CHAPTER III

HYPERRINGS HAVING THE INTERSECTION PROPERTY OF QUASI-HYPERIDEALS

The purpose of this chapter is to generalize Proposition 1.8 to Proposition 1.11 by characterizing when quasi-hyperideals in hyperrings have the intersection property and when hyperrings have the intersection property of quasi-hyperideals.

The first theorem of this chapter is to generalize Proposition 1.8. We first give a lemma which follows directly from Proposition 1.29 and Lemma 2.3(iv).

Lemma 3.1. *If S is a subhyperring of a hyperring A, then $S+ < AS >$ and $S+ < SA >$ are respectively a left hyperideal and a right hyperideal of A containing S.*

Theorem 3.2. *If Q is a quasi-hyperideal of a hyperring A such that $Q \subseteq < AQ >$ or $Q \subseteq < QA >$ then*

$$Q = (Q + < AQ >) \cap (Q + < QA >).$$

In this case, Q has the intersection property.

Proof. Suppose that Q is a quasi-hyperideal of a hyperring A such that $Q \subseteq < AQ >$ or $Q \subseteq < QA >$. Let $D = (Q + < AQ >) \cap (Q + < QA >)$. The inclusion $Q \subseteq D$ is evident. Now assume that $Q \subseteq < AQ >$. Then $Q + < AQ > = < AQ >$, so $D = < AQ > \cap (Q + < QA >)$. Let $d \in D$. Then $d \in < AQ >$ and $d \in k + c$ for some $k \in Q$ and $c \in < QA >$. Since $(A, +)$ is reversible, $c \in -k + d \subseteq Q + < AQ >$. Since $Q \subseteq < AQ >$, then c belongs to both

$< AQ >$ and $< QA >$, whence $c \in < AQ > \cap < QA > \subseteq Q$ since Q is a quasi-hyperideal of A. Then $d \in k + c \subseteq Q$. Therefore $D \subseteq Q$. Hence $D = Q$, as required. If $Q \subseteq < QA >$, then we obtain $D = Q$ similarly. We conclude that $Q = (Q + < AQ >) \cap (Q + < QA >)$. By Lemma 3.1, Q has the intersection property. □

Proposition 1.8 becomes a corollary of Theorem 3.2.

Corollary 3.3. *Let Q be a quasi-ideal of a ring A. If $Q \subseteq QA$ or $Q \subseteq AQ$, then*

$$Q = (Q + AQ) \cap (Q + QA).$$

In this case, Q has the intersection property.

The following theorem gives some equivalent conditions for a quasi-hyperideal of a hyperring to have the intersection property.

Theorem 3.4. *Let Q be a quasi-hyperideal of a hyperring A. Then the following statements are equivalent.*

(i) *Q has the intersection property.*

(ii) $(Q + < AQ >) \cap (Q + < QA >) = Q$.

(iii) $< AQ > \cap (Q + < QA >) \subseteq Q$.

(iv) $< QA > \cap (Q + < AQ >) \subseteq Q$.

Proof. (i)⇒(ii). Since Q has the intersection property, there exist a left hyperideal L and a right hyperideal R of A such that $Q = L \cap R$. Then $Q \subseteq L$ and $Q \subseteq R$ and so $< AQ > \subseteq < AL > \subseteq L$ and $< QA > \subseteq < RA > \subseteq R$. Thus $Q + < AQ > \subseteq L$ and $Q + < QA > \subseteq R$. Consequently, $(Q + < AQ >) \cap (Q + < QA >) \subseteq L \cap R = Q$. But $Q \subseteq (Q + < AQ >) \cap (Q + < QA >)$, so (ii) holds.

(ii)⇒(i). This follows from Lemma 3.1.

(ii)⇒(iii). It is obvious since $< AQ > \subseteq Q + < AQ >$.

(iii)$\Rightarrow$(ii). Assume that $< AQ > \cap (Q + < QA >) \subseteq Q$. We have that $Q \subseteq (Q + < AQ >) \cap (Q + < QA >)$. To prove the reverse inclusion, let $x \in (Q + < AQ >) \cap (Q + < QA >)$. Then $x \in t + c$ and $x \in s + d$ for some $s, t \in Q$, $c \in < AQ >$ and $d \in < QA >$. Since $(A, +)$ is reversible, $c \in x - t \subseteq s + d - t = (s - t) + d \subseteq Q + < QA >$. Now, we have that $c \in < AQ > \cap (Q + < QA >)$, so $c \in Q$. This implies that $x \in t + c \subseteq Q$. Hence (ii) holds.

Similarly, we can prove that (ii)$\Leftrightarrow$(iv). □

The following theorem strengthens the result in Theorem 3.4.

Theorem 3.5. *Let X be a nonempty subset of a hyperring A. Then the following statements are equivalent.*

(i) *$(X)_q$ has the intersection property.*

(ii) $(\mathbb{Z}X + < AX >) \cap (\mathbb{Z}X + < XA >) = (X)_q$.

(iii) $< AX > \cap (\mathbb{Z}X + < XA >) \subseteq (X)_q$.

(iv) $< XA > \cap (\mathbb{Z}X + < AX >) \subseteq (X)_q$.

Proof. (i)$\Rightarrow$(ii). Since $(X)_q$ has the intersection property, there exist a left hyperideal L and a right hyperideal R of A such that $(X)_q = L \cap R$. Then $X \subseteq L$ and $X \subseteq R$. This implies that $\mathbb{Z}X \subseteq L$, $\mathbb{Z}X \subseteq R$, $< AX > \subseteq < AL > \subseteq L$ and $< XA > \subseteq < RA > \subseteq R$. Hence $\mathbb{Z}X + < AX > \subseteq L$ and $\mathbb{Z}X + < XA > \subseteq R$. Therefore $(\mathbb{Z}X + < AX >) \cap (\mathbb{Z}X + < XA >) \subseteq L \cap R = (X)_q$. By Theorem 2.9, $(X)_q = \mathbb{Z}X + (< AX > \cap < XA >)$, so $(X)_q = \mathbb{Z}X + (< AX > \cap < XA >) \subseteq (\mathbb{Z}X + < AX >) \cap (\mathbb{Z}X + < XA >)$. This proves that (ii) holds.

(ii)$\Rightarrow$(i). It is true because of Lemma 2.3(iv).

(ii)$\Rightarrow$(iii). This implication is clear.

(iii)$\Rightarrow$(ii). Assume that $< AX > \cap (\mathbb{Z}X + < XA >) \subseteq (X)_q$. By Theorem 2.9, $(X)_q = \mathbb{Z}X + (< AX > \cap < XA >)$. It then follows that $(X)_q \subseteq (\mathbb{Z}X +$

$< AX >) \cap (\mathbb{Z}X + < XA >)$. To show that $(\mathbb{Z}X + < AX >) \cap (\mathbb{Z}X + < XA >) \subseteq (X)_q$, let $t \in (\mathbb{Z}X + < AX >) \cap (\mathbb{Z}X + < XA >)$. Then $t \in t_1 + t_2$ and $t \in q_1 + q_2$ for some $t_1, q_1 \in \mathbb{Z}X$, $t_2 \in < AX >$ and $q_2 \in < XA >$. Since $(A, +)$ is reversible, $t_2 \in -t_1 + t \subseteq -t_1 + (q_1 + q_2) = (-t_1 + q_1) + q_2 \subseteq \mathbb{Z}X + < XA >$. Hence $t_2 \in < AX > \cap (\mathbb{Z}X + < XA >)$. By the assumption, $t_2 \in (X)_q$. This implies $t \in t_1 + t_2 \subseteq \mathbb{Z}X + (X)_q = (X)_q$. Therefore (ii) holds.

We obtain (ii)⇔(iv) similarly. □

Proposition 1.9 and Proposition 1.10 become special cases of Theorem 3.4 and Theorem 3.5, respectively.

Corollary 3.6. *Let Q be a quasi-ideal of a ring A. Then the following statements are equivalent.*

(i) *Q has the intersection property.*

(ii) $(Q + AQ) \cap (Q + QA) = Q$.

(iii) $AQ \cap (Q + QA) \subseteq Q$.

(iv) $QA \cap (Q + AQ) \subseteq Q$.

Corollary 3.7. *Let X be a nonempty subset of a ring A. Then the following statements are equivalent.*

(i) *$(X)_q$ has the intersection property.*

(ii) $(\mathbb{Z}X + AX) \cap (\mathbb{Z}X + XA) = (X)_q$.

(iii) $AX \cap (\mathbb{Z}X + XA) \subseteq (X)_q$.

(iv) $XA \cap (\mathbb{Z}X + AX) \subseteq (X)_q$.

The following theorem gives some equivalent conditions for a hyperring to have the intersection property of quasi-hyperideals.

Theorem 3.8. *Let A be a hyperring. Then the following statements are equivalent.*

(i) *A has the intersection property of quasi-hyperideals.*

(ii) *For any finite nonempty subset X of A,*

$$< AX > \cap (\mathbb{Z}X + < XA >) \subseteq \mathbb{Z}X + (< AX > \cap < XA >)(= (X)_q).$$

(iii) *For a finite subset $X = \{x_1, x_2, \ldots, x_n\}$ of A and $a_1, a_2, \ldots, a_n \in A$, if*

$$y \in (\sum_{i=1}^{n} a_i x_i) \cap (\sum_{i=1}^{n} (k_i x_i + x_i a_i'))$$

for some $a_i' \in A, k_i \in \mathbb{Z}$, then $y \in (X)_q$.

Proof. (i)$\Rightarrow$(ii). Suppose that A has the intersection property of quasi-hyperideals and let X be a finite nonempty subset of A. Then $(X)_q$ has the intersection property. Therefore (ii) holds by Theorem 3.5.

(ii)$\Rightarrow$(i). Assume that (ii) is true. Let Q be any quasi-hyperideal of A. We need to show that $< AQ > \cap (Q + < QA >) \subseteq Q$. Let $y \in < AQ > \cap (Q + < QA >)$. Then $y \in \sum_{i=1}^{n} a_i q_i$ and $y \in q + \sum_{j=1}^{m} q_j' b_j$ for some $a_i, b_j \in A$ and $q, q_i, q_j' \in Q$. Consider $X = \{q, q_1, \ldots, q_n, q_1', \ldots, q_m'\}$. Then $X \subseteq Q$ and $|X| < \infty$. By (ii), $< AX > \cap (\mathbb{Z}X + < XA >) \subseteq \mathbb{Z}X + (< AX > \cap < XA >)$, so we have

$$\begin{aligned} y \in < AX > \cap (\mathbb{Z}X + < XA >) &\subseteq \mathbb{Z}X + (< AX > \cap < XA >) \\ &\subseteq Q + (< AQ > \cap < QA >) \subseteq Q + Q \subseteq Q. \end{aligned}$$

This shows that $< AQ > \cap (Q + < QA >) \subseteq Q$. Therefore Q has the intersection property by Theorem 3.4. Hence (i) is proved.

(ii)$\Rightarrow$(iii). Assume that (ii) holds. Let $X = \{x_1, x_2, \ldots, x_n\} \subseteq A$ and $a_1, a_2, \ldots, a_n \in A$, and let $y \in (\sum_{i=1}^{n} a_i x_i) \cap (\sum_{i=1}^{n} (k_i x_i + x_i a_i'))$ for some $a_i' \in A$ and $k_i \in \mathbb{Z}$. Then $y \in (\sum_{i=1}^{n} a_i x_i) \cap (\sum_{i=1}^{n} k_i x_i + \sum_{i=1}^{n} x_i a_i') \subseteq < AX > \cap (\mathbb{Z}X + < XA >)$. But $< AX > \cap (\mathbb{Z}X + < XA >) \subseteq (X)_q$ by (ii), so $y \in (X)_q$.

(iii)$\Rightarrow$(ii). Assume that (iii) is true. To prove (ii), let X be a finite nonempty subset of A, say $X = \{x_1, x_2, \ldots, x_n\}$, and $y \in< AX > \cap (\mathbb{Z}X + < XA >)$. Then $y \in \sum_{i=1}^{n} a_i x_i$ and $y \in \sum_{i=1}^{n} k_i x_i + \sum_{i=1}^{n} x_i a_i'$ for some $a_i, a_i' \in A$ and $k_i \in \mathbb{Z}$. This implies that $y \in \left(\sum_{i=1}^{n} a_i x_i \right) \cap \left(\sum_{i=1}^{n} (k_i x_i + x_i a_i') \right)$. It then follows from (iii) that $y \in (X)_q$. Hence (ii) is proved. □

A corollary of Theorem 3.8 is Proposition 1.11.

Corollary 3.9. *The following statements for a ring A are equivalent.*

(i) *A has the intersection property of quasi-ideals.*

(ii) *For any finite nonempty subset X of A,*

$$AX \cap (\mathbb{Z}X + XA) \subseteq \mathbb{Z}X + (AX \cap XA)(= (X)_q).$$

(iii) *For any finite subset $X = \{x_1, x_2, \ldots, x_n\}$ of A and $a_1, a_2, \ldots, a_n \in A$, if*

$$\sum_{i=1}^{n} (a_i x_i + k_i x_i + x_i a_i') = 0,$$

for some $a_i' \in A$ and $k_i \in \mathbb{Z}$, then $\sum_{i=1}^{n} a_i x_i \in (X)_q$.