CHAPTER V

BOREL MEASURES

In this chapter we study some properties of the semicontinuous functions and the representation of Radon measures in terms of Borel measures. Later on some well-known integration theorems will also be discussed .

The materials of this chapter are drawn from references [2], [3], [4], [6], [8] and [9].

5.1 Semicontinuous Functions

Let Ω be an open subset of $\mathbb{R}^n$ and f a mapping of Ω into the extended real line $\bar{\mathbb{R}}$. For each $x \in \Omega$ let N(x) be the collection of neighborhoods of x. If $A \subset \Omega$ and x_o is any point of $\bar{A}$, we define

$$\liminf_{\substack{x \to x_o \\ x \in A}} f(x) = \sup_{U \in N(x_o)} \left[\inf_{x \in U \cap A} f(x) \right]$$

(5.1.a)

$$\limsup_{\substack{x \to x_o \\ x \in A}} f(x) = \inf_{U \in N(x_o)} \left[\sup_{x \in U \cap A} f(x) \right];$$

If $A = \Omega$ we simply write $\liminf_{x \to x_o} f(x)$ and $\limsup_{x \to x_o} f(x)$.

5.1.1 Definition The function f is said to be lower semi-continuous (l.s.c.) at a point $x_0 \in \Omega$ if $f(x_0) = \lim\limits_{x \to x_0} \inf f(x)$ and upper semicontinuous (u.s.c.) at a point $x_0 \in \Omega$ if $f(x_0) = \lim\limits_{x \to x_0} \sup f(x)$. The function f is said to be l.s.c. on Ω (u.s.c. on Ω) if l.s.c. at each point of Ω (u.s.c. at each point of Ω).

Clearly, if f is l.s.c. at x_0, then $-f$ is u.s.c. at this point. Hence we need consider only lower semicontinuous functions.

5.1.2 Theorem. A mapping $f: \Omega \to \bar{\mathbb{R}}$ is l.s.c. at $x_0 \in \Omega$ if and only if, for each $\alpha \in \bar{\mathbb{R}}$ such that $\alpha < f(x_0)$, there exists a neighborhood U of x_0 such that, for all $x \in U$, we have $\alpha < f(x)$.

Proof : If the condition is satisfied, we have that $\alpha \leq \inf\limits_{x \in U} f(x)$, so that $f(x_0) \leq \sup\limits_{U} \inf\limits_{x \in U} f(x) = \lim\limits_{x \to x_0} \inf f(x)$. But in fact,

$\lim\limits_{x \to x_0} \inf f(x) \leq f(x_0)$, so we get $f(x_0) = \lim\limits_{x \to x_0} \inf f(x)$. Therefore f is l.s.c. at x_0(5.1.1). The converse is immediately true from (5.1.1) and (5.1.a).

5.1.3 Theorem. A mapping $f: \Omega \to \bar{\mathbb{R}}$ is l.s.c. on Ω if and only if, for each $\alpha \in \bar{\mathbb{R}}$, the set $f^{-1}((\alpha, +\infty])$ of points x at which $f(x) > \alpha$ is open in Ω (or, equivalently, if and only if, for each $\alpha \in \bar{\mathbb{R}}$, the set $f^{-1}([-\infty, \alpha])$ of points of x at which $f(x) \leq \alpha$ is closed in Ω).

Proof : (5.1.2) implies that, for each $\alpha \in \bar{\mathbb{R}}$, the set $f^{-1}((\alpha,+\infty])$ is a neighborhood of each of its points, so we get the result. The equivalent assertion follows by taking complements.

5.1.4 Theorem. Let f, g be two mappings of Ω into $\bar{\mathbb{R}}$ each of which is l.s.c. at a point $x_o \in \Omega$. Then

(i) f+g is l.s.c. at x_o if $f(x)+g(x)$ is defined for all $x \in \Omega$, and

(ii) sup (f,g) and inf (f,g) are l.s.c. at x_o.

Proof : (i) The result is obvious if $f(x_o)$ or $g(x_o)$ is equal to $-\infty$. If not, then we have $f(x_o)+g(x_o) > -\infty$. Every number $\alpha \in \mathbb{R}$ such that $\alpha < f(x_o)+g(x_o)$ can be written in the form $\alpha = \beta + \gamma$, with $\beta < f(x_o)$ and $\gamma < g(x_o)$ (it is enough to choose γ such that $\alpha - f(x_o) < \gamma < g(x_o)$). By the hypothesis, there exists a neighborhood U of x_o such that, for all $x \in U$, we have $\beta < f(x)$ and $\gamma < g(x)$ (5.1.2). It follows that $\alpha = \beta + \gamma < f(x)+g(x)$ for all $x \in U$. Hence the result (5.1.2).

(ii) For every number $\alpha \in \bar{\mathbb{R}}$ such that $\alpha < \sup(f(x_o),g(x_o))$, we have $\alpha < f(x_o)$ or $\alpha < g(x_o)$. By the hypothesis, there exists a neighborhood U of x_o such that for all $x \in U$, we have $\alpha < f(x)$ or $\alpha < g(x)$(5.1.2). If follows that $\alpha < \sup(f(x),g(x))$ for all $x \in U$. Hence the result (5.1.2). The other case is analogous.

5.1.5 <u>Definition</u>. Given a set Ω and any family $(f_\lambda)_{\lambda \in \Lambda}$ of mappings of Ω into $\bar{\mathbb{R}}$, the <u>upper</u> (resp. lower) <u>envelope</u> of the family is defined to be the mapping $x \mapsto \sup_{\lambda \in \Lambda} f_\lambda(x)$ (resp. $x \mapsto \inf_{\lambda \in \Lambda} f_\lambda(x)$) of Ω into $\bar{\mathbb{R}}$. It is denoted by $\sup_{\lambda \in \Lambda} f_\lambda$ (resp. $\inf_{\lambda \in \Lambda} f_\lambda$). We have $\sup_{\lambda \in \Lambda} (-f_\lambda) = -\inf_{\lambda \in \Lambda} f_\lambda$.

5.1.6 <u>Theorem</u>. Let Ω be an open subset of $\mathbb{R}^n$ and let $(f_\lambda)_{\lambda \in \Lambda}$ be a family of mappings of Ω into $\bar{\mathbb{R}}$. If each f_λ is l.s.c. at a point $x_0 \in \Omega$, then the upper envelope $f = \sup_{\lambda \in \Lambda} f_\lambda$ is l.s.c. at x_0.

<u>Proof</u> : Given any $\alpha < f(x_0) = \sup_{\lambda \in \Lambda} f_\lambda(x_0)$, there exists by the hypothesis a $\lambda_0 \in \Lambda$ such that $\alpha < f_{\lambda_0}(x_0)$. Since f_{λ_0} is l.s.c. at the point x_0, there exists a neighborhood U of x_0 such that $\alpha < f_{\lambda_0}(x)$ for all $x \in U$ (5.1.2), and therefore $\alpha < f_{\lambda_0}(x) \leq f(x)$ for all $x \in U$. Hence the result (5.1.2).

5.1.7 <u>Theorem</u>. A mapping $f: \Omega \longrightarrow \bar{\mathbb{R}}$ is continuous on Ω if and only if it is both u.s.c. and l.s.c. on Ω.

<u>Proof</u> : If f is continuous on Ω, then for any $x_0 \in \Omega$, $\liminf_{x \to x_0} f(x) = f(x_0) = \limsup_{x \to x_0} f(x)$. Hence, by (5.1.1), f is both u.s.c. and l.s.c. on Ω. The converse follows by reversing.

5.1.8 <u>Theorem</u>. The upper (resp. lower) envelope of a family of continuous mappings of Ω into $\bar{\mathbb{R}}$ is lower (resp. upper) semicontinuous.

<u>Proof</u> : It follows immediately from (5.1.7) and (5.1.6).

5.1.9 <u>Definition</u>. If A is any subset of a set Ω , the <u>characteristic function</u> of A (usually denoted by χ_A) is the mapping of Ω into $\mathbb{R}$ such that $\chi_A(x) = 1$ for all $x \in A$ and $\chi_A(x) = 0$ for all $x \in \Omega - A$. So we have $\chi_\Omega = 1$, $\chi_\emptyset = 0$, and $\chi_{\Omega-A} = 1 - \chi_A$.

5.1.10 <u>Theorem</u>. A subset A of Ω is open (resp. closed) in Ω if and only if χ_A is l.s.c. (resp. u.s.c.) on Ω .

<u>Proof</u> : This follows immediately from (5.1.3).

5.1.11 <u>Theorem</u>. If f is l.s.c. on $A \subset \mathbb{R}^n$ and there is a real-valued continuous function g on $\mathbb{R}^n$ such that $f \geq g$ on A, then there is an increasing sequence of continuous functions (g_m) on $\mathbb{R}^n$ such that $\lim_{m \to +\infty} g_m = f$ on A.

<u>Proof</u> : Replacing f if necessary by f-g (which is everywhere defined), we may assume that $f \geq 0$ and somewhere finite (the case $f = +\infty$ is trivial). For each positive integer m and each $x \in \mathbb{R}^n$ define

$$g_m(x) = \inf \{ f(y) + m|y-x| : y \in A \}.$$

This is finite for each x. Clearly $f(y)+m|y-x| \leq f(y)+(m+1)|y-x|$ for all y, so $g_m(x) \leq g_{m+1}(x)$ for all x. Also $g_m(x) \geq \inf f(A) \geq g(x)$. Let x_1 and x_2 be any points of $\mathbb{R}^n$. For every y in A we have $f(y)+m|y-x_1| \leq f(y)+m\{|x_1-x_2|+|x_2-y|\} = (f(y)+m|x_2-y|)+m|x_1-x_2|$, So $g_m(x_1) \leq g_m(x_2)+m|x_1-x_2|$. Interchanging x_1 and x_2, we get $g_m(x_2) \leq g_m(x_1)+m|x_1-x_2|$. Therefore $|g_m(x_1)-g_m(x_2)| \leq m|x_1-x_2|$, which proves that g_m is continuous on $\mathbb{R}^n$.

Finally, let x_o be a point in A; we must show that $g_m(x_o) \to f(x_o)$ as $m \to +\infty$. Let k be any number less than $f(x_o)$. By the l.s.c. of f, there is an $\varepsilon > 0$ such that $f(x) > k$ for all $x \in B(x_o,\varepsilon)$. Now choose a number m_o large enough so that $g(x)+m_o\varepsilon > k$. If $m > m_o$, then in the expression $f(y)+m|x_o-y|$ either $|x_o-y| \geq \varepsilon$ or $y \in B(x_o,\varepsilon)$. In the first case, $f(y)+m|x_o-y| \geq g(x)+m\varepsilon > g(x)+m_o\varepsilon > k$. In the second case, $f(y)+m|x_o-y| \geq f(y) > k$ by the choice of ε. Then k is a lower bound for $f(y)+m|x_o-y|$, and by definition $g_m(x_o) \geq k$. Since m was any number greater than m_o, and $\lim_{m\to+\infty} g_m(x_o)$ exists, $\lim_{m\to+\infty} g_m(x_o) = k$. But k was any number $< f(x_o)$, so $\lim_{m\to+\infty} g_m(x_o) \geq f(x_o)$.

On the other hand, in the definition of $g_m(x_o)$ we can take $y = x_o$, because $x_o \in A$, So one possible value of $f(y)+m|x_o-y|$ is $f(x_o)$, and $g_m(x_o) \leq f(x_o)$. Since this is true for all m, we have $\lim_{m\to+\infty} g_m(x_o) \leq f(x_o)$. Thus, with the preceding inequality, proves $g_m(x_o) \to f(x_o)$ as $m \to +\infty$.

5.1.13 Theorem. Let $f \geq 0$ be a l.s.c. function on an open set $\Omega \subset \mathbb{R}^n$. Then for all $x_0 \in \Omega$,

$$f(x_0) = \sup_{\varphi \leq f,\, \varphi \in \mathcal{D}(\Omega)} \varphi(x_0).$$

Proof : The inequelity $f(x_0) \geq \sup_{\varphi \leq f,\, \varphi \in \mathcal{D}(\Omega)} \varphi(x_0)$ is clear. Let us prove the reverse inequality. For any $x_0 \in \Omega$ and any $k < f(x_0)$, by the l.s.c. of f, there exists a neighborhood U of x_0 such that $k < f(x)$ for all $x \in U$. Let $B = B(x_0, \varepsilon)$ be a ball with the compact closure $\bar{B} \subset U$. Then, by (2.2.5), there exists a function $\psi \in \mathcal{D}(\Omega)$ such that $\psi(x) = 1$ for all $x \in \bar{B}$. Let $\varphi(x) = k\,\psi(x)$ for all $x \in \bar{B}$ and equal to zero otherwise. Then $\varphi \in \mathcal{D}(\Omega)$, $\varphi \leq f$ and $\varphi(x_0) \geq k$. Since k is arbitrary, we conclude that

$$\sup_{\varphi \leq f,\, \varphi \in \mathcal{D}(\Omega)} \varphi(x_0) \geq f(x_0),$$

whence the result.

5.2 The Representation of a Radon Measure and Some Fundamental Results on Integration Theory

In this section we will show that there is a one to one correspondence between a positive Radon measure and a positive Borel measure. Finally we will state some well-known integration theorems without proof.

Some definitions and facts from measure theory are assumed, but we will recall some definitions and facts of Borel measure.

5.2.1 Definitions. Let Ω be an open subset of $\mathbb{R}^n$, and let $\mathcal{B}$ be the σ- algebra generated by the class of all open (or closed) subsets of Ω. The elements of $\mathcal{B}$ are called the Borel sets of Ω.

A positive Borel measure ν on Ω is a positive real-valued function on $\mathcal{B}$ such that

(i) $\nu(\emptyset) = 0$,

(ii) if $A_1, A_2, \ldots \in \mathcal{B}$ is a sequence of disjoint sets, then $\nu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \nu(A_j)$, and

(iii) $\nu(K) < +\infty$ for every compact set $K \subset \Omega$.

If f is a mapping of Ω into $\mathbb{R}$, then f is said to be Borel function provided that $f^{-1}(V)$ is a Borel set in Ω for every open set V in $\mathbb{R}$.

5.2.2 Examples. (i) Every continuous function of Ω is Borel function.

(ii) Every semicontinuous function of Ω is Borel function.

(iii) If A is a Borel set in Ω, then the characteristic function χ_A is a Borel function.

5.2.3 Remark. If f and g are Borel functions on Ω, then so are $|f|$, $f+g$, fg, $\max(f,g)$ and $\min(f,g)$.

The following theorem will show the 1-1 corresponding between a positive Radon measure and a positive Borel measure.

5.2.4 Theorem. (The Riesz Representation Theorem) Let Ω be an open subset of $\mathbb{R}^n$, and let μ be a positive Radon measure on $K(\Omega)$. Then there exists a σ- algebra $\mathcal{B}$ in Ω which contains all Borel sets in Ω, and there exists a unique positive Borel measure ν on $\mathcal{B}$ with represents μ in the sense that

$$\mu(f) = \int_{\Omega} f \, d\nu$$

for every $f \in K(\Omega)$ and which has the following additional properties :

(i) $\nu(K) < +\infty$ for every compact set $K \subset \Omega$,

(ii) For every $E \in \mathcal{B}$, we have

$$\nu(E) = \inf \{ \nu(V) : E \subset V, V \text{ open} \}.$$

(iii) The relation

$$\nu(E) = \sup \{ \nu(K) : K \subset E, K \text{ compact} \}$$

holds for every open set E, and for every $E \in \mathcal{B}$ with $\nu(E) < +\infty$.

(iv) If $E \in \mathcal{B}$, $A \subset E$, and $\nu(E) = 0$, then $A \in \mathcal{B}$.

The proof of the theorem will be omitted (for the completed proof see [9], p.40).

Convention : By the uniqueness of the positive Borel measure ν in the theorem, we shall simply write $\int_\Omega f\, d\mu$ instead of $\int_\Omega f\, d\nu$.

Now we will state some integration theorems which will be used repeatedly in the next chapter. Since they are already well-known, we will state them without proof (for the completed proofs see e.g., [9]).

5.2.5 Definition. We define $L(\mu)$ to be the collection of all Borel functions f on Ω for which

$$\int_\Omega |f|\, d\mu \quad < \quad +\infty .$$

The numbers of $L(\mu)$ are called Lebesgue integrable functions (with respect to μ).

5.2.6 Theorem. (Lebesgue's Monotone Convergence Theorem) Let (f_m) be a sequence of Borel functions on Ω, and suppose that

(i) $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq +\infty$ for every $x \in \Omega$,

(ii) $f_m(x) \longrightarrow f(x)$ as $m \longrightarrow +\infty$, for every $x \in \Omega$.

Then f is a Borel function, and

$$\lim_{m \to +\infty} \int_\Omega f_m\, d\mu = \int_\Omega f\, d\mu .$$

5.2.7 Theorem. (Lebesgue's Dominated Convergence Theorem). Suppose (f_m) is a sequence of Borel functions on Ω such that

$$f(x) = \lim_{m \to +\infty} f_m(x)$$

exists for every $x \in \Omega$. If there is a Borel function $g \in L(\mu)$ such that

$$|f_m(x)| \leq g(x)$$

for all m and for all $x \in \Omega$, then $f \in L(\mu)$, and

$$\lim_{m \to +\infty} \int_\Omega f_m \, d\mu = \int_\Omega f \, d\mu .$$

5.2.8 Theorem. Let Ω and K be open and compact subsets of $\mathbb{R}^n$, respectively, let $\mathrm{Supp}(\mu) = K$, and let $(x,y) \mapsto f(x,y)$ be a real-valued function on $\Omega \times K$ with the property that for each $y \in K$, the function $x \mapsto f(x,y)$ is continuous on Ω. Then the function h given by

$$h(x) = \int_K f(x,y) \, d\mu(y) \qquad (x \in \Omega)$$

is continuous on Ω.

If further for each $y \in K$ the function $x \mapsto \frac{\partial f}{\partial x_j}(x,y)$ is continuous on Ω, then

$$\frac{\partial h(x)}{\partial x_j} = \int_K \frac{\partial}{\partial x_j} f(x,y) \, d\mu(y).$$

(" differentiation under the integral sign ").

5.2.9 Theorem. (Lebesgue-Fubini Theorem). Let X, Y be either open sets or spheres in $\mathbb{R}^n$, let λ, μ be positive Borel measures on X and Y, respectively, and $\lambda \otimes \mu$ their product.

If $f(x,y) \in L(\lambda \otimes \mu)$, then

$$\text{(i)} \quad G(x) = \int_Y f_x(x,y)\,d\mu(y) \qquad (x \in X),$$

and

$$H(y) = \int_X f_y(x,y)\,d\lambda(x) \qquad (y \in Y)$$

belongs to $L(\lambda)$ and $L(\mu)$, respectively, and

$$\text{(ii)} \quad \int_X G(x)\,d\lambda(x) = \int_{X\times Y} f(x,y)\,d(\lambda \otimes \mu) = \int_Y H(y)\,d\mu(y)$$

which can also be written in the more usual form

$$\text{(iii)} \quad \int_X d\lambda(x) \int_Y f(x,y)\,d\mu(y) = \int_Y d\mu(y) \int_X f(x,y)\,d\lambda(x).$$

These are the so-called " iterated integrals " of f.

The following is the useful consequence of the theorem : If f is a Borel function on $X\times Y$, and if

$$\int_X d\lambda(x) \int_Y |f(x,y)|\,d\mu(y) < +\infty,$$

then the two iterated integrals in (iii) are finite and equal.

In other words " The order of integration may be reversed " for a Borel function f on $X \times Y$ whenever $f \geq 0$ and also whenever one of the iterated integrals of $|f|$ is finite.