CHAPTER II

PRELIMINARIES

In this thesis, we assume a basic knowledge of logic. The materials of this chapter are drawn from [1] and [3] .

2.1 Definition. A first-order language L is a finite collection of symbols.

These symbols are separated into three groups; relation symbols, function symbols and (individual) constant symbols. The relation and function symbols of L will be denoted by capital letters P, F with superscripts and subscripts. Lower case letters c, with subscripts, range over the constant symbols of L.

We may write the symbols of L as follows :

$$L = \{ P_1^{i_1}, \ldots, P_n^{i_n}, F_1^{j_1}, \ldots, F_m^{j_m}, c_1, \ldots, c_q \}$$

Eventually, each relation symbol P_j^n will seen as representing an n-placed relation, similarly, each function symbol F_j^m of L, an m-placed function. Subsequently, the superscripts of these symbols will be omitted in cases where it is clear what they are, e.g. if we write $P_1 (v_1 \ldots v_n)$, this means that P_1 is P_1^n.

When dealing with several languages at the same time, we use the letters L, L', L'', etc. If the symbols of the language are quite standard, as for example + for addition, $\leq$ for an order relation, etc., we shall simply write

$$L = \{ \leq \},\ L' = \{\leq,+,\cdot,0\},\ L'' = \{+,\cdot,-,0,1\},\ \text{etc.},$$

for such languages. The number of places of the various kinds of symbols is understood to follow the standard usage.

2.2 Definition. The cardinal, or power, of a first-order language L, denoted by $||L||$, is defined as

$$||L|| = \omega \cup |L| \quad \text{where } |L| \text{ is the cardinal of set of}$$

symbols of L.

2.3 Definition. A first-order language L' is an expansion of a first-order language L if and only if L' has all the symbols of L plus some additional symbols. We use the notation $L \subset L'$.

Since L and L' are just sets of symbols, the expansion L' may be written as $L' = L \cup X$, where X is the set of new symbols.

2.4 Definition. A model of a first-order language L consists of

(1) a nonempty set A called universe,

(2) interpretations of relation, function and constant symbols where

(2.1) each relation symbol P_j^n corresponds to an n-placed relation $R_j \subseteq A^n$,

(2.2) each function symbol F_j^m corresponds to an m-placed function G_j from A^m into A, (2.3) each constant symbol c corresponds to an element x in A.

Hence if $L = \{ P_1^{i_1}, \ldots, P_n^{i_n}, F_1^{j_1}, \ldots, F_m^{j_m}, c_1, \ldots, c_q \}$, then a model M of L is written as $M = < A, R_1^{i_1}, \ldots, R_n^{i_n}, G_1^{j_1}, \ldots, G_m^{j_m}, x_1, \ldots, x_q >$. When the symbols of L are familiar, we shall agree to use, for instance, $M = < A, \leq, +, . >$ for model of the language $L = \{\leq, +, . \}$. Sometimes; we use the shorter notation $< A, \mathcal{I} >$ for the model of L, where $\mathcal{I}$ is an interpretation function mapping the symbols of L to appropriate relations, functions and constants in A.

2.5 Definition. If M is a model of L and $L' = L \cup X$, then M can be expanded to a model M' of L' by giving appropriate interpretations for the symbols in X. We call M' an expansion of M to L' and M is the reduct of M' to L.

If $\mathcal{I}'$ is any interpretation for the symbols in X, then $M' = < A, \mathcal{I} \cup \mathcal{I}' >$ or $< M, \mathcal{I}' >$ is a model of L'.

2.6 Remark. Let $L \subset L'$ and $L' = L \cup X$.

(i) There are many ways that a model M of L can be expanded to a model M' of L'.

(ii) There is only one reduct M of M' to L, namely, by restricting the interpretation function $\mathcal{I}'$ on $L \cup X$ to L.

(iii) Expansion and reduction do not change the universe of the model.

2.7 <u>Definition</u>. The cardinal, or power, of the model M is the cardinal $|A|$.

M is said to be finite, countable or uncountable if A is finite, countable or uncountable.

To formalize a first-order language L, we need the following logical symbols :

parentheses), (;

a denumerable list of individual variables $v_1, v_2, \ldots, v_n, \ldots$;

connectives $\wedge$, $\sim$;

quantifier $\forall$;

and identity symbol = .

2.8 <u>Definition</u>. Terms of L are defined as follows :

(i) An individual variable is a term.

(ii) A constant symbol is a term.

(iii) If F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms, then $F(t_1 \ldots t_m)$ is a term.

(iv) A string of symbols is a term only if it can be shown to be a term by a finite number of applications of (i) - (iii).

2.9 <u>Definition</u>. Atomic formulas of L are defined as follows :

(i) $t_1 = t_2$ is an atomic formula, where t_1 and t_2 are terms of L.

(ii) If P is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms, then $P\ (t_1 \ldots t_n)$ is an atomic formula.

(iii) A string of symbols is an atomic formula only by (i) and (ii).

2.10 <u>Definition</u>. Formulas of L are defined as follows :

(i) An atomic formula is a formula.

(ii) If ϕ and ψ are formulas, then $(\phi \wedge \psi)$, $(\sim \phi)$ and $(\sim \psi)$ are formulas.

(iii) If v is an individual variable and ϕ is a formula, then $(\forall v)\ \phi$ is a formula.

(iv) A sequence of symbols is a formula by a finite number of applications of (i) - (iii).

2.11 <u>Definition</u>. The defined connectives $\vee$, $\rightarrow$, $\longleftrightarrow$, and $\exists$ are introduced as abbreviations defined as :

$\phi \vee \psi$	for	$\sim(\sim \phi \wedge \sim \psi)$.
$\phi \longrightarrow \psi$	for	$\sim \phi \vee \psi$
$\phi \longleftrightarrow \psi$	for	$(\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi)$.
$(\exists v)\ \phi$	for	$\sim(\forall v) \sim \phi$

2.12 <u>Definition</u>. Length of a term t is the number of occurrences of function symbols in t.

2.13 <u>Definition</u>. Length of a formula is the number of connectives and quantifiers.

2.14 <u>Note</u>. An atomic formula is a formula of length zero.

2.15 Definition. Subformulas of a formula ϕ are defined as follows :

(i) ϕ is a subformula of ϕ.

(ii) If $\psi \wedge \theta$ is a subformula of ϕ, then both ψ and θ are subformulas of ϕ.

(iii) If $\sim \psi$ is a subformula of ϕ, then ψ is a subformula of ϕ.

(iv) If $(\forall v)\ \psi$ is a subformula of ϕ, then ψ is a subformula of ϕ.

2.16 Definition. The scope of $(\forall v)$ in $(\forall v)\ \phi$ is ϕ.

2.17 Definition. An occurrence of an individual variable v is bound in a formula ϕ if and only if it is the variable of a quantifier $(\forall v)$ in ϕ, or it is within the scope of a quantifier $(\forall v)$ in ϕ.

2.18 Definition. An occurrence of an individual variable is free in a formula ϕ if and only if it is not bound in ϕ.

2.19 Definition. An individual variable is free (bound) in a formula ϕ if and only if it has a free (bound) occurrence in ϕ.

2.20 Definition. $\phi(v_1 \ldots v_k)$ means that some of $v_1, \ldots v_k$ are free in ϕ.

2.21 Definition. A sentence is a formula with no free variables.

To make all the above syntactical notions into a formal system we need logical axioms and rules of inference.

Let ϕ, ψ and θ be formula of L.

2.22 Logical axioms of L.

(i) $\phi \longrightarrow (\psi \longrightarrow \phi)$.

(ii) $(\phi \longrightarrow (\psi \longrightarrow \theta)) \longrightarrow ((\phi \longrightarrow \psi) \longrightarrow (\phi \longrightarrow \theta))$.

(iii) $(\sim \phi \longrightarrow \sim \psi) \longrightarrow ((\sim \phi \longrightarrow \psi) \longrightarrow \phi)$.

(iv) $(\forall v)\ (\phi \longrightarrow \psi) \longrightarrow (\phi \longrightarrow (\forall v)\ \psi)$; where v is a variable not free in ϕ.

(v) $(\forall v)\ \phi \longrightarrow \psi$; where ψ is a formula obtained from ϕ by freely substituting each free occurrence of v in ϕ by a term t. (i.e. no variable x in t shall occur bound in ψ at the place where it is introduced.).

(vi) $x = x$; x is a variable.

(vii) $x = y \longrightarrow t(v_1 \ldots v_{i-1} x v_{i+1} \ldots v_n) = t(v_1 \ldots v_{i-1} y v_{i+1} \ldots v_n)$; where x, y are variables and $t(v_1 \ldots v_n)$ is a term.

(viii) $x = y \longrightarrow (\phi(v_1 \ldots v_{i-1} x v_{i+1} \ldots v_n) \longrightarrow \phi(v_1 \ldots v_{i-1} y v_{i+1} \ldots v_n))$; where x, y are variables and $\phi(v_1 \ldots v_n)$ is a formula.

2.23 Rules of Inference.

(i) Rule of Detachment (or Modus Ponen or MP.) : From ϕ and $\phi \longrightarrow \psi$ infer ψ.

(ii) Rule of Generalization : From ϕ infer $(\forall v)\ \phi$.

2.24 Definition. A proof is a finite sequence of formulas $\psi_1, \ldots, \psi_n$ such that each ψ_i, $1 \leq i \leq n$, is

(i) a logical axiom of L, or

(ii) a conclusion from ψ_j, ψ_k (j, k < i) by MP., or

(iii) a conclusion from ψ_j (j < i) by generalization.

2.25 Definition. Let Σ be a set of sentences of L and ϕ be a formula. A proof of ϕ from Σ is a finite sequence of formulas $\psi_1, \ldots, \psi_n$ such that $\psi_n = \phi$ and each ψ_i, $1 \leq i \leq n$, is

(i) a logical axiom of L, or

(ii) a conclusion from ψ_j, ψ_k (j, k < i) by MP., or

(iii) a conclusion from ψ_j (j < i) by generalization, or

(iv) a member of Σ.

2.26 Definition. ϕ is deducible from Σ (in notation $\Sigma \vdash \phi$) if and only if there exists a proof of ϕ from Σ. If it is not the case that ϕ is deducible from Σ, then we use $\Sigma \nvdash \phi$.

If $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$, we write $\sigma_1 \ldots \sigma_n \vdash \phi$ for $\Sigma \vdash \phi$.

2.27 Definition. ϕ is a theorem ($\vdash \phi$) if and only if ϕ is deducible from empty set. If ϕ is not a theorem, we then use $\nvdash \phi$.

2.28 Definition. Let Σ be a set of sentences of L. Σ is inconsistent if and only if $\Sigma \vdash \phi \wedge \sim \phi$, for any formula ϕ of L. Otherwise Σ is consistent.

A sentence σ is consistent if and only if $\{\sigma\}$ is.

2.29 Definition. Let Σ be a set of sentences of L. Σ is maximal consistent (in L) if and only if Σ is consistent and no set of sentences (of L) properly containing Σ is consistent.

2.30 Lemma. Let ϕ be a formula of L, then $\phi \to \phi$ is a theorem, i.e. $\vdash \phi \to \phi$.

proof. (1) $(\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$ by axiom (ii)

(2) $\phi \to ((\phi \to \phi) \to \phi)$ by axiom (i)

(3) $(\phi \to (\phi \to \phi)) \to (\phi \to \phi)$ by (1), (2) and MP.

(4) $\phi \to (\phi \to \phi)$ by axiom (i)

(5) $\phi \to \phi$ by (3), (4) and MP.

Hence $\phi \to \phi$ is a theorem.

2.31 Theorem. (Deduction Theorem.) Let Σ be a set of sentences of L, ϕ a sentence and ψ a formula. $\Sigma \cup \{\phi\} \vdash \psi$ if and only if $\Sigma \vdash \phi \to \psi$. In particular, $\phi \vdash \psi$ if and only if $\vdash \phi \to \psi$.

proof. Assume $\Sigma \cup \{\phi\} \vdash \psi$, therefore, there exists a finite sequence of formulas $\theta_1, \ldots, \theta_n$ such that $\theta_n = \psi$ and each θ_i, $1 \leq i \leq n$, is a logical axiom of L, or $\theta_i \in \Sigma \cup \{\phi\}$, or θ_i is a conclusion from θ_j, θ_k $(j, k < i)$ by MP., or θ_i is a conclusion from θ_j $(j < i)$ by generalization.

Claim that $\Sigma \vdash \phi \to \theta_i$, $1 \leq i \leq n$. We must show this by induction on i. Suppose $i = 1$, therefore θ_1 is a logical axiom or $\theta_1 \in \Sigma$ or $\theta_1 = \phi$. Suppose θ_1 is a logical axiom or $\theta_1 \in \Sigma$. Since $\vdash \theta_1 \to (\phi \to \theta_1)$, we get $\Sigma \vdash \phi \to \theta_1$. Suppose $\theta_1 = \phi$. Since $\vdash \phi \to \phi$ by Lemma 2.30, we have $\Sigma \vdash \phi \to \theta_1$.

Assume $\Sigma \vdash \phi \rightarrow \theta_j$, for all $j < k \leq n$. Then θ_k is a logical axiom, or $\theta_k \in \Sigma$, or $\theta_k = \phi$, or θ_k is a conclusion from θ_j, $\theta_j \rightarrow \theta_k$, $(j < k)$ by MP., or θ_k is a conclusion from θ_j, $(j < k)$ by generalization. For the first three possibilities, $\Sigma \vdash \phi \rightarrow \theta_k$ by the proof for θ_1. If θ_k is a conclusion from θ_j, $\theta_j \rightarrow \theta_k$, $(j < k)$ by MP., then for some $\ell < k$, $\theta_\ell = \theta_j \rightarrow \theta_k$. By induction hypothesis; $\Sigma \vdash \phi \rightarrow \theta_j$ and $\Sigma \vdash \phi \rightarrow (\theta_j \rightarrow \theta_k)$ and since $\vdash (\phi \rightarrow (\theta_j \rightarrow \theta_k)) \rightarrow ((\phi \rightarrow \theta_j) \rightarrow (\phi \rightarrow \theta_k))$, we get $\Sigma \vdash \phi \rightarrow \theta_k$. If θ_k is a conclusion from θ_j, $(j < k)$ by generalization, then by induction hypothesis; $\Sigma \vdash \phi \rightarrow \theta_j$, and $\Sigma \vdash (\forall v)\ (\phi \rightarrow \theta_j)$. Since $\vdash ((\forall v)\ (\phi \rightarrow \theta_j) \rightarrow (\phi \rightarrow (\forall v)\ \theta_j))$, we get $\Sigma \vdash \phi \rightarrow (\forall v)\ \theta_j$, i.e. $\Sigma \vdash \phi \rightarrow \theta_k$. Therefore $\Sigma \vdash \phi \rightarrow \theta_i$, $1 \leq i \leq n$, so $\Sigma \vdash \phi \rightarrow \theta_n$. Hence $\Sigma \vdash \phi \rightarrow \psi$.

To prove the converse, assume that $\Sigma \vdash \phi \rightarrow \psi$, then there exists a finite sequence of formulas $\theta_1, \ldots, \theta_n$ such that $\theta_n = \phi \rightarrow \psi$; which is a proof of $\phi \rightarrow \psi$ from Σ. Add ϕ to the proof, we then get ψ by MP. Hence $\Sigma \cup \{\phi\} \vdash \psi$.

2.32 <u>Proposition</u>. Let Σ be a set of sentences of L and ϕ be a sentence.

(i) If $\Sigma \cup \{\phi\}$ is inconsistent, then $\Sigma \vdash \sim \phi$.

(ii) If $\Sigma \nvdash \phi$, then $\Sigma \cup \{\sim \phi\}$ is consistent.

<u>proof</u>. (i) Assume $\Sigma \cup \{\phi\}$ is inconsistent, so $\Sigma \cup \{\phi\} \vdash \psi \wedge \sim \psi$ for any formula ψ of L. Then $\Sigma \cup \{\phi\} \vdash \psi$ and $\Sigma \cup \{\phi\} \vdash \sim \psi$. By Deduction Theorem, we get $\Sigma \vdash \phi \rightarrow \psi$ and $\Sigma \vdash \phi \rightarrow \sim \psi$. Since $\vdash (\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \sim \psi) \rightarrow \sim \phi)$, we get $\Sigma \vdash \sim \phi$.

(ii) Assume $\Sigma \nvdash \phi$. Suppose $\Sigma \cup \{\sim \phi\}$ is inconsistent, then $\Sigma \cup \{\sim \phi\} \vdash \psi \wedge \sim \psi$ for any formula ψ of L, i.e. $\Sigma \cup \{\sim\phi\} \vdash \psi$ and $\Sigma \cup \{\sim \phi\} \vdash \sim \psi$. By Deduction Theorem, we get $\Sigma \vdash \sim \phi \longrightarrow \psi$ and $\Sigma \vdash \sim \phi \longrightarrow \sim \psi$. Since $\vdash (\sim \phi \longrightarrow \sim \psi) \longrightarrow ((\sim \phi \longrightarrow \psi) \longrightarrow \phi)$, we get $\Sigma \vdash \phi$ which is a contradiction. Hence $\Sigma \cup \{\sim \phi\}$ is consistent.

2.33 Proposition. Let Σ be a set of sentences of L. If Σ is maximal consistent, then for any sentences ϕ and ψ of L,

(i) $\Sigma \vdash \phi$ if and only if $\phi \varepsilon \Sigma$,

(ii) $\phi \not\varepsilon \Sigma$ if and only if $\sim \phi \varepsilon \Sigma$,

and (iii) $\phi \wedge \psi \varepsilon \Sigma$ if and only if both ϕ and ψ belong to Σ .

proof. (i) Assume $\Sigma \vdash \phi$. Consider $\Sigma \cup \{\phi\} = \Sigma_1$. Suppose Σ_1 is inconsistent. By Proposition 2.32 (i) we get $\Sigma \vdash \sim \phi$ and so $\Sigma \vdash \phi \wedge \sim \phi$, then Σ is inconsistent which is a contradiction. Thus Σ_1 is consistent and since Σ is maximal consistent, we get $\Sigma = \Sigma_1$. Hence $\phi \varepsilon \Sigma$.

To prove the converse, assume that $\phi \varepsilon \Sigma$. By Definition 2.26, we get $\Sigma \vdash \phi$.

(ii) Assume $\phi \not\varepsilon \Sigma$. By (i), $\Sigma \nvdash \phi$, so $\Sigma \cup \{\sim \phi\}$ is consistent. Since Σ is maximal consistent, we get $\Sigma \cup \{\sim \phi\} = \Sigma$. Hence $\sim \phi \varepsilon \Sigma$.

To prove the converse, assume that $\sim \phi \varepsilon \Sigma$, by (i) we get $\Sigma \vdash \sim \phi$. Suppose $\phi \varepsilon \Sigma$, then $\Sigma \vdash \phi$. Thus $\Sigma \vdash \phi \wedge \sim \phi$ and so Σ is inconsistent which is a contradiction. Hence $\phi \not\varepsilon \Sigma$.

(iii) Assume $\phi \wedge \psi \varepsilon \Sigma$, by (i) we get $\Sigma \vdash \phi \wedge \psi$, i.e. $\Sigma \vdash \phi$ and $\Sigma \vdash \psi$. Hence $\phi \varepsilon \Sigma$ and $\psi \varepsilon \Sigma$.

To prove the converse, assume that $\phi \varepsilon \Sigma$ and $\psi \varepsilon \Sigma$, so $\Sigma \vdash \phi$ and $\Sigma \vdash \psi$. Hence $\phi \varepsilon \Sigma$ and $\psi \varepsilon \Sigma$, i.e. $\phi \wedge \psi \varepsilon \Sigma$.

2.34 Theorem. (Lindenbaum's Theorem). Any consistent set of sentences Σ of L can be extended to a maximal consistent set of sentences Γ of L.

proof. Let us arrange all the sentences of L in a list, ϕ_0, $\phi_1, \ldots, \phi_\alpha, \ldots$. The order in which we list them is immaterial, as long as the list associates in a one-one fashion an ordinal number with each sentence. If $\Sigma \cup \{\phi_0\}$ is consistent, define $\Sigma_1 = \Sigma \cup \{\phi_0\}$. Otherwise define $\Sigma_1 = \Sigma$. At the α^{th} stage, we define $\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{\phi_\alpha\}$ if $\Sigma_\alpha \cup \{\phi_\alpha\}$ is consistent, and otherwise define $\Sigma_{\alpha+1} = \Sigma_\alpha$. At limit ordinals α take unions $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$. So we shall form an increasing chain $\Sigma = \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma_\alpha \subset \ldots$ of consistent set of sentences. Now let Γ be the union of all the sets Σ_α .

Claim that Γ is consistent. Suppose not. Then there is a deduction $\psi_1, \ldots, \psi_p$ of the formula $\phi \wedge \sim \phi$ from Γ. Let $\theta_1, \ldots, \theta_q$ be all the formulas in Γ which are used in this deduction. We may choose α so that all of $\theta_1, \ldots, \theta_q$ belong to Σ_α. But this means that Σ_α is inconsistent, which is a contradiction.

Having shown that Γ is consistent, we next claim that Γ is maximal consistent. Suppose Δ is consistent and $\Gamma \subseteq \Delta$. Let $\phi_\alpha \varepsilon \Delta$. Claim that $\Sigma_\alpha \cup \{\phi_\alpha\}$ is consistent. To prove this, suppose $\Sigma_\alpha \cup \{\phi_\alpha\}$

is inconsistent, therefore $\Sigma_\alpha \vdash \sim \phi_\alpha$. Since $\Sigma_\alpha \in \Gamma \subseteq \Delta$, we get $\Delta \vdash \sim \phi_\alpha$ and since $\phi_\alpha \in \Delta$, it follows that $\Delta \vdash \phi_\alpha$. Therefore $\Delta \vdash \phi_\alpha \wedge \sim\phi_\alpha$, and so Δ is inconsistent which is a contradiction. Hence $\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{\phi_\alpha\}$. Thus $\phi_\alpha \in \Gamma$ and so $\Delta = \Gamma$.

Satisfaction of formulas of L.

Let ϕ be any formula of L,

$M = \langle A, \mathcal{G} \rangle$ be a model of L,

and $s = (s_1, s_2, \ldots)$ be any sequence of elements of A.

2.35 Definition. The value of a term t at the sequence s, denoted by $t[s]$, is defined as follows :

(i) If $t = v_i$, then $t[s] = s_i$.

(ii) If t is a constant symbol c, then $t[s]$ is the interpretation of c in M, denoted by $\mathcal{G}(c)$.

(iii) If $t = F(t_1 \ldots t_m)$ where F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms, then $t[s] = G(t_1[s] \ldots t_m[s])$ where G is the interpretation of F in M.

2.36 Definition. Satisfaction of an atomic formula ϕ by a sequence s in M is defined as follows :

(i) If ϕ is $t_1 = t_2$ where t_1, t_2 are terms, then s satisfies ϕ in M if and only if $t_1[s] = t_2[s]$.

(ii) If ϕ is $P(t_1 \ldots t_n)$ where P is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms, then s satisfies ϕ in M if and only if

$(t_1[s], \ldots, t_n[s]) \in R$ where R is the interpretation of P in M.

2.37 <u>Definition</u>. Satisfaction of a formula ϕ by a sequence s in M is defined as follows :

(i) If ϕ is $\theta_1 \wedge \theta_2$ where θ_1 and θ_2 are formulas, then s satisfies ϕ in M if and only if s satisfies both θ_1 and θ_2 in M.

(ii) If ϕ is $\sim \theta$ where θ is a formula, then s satisfies ϕ if and only if s does not satisfy θ in M.

(iii) If ϕ is $(\forall v_i)\,\theta$ where v_i is an individual variable and θ is a formula, then s satisfies ϕ in M if and only if every sequence of elements of A differing from s in at most i^{th} place satisfies θ .

2.38 <u>Lemma</u>. If the free variables of a formula ϕ occur in the list $v_{i_1}, \ldots, v_{i_k}$ and if the sequences s and s' have the same components in the $i_1^{th}, \ldots, i_k^{th}$ places, then s satisfies ϕ if and only if s' satisfies ϕ .

<u>proof</u>. We must prove this lemma by induction on length of a formula ϕ .

First we must prove that if t is a term with variables among $v_{i_1}, \ldots, v_{i_k}$ and if s and s' have the same components in the $i_1^{th}, \ldots, i_k^{th}$ places, then $t[s] = t[s']$ .... ⊛

If t is an individual variable v_{i_j} for some j, $i \leq j \leq k$ then $t[s] = a_{i_j}$, $t[s'] = a_{i_j}$ where a_{i_j} is the i_j^{th} element of the sequence s and s', hence $t[s] = t[s']$.

If t is a constant symbol c, then t contains no variables at all. So $t[s_1] = t[s_2]$ for any sequences s_1 and s_2.

Suppose ⊛ is true for all terms t such that length of $t < k$. If t is $F(t_1 \ldots t_m)$ of length k, where F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms with variables among $v_{i_1}, \ldots, v_{i_k}$, such that length of $t_j < k$, then $t[s] = G(t_1[s] \ldots t_m[s])$ and $t[s'] = G(t_1[s'] \ldots t_m[s'])$ where G is the interpretation of F. By induction hypothesis, $t_1[s] = t_1[s'], \ldots, t_m[s] = t_m[s']$, so $G(t_1[s] \ldots t_m[s]) = G(t_1[s'] \ldots t_m[s'])$. Hence $t[s] = t[s']$.

If ϕ is an atomic formula $t_1 = t_2$ where t_1, t_2 are terms and t_1 and t_2 are terms with variables among $v_{i_1}, \ldots, v_{i_k}$, then $t_1[s] = t_1[s']$ and $t_2[s] = t_2[s']$. Assume s satisfies $t_1 = t_2$ then $t_1[s] = t_2[s]$ and so $t_1[s'] = t_2[s']$. Therefore s' satisfies $t_1 = t_2$. Similarly, if s' satisfies $t_1 = t_2$ then s satisfies $t_1 = t_2$.

If ϕ is an atomic formula $P(t_1 \ldots t_m)$ where P is an n-placed relation symbol and t_j, $1 \leq j \leq n$, is a term with variables among $v_{i_1}, \ldots, v_{i_k}$, then $t_1[s] = t_1[s'], \ldots, t_n[s] = t_n[s']$. Assume s satisfies $P(t_1 \ldots t_n)$ then $(t_1[s], \ldots, t_n[s]) \in R$ where R is the interpretation of P, so $(t_1[s'], \ldots, t_n[s']) \in R$. Hence s' satisfies $P(t_1 \ldots t_n)$. Similarly, if s' satisfies $P(t_1 \ldots t_n)$ then s satisfies $P(t_1 \ldots t_n)$.

Suppose this lemma is true for all formulas ψ such that length of $\psi <$ length of ϕ.

If ϕ is $\sim \psi$, then s satisfies ψ if and only if s' satisfies ψ.

Therefore s does not satisfy ψ if and only if s' does not satisfy ψ, i.e. s satisfies ϕ if and only if s' satisfies ϕ.

If ϕ is $\psi_1 \wedge \psi_2$, then ψ_1 and ψ_2 are formulas whose lengths < length of ϕ. Therefore s satisfies ψ_1 if and only if s' satisfies ψ_1 and s satisfies ψ_2 if and only if s' satisfies ψ_2. Hence s satisfies $\psi_1 \wedge \psi_2$ if and only if s' satisfies $\psi_1 \wedge \psi_2$.

If ϕ is $(\forall v_r)\ \psi$ where $v_r \notin \{v_{i_1}, \ldots, v_{i_k}\}$, then ψ is a formula of length < length of ϕ. Assume s satisfies $(\forall v_r)\ \psi$, then s satisfies ψ and $\bar{s}$ satisfies ψ where $\bar{s}$ is a sequence differing from s in at most r^{th} place. By induction hypothesis, s' satisfies ψ. Let $\bar{s}'$ be any sequence differing from s' in at most r^{th} place, so $\bar{s}$ and $\bar{s}'$ have the same components in the $i_1^{th}, \ldots, i_k^{th}$ places, therefore $\bar{s}'$ satisfies ψ. Hence s' satisfies $(\forall v_r)\ \psi$. Similarly, if s' satisfies $(\forall v_r)\ \psi$ then s satisfies $(\forall v_r)\ \psi$.

If ϕ is $(\forall v_r)\ \psi$ and $v_r \in \{v_{i_1}, \ldots, v_{i_k}\}$, so v_r must be v_{i_j} for some j, $1 \leq j \leq k$. Assume $s = (b_1, \ldots, b_{i_1-1}, a_{i_1}, \ldots, a_{i_k}, b_{i_k+1}, \ldots)$ satisfies $(\forall v_r)\ \psi$. Suppose $s' = (c_1, \ldots, c_{i_1-1}, a_{i_1}, \ldots, a_{i_k}, c_{i_k+1}, \ldots)$ does not satisfy $(\forall v_r)\ \psi$, then there exists a sequence $\bar{s}' = (c_1, \ldots, c_{i_1-1}, a_{i_1}, \ldots, d_{i_j}, \ldots, a_{i_k}, c_{i_k+1}, \ldots)$, which is differing from s' in at most i_j^{th} place does not satisfy ψ. By induction hypothesis, $\bar{s} = (b_1, \ldots, b_{i_1-1}, a_{i_1}, \ldots, d_{i_j}, \ldots, a_{i_k}, b_{i_k+1}, \ldots)$, which is a sequence differing from s in at most i_j^{th} place does not satisfy ψ. Contradicts to the assumption, hence s' satisfies $(\forall v_r)\ \psi$. Similarly, if s' satisfies $(\forall v_r)\ \psi$ then s satisfies $(\forall v_r)\ \psi$.

Hence this Lemma is true for all formulas ϕ.

2.39 <u>Definition.</u> A sentence ϕ of L is true in a model $M = \langle A, \mathcal{J} \rangle$ or M is a model of ϕ ($M \models \phi$) if and only if every sequence of elements of A satisfies ϕ in M.

2.40 <u>Lemma.</u> Let ϕ be a sentence of L and $M = \langle A, \mathcal{J} \rangle$ a model of L. If there exists a sequence of elements of A satisfies ϕ in M, then every sequence of elements of A satisfies ϕ in M.

<u>proof.</u> Assume $(a_1, a_2, \ldots)$, a sequence of elements of A satisfies ϕ in M. Let $(b_1, b_2, \ldots)$ be any sequence of elements of A. Want to show that $(b_1, b_2, \ldots)$ satisfies ϕ in M. We must prove this by induction on length of sentence ϕ.

continue

First, we must prove that if t is a term with no free variables, then $t[s] = t[s']$.

If $t = c$, where c is a constant symbol, then $t[s] = \mathcal{I}(c) = t[s']$.

Assume this is true for all terms t with no free variables of lengths < k. If t is $F(t_1 \ldots t_m)$ of length k, where F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms with no free variables of lengths < k, then $t[s] = G(t_1[s], \ldots, t_m[s])$ and $t[s'] = G(t_1[s'], \ldots, t_m[s'])$ where G is the interpretation of F in M. By induction hypothesis, we get $t_1[s] = t_1[s'], \ldots, t_m[s] = t_m[s']$ and so $G(t_1[s], \ldots, t_m[s]) = G(t_1[s'], \ldots, t_m[s'])$. Hence $t[s] = t[s']$.

If ϕ is an atomic formula $t_1 = t_2$ where t_1, t_2 are terms with no free variables and s satisfies $t_1 = t_2$, then $t_1[s] = t_2[s]$. Since $t_1[s] = t_1[s']$ and $t_2[s] = t_2[s']$, hence $t_1[s'] = t_2[s']$, i.e. s' satisfies $t_1 = t_2$.

If ϕ is $P(t_1 \ldots t_n)$ where P is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms with no free variables and s satisfies $P(t_1 \ldots t_n)$, then $(t_1[s], \ldots, t_n[s]) \in R$, R is the interpretation of P in M. Therefore, $t_1[s] = t_1[s'], \ldots, t_n[s] = t_n[s']$ and so $(t_1[s'], \ldots, t_n[s']) \in R$, i.e. s' satisfies $P(t_1 \ldots t_n)$.

Assume this lemma is true for all sentences ψ such that length of $\psi <$ length of ϕ.

If ϕ is $\psi_1 \wedge \psi_2$, then ψ_1 and ψ_2 are sentences of lengths < length of ϕ. By induction hypothesis, we get $(b_1, b_2, \ldots)$ satisfies ψ_1 and

$(b_1, b_2,\ldots)$ satisfies ψ_2. Hence $(b_1, b_2,\ldots)$ satisfies $\psi_1 \wedge \psi_2$.

If ϕ is $\sim \psi$, then from the assumption, there exists a sequence satisfies $\sim \psi$, i.e. this sequence does not satisfy ψ. Suppose not, so there exists a sequence satisfies ψ which contradicts to the assumption. Hence every sequence satisfies $\sim \psi$.

If ϕ is $(\forall v_i)\ \psi$, then ψ is a sentence of length < length of ϕ. Therefore every sequence differing from $(a_1, a_2,\ldots)$ in at most i^{th} place satisfies ψ, i.e. $(a_1,\ldots a_{i-1}, c,\ldots a_{i+1},\ldots)$, for any c, satisfies ψ.

<u>case 1</u> : v_i is not free in ψ. Since ψ is a sentence whose length < length of ϕ, we get $(b_1, b_2,\ldots)$ satisfies ϕ .

<u>case 2</u> : v_i is free in ψ. By Lemma 2.38, we get $(b_1,\ldots b_{i-1}, c, b_{i+1},\ldots)$, for any c, satisfies ψ. Hence every sequence differing from $(b_1, b_2,\ldots)$ in at most i^{th} place satisfies ψ. Thus $(b_1, b_2,\ldots)$ satisfies $(\forall v_i)\ \psi$.

Hence, we get $(b_1, b_2,\ldots)$ satisfies for all sentences ϕ. Since $(b_1, b_2,\ldots)$ is arbitrary sequence, we get every sequence of elements of A satisfies for all sentences ϕ of L.

2.41 <u>Theorem</u>. Let ϕ be a sentence in L and $M = \langle A, \mathcal{J} \rangle$ a model of L. If M is not a model of ϕ, then M is a model of $\sim \phi$.

<u>proof</u>. Assume M is not a model of ϕ, then there exists a sequence of elements of A does not satisfy ϕ, i.e. a sequence satisfies

$\sim\phi$. Since ϕ is a sentence, we get $\sim\phi$ is a sentence. Hence, by Lemma 2.40, we get every sequence of elements of A satisfies $\sim\phi$. Thus M is a model of $\sim\phi$.

2.42 Note. If M is not a model of ϕ, we then use the notation $M \not\models \phi$.

2.43 Definition. Let Σ be a set of sentences. M is a model of Σ $(M \models \Sigma)$ if and only if M is a model of each sentence ϕ in Σ.

2.44 Definition. A sentence ϕ of L is valid $(\models \phi)$ if and only if ϕ is true in every model of L. If ϕ is not valid, we use the notation $\not\models \phi$.

2.45 Definition. A sentence ψ is a consequence of another sentence ϕ, in symbols $\phi \models \psi$, if and only if every model of ϕ is a model of ψ. A sentence ϕ is a consequence of a set of sentences Σ, in symbols $\Sigma \models \phi$, if and only if every model of Σ is a model of ϕ.

2.46 Definition. Two models M and M' of L are elementarily equivalent, in symbols $M \equiv M'$, if and only if every sentence that is true in M is true in M', and vice versa.

2.47 Lemma. If t and u are terms and s is a sequence of model M, and t' results from t by substitution of u for all occurrences of v_i and s' results from s by substituting $u[s]$ for the i^{th} component of s, then $t'[s] = t[s']$.

proof. We must prove this lemma by induction on length of a term t.

(i) $t = v_j$ where v_j is an individual variable.

If $v_i \neq v_j$, then $t' = t$. Since v_i is not in t, we get $t'[s] = t[s']$ by Lemma 2.38.

If $v_i = v_j$, then $t' = u$ and so $t'[s] = u[s] = t[s']$.

(ii) t = constant symbol c, therefore $t' = t$ and $t'[s] = t[s'] = \mathcal{I}(c)$.

Assume this lemma is true for all terms t of length $< k$.

Let t be of the form $F(t_1 \ldots t_m)$ of length k, where F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms of length $< k$. Then $t[s'] = G(t_1[s'] \ldots t_m[s'])$ where G is the interpretation of F in M. Since $t' = F(t_1' \ldots t_m')$, we get $t'[s] = G(t_1'[s] \ldots t_m'[s])$. By induction hypothesis, we get $t_i[s'] = t_i'[s]$, $1 \leq i \leq m$. Thus $t[s'] = t'[s]$.

Hence this lemma is true for all terms t.

2.48 <u>Lemma</u>. Let $\phi(v_i)$ be a formula, and $\phi(t)$ results from $\phi(v_i)$ by replacing free occurrences of v_i with a term t, where t is a term such that no variable x in t shall occur bound in $\phi(t)$ at the place where it is introduced. Then $s = (a_1, a_2, \ldots)$ satisfies $\phi(t)$ if and only if $s' = (a_1, \ldots, a_{i-1}, t[s], a_{i+1}, \ldots)$ satisfies $\phi(v_i)$.

<u>proof</u>. We must prove this lemma by induction on length of a formula ϕ.

Suppose ϕ is an atomic formula $t_1 = t_2$ where t_1, t_2 are terms. If $v_i \notin t_1 = t_2$ then $\phi(v_i) = \phi(t) = \phi$. Therefore s satisfies $\phi(t)$ if and only if s' satisfies $\phi(v_i)$ by Lemma 2.38. If $v_i \in t_1 = t_2$, then $\phi(t)$

is $(t_1 = t_2)\binom{v}{t}i)$ and $(t_1 = t_2)\binom{v}{t}i)$ is $t_1\binom{v}{t}i) = t_2\binom{v}{t}i)$ where $t_i\binom{v}{t}i)$ is a term obtained from t_i by replacing v_i with t. Assume s satisfies $\phi(t)$, then $t_1\binom{v}{t}i)[s] = t_2\binom{v}{t}i)[s]$. By Lemma 2.47, we get $t_1[s'] = t_2[s']$, therefore s' satisfies $\phi(v_i)$. Similarly, if s' satisfies $\phi(v_i)$ then s satisfies $\phi(t)$.

Suppose ϕ is $P(t_1 \ldots t_n)$ where P is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms. If $v_i \notin \{v/v \in t_1$ or $\ldots, v \in t_n\}$, then $\phi(t) = \phi(v_i) = \phi$. By Lemma 2.38, we get s satisfies $\phi(t)$ if and only if s' satisfies $\phi(v_i)$. If $v_i \in \{v/v \in t_1$ or $\ldots, v \in t_n\}$, then $\phi(t) = P(t_1\binom{v}{t}i) \ldots t_n\binom{v}{t}i))$. Assume s satisfies $\phi(t)$, then $(t_1\binom{v}{t}i)[s], \ldots, t_n\binom{v}{t}i)[s]) \in R$ where R is the interpretation of P. By Lemma 2.47, we get $(t_1[s'], \ldots, t_n[s']) \in R$, so s' satisfies $P(t_1 \ldots t_n)$, i.e. s' satisfies $\phi(v_i)$. Similarly, if s' satisfies $\phi(v_i)$ then s satisfies $\phi(t)$.

Assume this lemma is true for all formulas ψ such that length of $\psi <$ length of ϕ.

If ϕ is $\sim \psi$, then lemma is true for ψ. Therefore s satisfies $\psi(t)$ if and only if s' satisfies $\psi(v_i)$. Thus s does not satisfy $\psi(t)$ if and only if s' does not satisfy $\psi(v_i)$, i.e. s satisfies $\sim\psi(t)$ and only if s' satisfies $\sim\psi(v_i)$.

If ϕ is $\psi_1 \wedge \psi_2$, then ψ_1 and ψ_2 are formulas of lengths < length of ϕ. Therefore s satisfies $\psi_1(t)$ if and only if s' satisfies $\psi_1(v_i)$ and s satisfies $\psi_2(t)$ if and only if s' satisfies $\psi_2(v_i)$. Hence s satisfies $\psi_1(t) \wedge \psi_2(t)$ if and only if s' satisfies $\psi_1(v_i) \wedge \psi_2(v_i)$, i.e. s satisfies $(\psi_1 \wedge \psi_2)(t)$ if and only if s' satisfies $(\psi_1 \wedge \psi_2)(v_i)$.

If ϕ is $(\forall v_j)\ \psi$; $v_j \neq v_i$, and assume s satisfies $\phi(t)$, then by induction hypothesis, s satisfies $\psi(t)$ if and only if s' satisfies $\psi(v_i)$. Let $\bar{s}$ be any sequence differing from s in at most j^{th} place, then $\bar{s}$ satisfies $\psi(t)$. Thus $\bar{s}'$ satisfies $\psi(v_i)$ where $\bar{s}'$ is any sequence differing from s' in at most j^{th} place. Therefore s' satisfies $\phi(v_i)$. Similarly, if s' satisfies $\phi(v_i)$ then s satisfies $\phi(t)$.

If ϕ is $(\forall v_j)\ \psi$; $v_j = v_i$, then $\phi(t) = \phi(v_i)$ and v_i is not free in ϕ.

Hence this lemma is true for all formulas ϕ.

2.49 Theorem. (i) Logical axioms of L are valid.

(ii) Rules of inference preserve validity.

proof. (i) To show logical axioms (i) - (viii) are valid.

Axiom (i) : $\phi \longrightarrow (\psi \longrightarrow \phi)$.

Let $M = \langle A, \mathcal{I} \rangle$ be any model of L and $(a_1, a_2, \ldots)$ be any sequence of elements of A. Suppose $(a_1, a_2, \ldots)$ does not satisfy $\phi \rightarrow (\psi \longrightarrow \phi)$. Therefore $(a_1, a_2, \ldots)$ satisfies ϕ but does not satisfy $\psi \longrightarrow \phi$, i.e. satisfies ψ but does not satisfy ϕ which is a contradiction. Then $(a_1, a_2, \ldots)$ satisfies $\phi \longrightarrow (\psi \longrightarrow \phi)$. Thus $\phi \longrightarrow (\psi \longrightarrow \phi)$ is true in M and M is arbitrary model, so $\phi \rightarrow (\psi \rightarrow \phi)$ is valid.

Axiom (ii) : $(\phi \longrightarrow (\psi \longrightarrow \theta)) \longrightarrow ((\phi \longrightarrow \psi) \longrightarrow (\phi \longrightarrow \theta))$.

Let $M = \langle A, \mathcal{I} \rangle$ be any model of L and $(a_1, a_2, \ldots)$ be any sequence of elements of A. Suppose $(a_1, a_2, \ldots)$ does not satisfy axiom (ii), then

$(a_1, a_2, \ldots)$ satisfies $\phi \longrightarrow (\psi \longrightarrow \theta)$ but does not satisfy $(\phi \longrightarrow \psi) \longrightarrow (\phi \longrightarrow \theta)$. From this, we get $(a_1, a_2, \ldots)$ does not satisfy ϕ and satisfies ϕ which is a contradiction. Thus $(a_1, a_2, \ldots)$ satisfies axiom (ii). Then axiom (ii) is true in M and M is arbitrary model, thus axiom (ii) is valid.

Axiom (iii) : $(\sim \phi \longrightarrow \sim \psi) \longrightarrow ((\sim \phi \longrightarrow \psi) \longrightarrow \phi)$.

Let $M = < A, \mathcal{J} >$ be any model of L and $(a_1, a_2, \ldots)$ be any sequence of elements of A. Suppose $(a_1, a_2, \ldots)$ does not satisfy axiom (iii), then $(a_1, a_2, \ldots)$ satisfies $\sim \phi \longrightarrow \sim \psi$ but does not satisfy $(\sim \phi \longrightarrow \psi) \longrightarrow \phi$. From this, we get $(a_1, a_2, \ldots)$ satisfies ϕ and does not satisfy ϕ which is a contradiction, so $(a_1, a_2, \ldots)$ satisfies axiom (iii). Then axiom (iii) is true in M and M is arbitrary model, thus axiom (iii) is valid.

Axiom (iv) : $(\forall v_i)\ (\phi \longrightarrow \psi) \longrightarrow (\phi \longrightarrow (\forall v_i)\ \psi)$, where v_i is a variable not free in ϕ.

Suppose there exists a model $M = < A, \mathcal{J} >$ and a sequence $(a_1, a_2, \ldots)$ of elements of A such that $(a_1, a_2, \ldots)$ does not satisfy axiom (iv) in M. Therefore $(a_1, a_2, \ldots)$ satisfies $(\forall v_i)\ (\phi \longrightarrow \psi)$ but does not satisfy $\phi \longrightarrow (\forall v_i)\ \psi$, i.e. $(a_1, a_2, \ldots)$ satisfies ϕ but does not satisfy $(\forall v_i)\ \psi$. Since v_i is not free in ϕ; we get, $(a_1, a_2, \ldots\ a_{i-1}, b, a_{i+1}, \ldots)$ satisfies ϕ for any b, by Lemma 2.38. Hence $(a_1, a_2, \ldots, a_{i-1}, b, a_{i+1}, \ldots)$ satisfies ψ for any b. Thus $(a_1, a_2, \ldots)$ satisfies $(\forall v_i)\ \psi$, contradiction. Therefore for any model $M = < A, \mathcal{J} >$ of L and any sequence of elements of A satisfies axiom (iv). Thus axiom (iv) is valid.

Axiom (v) : $(\forall v_i)\phi \longrightarrow \psi$ where ψ is a formula obtained from ϕ by freely substituting each free occurrence of v_i in ϕ by a term t.

Let $M = < A, \mathcal{J} >$ be any model of L and $s = (a_1, a_2, \ldots)$ be any sequence of elements of A. Suppose s does not satisfy axiom (v), then s satisfies $(\forall v_i)\ \phi$ but does not satisfy ψ. Let $s' = (a_1, \ldots, a_{i-1}, t[s], a_{i+1}, \ldots)$ be any sequence differing from s in at most i^{th} place, then s' satisfies ϕ. By Lemma 2.48, s satisfies ψ, which is a contradiction. Thus s satisfies axiom (v), so axiom (v) is true in M and M is arbitrary model, then axiom (v) is valid.

Axiom (vi) : $v_i = v_i$, v_i is variable.

Let $M = < A, \mathcal{J} >$ be any model of L and $s = (a_1, a_2, \ldots)$ be any sequence of elements of A. Suppose s does not satisfy axiom (vi), then there exists a_i such that $a_i \neq a_i$ which is impossible. Thus s satisfies axiom (vi) and axiom (vi) is true in M, and M is arbitrary model, then axiom (vi) is valid.

Axiom (vii) : $x_i = x_j \longrightarrow t(v_1, ..v_{i-1} x_i v_{i+1} \ldots v_n) = t(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$ where x_i, x_j are variables and $t(v_1 \ldots v_n)$ is a term.

Let $M = < A, \mathcal{J} >$ be any model of L and $s = (a_1, a_2, \ldots)$ be any sequence of elements of A such that $a_i = a_j$.

If t is v_i, then $t[s]_i = a_i = a_j = t[s]_j$, where $t[s]_i$ is the value of t at $(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots)$ and $t[s]_j$ is the value of t at $(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots)$

If t is a constant symbol c and x is the interpretation of c in M, then $t[s]_i = x = t[s]_j$.

Assume this axiom is true for all terms t of length < k. Let t be of the form $F(t_1 \ldots t_m)$ of length k, where F is an m-placed function symbol and $t_1, \ldots, t_m$ are terms of length < k. By induction hypothesis, $t_k[s]_i = t_k[s]_j$ for k = 1, 2,..., m. Thus $t[s]_i = G(t_1[s]_i \ldots t_m[s]_i)$ $= G(t_1[s]_j \ldots t_m[s]_j) = t[s]_j$, where G is the interpretation of F in M. Hence s satisfies axiom (vii), so axiom (vii) is true in M, and M is arbitrary model, then axiom (vii) is valid.

Axiom (viii) : $x_i = x_j \longrightarrow (\phi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n) \longrightarrow \phi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n))$ where x_i, x_j are variables and $\phi(v_1 \ldots v_n)$ is a formula.

Let $M = \langle A, \mathcal{J} \rangle$ be any model of L and $s = (a_1, a_2, \ldots)$ be any sequence of elements of A such that $a_i = a_j$.

If ϕ is an atomic formula $t_1 = t_2$ where t_1, t_2 are terms, then by axiom (vii), $t_1[s]_i = t_1[s]_j$ and $t_2[s]_i = t_2[s]_j$. Assume s satisfies $\phi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$, then $t_1[s]_i = t_2[s]_i$. Therefore $t_1[s]_j = t_2[s]_j$, so s satisfies $\phi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$.

If ϕ is $P(t_1 \ldots t_n)$ where P is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms, then by axiom (vii), $t_k[s]_i = t_k[s]_j$, for k = 1, 2,..., n. Assume s satisfies $\phi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$, then $(t_1[s]_i, \ldots, t_n[s]_i) \in R$ where R is the interpretation of P in M. Hence $(t_1[s]_j, \ldots, t_n[s]_j) \in R$, and so s satisfies $\phi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$.

Assume this axiom is true for all formulas ψ such that length of ψ < length of ϕ.

If ϕ is $\sim \psi$, then ψ is a formula of length < length of ϕ. Assume s satisfies $\sim \psi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$. Since $x_i = x_j$, we get $\psi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$ is $\psi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$. Hence s satisfies $\sim \psi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$.

If ϕ is $\psi_1 \wedge \psi_2$, then ψ_1 and ψ_2 are formulas of lengths < length of ϕ. Thus, if s satisfies $\psi_1(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$ then s satisfies $\psi_1(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$, and if s satisfies $\psi_2(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$ then s satisfies $\psi_2(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$. Hence, if s satisfies $(\psi_1 \wedge \psi_2)(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$ then s satisfies $(\psi_1 \wedge \psi_2)(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$.

If ϕ is $(\forall v_r)\ \psi$, $v_r \neq v_i$, then ψ is a formula of length < length of ϕ. Assume s satisfies $\phi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$. By induction hypothesis, s satisfies $\psi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$. If $s' = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_r', \ldots)$ be any sequence differing from s in at most r^{th} place, then s' satisfies $\psi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$. Since $a_i = a_j$, we get $s'' = (a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_r', \ldots)$ satisfies $\psi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$. Since s'' is any sequence differing from s in at most r^{th} place, we get s satisfies $\phi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$.

If ϕ is $(\forall v_r)\ \psi$, $v_r = v_i$, and assume that s satisfies $\phi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots v_n)$, then by induction hypothesis, s satisfies $\psi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$. Let $s' = (a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots)$ be any sequence differing from s in at most i^{th} place and s' satisfies $\psi(v_1 \ldots v_{i-1} x_i v_{i+1} \ldots$

v_n). Since $a_i = a_j$, we get s' satisfies $\psi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$. Hence s satisfies $\phi(v_1 \ldots v_{i-1} x_j v_{i+1} \ldots v_n)$.

Thus s satisfies this axiom for all formulas ϕ, so s satisfies axiom (viii), and axiom (viii) is true in M, and M is arbitrary model, then axiom (viii) is valid.

(ii) (a) To show MP. preserves validity, i.e. if $M \models \psi$ and $M \models \psi \rightarrow \phi$ then $M \models \phi$, for any model M and any formulas ϕ, ψ of L.

Let ϕ , ψ be any formulas of L, M = $< A, \mathcal{J} >$ any model of L and s any sequence of elements of A. Assume $M \models \psi$ and $M \models \psi \rightarrow \phi$, i.e. s satisfies ψ and s satisfies $\psi \longrightarrow \phi$. From s satisfies $\psi \longrightarrow \phi$, we get s does not satisfy ψ or s satisfies ϕ. Thus s satisfies ϕ. Therefore ϕ is true in M and hence $M \models \phi$.

(b) To show Generalization preserves validity, i.e. if $M \models \phi$ then $M \models (\forall v_i) \phi$, for any model M and any formula ϕ of L.

Let ϕ be any formula of L, M = $< A, \mathcal{J} >$ any model of L and s any sequence of elements of A. Assume $M \models \phi$, i.e. s satisfies ϕ. Let s' be any sequence differing from s in at most i^{th} place, so s' is sequence of elements of A. Thus s' satisfies ϕ. Therefore s satisfies $(\forall v_i) \phi$, i.e. $(\forall v_i)\phi$ is true in M. Hence $M \models (\forall v_i) \phi$.

One of the important theorems of first-order Model Theory is Gödel's Completeness Theorem. Before we prove this theorems, we need a new definition, two lemmas and the Extended Completeness Theorem.

2.50 Definition. Let T be a set of sentences of L and let C be a set of constant symbols of L (C might be a proper subset of the set of all constant symbols of L). We say that C is a set of witnesses for T in L if and only if for every formula ϕ of L with at most one free variable, say v, there is a constant $c \in C$ such that

$$T \vdash (\exists v)\ \phi \longrightarrow \phi(c),$$

where $\phi(c)$ is obtained from ϕ by replacing simultaneously all free occurrences of v in ϕ by the constant c.

We say that T has witnesses in L if and only if T has some set C of witnesses in L.

2.51 Lemma. Every maximal consistent set of sentences T of L, which has witnesses C in L, has a model.

proof. Let T be a maximal consistent set of sentences of L, and C be a set of witnesses for T in L.

Define a relation $\sim$ on C as follows :

for all $c, d \in C$, $c \sim d$ if and only if $c = d \in T$. Since T is maximal consistent, we see that for $c, d, e, \in C$;

$c \sim c$,

if $c \sim d$ and $d \sim e$, then $c \sim e$,

if $c \sim d$ then $d \sim c$.

So $\sim$ is an equivalence relation on C. For each $c \in C$, let $\tilde{c}$ =

$\{d \in C \mid d \sim c\}$ be an equivalence class of c. We purpose to construct a model $M = \langle A, \mathcal{I} \rangle$ whose set of elements A is the set of all these equivalence classes $\tilde{c}$, for $c \in C$; so we define

(1) $A = \{\tilde{c} \mid c \in C\}$.

We now define the relations, constants and functions of M.

(i) For each n-placed relation symbol P in L, we define an n-placed relation R' on the set C by : for all $c_1, \ldots, c_n \in C$,

(2) $R'(c_1 \ldots c_n)$ if and only if $P(c_1 \ldots c_n) \in T$.

By the axiom of L, we have

$$\vdash P(c_1 \ldots c_n) \wedge c_1 = d_1, \wedge \ldots \wedge c_n = d_n \longrightarrow P(d_1 \ldots d_n).$$

If follows that we may define a relation R on A by

(3) $R(\tilde{c}_1 \ldots \tilde{c}_n)$ if and only if $P(c_1 \ldots c_n) \in T$. This relation R is the interpretation of the symbol P in M.

(ii) Consider a constant symbol d of L. Since $\vdash d = d$, we see that $\vdash (\exists v_0)(d = v_0)$ and so $T \vdash (\exists v_0)(d = v_0)$. Since T has witnesses, there is a constant $c \in C$ such that $T \vdash (\exists v_0)(d = v_0) \longrightarrow d = c$. Thus $T \vdash d = c$, and hence $d = c \in T$. The constant c may not be uniqued, but its equivalence class is unique because $\vdash (d = c \wedge d = c' \longrightarrow c = c')$. The constant d is interpreted in the model M by the (uniquely determined) element $\tilde{c}$ of A. In particular, if $d \in C$, then d is interpreted by its own equivalence class $\tilde{d}$ in M, because $(d = d) \in T$.

(iii) We handle the function symbols in a similar way. Let F be any m-placed function symbol of L, and let $c_1, \ldots, c_m \in C$. As before, we have $T \vdash (\exists v_0)(F(c_1 \ldots c_m) = v_0)$ and because T has witnesses, there is a constant $c \in C$ such that $(F(c_1 \ldots c_m) = c) \in T$. Once more, we have a slight difficulty because c may not be unique, and use our axiom to obtain :

$\vdash (F(c_1 \ldots c_m) = c \wedge c_1 = d_1 \wedge \ldots \wedge c_m = d_m \wedge c = d) \rightarrow F(d_1 \ldots d_m) = d.$
This shows that a function G can be defined on the set A of equivalence classes by the rule.

(4) $G(\tilde{c}_1 \ldots \tilde{c}_m) = \tilde{c}$ if and only if $(F(c_1 \ldots c_m) = c) \in T$. We interpret the function symbol F by the function G in the model M.

We have now specified the universe set and the interpretation of each symbol of L in M, so we have completed the definition of the model M.

We proceed to prove that M is a model of T. We will prove $M \models \phi$ if and only if $\phi \in T$ by induction on length of sentence ϕ.

First of all, using (4), we get : for every term t of L with no free variables and for every constant $c \in C$,

(5) $M \models t = c$ if and only if $(t = c) \in T$.

Using the fact that C is a set of witnesses for T, we have :

for any two terms t_1, t_2 of L with no free variables,

(6) $M \models t_1 = t_2$ if and only if $t_1 = t_2 \in T$, and

for any $P(t_1 \ldots t_n)$ of L containing no free variables,

(7) $M \models P(t_1 \ldots t_n)$ if and only if $P(t_1 \ldots t_n) \in T$.

Suppose $M \models \psi$ if and only if $\psi \in T$ for all sentences ψ such that length of $\psi <$ length of ϕ.

If ϕ is $\sim\psi$, then $M \models \psi$ if and only if $\psi \in T$, and so $M \models \sim\psi$ if and only if $\sim\psi \in T$.

If ϕ is $\psi_1 \wedge \psi_2$, then ψ_1, ψ_2 are sentences of lengths < length of ϕ. Therefore $M \models \psi_1$ if and only if $\psi_1 \in T$ and $M \models \psi_2$ if and only if $\psi_2 \in T$. Thus $M \models \psi_1 \wedge \psi_2$ if and only if $\psi_1 \wedge \psi_2 \in T$.

Suppose ϕ is $(\exists v)\ \psi$. If $M \models \phi$, then for some $\tilde{c} \in A$, $M \models \psi[\tilde{c}]$. This means that $M \models \psi(c)$, where $\psi(c)$ is obtained from ψ by replacing all free occurrences of v by c. Thus $\psi(c) \in T$ and because $\vdash \psi(c) \longrightarrow (\exists v)\ \psi$, we have $\phi \in T$. On the other hand, if $\phi \in T$, then because T has witnesses, there exists a constant $c \in C$ such that $T \vdash (\exists v)\psi \longrightarrow \psi(c)$. As T is maximal consistent, $\psi(c) \in T$, so $M \models \psi(c)$. This gives $M \models \psi[\tilde{c}]$ and $M \models \phi$.

This shows that M is a model of T.

2.52 <u>Lemma</u>. Every consistent set of sentences T of L can be extended to a consistent set of sentences $\overline{T}$ of $\overline{L} = L \cup C$, where C is a set of new constant symbols of power $|C| = ||L||$, such that $\overline{T}$ has witnesses in $\overline{L}$.

<u>proof</u>. Let $\omega = ||L||$. For each $\alpha < \omega$. Let c_α be a constant symbol which does not occur in L and such that $c_\alpha \neq c_\gamma$ if $\alpha < \gamma < \omega$. Let $C = \{c_\alpha \mid \alpha < \omega\}$, $\overline{L} = L \cup C$. Clearly $||\overline{L}|| = \omega$, so we may arrange all formulas of $\overline{L}$ with at most one free variable in a sequence ϕ_ξ, $\xi < \omega$. We now define an increasing sequence of sets of sentences of $\overline{L}$: $T = T_0 \subset T_1 \subset \ldots \subset T_\xi \subset \ldots, \xi < \omega$, and a sequence d_ξ, $\xi < \omega$, of constants from C such that :

(i) each T_ξ is consistent in $\overline{L}$;

(ii) if $\xi = \zeta + 1$, then $T_\xi = T_\zeta \cup \{(\exists v_\zeta)\phi_\zeta \longrightarrow \phi_\zeta(d_\zeta)\}$; v_ζ is the free variable in ϕ_ζ if it has one, otherwise $v_\zeta = v_0$;

(iii) if ξ is a limit ordinal different from zero, then $T_\xi = \bigcup_{\zeta<\xi} T_\zeta$.

Suppose that T_ζ has been defined. Note that the number of sentences in T_ζ which are not sentences of L is smaller than ω, i.e. the cardinal of the set of such sentences is less than ω. Furthermore, each such sentence contains at most a finite number of constants from C. Therefore, let d_ζ be the first element of C which has not yet occurred in T_ζ. We show that

$$T_{\zeta+1} = T_\zeta \cup \{(\exists v_\zeta)\phi_\zeta \longrightarrow \phi_\zeta(d_\zeta)\}$$

is consistent. If this were not the case, then by proposition 2.32 (i), we get

$$T_\zeta \vdash \sim ((\exists v_\zeta)\phi_\zeta \longrightarrow \phi_\zeta(d_\zeta)).$$

Therefore $T_\zeta \vdash (\exists v_\zeta)\phi_\zeta \wedge \sim \phi_\zeta(d_\zeta)$. As d_ζ does not occur in T_ζ, so

$$T_\zeta \vdash (\exists v_\zeta)\ \phi_\zeta \wedge \sim \phi_\zeta\ (v_\zeta).$$

Hence $T_\zeta \vdash (\forall v_\zeta)\ ((\exists v_\zeta)\ \phi_\zeta \wedge \sim \phi_\zeta(v_\zeta))$, and so

$$T_\zeta \vdash (\exists v_\zeta)\ \phi_\zeta \wedge \sim (\exists v_\zeta)\ \phi_\zeta \quad ,$$

which contradicts the consistency of T_ζ . If ξ is a nonzero limit ordinal, and each member of the increasing chain T_ζ , $\zeta < \xi$, is consistent, then obviously $T_\xi = \bigcup_{\zeta<\xi} T_\zeta$ is consistent. This complete the induction.

Now we let $\overline{T} = \bigcup_{\xi<\omega} T_\xi$. It is evident that $\overline{T}$ is consistent in $\overline{L}$ and $\overline{T}$ is an extension of T. Next, we want to show that C is a set of witnesses for $\overline{T}$ in $\overline{L}$. Suppose ϕ is a formula of $\overline{L}$ with at most one free variable v. Then we may suppose that $\phi = \phi_\xi$ and $v = v_\xi$ for some $\xi < \omega$. Since $T_{\xi+1} = T_\xi \cup \{(\exists v_\xi)\ \phi_\xi \rightarrow \phi_\xi(d_\xi)\}$, we get $(\exists v_\xi)\ \phi_\xi \longrightarrow \phi_\xi (d_\xi) \in T_{\xi+1}$, and so $\in \overline{T}$. Then $\overline{T} \vdash (\exists v)\phi \longrightarrow \phi(c)$ for some $c \in C$. Thus C is a set of witnesses for $\overline{T}$ in $\overline{L}$.

2.53 <u>Theorem</u>. (Extended Completeness Theorem). Let Σ be a set of sentences of L. Then Σ is consistent if and only if Σ has a model.

<u>proof</u>. Assume Σ is consistent. By Lemma 2.52, we can extend Σ to $\overline{\Sigma}$ which is consistent and has witnesses in $\overline{L}$. By Lindenbaum's Theorem, we can extend $\overline{\Sigma}$ to a maximal consistent $\overline{\overline{\Sigma}}$ which has witnesses in $\overline{L}$. Therefore, by Lemma 2.51, $\overline{\overline{\Sigma}}$ has a model $\overline{M} = \langle A, \mathcal{J} \cup \mathcal{J}' \rangle$ for $\overline{L}$, so let $M = \langle A, \mathcal{J} \rangle$ be the model of L which is the reduct of $\overline{M}$ to L. Because sentences in Σ do not involve constants of $\overline{L}$ not in L, we see that M is a model of Σ.

To prove the converse, assume that Σ has a model M. Therefore $M \models \phi$ for each sentence ϕ of Σ. Suppose Σ is inconsistent, so $\Sigma \vdash \psi \wedge \sim\psi$ for any formula ψ of L. Then there exists a finite sequence of formulas $\theta_1, \ldots, \theta_n$ such that $\theta_n = \psi \wedge \sim \psi$ in which each θ_i, $1 \leq i \leq n$, is a logical axiom, or a member of Σ, or a conclusion from θ_j, θ_k $(j, k < i)$ by MP., or a conclusion from θ_j $(j < i)$ by generalization. By Lemma 2.49 (i), if θ_i is a logical axiom, then $M \models \theta_i$, and if $\theta_i \in \Sigma$, then $M \models \theta_i$. By Lemma 2.49 (ii); if $M \models \theta_j$ and $M \models \theta_j \rightarrow \theta_i$ then $M \models \theta_i$, and if $M \models \theta_j$ then $M \models (\forall v_i)\ \theta_j$. Therefore $M \models \theta_i$, $1 \leq i \leq n$, so we get $M \models \psi \wedge \sim\psi$. Hence $M \models \psi$ and $M \models \sim \psi$ which is impossible. Thus Σ is consistent.

2.54 Theorem. (Gödel's Completeness Theorem.) Let Σ be a set of sentences of L and ϕ a sentence. Then $\Sigma \vdash \phi$ if and only if $\Sigma \models \phi$. In particular, $\vdash \phi$ if and only if $\models \phi$.

proof. Assume $\Sigma \vdash \phi$. Let M be any model of Σ, i.e. $M \models \psi$ for each sentence ψ of Σ. Since $\Sigma \vdash \phi$, there exists a finite sequence of formulas $\theta_1, \ldots, \theta_n$ such that $\theta_n = \phi$ and each i, $1 \leq i \leq n$, θ_i is a logical axiom, or θ_i is a member of Σ, or θ_i is a conclusion from θ_j, θ_k $(j, k < i)$ by MP., or θ_i is a conclusion from θ_j, $(j < i)$ by generalization. If θ_i is a logical axiom, then $M \models \theta_i$, and if $\theta_i \in \Sigma$, then $M \models \theta_i$, by Lemma 2.49 (i). If $M \models \theta_j$ and $M \models \theta_j \rightarrow \theta_i$ then $M \models \theta_i$ and if $M \models \theta_j$ then $M \models (\forall v_i)\ \theta_j$ by Lemma 2.49 (ii). Therefore $M \models \theta_i$, $1 \leq i \leq n$, i.e. $M \models \phi$. Hence $\Sigma \models \phi$.

To prove the converse, assume that $\Sigma \models \phi$. Suppose $\Sigma \nvdash \phi$. By proposition 2.32 (ii), $\Sigma \cup \{\sim \phi\}$ is consistent. By Lemma 2.52, $\Sigma \cup$

$\{\sim\phi\}$ has a model M, i.e. $M \models \Sigma$ and $M \models \sim \phi$. Since $\Sigma \models \phi$, it follows that if $M \models \Sigma$, then $M \models \phi$. Therefore $M \models \phi$ and $M \models \sim \phi$ which is impossible. Thus $\Sigma \vdash \phi$.

2.55 Definition. A first-order theory T of L is a collection of sentences of L.

Since theories are sets of sentences of L, we can define a model of a theory and a consistent theory as before

2.56 Definition. A set of axioms of a theory T is a set of sentences with the same consequences as T.

The most convenient and standard way of giving a theory T is by listing a finite or infinite set of axioms for it. Another way to give a theory is as follows : Let M be a model of L; then the theory of M is the set of all sentences which is true in M.

2.57 Theorem. (Löwenheim's Theorem.) Every consistent theory T in L has a model of power at most $||L||$, i.e. if T has a model, then T has a countable model.

proof. In the proof of Theorem 2.53, we may choose a model $\overline{M}$ of $\overline{L}$ such that every element is a constant, and we have $|A| \leq ||\overline{L}|| = ||L||$.