CHAPTER IV

HYPERRINGS WHOSE BI-HYPERIDEALS AND QUASI-HYPERIDEALS COINCIDE

In this chapter, we shall study hyperrings whose bi-hyperideals and quasi-hyperideals coincide in order to generalize Proposition 1.12 to Proposition 1.14.

We know that quasi-hyperideals are bi-hyperideals. Example 1.2 shows that a bi-ideal of a ring need not be a quasi-ideal. The following example shows that in a hyperring which is not a ring, the quasi-hyperideals and the bi-hyperideals need not coincide.

Example 4.1. Consider the ring $(SU_n(F), +, \cdot)$ and the hyperring $(SU_n(F)/\rho, \oplus, \circ)$ as in Example 1.2 and Theorem 2.18, respectively. Let

$$B = \left\{ \begin{bmatrix} 0 & \dots & 0 & x & 0 \\ 0 & \dots & 0 & 0 & y \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \middle| \; x,\ y \in F \right\} \text{ and } B' = \{C\rho \mid C \in B\}.$$

From Example 1.2, we have B is a subring of $(SU_n(F), +, \cdot)$. By Lemma 2.20, $B' = \{C\rho \mid C \in B\}$ is a subhyperring of $(SU_n(F)/\rho, \oplus, \circ)$. Since $BSU_n(F)B = \{0\}$, it follows that for all $C, D \in B$ and $E \in SU_n(F)$, $(C\rho)\circ(E\rho)\circ(D\rho) = (CED)\rho = 0\rho$. Then we deduce that $< B'(SU_n(F)/\rho)B' >= \{0\rho\}$. Hence B' is a bi-hyperideal of $(SU_n(F)/\rho, \oplus, \circ)$. From Example 1.2, B is not a quasi-ideal of $(SU_n(F), +, \cdot)$ and hence B' is not a quasi-hyperideal of $(SU_n(F)/\rho, \oplus, \circ)$ by Theorem 2.21.

The next theorem gives a generalization of Proposition 1.12.

Theorem 4.2. *Let A be a hyperring and B a bi-hyperideal of A. If every element of B is regular in A, then B is a quasi-hyperideal of A.*

Proof. Assume that every element of B is regular in A. We shall show that B is a quasi-hyperideal of A, that is, $< AB > \cap < BA > \subseteq B$. Let $x \in < AB > \cap < BA >$. Then

$$x \in \sum_{i=1}^{n} b_i a_i \tag{1}$$

for some $b_i \in B$, $a_i \in A$ and $x \in \ < AB >$. We proceed the proof inductively. By regularity of B in A, $b_1 = b_1 t_1 b_1$ for some $t_1 \in A$. By (1), Proposition 1.19 and reversibility of $(A, +)$, we have that $b_1 a_1 \in x - b_2 a_2 - b_3 a_3 - \cdots - b_n a_n$. Thus

$$\begin{aligned} b_1 a_1 = b_1 t_1 b_1 a_1 \ &\in b_1 t_1 (x - b_2 a_2 - b_3 a_3 - \cdots - b_n a_n) \\ &= b_1 t_1 x - b_1 t_1 b_2 a_2 - \cdots - b_1 t_1 b_n a_n \\ &= b_1^{(1)} - b_1 t_1 b_2 a_2 - \cdots - b_1 t_1 b_n a_n \end{aligned}$$

where $b_1^{(1)} = b_1 t_1 x$. By Proposition 1.22(9), $b_1^{(1)} \in BA < AB > \subseteq < BAB > \subseteq B$ since B is a bi-hyperideal of A. From (1), we have

$$\begin{aligned} x \ &\in b_1 a_1 + b_2 a_2 + \cdots + b_n a_n \\ &\subseteq (b_1^{(1)} - b_1 t_1 b_2 a_2 - \cdots - b_1 t_1 b_n a_n) + b_2 a_2 + \cdots + b_n a_n \\ &= b_1^{(1)} + (-b_1 t_1 b_2 + b_2) a_2 + \cdots + (-b_1 t_1 b_n + b_n) a_n. \end{aligned}$$

Since for each $i = 1, 2, \ldots, n$, $-b_1 t_1 b_i \in BAB \subseteq B$, we have that $-b_1 t_1 b_i + b_i \subseteq B$ for all $i \in \{1, 2, \ldots, n\}$. Thus

$$x \in b_1^{(1)} + b_2^{(1)} a_2 + \cdots + b_n^{(1)} a_n \tag{2}$$

for some $b_i^{(1)} \in -b_1 t_1 b_i + b_i$ where $i = 2, 3, \ldots, n$. By the regularity of B in A, $b_2^{(1)} = b_2^{(1)} t_2 b_2^{(1)}$ for some $t_2 \in A$. From (2), Proposition 1.19 and reversibility of

$(A,+)$, we have that $b_2^{(1)}a_2 \in x - b_1^{(1)} - b_3^{(1)}a_3 - \cdots - b_n^{(1)}a_n$. Thus

$$\begin{aligned} b_2^{(1)}a_2 = b_2^{(1)}t_2b_2^{(1)}a_2 &\in b_2^{(1)}t_2(x - b_1^{(1)} - b_3^{(1)}a_3 - \cdots - b_n^{(1)}a_n) \\ &= b_2^{(1)}t_2x - b_2^{(1)}t_2b_1^{(1)} - b_2^{(1)}t_2b_3^{(1)}a_3 - \cdots - b_2^{(1)}t_2b_n^{(1)}a_n. \end{aligned}$$

Since $b_2^{(1)}t_2x \in BA < AB > \subseteq < BAB > \subseteq B$ and $b_2^{(1)}t_2b_1^{(1)} \in BAB \subseteq B$, we have $b_2^{(1)}t_2x - b_2^{(1)}t_2b_1^{(1)} \subseteq B$. Then $b_2^{(1)}a_2 \in b_2^{(2)} - b_2^{(1)}t_2b_3^{(1)}a_3 - \cdots - b_2^{(1)}t_2b_n^{(1)}a_n$ for some $b_2^{(2)} \in b_2^{(1)}t_2x - b_2^{(1)}t_2b_1^{(1)} \subseteq B$. It then follows from (2) that,

$$\begin{aligned} x &\in b_1^{(1)} + b_2^{(1)}a_2 + \cdots + b_n^{(1)}a_n \\ &\subseteq b_1^{(1)} + \left(b_2^{(2)} - b_2^{(1)}t_2b_3^{(1)}a_3 - \cdots - b_2^{(1)}t_2b_n^{(1)}a_n\right) \\ &\quad + b_3^{(1)}a_3 + \cdots + b_n^{(1)}a_n \\ &= b_1^{(1)} + b_2^{(2)} + \left(-b_2^{(1)}t_2b_3^{(1)} + b_3^{(1)}\right)a_3 \\ &\quad + \cdots + \left(-b_2^{(1)}t_2b_n^{(1)} + b_n^{(1)}\right)a_n. \end{aligned}$$

Thus $x \in b_1^{(1)} + b_2^{(2)} + b_3^{(2)}a_3 + \cdots + b_n^{(2)}a_n$ for some $b_i^{(2)} \in -b_2^{(1)}t_2b_i^{(1)} + b_i^{(1)} \subseteq BAB + B \subseteq B + B \subseteq B$ where $i = 3, 4, \ldots, n$.

We continue in the same above argument. Finally, we obtain $x \in b_1^{(1)} + b_2^{(2)} + \cdots + b_n^{(n)}$ for some $b_i^{(i)} \in B$ where $i = 1, 2, \ldots, n$. Since $(B,+)$ is a canonical hypergroup, $x \in B$. Thus $< AB > \cap < BA > \subseteq B$. Therefore B is a quasi-hyperideal of A. □

Clearly, Proposition 1.12 becomes a special case of Theorem 4.2.

Corollary 4.3. *Let B be a bi-ideal of a ring A. If every element of B is regular in A, then B is a quasi-ideal of A.*

To be convenient, let us call a hyperring A a *BQ-hyperring* if its bi-hyperideals and quasi-hyperideals coincide, that is, every bi-hyperideal of A is a quasi-hyperideal. Then from Theorem 4.2 we have

Theorem 4.4. *Every regular hyperring is a BQ-hyperring.*

Proposition 1.13 can be considered as a corollary of Theorem 4.4 as follows:

Corollary 4.5. *Every regular ring is a BQ-ring.*

Next, a necessary and sufficient condition for a hyperring to be a BQ-hyperring is given.

Theorem 4.6. *Let A be a hyperring. Then A is a BQ-hyperring if and only if for any finite subset X of A, $(X)_b = (X)_q$.*

Proof. We know in general that $(X)_b \subseteq (X)_q$ for every subset X of A(page 23). To prove the theorem, it suffices to prove that A is a BQ-hyperring if and only if for every finite subset X of A, $(X)_b$ is a quasi-hyperideal of A.

Assume that A is a BQ-hyperring. Since every bi-hyperideal of A is a quasi-hyperideal of A, $(X)_b$ is a quasi-hyperideal of A for every finite subset X of A.

Conversely, suppose that $(X)_b$ is a quasi-hyperideal of A for every finite subset X of A. Let B be a bi-hyperideal of A. Claim that $< AB > \cap < BA > \subseteq B$. Let $x \in < AB > \cap < BA >$. Then $x \in \sum_{i=1}^{n} a_i b_i$ and $x \in \sum_{j=1}^{m} b'_j a'_j$ for some $a_i, a'_j \in A$ and $b_i, b'_j \in B$. Set $X = \{b_1, b_2, \ldots, b_n, b'_1, b'_2, \ldots, b'_m\}$. Then $(X)_b \subseteq B$ and by the assumption, $(X)_b$ is a quasi-hyperideal of A. Then $< A(X)_b > \cap < (X)_b A > \subseteq (X)_b$. Since $x \in \sum_{i=1}^{n} a_i b_i \subseteq < A(X)_b >$ and $x \in \sum_{j=1}^{m} b'_j a'_j \subseteq < (X)_b A >$, we have $x \in < A(X)_b > \cap < (X)_b A > \subseteq (X)_b \subseteq B$. Hence B is a quasi-hyperideal of A. □

The following corollary is Proposition 1.14.

Corollary 4.7. *A ring A is a BQ-ring if and only if for every finite subset X of A, $(X)_b = (X)_q$.*