau_2 = \sqrt{1 - \tau_1^2}$.

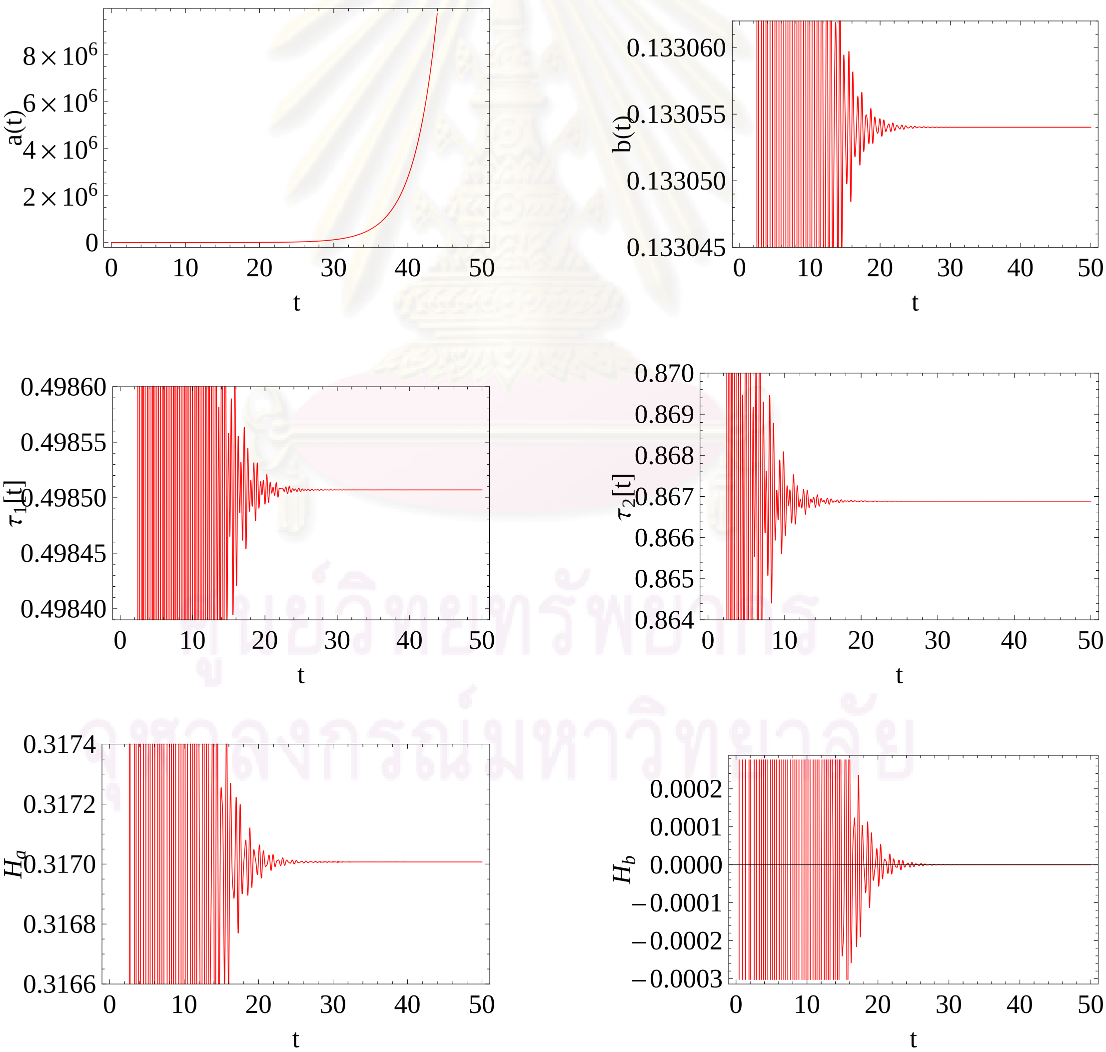

Figure 2.2: Cosmological dynamics for the universe is initially tossed very close to the saddle point.

Chapter III

Æther Field in Extra Dimensions

3.1 Kaluza-Klein Theory

In the 1920's Kaluza first proposed an additional dimension in his attempt to unify gravity and electromagnetism [14]. He assumed that our world is a direct product of the four-dimensional Minkowski space M_4 and a circle S_1 with the radius R. In 1926, Klein showed that the size of this additional dimension determined by the Plank scale 10^{-33} cm is completely unobservable [15].

In this section we will review in some details what the effective $4D$ theory looks like, let us consider a $5D$ toy model where the fifth direction has been compactified on a circle of radius R [16]. The five dimensional Klein-Gordon equation for bulk massless scalar field is given by

$$\partial^a \partial_a \Phi(x, y) = (\partial^\mu \partial_\mu - \frac{\partial^2}{\partial y^2})\Phi(x, y) = 0, \tag{3.1}$$

where $a = 0, 1, 2, 3, 5$, and y denotes the fifth direction. Impose the periodic boundary condition in the fifth direction as

$$\Phi(x, y) = \Phi(x, y + 2\pi R), \tag{3.2}$$

which allow for a Fourier expansion as

$$\Phi(x, y) = \sum_{k=0,\pm1,\ldots} \phi_k(x) e^{iky/R}. \tag{3.3}$$

The expansion coefficients ϕ_k are referred to as modes. The zero mode ϕ_0 corresponding to $k = 0$ is a ground state. Other Fourier modes $k \neq 0$ are called the excited mode or Kaluza-Klein (KK) modes, this gives rise to a tower of states dubbed KK tower. By substituting the last expansion into the Eq.(3.1) we see that each mode ϕ_k satisfies the four dimensional Klein-Gordon equation

$$(\partial^\mu \partial_\mu + \frac{k^2}{R^2})\phi_k(x) = 0, \tag{3.4}$$

with the mass term replace by

$$m_k^2 = \frac{k^2}{R^2}. \tag{3.5}$$

The zero-mode ϕ_0 remain massless while all other modes become massive four dimensional scalar field with mass term $m_k = \frac{k}{R}$.

In the KK scenario we assume that R is small and then $\frac{1}{R}$ is large compared to currently available energy scale. Thus the appreciation of the KK tower depend on the relevant energy of the experiment and on the compactification scale: (i) Given the energy $E \ll \frac{1}{R}$ the non-zero mode quanta can not be produced and physics would behave as four dimensional world. (ii) At accessible energy become higher than $\frac{1}{R}$ or equivalently as we do measurements at shorter distances. We can discover the KK excitation and hence it is a signature of the extra dimensions. However, the shape moduli τ_1, τ_2 can be included in the Kaluza-Klein theory. The phenomenological implications of nontrivial shape moduli were pointed out in [32, 33, 34].

3.1.1 Large Extra Dimensions and Deviation From Newton Gravitational Force Law

The Standard Model of particle physics is a successful theory because it is consistent with several experimental facts. On the other hand, we believe that the Standard Model is incomplete because it has a serious problem. Consider the two fundamental energy scales in the Standard Model, the electroweak scale $m_{EW} \sim 10^3$ GeV and the Plank scale $M_{Pl} \sim 10^{18}$ GeV. Their ratio $\frac{m_{EW}}{M_{Pl}} \sim 10^{-17}$ is very small. The large hierarchy between the electroweak and the Plank scales is highly unnatural and it is called the hierarchy problem.

A solution to solve the hierarchy problem which does not rely on either supersymmetry or technicolor is the ADD scenarios [17] where ADD stands for Arkani-Hamed, Dimopoulos and Dvali, its inventors. The model is to suppose that our world has a (4+n)-dimensional spacetime with there are n extra compact spatial dimensions of radius $\sim R$. And all SM (Standard Model) particles are localized on a 3-brane (a 4-dimensional spacetime object). In the other hand, gravity can escapes in the bulk (all 4+n dimensions), it becomes 4-dimensional only at the distance far away from the size of the extra dimensions, $r \gg R$. The Plank scale $M_{Pl(4+n)}$ of this (4+n) dimensional theory is taken to be $\sim m_{EW}$ according to ADD scenario. In a distance $r \ll R$ the gravitational potential take the form

$$V(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2}} \frac{1}{r^{n+1}}. \tag{3.6}$$

However, if the two test mass m_1, m_2 placed at distance $r \gg R$, at such distance the gravitational potential take the standard Newton form

$$V(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2} R^n} \frac{1}{r}, \tag{3.7}$$

so the effective 4 dimensional M_{Pl} is

$$M_{Pl}^2 \sim M_{Pl(4+n)}^{2+n} R^n. \tag{3.8}$$

We impose $M_{Pl(4+n)} \sim m_{EW}$ and then extra compact spatial dimensions of radius R be chosen to reproduce the M_{Pl} yields

$$R \sim 10^{30/n-17} cm \times (\frac{1\mathrm{TeV}}{m_{EW}})^{1+2/n}. \tag{3.9}$$

For $n = 1$ then $R \sim 10^{13}$ cm which is definitely inconsistent with the Newton gravitational force law so this case is ruled out in the ADD scenario. For all $n \geq 2$ the modification of gravity become noticeable at distances smaller than currently probed experiments [35, 36, 37, 38]. From the above formula if $n = 2$ then $R \sim 0.1mm$ this case is very exciting because experiments can be performed in the future to look for deviations from the Newton gravitational force law in precisely this range scale.

In conclusion, ADD scenario takes the electroweak scale as the only fundamental scale in the Standard Model of particle physics. The 4-dimensional Plank scale is not a fundamental scale. The effective Plank scale is so big with respect to electroweak scale because of the large size of the extra dimensions. In this framework gravity is so weak (the Plank scale M_{Pl} is so big) because the graviton is the only field that can propagate in the bulk therefore gravity spreads its strength over more than 4 dimensional spacetime.

3.2 Æther Field in 5-Dimensions

In this section we investigate in some details the aether compactification model [18]. This model based on Lorentz violating vector fields with vacuum expectation value (vev) along the extra directions to hide large extra dimensions which does not invoke brane as in the ADD scenario.

3.2.1 Model Building

Let us consider the aether field in a 5-dimensional flat spacetime with coordinates $x^a = \{x^\mu, x^5\}$ and metric signature $(-,+,+,+,+)$. The fifth direction is compactified on a circle of radius R. Now we consider the aether u^a is a spacelike-vector type, and we can define a field strength tensor as

$$V_{ab} = \nabla_a u_b - \nabla_b u_a. \tag{3.10}$$

Even though the aether field is not related to the electromagnetic potential A^α, its field strength tensor is in the Maxwell type. Demand that the norm of the aether vector field is fixed then we write the action with constraint

$$u_a u^a = v^2, \tag{3.11}$$

as

$$S = \int d^5x\sqrt{g}\left[-\frac{1}{4}V_{ab}V^{ab} - \lambda\left(u_a u^a - v^2\right) + \sum_i \mathcal{L}_i\right], \tag{3.12}$$

where $\mathcal{L}_i$ represent various interaction terms between aether field and other fields. Note that components of this vector field has a dimension of mass then v^2 has a dimension of $mass^2$ and λ is a Lagrange multiplier enforcing the fixed-norm. With this action, the equation of motion obtained by varying action with respect to u^a is

$$\nabla_a V^{ab} = 2\lambda u^b. \tag{3.13}$$

The value of λ is determined by decomposing the vector equation Eq.(3.13) into component along u^b. To do that in mathematical context, we multiply both sides of Eq.(3.13) by u_c

$$u_c \nabla_a V^{ab} = 2\lambda u_c u^b, \tag{3.14}$$

then contract indices b and c, we obtain a Lagrange multiplier by used the constraint Eq.(3.10)

$$\lambda = \frac{1}{2v^2} u_c \nabla_d V^{dc}. \tag{3.15}$$

Substitute it into Eq.(3.13) we obtain the equation of motion for u^a is

$$\nabla_a V^{ab} + \frac{1}{v^2} u^b u_c \nabla_d V^{cd} = 0. \tag{3.16}$$

We impose the aether field points along the extra dimensional direction, there is a background solution in the form

$$u^a = (0,0,0,0,v). \tag{3.17}$$

This background solution implies the field strength tensor $V_{ab} = 0$ so it is a solution for the equation of motion Eq.(3.16).

3.2.2 Energy-Momentum Tensor and Compactification

By varying the action in Eq.(3.12) with respect to the inverse metric tensor g^{ab}, we obtain the energy-momentum tensor for the aether field in the following form,

$$T_{ab} = V_{ac}V_b^c - \frac{1}{4}V_{cd}V^{cd}g_{ab} + \frac{1}{v^2}u_a u_b u_c \nabla_d V^{dc}. \tag{3.18}$$

For our background solution in Eq.(3.17) T_{ab} is zero while the expectation value of the aether field does not vanish. When the background spacetime is not Minkowski then a fixed aether field can give a non-vanishing energy-momentum tensor [39, 40].

Consider the metic

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + b(x)^2 dx_5^2, \tag{3.19}$$

where x is the 4-dimensional coordinates and $b(x)$ is the radion field that parameterize the size of the fifth direction. In this spacetime background, there is a background solution for aether field

$$u^a = (0, 0, 0, 0, \frac{v}{b(x)}). \tag{3.20}$$

This configuration satisfies the equation of motion Eq.(3.16), as well as the constraint Eq.(3.11) and non-vanishing field strength tensor is

$$V_{\mu 5} = -V_{5\mu} = v\nabla_\mu b. \tag{3.21}$$

And the non-zero energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu}|^u &= \frac{v^2}{b^2}\left(\nabla_\mu b \nabla_\nu b - \frac{1}{2}g_{\mu\nu}\nabla_\sigma b \nabla^\sigma b\right), \\ T_{55}|^u &= v^2\left(\nabla_\sigma b \nabla^\sigma b - \frac{1}{2}\nabla_\sigma b \nabla^\sigma b\right). \end{aligned} \tag{3.22}$$

The important feature is the energy-momentum tensor will vanish when the extra dimension is stabilized. There will be no contribution to the effective 4-dimensional spacetime.

3.2.3 Interaction of the Æther on Scalar Fields

We now consider the effect of the interaction term $\sum_i \mathcal{L}_i$ in Eq.(3.12) which in general can include the terms aether field coupled to scalars, fermion and gravity. Now we will investigate the effect of the aether coupled to real massive scalar field. The simplest Lagrangian is

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2\mu_\phi^2}u^a u^b \partial_a\phi\partial_b\phi, \tag{3.23}$$

this imposes a Z_2 symmetry, $u^a \to -u^a$ because if we have not imposed it the lowest order coupling is $-\mu_\phi^{-1} u^a \partial_a \phi = -\mu_\phi^{-1}(\partial_a u^a)\phi$ by integration by parts which vanishes in our background solution for u^a. The equation of motion for this Lagrangian is

$$\partial_a \partial^a \phi - m^2 \phi = -\mu_\phi^{-2} \partial_a (u^a u^b \partial_b \phi). \tag{3.24}$$

Expanding the scalar field in Fourier modes

$$\phi \sim e^{ik_a x^a} = e^{ik_\mu x^\mu + ik_5 x^5}, \tag{3.25}$$

we obtain the modified dispersion relation

$$-k_\mu k^\mu = m^2 + (1 + \alpha_\phi^2) k_5^2, \tag{3.26}$$

where the dimensionless parameter $\alpha_\phi = v/\mu_\phi$ is the ratio of the aether vev to the coupling μ_ϕ.

Consequently, we know that compactifying the fifth dimension on a circle of radius R lead to the quantization of the momentum in that direction, $k_5 = n/R$. In Kaluza-Klein theory all this excited modes gives rise to a KK tower of masses

$$m_{KK}^2 = m^2 + (\frac{n}{R})^2. \tag{3.27}$$

While the coupling with aether field gives the mass spacing between different states in the KK tower is reinforced

$$m_{AC}^2 = m^2 + (1 + \alpha_\phi^2)(\frac{n}{R})^2. \tag{3.28}$$

We will examine the effect of aether compactification by imposed the vev is $v \sim M_{Pl}$ and the $\mu_\phi \sim$ TeV then the masses of the excited modes are reinforced by a factor of 10^{15} [18]. By comparison with Kaluza-Klein theory the first excited state $n = 1$ would have a mass of order TeV while the extra dimensions could be as large as $R \sim 1$ mm. This means that at an accessible energy higher than $\frac{1}{R}$ it can not discover the KK excitation although our world have one extra compact direction.

3.3 Æther Field in $M^{1+3} \times T^2$ Spacetime

In this thesis we consider the aether field in a product space between a 4-dimensional spacetime and 2-toroidally-compactified space. Let define the coordinates $x^a = \{x^\mu, x^5, x^6\}$ where x^μ, with $\mu = 0, ..., 3$ are non-compact coordinates, while $x^5, x^6 \in$

$[0, 2\pi]$ are compact coordinates. Similarly as in the previous section we consider the aether u^a is a spacelike-vector type, and we can define a field strength tensor as

$$V_{ab} = \nabla_a u_b - \nabla_b u_a. \tag{3.29}$$

Here we investigate the action in 6-dimensions with Maxwell-type kinetic term

$$S = \int d^6x\sqrt{g}\left[-\frac{1}{4}V_{ab}V^{ab} - \lambda\left(u_a u^a - v^2\right) + \sum_i \mathcal{L}_i\right], \tag{3.30}$$

where λ is a Lagrange multiplier enforcing the constraint

$$u_a u^a = v^2. \tag{3.31}$$

The equation of motion for u^a is

$$\nabla_a V^{ab} + \frac{1}{v^2}u^b u_c \nabla_d V^{cd} = 0. \tag{3.32}$$

From previous chapter we assume the cosmological ansatz

$$ds^2 = -dt^2 + a\left(t\right)^2 dx^i dx^j \delta_{ij} + \frac{b\left(t\right)^2}{\tau_2\left(t\right)}\left[dx^5 dx^5 + 2\tau_1 dx^5 dx^6 + |\tau\left(t\right)|^2 dx^6 dx^6\right]. \tag{3.33}$$

To find the background solution we impose the aether field points along the extra dimensions, so that

$$u^a = \left(0, 0, 0, 0, u^5, u^6\right), \tag{3.34}$$

this solution must satisfy the constraint equation, that is

$$v^2 = \frac{b^2}{\tau_2}\left(u^5\right)^2 + \frac{2b^2\tau_1}{\tau_2}u^5 u^6 + \frac{b^2|\tau|^2}{\tau_2}\left(u^6\right)^2. \tag{3.35}$$

Using the complete square method we parameterized the solution in our background as

$$u^a = \left(0, 0, 0, 0, \frac{v\sqrt{\tau_2}}{b}\left(\cos\theta - \frac{\tau_1}{\tau_2}\sin\theta\right), \frac{v\sin\theta}{b\sqrt{\tau_2}}\right), \tag{3.36}$$

where θ is an angle parameter. And components of a covariant vector define as $u_a = g_{ab}u^b$ then

$$u_a = \left(0, 0, 0, 0, \frac{vb\cos\theta}{\sqrt{\tau_2}}, vb\sqrt{\tau_2}\left(\sin\theta + \frac{\tau_1}{\tau_2}\cos\theta\right)\right). \tag{3.37}$$

By varying the action with respect to the metric, we obtain the energy-momentum tensor from the aether field in the following form,

$$T_{ab} = V_{ac}V_b^c - \frac{1}{4}V_{cd}V^{cd}g_{ab} + \frac{1}{v^2}u_a u_b u_c \nabla_d V^{dc}. \tag{3.38}$$

Raising an index we obtain

$$
\begin{aligned}
T^0_0|_u &= -\frac{v^2}{8\tau_2^2}\left(4H_b^2\tau_2^2 + 4\cos^2\theta\dot{\tau}_1^2 + 2\sin\theta\dot{\tau}_1\dot{\tau}_2 + \dot{\tau}_2^2 + 4H_b\tau_2\left(\sin 2\theta\dot{\tau}_1 - \cos 2\theta\dot{\tau}_2\right)\right),\\
T^i_j|_u &= \frac{v^2}{8\tau_2^2}\left(4H_b^2\tau_2^2 + 4\cos^2\theta\dot{\tau}_1^2 + 2\sin\theta\dot{\tau}_1\dot{\tau}_2 + \dot{\tau}_2^2 + 4H_b\tau_2\left(\sin 2\theta\dot{\tau}_1 - \cos 2\theta\dot{\tau}_2\right)\right)\delta^i_j,\\
T^5_5|_u &= -\frac{v^2}{8\tau_2^3}(4H_b^2\tau_2^2(\cos 2\theta\tau_2 - \sin 2\theta\tau_1) - 4\tau_2(2\cos\theta(\tau_2(\sin\theta\tau_1 - \cos\theta\tau_2)\\
&\quad (3H_aH_b + H_b^2 + \dot{H}_b) + H_b(\cos\theta\tau_1 + \sin\theta\tau_2)\dot{\tau}_1) + H_b\tau_2\dot{\tau}_2) + \tau_2(-4\cos^2\theta\\
&\quad (2+\cos 2\theta)\dot{\tau}_1^2 - 2(3\sin 2\theta + \sin 4\theta)\dot{\tau}_1\dot{\tau}_2 + (2\cos 2\theta + \cos 4\theta)\dot{\tau}_2^2\\
&\quad +4\cos^2\theta\tau_2(\sin 2\theta(3H_a\dot{\tau}_1 + \ddot{\tau}_1) - \cos 2\theta(3H_a\dot{\tau}_2 + \ddot{\tau}_2))) + 2\tau_1(2\cos\theta\\
&\quad (2\cos^2\theta\sin\theta\dot{\tau}_1^2 - (\cos 3\theta - 2\cos\theta)\dot{\tau}_1\dot{\tau}_2 + 2\sin^3\theta\dot{\tau}_2^2) + \sin 2\theta\tau_2(-\sin 2\theta\\
&\quad (3H_a\dot{\tau}_1 + \ddot{\tau}_1) + \cos 2\theta(3H_a\dot{\tau}_2 + \ddot{\tau}_2)))),\\
T^5_6|_u &= \frac{v^2}{4\tau_2^3}((\cos\theta\tau_2 - \sin\theta\tau_1)(\cos\theta\tau_1 + \sin\theta\tau_2)\dot{\tau}_1^2 - (\sin\theta\tau_1 - \cos\theta\tau_2)\dot{\tau}_1(2H_b\tau_2\\
&\quad (\sin\theta\tau_1 - \cos\theta\tau_2) + (\cos\theta\tau_1 - \sin\theta\tau_2)\dot{\tau}_1 + (\sin\theta\tau_1 + \cos\theta\tau_2)\dot{\tau}_2)\\
&\quad +(\cos\theta\tau_1 + \sin\theta\tau_2)\dot{\tau}_1(2H_b\tau_2(\cos\theta\tau_1 + \sin\theta\tau_2) + (\sin\theta\tau_1 + \cos\theta\tau_2)\dot{\tau}_1\\
&\quad +(\sin\theta\tau_2 - \cos\theta\tau_1)\dot{\tau}_2) + (2H_b\tau_2(\sin\theta\tau_1 - \cos\theta\tau_2) + (\cos\theta\tau_1 - \sin\theta\tau_2)\dot{\tau}_1\\
&\quad +(\sin\theta\tau_1 + \cos\theta\tau_2)\dot{\tau}_2)(2H_b\tau_2(\cos\theta\tau_1 + \sin\theta\tau_2) + (\sin\theta\tau_1 + \cos\theta\tau_2)\dot{\tau}_1\\
&\quad +(\sin\theta\tau_2 - \cos\theta\tau_1)\dot{\tau}_2) - (\sin\theta\tau_1 - \cos\theta\tau_2)(\cos\theta\tau_1 + \sin\theta\tau_2)(4\cos^2\theta\dot{\tau}_1^2\\
&\quad +4\sin 2\theta\dot{\tau}_1\dot{\tau}_2 + (1 - 2\cos 2\theta)\dot{\tau}_2^2 - 6H_a\tau_2(2H_b\tau_2 + \sin 2\theta\dot{\tau}_1 - \cos 2\theta\dot{\tau}_2)\\
&\quad -2\tau_2(2\tau_2(H_b^2 + \dot{H}_b) + \sin 2\theta\ddot{\tau}_1 - \cos 2\theta\ddot{\tau}_2))),\\
T^6_5|_u &= \frac{v^2\cos\theta}{2\tau_2^3}(-4H_b^2\sin\theta\tau_2^2 - 2\sin\theta\tau_2^2\dot{H}_b - 2\cos\theta H_b\tau_2\dot{\tau}_1 + 2\cos^2\theta\sin\theta\dot{\tau}_1^2\\
&\quad +2\cos\theta\dot{\tau}_1\dot{\tau}_2 - \cos 3\theta\dot{\tau}_1\dot{\tau}_2 + 2\sin^3\theta\dot{\tau}_2^2 - 3H_a\sin\theta\tau_2(2H_b\tau_2 + \sin 2\theta\dot{\tau}_1\\
&\quad -\cos 2\theta\dot{\tau}_2) - 2\cos\theta\sin^2\theta\tau_2\ddot{\tau}_1 + \cos 2\theta\sin\theta\tau_2\ddot{\tau}_2),\\
T^6_6|_u &= \frac{v^2}{8\tau_2^3}(4H_b^2\tau_2^2(\cos 2\theta\tau_2 - \sin 2\theta\tau_2) - 4\tau_2(2(\cos\theta\tau_1 + \sin\theta\tau_2)(\sin\theta\tau_2(3H_aH_b\\
&\quad +H_b^2 + \dot{H}_b) + \cos\theta H_b\dot{\tau}_1) + H_b\tau_2\dot{\tau}_2) + \tau_2(-4\cos^2\theta\cos 2\theta\dot{\tau}_1^2\\
&\quad +2(\sin 2\theta - \sin 4\theta)\dot{\tau}_1\dot{\tau}_2 + (2 - 2\cos 2\theta + \cos 4\theta)\dot{\tau}_2^2 + 4\sin^2\theta\tau_2\\
&\quad (-\sin 2\theta(3H_a\dot{\tau}_1 + \ddot{\tau}_1) + \cos 2\theta(3H_a\dot{\tau}_2 + \ddot{\tau}_2))) + 2\tau_1(2\cos\theta(2\cos^2\theta\sin\theta\dot{\tau}_1^2\\
&\quad -(-2\cos\theta + \cos 3\theta)\dot{\tau}_1\dot{\tau}_2 + 2\sin^3\theta\dot{\tau}_2^2) + \sin 2\theta\tau_2(-\sin 2\theta(3H_a\dot{\tau}_1 + \ddot{\tau}_1)\\
&\quad +\cos 2\theta(3H_a\dot{\tau}_2 + \ddot{\tau}_2)))),
\end{aligned}
\tag{3.39}
$$

where $H_a = \frac{\dot{a}}{a}$ and $H_b = \frac{\dot{b}}{b}$. The important result from the energy-momentum tensor is that $T^a_b|_u$ vanish when the moduli fields is stabilized, $\dot{b} = \dot{\tau}_1 = \dot{\tau}_2 = 0$. Therefore the aether field will be no contribution to the accelerated expansion of the late-time universe.

3.4 Interaction of the Aether on Scalar and Fermionic Fields

We now consider the effect of the aether coupled to real massive scalar field. The simplest Lagrangian is

$$\mathcal{L}_\phi = -\frac{1}{2}\left(\nabla\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2\mu_\phi^2}u^a u^b \nabla_a\phi\nabla_b\phi, \tag{3.40}$$

this imposes a Z_2 symmetry, $u^a \to -u^a$ because if we have not imposed it the lowest order coupling is $-\mu_\phi^{-1}u^a\nabla_a\phi = -\mu_\phi^{-1}\phi\nabla_a u^a$ by integration by parts, which vanishes in our background solution for aether field. The equation of motion for this Lagrangian is

$$\nabla_a\nabla^a\phi - m^2\phi = -\mu_\phi^{-2}\nabla_a\left(u^a u^b\nabla_b\phi\right). \tag{3.41}$$

Expanding the scalar field in Fourier modes as the previous section, we obtain the modified dispersion relation,

$$\begin{aligned} k_a k^a \;&=\; m^2 + \frac{|\tau|^2}{b^2\tau_2}n_1^2 - \frac{2\tau_1}{b^2\tau_2}n_1n_2 + \frac{1}{b^2\tau_2}n_2^2 \\ &+\alpha_\phi^2((\frac{\tau_1^2}{b^2\tau_2} - \frac{\tau_1\sin 2\theta}{b^2} + \frac{\tau_2\cos^2\theta}{b^2} - \frac{\tau_1^2\cos^2\theta}{b^2\tau_2})n_1^2 \\ &+(\frac{\sin 2\theta}{b^2} - \frac{2\tau_1\sin^2\theta}{b^2\tau_2})n_1n_2 + \frac{\sin^2\theta}{b^2\tau_2}n_2^2), \end{aligned} \tag{3.42}$$

where the dimensionless parameter $\alpha_\phi = \frac{v}{\mu_\phi}$ is the ratio of the aether vacuum expectation value (vev) to the coupling μ_ϕ. The momentum of the scalar field along the compactified extra dimensions will be quantized as $k_5 = n_1$ and $k_6 = n_2$ in our spacetime geometry. Eqs.(3.42) suggests that the mass gap between the different states in the KK tower is enhanced by the interaction with the aether field and the distortion of the extra dimensions parameterized by b, τ_1, τ_2. Moreover, the mass also depends crucially on the angle parameter θ.

Next we consider the fermionic terms. The Lagrangian for fermionic field with the first non-trivial coupling is

$$\mathcal{L}_\psi = i\bar{\psi}\gamma^a\nabla_a\psi - m\bar{\psi}\psi - \frac{i}{\mu_\psi}u^a\bar{\psi}\nabla_a\psi, \tag{3.43}$$

where μ_ψ is the fermionic coupling constant with the aether. From the supersymmetry the corresponding modification of the dispersion relation for the fermion case can be written as the Eq.(3.42).

Chapter IV

Effect of Æther Field on Casimir Dark Energy Model

In this chapter, we investigate the role of a Lorentz violating vector field called an aether field on the moduli stabilization mechanism. We consider the space-like aether field with Maxwell-type kinetic term on the compact extra dimensions [4]. From chapter II we know that the Casimir energy of certain combinations of fields with different masses and spins can give a minimum that stabilizes the size of extra dimensions while the large spatial dimensions feel the Casimir energy as a sort of vacuum energy to accelerate its expansion. It is shown here that the aether field can slow down the oscillation of the moduli fields τ_1, τ_2 and radion field b. In certain cases, it leads to the stabilization of the extra dimensional torus of the universe even in the matter dominated era.

We consider cosmological dynamics of a factorizable geometry with 4-dimensional FRW metric and a two torus T^2

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + h_{ij}dy^i dy^j, \tag{4.1}$$

where the four dimensional metric $g_{\mu\nu}$ with $\mu, \nu = 0, ..., n$ is the Friedmann-Robertson-Walker metric in flat universe ($k = 0$), while the metric h_{ij} represent the p-dimensional compact space with compact coordinate $y^i \in [0, 2\pi]$. Note that $i, j = 1, ..., p$.

We will investigate the cosmological dynamics of a 4-dimensional flat space-time with two extra dimensions ($n = 3, p = 2$). The metric h_{ij} on the torus is

$$(h_{ij}) = \frac{b^2}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \tag{4.2}$$

where $\tau = \tau_1 + i\tau_2$ and b is the volume moduli or the scale factor for the extra dimensions. For our metric Eq.(4.1) the background solution of an aether field Eq.(3.36) can be written as

$$u^a = \left(0, 0, 0, 0, \frac{v\sqrt{\tau_2}}{b}\left(\cos\theta - \frac{\tau_1}{\tau_2}\sin\theta\right), \frac{v\sin\theta}{b\sqrt{\tau_2}}\right). \tag{4.3}$$

The energy-momentum tensor for this background solution are given in Eq.(3.39). Use the combinations of the Einstein tensors in chapter II, we define

$$\begin{aligned}
\tilde{T}|_0 &= -\frac{v^2}{8\tau_2^2}\left(4H_b^2\tau_2^2 + 4\cos^2\theta\dot{\tau}_1^2 + 2\sin\theta\dot{\tau}_1\dot{\tau}_2 + \dot{\tau}_2^2 + 4H_b\tau_2\left(\sin 2\theta\dot{\tau}_1 - \cos 2\theta\dot{\tau}_2\right)\right), \\
\tilde{T}|_1 &= \frac{4\pi G v^2}{\tau_2^2}\{2\tau_2^2(3H_aH_b + 2H_b^2 + \dot{H}_b) + \sin 2\theta(\dot{\tau}_1((3H_a + 2H_b)\tau_2 - \dot{\tau}_2) + \tau_2\ddot{\tau}_1) \\
&\quad + \cos 2\theta(\dot{\tau}_2(-(3H_a + 2H_b^2)\tau_2 + \dot{\tau}_2) - \tau_2\ddot{\tau}_2)\}, \\
\tilde{T}|_2 &= \frac{4\pi G v^2}{\tau_2^2}\{2\tau_2^2(3H_aH_b + 2H_b^2 + \dot{H}_b) + \sin 2\theta(\dot{\tau}_1((3H_a + 2H_b)\tau_2 - \dot{\tau}_2) + \tau_2\ddot{\tau}_1) \\
&\quad + \cos 2\theta(\dot{\tau}_2(-(3H_a + 2H_b^2)\tau_2 + \dot{\tau}_2) - \tau_2\ddot{\tau}_2)\}, \\
\tilde{T}|_3 &= \frac{4\pi G v^2}{\tau_2^3}\cos\theta\{-4H_b^2\sin\theta\tau_2^2 - 2\sin\theta\tau_2^2\dot{H}_b - 2\cos\theta H_b\tau_2\dot{\tau}_1 \\
&\quad +2\cos^2\theta\sin\theta\dot{\tau}_1^2 + 2\cos\theta\dot{\tau}_1\dot{\tau}_2 - \cos 3\theta\dot{\tau}_1\dot{\tau}_2 + 2\sin^3\theta\dot{\tau}_2^2 \\
&\quad -3H_a\sin\theta\tau_2(2H_b\tau_2 + \sin 2\theta\dot{\tau}_1 - \cos 2\theta\dot{\tau}_2) - 2\cos\theta\sin^2\theta\tau_2\ddot{\tau}_1 \\
&\quad + \cos 2\theta\sin\theta\tau_2\ddot{\tau}_2\}, \\
\tilde{T}|_4 &= \frac{2\pi G v^2\tau_1}{\tau_2^2}\{3\dot{\tau}_1^2 + 4\cos 2\theta(-\tau_2^2(3H_aH_b + 2H_b^2 + \dot{H}_b) + \dot{\tau}_1^2) \\
&\quad +3H_a\tau_2\dot{\tau}_2 + 4H_b\tau_2\dot{\tau}_2 - \dot{\tau}_2^2 + 2\sin 2\theta\dot{\tau}_1(2H_b\tau_2 + \dot{\tau}_2) - \sin 4\theta \\
&\quad (\dot{\tau}_1(3H_a\tau_2 - 2\dot{\tau}_2) + \tau_2\ddot{\tau}_1) + \tau_2\ddot{\tau}_2 + \cos 4\theta(\dot{\tau}_1^2 + 3H_a\tau_2\dot{\tau}_2 - \dot{\tau}_2^2 + \tau_2\ddot{\tau}_2)\}. \quad (4.4)
\end{aligned}$$

Add the aether energy-momentum tensor into the Einstein's field equation Eq.(2.29), we obtain the following equations governing the cosmological dynamics between Casimir energy and the aether field:

$$3H_a^2 + H_b^2 + 6H_aH_b - \frac{1}{4\tau_2^2}\left(\dot{\tau}_1^2 + \dot{\tau}_2^2\right) = 8\pi G\rho_{6D} + \tilde{T}|_0,$$

$$\begin{aligned}
\dot{H}_a + 3H_a^2 + 2H_aH_b &= \frac{8\pi G}{4}\left\{2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2}\right)^2\right] b\partial_b\rho_{6D} - 2\tau_1\partial_{\tau_1}\rho_{6D} + \frac{2\tau_1^2}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\} \\
&\quad + \frac{1}{4}\tilde{T}|_1,
\end{aligned}$$

$$\begin{aligned}
\dot{H}_b + 2H_b^2 + 3H_aH_b &= -\frac{8\pi G}{4}\left\{-2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2}\right)^2\right] b\partial_b\rho_{6D} - 2\tau_1\partial_{\tau_1}\rho_{6D} + \frac{2\tau_1^2}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\} \\
&\quad + \frac{1}{4}\tilde{T}|_2,
\end{aligned}$$

$$\begin{aligned}
\ddot{\tau}_1 + \left(3H_a + 2H_b - 2\frac{\dot{\tau}_2}{\tau_2}\right)\dot{\tau}_1 &= -16\pi G\tau_2^2\left\{\frac{b\tau_1}{2\tau_2^2}\partial_b\rho_{6D} + 2\partial_{\tau_1}\rho_{6D} - \frac{\tau_1}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\} \\
&\quad + 2\tilde{T}|_3,
\end{aligned}$$

$$\frac{\ddot{\tau}_2}{\tau_2} + \frac{\dot{\tau}_1^2 - \dot{\tau}_2^2}{\tau_2^2} + 3H_a\frac{\dot{\tau}_2}{\tau_2} + 2H_b\frac{\dot{\tau}_2}{\tau_2} = 8\pi G\{\frac{b\tau_1^2}{\tau_2^2}\partial_b\rho_{6D} + 2\tau_1\partial_{\tau_1}\rho_{6D} - 2\tau_2\left[1 + \left(\frac{\tau_1}{\tau_2}\right)^2\right]\partial_{\tau_2}\rho_{6D}\} + \tilde{T}|_4. \quad (4.5)$$

4.1 Æther Field and Casimir Energy in $M^{1+3} \times T^2$ Spacetime

In this section, we will determine the Casimir energy associated with a scalar field of mass M in our spacetime background. The fermionic degree of freedom will contribute to the Casimir energy with the same expression as the bosonic degrees of freedom except for an extra minus sign. However, we consider the effect of aether field coupling to the Casimir energy. Let $V_n = L^n$ be the spatial volume of non-compact spacetime, and $V_p = l^n$ be the volume of compact space. For simplicity we define

$$\begin{aligned} A &= \tau_1 \sin\theta - \tau_2 \cos\theta, \\ B &= \tau_2 \cos\theta - \tau_1 \sin\theta, \\ C &= \sin\theta. \end{aligned}$$

If we assume $L \gg l$, the zero-point energy of scalar field can be evaluated to be

$$\hat{E}_{cas} = \frac{1}{2}(\frac{L}{2\pi})^n \sum_{n_i,n_j} \int_{-\infty}^{\infty} d^n k \sqrt{\delta^{ab}k_a k_b + h^{ij}n_i n_j + M^2 + \frac{\alpha_\phi^2}{b^2\tau_2}(A^2 n_1^2 + BCn_1 n_2 + C^2 n_2^2)}.$$

Using regularization method in appendix A, we obtain the Casimir energy density in the 4-dimensional spacetime

$$\begin{aligned} \rho_{4D} = & -(4\pi^2 b^2)^s \{ \frac{2\tau_2^s(\tau_2 b^2 M^2)^{-s/2+1/4}}{(1+\alpha^2 C^2)^{s/2+1/4}} \sum_{k=1}^{\infty} k^{s-1} K_{s-1}(2\pi k b M \sqrt{\frac{\tau_2}{1+\alpha^2 C^2}}) \\ & + \frac{2\tau^{s/2+1/2}(b^2 M^2)^{-s/2+1/2}}{\sqrt{1+\alpha^2 C^2}[\tau_1^2 + \tau_2^2 + \alpha^2 A^2 - \frac{(\tau_1+\alpha^2 A)^2}{1+\alpha^2 C^2}]^{s/2}} \sum_{k=1}^{\infty} k^{s-1/2} K_{s-1/2}(2\pi k b M \\ & \sqrt{\frac{\tau_2}{\tau_1^2 + \tau_2^2 + \alpha^2 A^2 - \frac{(\tau_1+\alpha^2 A)^2}{1+\alpha^2 C^2}}}) + \frac{4\tau_2^s[\tau_1^2 + \tau_2^2 + \alpha^2 A^2 - \frac{(\tau_1+\alpha^2 A)^2}{1+\alpha^2 C^2}]^{-s/2+1/4}}{(1+\alpha^2 C^2)^{s/2+1/4}} \\ & \sum_{k,m=1}^{\infty} k^{s-1/2} \frac{\cos(\frac{2\pi(\tau_1+\alpha^2 AC)km}{1+\alpha^2 C^2})}{(m^2 + \frac{\tau_2 b^2 M^2}{\tau_1^2+\tau_2^2+\alpha^2 A^2 - \frac{(\tau_1+\alpha^2 A)^2}{1+\alpha^2 C^2}})^{s/2-1/4}} K_{s-1/2}(\frac{2\pi k}{\sqrt{1+\alpha^2 C^2}} \\ & \sqrt{\tau_1^2 + \tau_2^2 + \alpha^2 A^2 - \frac{(\tau_1+\alpha^2 A)^2}{1+\alpha^2 C^2}})\}. \end{aligned} \quad (4.6)$$

In the case of massless scalar fields ($M = 0$), the Casimir energy density becomes

$$\begin{aligned}\rho_{4D} &= -(4\pi^2 b^2)^s \tau_2 \{ \frac{\pi^{s-1/2}\Gamma(\frac{1}{2}-s)\zeta(1-2s)}{(1+\alpha^2 C^2)^s} \\ &+ \frac{\pi^{s-1}\Gamma(1-s)\zeta(2-2s)}{\sqrt{1+\alpha^2C^2}[\tau_1^2+\tau_2^2+\alpha^2A^2-\frac{(\tau_1+\alpha^2A)^2}{1+\alpha^2C^2}]^{s-1/2}} \\ &+ \frac{4[\tau_1^2+\tau_2^2+\alpha^2A^2-\frac{(\tau_1+\alpha^2A)^2}{1+\alpha^2C^2}]^{-s/2+1/4}}{(1+\alpha^2C^2)^{s/2+1/4}} \sum_{k,m=1}^{\infty} (\frac{k}{n})^{s-1/2} \cos(\frac{2\pi(\tau_1+\alpha^2 AC)kn}{1+\alpha^2C^2}) \\ & K_{s-1/2}(\frac{2\pi k}{\sqrt{1+\alpha^2C^2}}\sqrt{(\tau_1^2+\tau_2^2+\alpha^2A^2-\frac{(\tau_1+\alpha^2A)^2}{1+\alpha^2C^2})n)}\}\end{aligned} \tag{4.7}$$

The Casimir energy density in 6-dimensions is given by $\rho_{6D} = \rho_{4D}/(2\pi b)^2$.

4.2 Effects of the Æther Field on the Stabilization of the Extra Dimensions

4.2.1 Moduli Stabilization in vacuum dominated universe

We first consider the universe when there is no non-relativistic matter field and the Casimir energy coupled to aether is the only contribution. The simplest model of the bulk fields in our spacetime consists of: (i) a massless boson, (ii) a massless fermion, (iii) a massive fermion of mass m_f (iv) a massive boson with mass $m_b = \lambda m_f$ and (v) an aether field. In this thesis we investigate the vacuum expectation value $v = 2, 4$ and setting the mass ratio and the coupling parameter to be $\lambda = 0.408$ and $\mu_\phi = 50$ respectively, in order to obtain the positive minimum so that the moduli fields can be stabilized. The numerical results are shown in Table IV.1 where subscripts *in* and *stab* mean it is an initial state and final state of the cosmological dynamics respectively. The cosmological dynamics when $v = 2, \mu = 50$, and $\theta = 0, \pi/4, \pi/2$ are shown in the figure 4.1, 4.2, and 4.3 respectively.

4.2.2 Moduli Stabilization in the Universe with Non-Relativistic Matter

In this section we investigate the effect of the aether field in the stabilizing mechanism of the extra dimensions in the universe with dominating non-relativistic matter. We allow the matter to live in the bulk,

$$S = S_{6D} + \int d^6x\sqrt{-g}\mathcal{L}_{matter}. \tag{4.8}$$

This is equivalent to adding the matter energy density $\rho_m \propto 1/a^3b^2$ into the cosmological equation of motion. By comparing with the observational data, the matter density today ρ_{m0} is 26% of the total energy density of the universe. The Casimir energy density today in the form of dark energy, $\rho_{\Lambda 0}$, will be 74% of the total energy density of the universe. The energy density of dark energy can be written in term of the minimum of the Casimir energy density and the stabilized size of the extra dimensions as $\rho_{\Lambda 0} = \rho_{min}(2\pi b)^2$. By using $(a_0/a = 1+z)$, where a_0 is the scale factor today and z is the red-shift, we obtain

$$\rho_m = \frac{2.6}{7.4}\rho_{min}(\frac{b_{min}}{b})^2(1+z)^3. \tag{4.9}$$

Therefore the energy-momentum tensor of matter component is $T^a_b|_{matter} = diag(\rho_m, 0, 0, 0, 0, 0)$. In this case, the cosmological equation of motion becomes

$$3H_a^2 + H_b^2 + 6H_aH_b - \frac{1}{4\tau_2^2}\left(\dot{\tau}_1^2 + \dot{\tau}_2^2\right) = 8\pi G\rho_{6D} + \tilde{T}|_0 + 8\pi G\rho_m,$$

$$\begin{aligned}\dot{H}_a + 3H_a^2 + 2H_aH_b \quad &= \quad \frac{8\pi G}{4}\left\{2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2}\right)^2\right] b\partial_b\rho_{6D} - 2\tau_1\partial_{\tau_1}\rho_{6D} + \frac{2\tau_1^2}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\} \\ &\quad + \frac{1}{4}\tilde{T}|_1 + \frac{8\pi G}{4}\rho_m,\end{aligned}$$

$$\begin{aligned}\dot{H}_b + 2H_b^2 + 3H_aH_b \quad &= \quad -\frac{8\pi G}{4}\left\{-2\rho_{6D} + \left[1 - \left(\frac{\tau_1}{\tau_2}\right)^2\right] b\partial_b\rho_{6D} - 2\tau_1\partial_{\tau_1}\rho_{6D} + \frac{2\tau_1^2}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\} \\ &\quad + \frac{1}{4}\tilde{T}|_2 - \frac{8\pi G}{4}\rho_m,\end{aligned}$$

$$\begin{aligned}\ddot{\tau}_1 + \left(3H_a + 2H_b - 2\frac{\dot{\tau}_2}{\tau_2}\right)\dot{\tau}_1 \quad &= \quad -16\pi G\tau_2^2\left\{\frac{b\tau_1}{2\tau_2^2}\partial_b\rho_{6D} + 2\partial_{\tau_1}\rho_{6D} - \frac{\tau_1}{\tau_2}\partial_{\tau_2}\rho_{6D}\right\} \\ &\quad + 2\tilde{T}|_3,\end{aligned}$$

$$\begin{aligned}\frac{\ddot{\tau}_2}{\tau_2} + \frac{\dot{\tau}_1^2 - \dot{\tau}_2^2}{\tau_2^2} + 3H_a\frac{\dot{\tau}_2}{\tau_2} + 2H_b\frac{\dot{\tau}_2}{\tau_2} \quad &= \quad 8\pi G\{\frac{b\tau_1^2}{\tau_2^2}\partial_b\rho_{6D} + 2\tau_1\partial_{\tau_1}\rho_{6D} \\ &\quad -2\tau_2\left[1 + \left(\frac{\tau_1}{\tau_2}\right)^2\right]\partial_{\tau_2}\rho_{6D}\} + \tilde{T}|_4 .\end{aligned} \tag{4.10}$$

We use some of the initial conditions such as $v = 1 - 10, b = 0.12 - 0.15, \tau_1 = 0.12 - 0.8$, and $\tau_2 = 0.5 - 0.9$ to solve the Einstein's field equations numerically. The numerical results show that the moduli fields b, τ_1, and τ_2 for these range of the initial conditions do not stabilize.

Let us consider the Casimir dark energy model when $\tau_1 = 0$ and $\tau_2 = 1$. We choose the background solution of the space-like aether point along the extra dimensions, $\theta = \pi/4$

$$u^a = (0, 0, 0, 0, \frac{v}{\sqrt{2}b}, \frac{v}{\sqrt{2}b}). \tag{4.11}$$

Using this background solution, the energy-momentum tensor of the aether field can be written as

$$\begin{aligned} T^0_0|_u &= -\frac{v^2}{2}(\frac{\dot{b}}{b})^2, & (4.12)\\ T^i_j|_u &= \frac{v^2}{2}(\frac{\dot{b}}{b})^2\delta^i_j, & (4.13)\\ T^5_5|_u &= T^6_6|_u = -\frac{\ddot{b}}{b} + \frac{1}{2}(\frac{\dot{b}}{b})^2 - 3\frac{\dot{a}\dot{b}}{ab}. & (4.14) \end{aligned}$$

We assume the total energy-momentum tensor can be decomposed as

$$T^a_b|_{total} = T^a_b|_{Cas} + T^a_b|_{matter} + T^a_b|_u. \tag{4.15}$$

In this case, the 6-dimensional cosmological equation of motion become

$$\begin{aligned} 3(\frac{\dot{a}}{a})^2 + 6\frac{\dot{a}\dot{b}}{ab} + (\frac{\dot{b}}{b})^2 &= 8\pi G(\rho^{(6)}_{Cas} + \rho_{matter} + \frac{v^2}{2}(\frac{\dot{b}}{b})^2), & (4.16)\\ 4\frac{\ddot{a}}{a} + 8(\frac{\dot{a}}{a})^2 + 8\frac{\dot{a}\dot{b}}{ab} &= 8\pi G(2\rho^{(6)}_{Cas} + b\partial_b\rho^{(6)}_{Cas} + \rho_{matter} + v^2 A), & (4.17)\\ -4\frac{\ddot{b}}{b} - 4(\frac{\dot{b}}{b})^2 - 12\frac{\dot{a}\dot{b}}{ab} &= 8\pi G(-2\rho^{(6)}_{Cas} + b\partial_b\rho^{(6)}_{Cas} - \rho_{matter} + v^2 A), & (4.18) \end{aligned}$$

where $A = \frac{\ddot{b}}{b} + (\frac{\dot{b}}{b})^2 + 3\frac{\dot{a}\dot{b}}{ab}$. The Casimir potential is shown as a function of b in Figure 4.4. Here we choose $\lambda = 0.408$ and ignore the interaction between SM fields with aether by setting $\alpha = 0$.

There is an interesting result obtained by carefully tuning the aether's norm, $v = 0.5$ and $b_{in} = 0.139$, we can show that the size of the extra dimensions b can be stabilized even if there is non-relativistic matter in the universe. The numerical results and the cosmological dynamics are shown in Figure 4.5.

For $m_b = 0.408m_f$ the total Casimir energy density in 6-dimensional space-time $\rho^{(6)}_{Cas}$ with no shape moduli is given in Figure 4.4. At the minimum, $b_{min} \sim 0.142(\frac{5}{m_f})$ and the potential depend on the mass by $\rho^{(6)}_{Cas} \sim m_f^6$, we obtain the 4-dimensional effective Casimir energy:

$$\begin{aligned} \rho^{(4)}_{min} &= (2\pi b_{min})^2\rho^{(6)}_{min}, & \\ &= [2\pi \times 0.142(\frac{5}{m_f})]^2 \times 0.00566(\frac{m_f}{5})^6, & (4.19) \end{aligned}$$

where the numbers 0.142 is the size of the extra dimensions and 0.00566 is the energy density at the minimum of the numerical results respectively. Compare the 4-dimensional effective Casimir energy to the observed value of the dark energy, $\rho_{DE}^{(4)obs} \sim (2.3 \times 10^{-3} eV)^4$, we obtain

$$\begin{aligned} (2.3 \times 10^{-3} eV)^4 &= [2\pi \times 0.142(\frac{5}{m_f})]^2 \times 0.00566(\frac{m_f}{5})^6 \\ m_f &= 4.4 \times 10^{-2} eV. \end{aligned} \quad (4.20)$$

It is follows that $b_{min} \sim 0.142(\frac{5}{4.4\times 10^{-2} eV}) = \frac{16.14}{eV} \times \frac{1.97\times 10^{-7} m}{eV^{-1}} = 3.18 \mu m$. The ADD scenario shows that the 4-dimensional Planck mass is

$$M_{Pl}^2 = M_{Pl(4+n)}^{2+n} (2\pi b)^n. \quad (4.21)$$

For $n = 2$ and $b = b_{min} \sim \frac{16.14}{eV}$, the Planck mass in the bulk from the Casimir dark energy model can be calculated by

$$\begin{aligned} (1.2 \times 10^{28})^2 &= M_{Pl(6)}^4 (2\pi \times 16.14)^2, \\ M_{Pl(6)} &= 4.34\, TeV \sim m_{EW}. \end{aligned} \quad (4.22)$$

Therefore the size of the extra dimensions needed to fix the Casimir energy to the dark energy at the observed value is consistent with the ADD scenario to solve the hierarchy problem.

We should mention here that the time scale, x_1, of the simulated figures can be calculated from the following method. Let we define $t = x_1 t_{sim}$, $G^{(6)} = x_2 G_{sim}^{(6)}$, $\rho^{(6)} = x_3 \rho_{sim}^{(6)}$ and $b = x_4 b_{sim}$ where subscripts sim mean it is a simulated value. From the first Einstein's field equation

$$3H_a^2 + H_b^2 + 6H_a H_b - \frac{1}{4\tau_2^2}\left(\dot{\tau}_1^2 + \dot{\tau}_2^2\right) = 8\pi G \rho_{6D},$$

we obtain

$$\frac{1}{x_1^2} = x_2 x_3. \quad (4.23)$$

It is interesting that in the Casimir dark energy model, the constancy of the 4-dimensional gravitational constant depend on the size of the compactified extra dimension is given by

$$G^{(4)} = \frac{1}{M_{Pl(4)}^2} = \frac{G^{(6)}}{4\pi^2 b^2}. \quad (4.24)$$

For $G_{sim}^{(6)} = 10$ we obtain

$$x_2 = \frac{4\pi^2 b_{sim}^2 x_4^2}{10 M_{Pl(4)}^2}. \quad (4.25)$$

By comparing a 4-dimensional effective Casimir energy density $\rho^{(4)}_{DE} = (2\pi b_{min})^2 \rho^{(6)}_{min}$ with the observed value of the dark energy we get

$$x_3 = \frac{\rho^{(4)}_{DE}}{4\pi^2 b^2_{sim} \rho^{(6)}_{sim} x_4^2}, \tag{4.26}$$

substitute x_2 and x_3 into the Eq.(4.23), then

$$x_1 = M_{Pl(4)} \sqrt{\frac{10\rho^{(6)}_{sim}}{\rho^{(4)}_{DE}}} \sim 10^{10} \; years. \tag{4.27}$$

It is interesting that we can defined the Hubble time as $t_H = H_{ao}^{-1} = \sqrt{3M_{Pl}^2/8\pi\rho_c} \sim 10^{10}$ years and it is coincident with the expansion time scale in the simulation, $x_1 \sim t_H$.

Let us compare the stabilization time t_{stab} of the extra dimension with the age of the universe. The age of the universe in cosmology can be calculated from the formula

$$t_{age} = \frac{1}{H_{a0}} \int_0^1 \frac{dx}{x\sqrt{\Omega_{Cas} + \Omega_m x^3}} = \frac{1.5376}{H_{a0}} \sim 10^{10}, \tag{4.28}$$

we know that in the late-time expansion of the universe $\Omega_\Lambda = \Omega_{Cas} = 0.76$ and $\Omega_m = 0.24$. From the Figure 4.5 the stabilization time $t_{stab} \sim 15 t_H$ Then $t_{stab} \sim 9.75 t_{age}$ is greater than the age of the universe.

v	α_ϕ	θ	b_{in}	τ_{1in}	τ_{2in}	b_{stab}	τ_{1stab}	τ_{2stab}
2	0.04	0	0.133	0.5	0.867	0.133	0.498	0.866
		$\pi/4$	0.133	0.5	0.867	0.133	0.499	0.867
		$\pi/2$	0.133	0.5	0.867	0.133	0.498	0.868
4	0.08	0	0.133	0.5	0.866	0.133	0.498	0.864
		$\pi/4$	0.133	0.5	0.866	0.133	0.501	0.867
		$\pi/2$	0.133	0.5	0.866	0.133	0.498	0.868

Table IV.1: Parameters for the stabilize solution in vacuum-dominated universe.

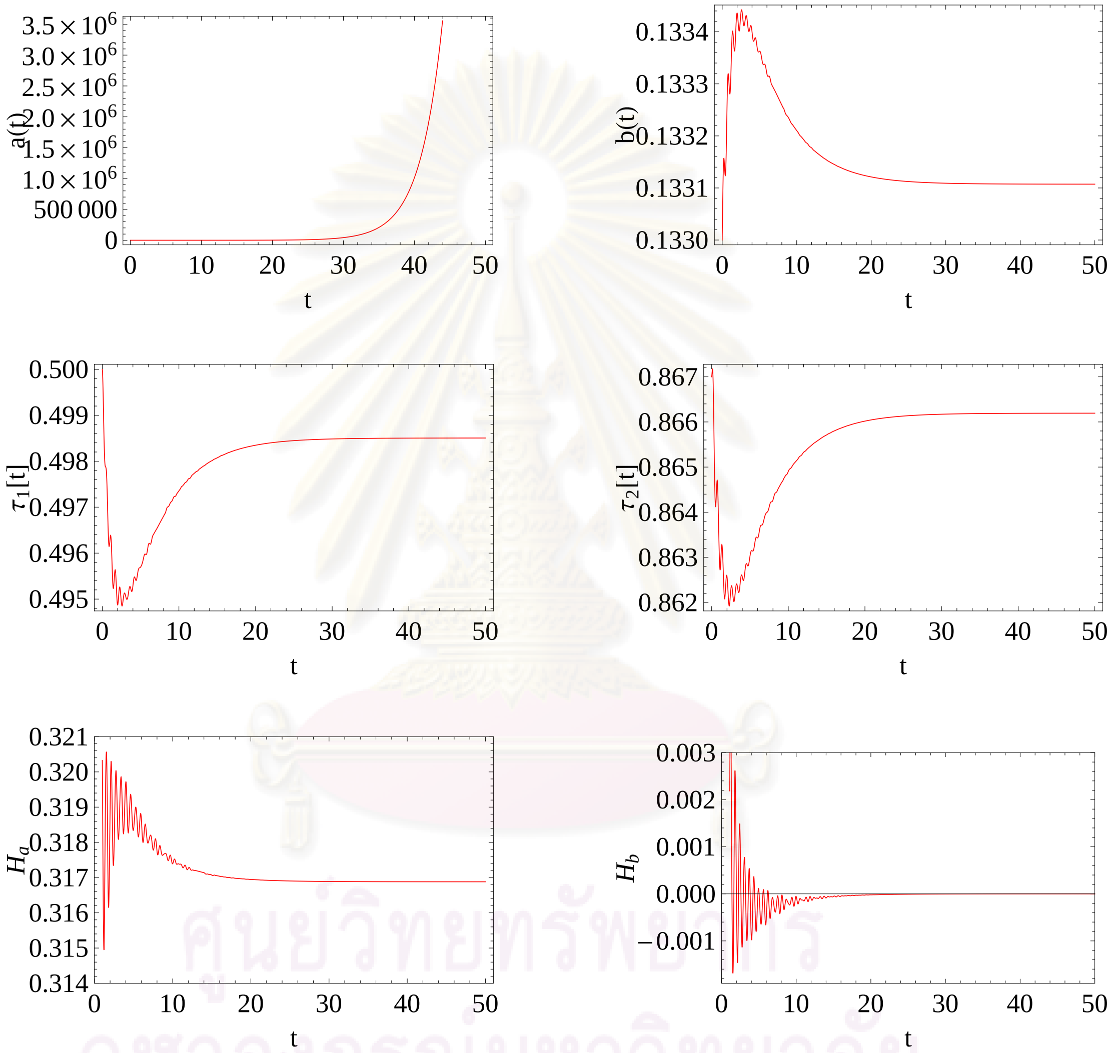

Figure 4.1: Cosmological dynamics in the vacuum dominated universe when $v = 2$, $\mu = 50$ and $\theta = 0$

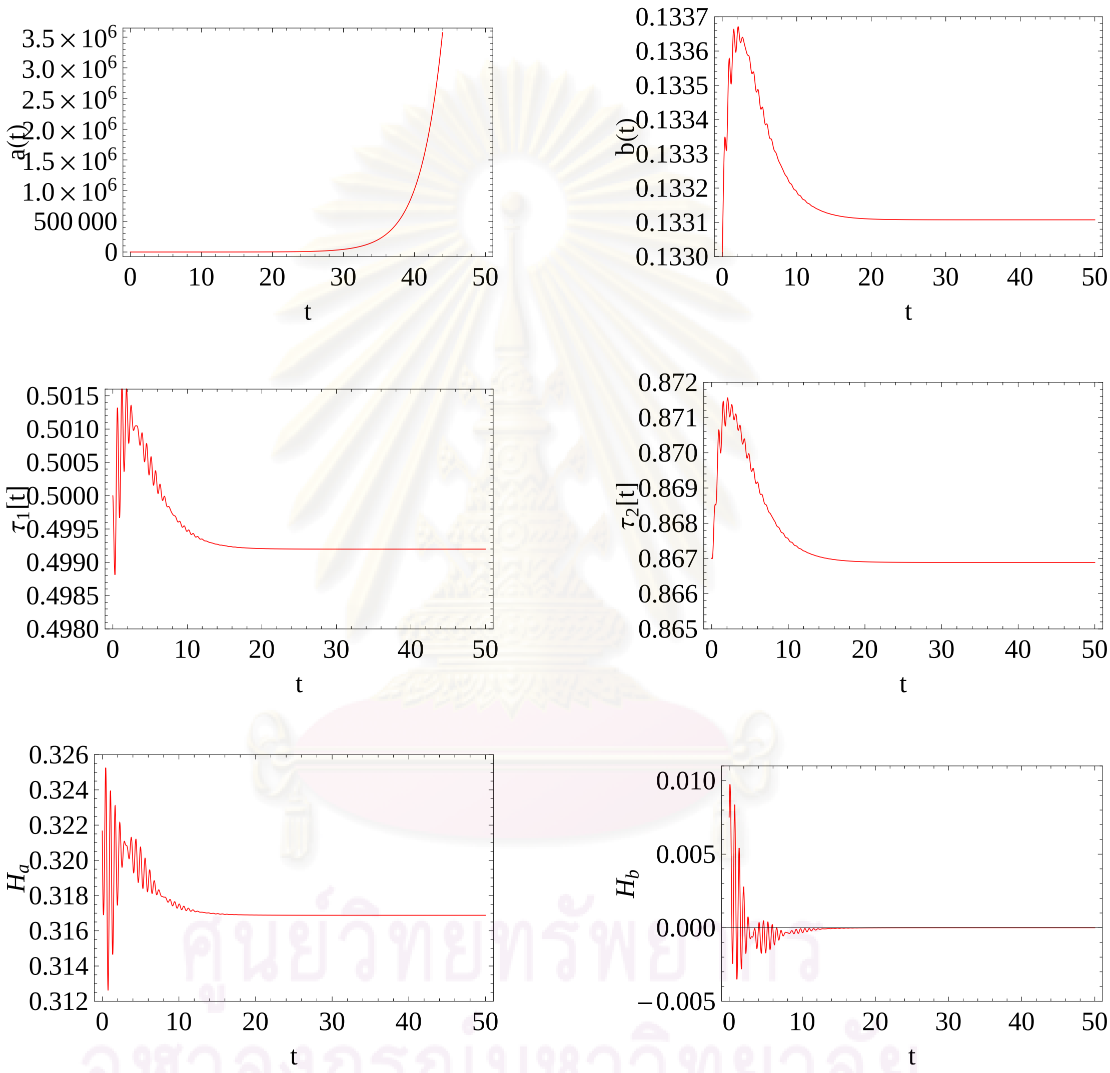

Figure 4.2: Cosmological dynamics in the vacuum dominated universe when $v = 2$, $\mu = 50$ and $\theta = \frac{\pi}{4}$

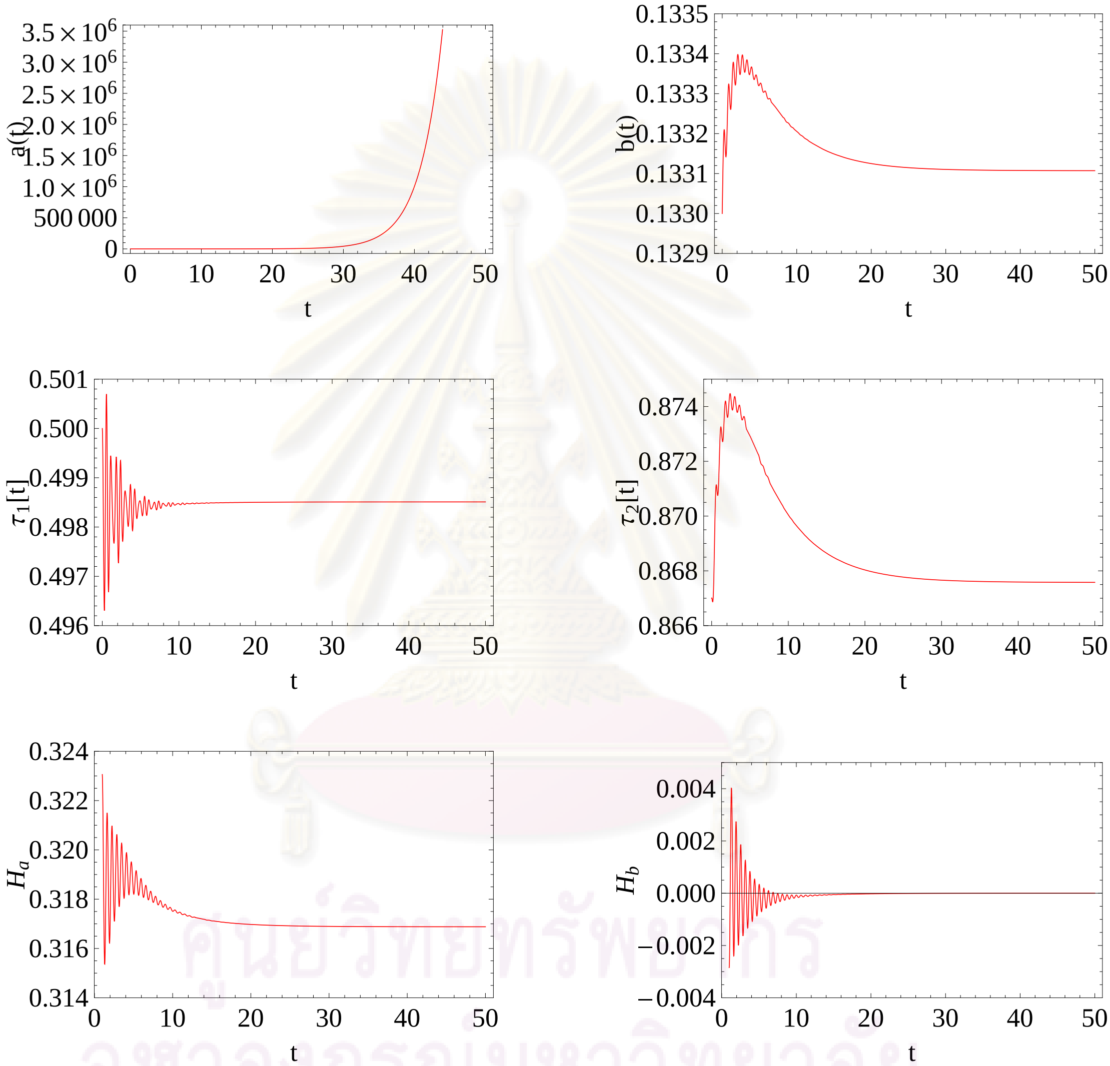

Figure 4.3: Cosmological dynamics in the vacuum dominated universe when $v = 2$, $\mu = 50$ and $\theta = \frac{\pi}{2}$

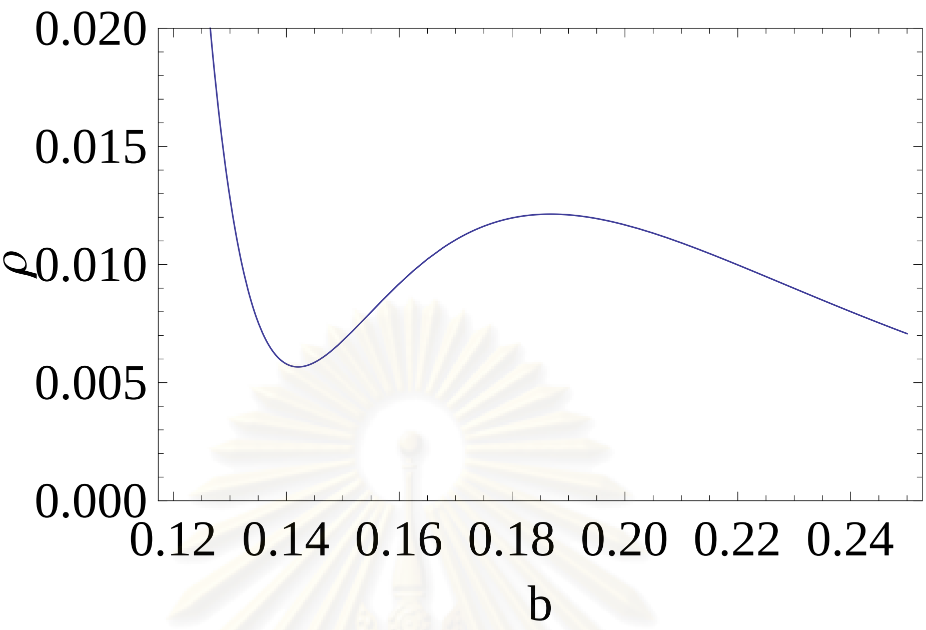

Figure 4.4: The total Casimir energy density in 6-dimensions for $M = 5, \lambda = 0.408$, $\tau_1 = 0$ and $\tau_2 = 1$.

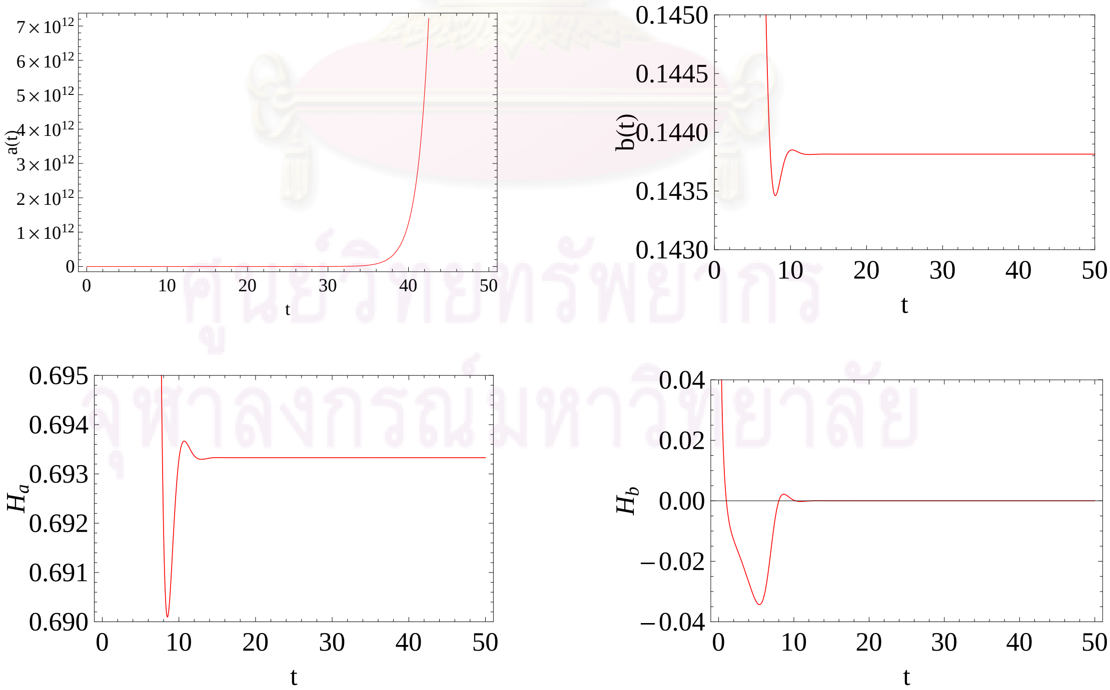

Figure 4.5: Cosmological dynamics of the matter-dominated universe when $v = 0.5$, $\tau_1 = 0$ and $\tau_2 = 1$

Chapter V

Conclusions

We have known that Casimir energies can act to stabilize the size of the extra dimensions while also giving a sort of vacuum energy in the accelerated expansion of the late-time universe [3]. The acceleration of the 4-dimensional universe occurs if the size of the compact dimensions is stabilize and the Casimir energy density becomes a constant at that stabilized value. As a result, the apparent cosmological constant that we can be observed in the 4-dimensional universe is effectively induced.

Shape moduli of the distorted torus can be added to the model [13]. The minimum of the Casimir energy density of the torus with shape moduli is located at $\tau_1 = \pm 1/2$, $\tau_2 = \sqrt{3}/2$. When the cosmological dynamics is initiate within a small perturbation of the saddle point, $\tau_1 = 0$, $\tau_2 = 1$, of the Casimir energy density. it will roll down to the true minimum of the total Casimir energy density at $\tau_1 = \pm 1/2$, $\tau_2 = \sqrt{3}/2$ even with minimal amount of perturbations.

In our model with the presence of a Lorentz violating vector aether field, we have shown that the aether field can reduce the oscillation frequency of the compact extra dimensions [4]. For vacuum dominated universe, the Casimir energy of the torus with shape moduli serves as a stabilizing potential for the moduli fields while also giving a sort of vacuum energy in the accelerated expansion of the 3-spatial directions. The aether field will also slow down the oscillation of the moduli fields.

For the universe dominated by non-relativistic matter with vector aether field pointing along the fifth direction ($\theta = 0$), the extra dimensions can not be stabilized for any sets of cosmological parameters. However, the extra dimensions can be stabilized in the universe which non-relativistic matter present, if we have not considered the shape moduli, $\tau_1 = 0$ and $\tau_2 = 1$.

In the Casimir dark energy model, there is a relationship between size of the compact extra dimensions and the observed 4-dimensional dark energy. This connection emerge because the Casimir energy depend only on the size of the compact extra dimension and we fix its minimum to the observed value of the cosmological constant. This leads to the size of the extra dimensions $b = 3.18 \mu m$ and 6-dimensional Planck mass $M_{Pl(6)} = 4.34 TeV \sim m_{EW}$. Therefore the size of the compact extra dimension from our numerical result is consistent with the ADD scenario to solve the hierarchy problem [17]. The Casimir dark energy show that the cosmological constant problem is naturally connect to the hierarchy problem.

The interesting theoretical idea have to guess that the early universe have all spatial dimensions are compactified. In the Casimir dark energy model show that the total Casimir energy of bosonic and fermionic fields in the bulk generate stabilized potential for the size of all compact directions. However, at the early time such as radiation or matter-dominated period the radiation and matter energy density will be dominated contribution to the total energy density. Therefore the Casimir energy can not be stabilized the size of the spatial dimensions and all directions will be expand. In the other hand, if the universe occur the spontaneous Lorentz symmetry breaking in some directions it will slow down the moduli fields associated with the broken directions. This broken directions will be stabilized by the Casimir energy while the other will be accelerated expand.

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APPENDICES

Appendix A

Regularization of the Two-Dimensional Inhomogeneous Zeta Function

The two-dimensional inhomogeneous zeta function series is defined by

$$F(s;a,b,c;q) = \sum_{m,n} \left(am^2 + bmn + cn^2 + q\right)^{-s}. \tag{A.1}$$

Starting with rewritten the $am^2 + bmn + cn^2 + q$ in the quadratic form plus the square of m

$$am^2 + bmn + cn^2 + q = c\left(n + \frac{bm}{2c}\right)^2 + \left(a - \frac{b^2}{4c}\right)m^2, \tag{A.2}$$

then

$$F(s;a,b,c;q) = \sum_{m,n} \left[c\left(n + \frac{bm}{2c}\right)^2 + \Delta m^2 + q\right]^{-s}, \tag{A.3}$$

where $\Delta = a - \frac{b^2}{4c}$. Use the Gamma function

$$F(s;a,b,c;q) = \frac{\int_0^\infty dx x^{s-1} e^{-x}}{\Gamma(s)} \sum_{m,n} \left[c\left(n + \frac{bm}{2c}\right)^2 + \Delta m^2 + q\right]^{-s}, \tag{A.4}$$

and change a variable $x = t\left[c\left(n + \frac{bm}{2c}\right)^2 + \Delta m^2 + q\right]$ so

$$\begin{aligned} F(s;a,b,c;q) &= \frac{1}{\Gamma(s)} \sum_{m,n} \int_0^\infty dt t^{s-1} exp\left\{-t\left[c\left(n + \frac{bm}{2c}\right)^2 + \Delta m^2 + q\right]\right\} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \sum_n exp\left[-ct\left(n + \frac{bm}{2c}\right)^2\right] \\ &\quad \times \sum_m exp\left(-\Delta m^2 t - qt\right). \end{aligned} \tag{A.5}$$

Consider the Fourier transform of the function $f(x) = exp\left[-ct\left(x - \frac{bm}{2c}\right)^2\right]$

$$\begin{aligned} F\left(\bar{k}\right) &= \int_{-\infty}^{\infty} dx e^{i\bar{k}x} exp\left(x - \frac{bm}{2c}\right)^2, \bar{k} = 2\pi k \\ &= \sqrt{\frac{\pi}{ct}} exp\left(-\frac{\bar{k}^2}{4ct} + \frac{i\bar{k}bm}{2c}\right). \end{aligned} \tag{A.6}$$

From the Poisson resummation formula,

$$\sum_n exp\left[-ct\left(x-\frac{bm}{2c}\right)^2\right]=\sqrt{\frac{\pi}{ct}}\sum_k exp\left(-\frac{\pi^2k^2}{ct}+i\frac{\pi bkm}{c}\right), \tag{A.7}$$

substitute it in Eqs.(A.5), we obtain

$$\begin{aligned} F\left(s;a,b,c;q\right) &= \frac{\sqrt{\pi}}{\sqrt{c}\,\Gamma\left(s\right)}\sum_{m,\ n=-\infty}^{\infty} e^{\frac{i\pi kmb}{c}}\int_0^\infty dtt^{s-1-\frac{1}{2}}e^{\{-\frac{\pi^2k^2}{ct}+\frac{i\pi bkm}{c}-t\left(\Delta m^2+q\right)\}} \\ &= \frac{\sqrt{\pi}}{\sqrt{c}\,\Gamma\left(s\right)}\{\sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\cos\left(\frac{\pi bkm}{c}\right)\int_0^\infty dtt^{s-1-\frac{1}{2}} \\ &\quad \times exp\left[-\frac{\pi^2k^2}{ct}+\frac{i\pi bkm}{c}-t\left(\Delta m^2+q\right)\right] \\ &\quad +\sum_{m=-\infty}^{\infty}\int_0^\infty dtt^{s-1-\frac{1}{2}}exp\left(-t\Delta m^2+q\right)\}. \end{aligned} \tag{A.8}$$

Change the variable $t'=t\left(\Delta m^2+q\right)$ and use the integral representation of the modified Bessel function of the second kind

$$K_\nu\left(z\right)=\frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty t^{-\nu-1}e^{-t-\frac{z^2}{4t}},$$

and the Epstein-Hurwitz zeta function

$$\begin{aligned} \zeta_{EH}\left(s;q\right) &= \frac{1}{2}\sum_{n\in z}\left(n^2+q\right)^{-s} \\ &= -\frac{q^{-s}}{2}+\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma\left(s\right)}q^{-s+\frac{1}{2}}+\sum_{n=1}^{\infty}\frac{2\pi^s q^{-\frac{s}{2}+\frac{1}{4}}}{\Gamma\left(s\right)}n^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi n\sqrt{q}\right), \end{aligned}$$

where the prime at the first sum indicates that the term $n=0$ is excluded, the above equation becomes

$$\begin{aligned} F\left(s;a,b,c;q\right) &= \frac{\sqrt{\pi}}{\sqrt{c}\,\Gamma\left(s\right)}\{\frac{4\pi^{s-\frac{1}{2}}}{c^{\frac{s}{2}-\frac{1}{4}}}\Delta^{-\frac{s}{2}+\frac{1}{4}}\sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{\pi kmb}{c}\right)}{\left(m^2+\frac{q}{\Delta}\right)^{\frac{s}{2}-\frac{1}{4}}}k^{s-\frac{1}{2}} \\ &\quad K_{s-\frac{1}{2}}\left(2\pi k\sqrt{\frac{\Delta}{c}m^2+\frac{q}{c}}\right)+\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Delta^{s-\frac{1}{2}}}\sum_{m=-\infty}^{\infty}\left(m^2+\frac{q}{\Delta}\right)^{-\left(s-\frac{1}{2}\right)}\} \\ &= \frac{\sqrt{\pi}}{\sqrt{c}\,\Gamma\left(s\right)}\{\frac{4\pi^{s-\frac{1}{2}}}{c^{\frac{s}{2}-\frac{1}{4}}}\Delta^{-\frac{s}{2}+\frac{1}{4}}\sum_{k=1}^{\infty}q^{-\frac{s}{2}+\frac{1}{4}}k^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi k\sqrt{\frac{q}{c}}\right) \\ &\quad +\sum_{m,k=1}^{\infty}\frac{8\pi^{s-\frac{1}{2}}}{c^{\frac{s}{2}-\frac{1}{4}}}\Delta^{-\frac{s}{2}+\frac{1}{4}}\frac{\cos\left(\frac{\pi bkm}{c}\right)}{\left(m^2+\frac{q}{\Delta}\right)^{\frac{s}{2}-\frac{1}{4}}}k^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi k\sqrt{\Delta\frac{m^2}{c}+\frac{q}{c}}\right) \\ &\quad +\Gamma(s-\frac{1}{2})q^{-s+\frac{1}{2}}+\frac{2\Gamma(s-\frac{1}{2})}{\Delta^{s-\frac{1}{2}}}\zeta_{EH}(s-\frac{1}{2},\frac{q}{\Delta})\}, \end{aligned}$$

rewrite again into the following form

$$\begin{aligned} F\left(s;a,b,c;q\right) &= \frac{\sqrt{\pi}}{\sqrt{c}\;\Gamma\left(s\right)}\{\Gamma(s-\frac{1}{2})q^{-s+\frac{1}{2}}+4\pi^{s-\frac{1}{2}}\sum_{k=1}^{\infty}\frac{q^{-\frac{s}{2}+\frac{1}{4}}}{c^{\frac{s}{2}-\frac{1}{4}}}k^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi k\sqrt{\frac{q}{c}}) \\ &+\frac{2\Gamma(s-\frac{1}{2})}{\Delta^{s-\frac{1}{2}}}\zeta_{EH}(s-\frac{1}{2},\frac{q}{\Delta})+\sum_{m,k=1}^{\infty}\frac{8\pi^{s-\frac{1}{2}}}{c^{\frac{s}{2}-\frac{1}{4}}}\Delta^{-\frac{s}{2}+\frac{1}{4}}\frac{\cos\left(\frac{\pi bkm}{c}\right)}{\left(m^2+\frac{q}{\Delta}\right)^{\frac{s}{2}-\frac{1}{4}}}k^{s-\frac{1}{2}} \\ &K_{s-\frac{1}{2}}\left(2\pi k\sqrt{\Delta\frac{m^2}{c}+\frac{q}{c}}\right)\}. \end{aligned}$$

Finally use the definition of Epstein-Hurwitz zeta function we obtain

$$\begin{aligned} F\left(s;a,b,c;q\right) &= q^{-s}+2c^{-s}\zeta_{EH}(s,\frac{q}{c})+\frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\sqrt{c}\Gamma(s)\Delta^{s-\frac{1}{2}}}\zeta_{EH}(s-\frac{1}{2},\frac{q}{\Delta}) \\ &+\sum_{m,k=1}^{\infty}\frac{8\pi^{s-\frac{1}{2}}}{c^{\frac{s}{2}-\frac{1}{4}}}\Delta^{-\frac{s}{2}+\frac{1}{4}}\frac{\cos\left(\frac{\pi bkm}{c}\right)}{\left(m^2+\frac{q}{\Delta}\right)^{\frac{s}{2}-\frac{1}{4}}}k^{s-\frac{1}{2}} \\ &K_{s-\frac{1}{2}}\left(2\pi k\sqrt{\Delta\frac{m^2}{c}+\frac{q}{c}}\right). \end{aligned} \tag{A.9}$$

Appendix B

The Casimir Effect

An experimentally verification of quantum vacuum is the Casimir effect. This effect is manifest as an attractive force between two parallel uncharged conducting plates. In fact, a realistic description requires quantization of the electromagnetic field in the presence of conducting plates. This section describes the calculation of the Casimir energy [41]. To simplify the calculations, we consider the simple case of a scalar field ϕ with mass m in d-dimensional space between two plates at $x = 0$ and $x = a$, introduce the boundary conditions

$$\phi(x)|_{x=0} = \phi(x)|_{x=a} = 0. \tag{B.1}$$

The scalar field satisfies the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0. \tag{B.2}$$

Under the above boundary conditions the modes of the field become

$$\phi(x, x_T, t) = \sin(\frac{n\pi x}{a})e^{ik_T x_T}e^{-i\omega_k t}, \tag{B.3}$$

where

$$\omega_k = \sqrt{(\frac{n\pi}{a})^2 + k_T^2 + m^2}, \tag{B.4}$$

and n is positive integer. The zero-point energy of the field between the plates is then

$$E = (\frac{L}{2\pi})^{d-1} \int d^{d-1}k_T \sum_{n=1}^{\infty} \frac{1}{2}\omega_k. \tag{B.5}$$

Using polar coordinate

$$\int d^d k f(k) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int dk k^{d-1} f(k), \tag{B.6}$$

substitute into Eqs.(B.5) we obtain

$$E = (\frac{L}{2\pi})^{d-1} \frac{2\pi^{d-1/2}}{\Gamma(\frac{d-1}{2})} \sum_{n=1}^{\infty} \int_0^{\infty} dk_T k_T^{d-2} \frac{1}{2} \sqrt{(\frac{n\pi}{a})^2 + k_T^2 + m^2}. \tag{B.7}$$

Define $u = k_T^2$ so

$$dk_T k_T^{d-2} = \frac{1}{2}(k_T^2)^{d-3/2} dk_T^2, \tag{B.8}$$

then

$$E = (\frac{L}{2\pi})^{d-1} \frac{2\pi^{d-1/2}}{\Gamma(\frac{d-1}{2})} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{2}(k_T^2)^{d-3/2} dk_T^2 \frac{1}{2} \sqrt{(\frac{n\pi}{a})^2 + k_T^2 + m^2}. \tag{B.9}$$

Using the Beta function relations

$$B(1+r, -s-r-1) = \int_0^{\infty} t^r (1+r)^s dt = \frac{\Gamma(r+1)\Gamma(-s-r-1)}{\Gamma(-s)}, \tag{B.10}$$

and imposing $t = \frac{k_T^2}{(\frac{n\pi}{a})^2 + m^2}$, so

$$E = (\frac{L}{2\pi})^{d-1} \frac{2\pi^{d-1/2}}{\Gamma(\frac{d-1}{2})} \ \frac{1}{2} \sum_{n=1}^{\infty} [(\frac{n\pi}{a})^2 + m^2]^{d/2} \int_0^{\infty} t^{d-3/2}(1+t)^{1/2} dt. \tag{B.11}$$

since $r = \frac{d-3}{2}$ and $s = \frac{1}{2}$ then yields

$$E = \frac{1}{2} \ \frac{\Gamma(-\frac{d}{2})}{\Gamma(-\frac{1}{2})} \ \pi^{d+1/2} \ (\frac{L}{2})^{d-1} \ \frac{1}{a^d} \ \sum_{n=1}^{\infty} [(\frac{am}{\pi})^2 + n^2]^{d/2}. \tag{B.12}$$

For the massless scalar field $m = 0$ the summation is

$$\sum_{n=1}^{\infty} n^d = \zeta(-d), \tag{B.13}$$

where $\zeta(s)$ is the usual Riemann zeta function. Using the reflection formula

$$\Gamma(\frac{s}{2})]\pi^{-s/2}\zeta(s) = \Gamma(\frac{1-s}{2})]\pi^{s-1/2}\zeta(1-s), \tag{B.14}$$

and change $s \to -d$ so

$$\zeta(-d) = \frac{\Gamma(\frac{d+1}{2})\pi^{-d-1/2}\zeta(d+1)}{\Gamma(-\frac{d}{2})}. \tag{B.15}$$

The total energy is

$$E = -\frac{L^{d-1}}{a^d}\Gamma(\frac{d+1}{2})(4\pi)^{-(d+1)/2}\zeta(d+1). \tag{B.16}$$

The force per unit area on the two plates is defined as

$$P = -\frac{\partial}{\partial a}(\frac{E}{L^{d-1}}) = -\frac{d}{a^{d+1}}\Gamma(\frac{d+1}{2})(4\pi)^{-(d+1)/2}\zeta(d+1). \tag{B.17}$$

In our 3-dimensional space $d = 3$, we obtain

$$P = -\frac{3}{a^4}\Gamma(2)(4\pi)^{-2}\zeta(4), \tag{B.18}$$

since $\zeta(4) = \frac{\pi^4}{90}$ therefore

$$P = -\frac{\pi^2}{480a^4}. \tag{B.19}$$

Vitae

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