# แบบจำลองฮิกซ์ขนาดย่อม

นายภาวิน อิทธิสมัย



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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ปีการศึกษา 2550

# LITTLE HIGGS MODELS

Mister Pawin Ittisamai

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| By | Mr. Pawin Ittisamai |
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree.

..............................................Dean of the Faculty of Science
(Professor Piamsak Menasveta, Ph.D.)

THESIS COMMITTEE

..............................................Chairman
(Assistant Professor Pornchai Pacharintanakul, Ph.D.)

..............................................Thesis Advisor
(Ahpisit Ungkitchanukit, Ph.D.)

..............................................Member
(Burin Asavapibhop, Ph.D.)

..............................................Member
(Auttakit Chatrabhuti, Ph.D.)

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ปัญหาลำดับชั้นขนาดย่อม (Little Hierarchy problem) คือความขัดแย้งระหว่างผลที่ได้จากการทดสอบความเที่ยงทางอิเล็คโทรวีก (Electroweak Precision Test) ซึ่งระบุว่าระดับพลังงานที่จำเป็นต้องใช้ฟิสิกส์นอกเหนือจากแบบจำลองมาตรฐานของฟิสิกส์อนุภาคมีค่าสูงกว่า 10 TeV ในขณะที่ผลที่ได้จากความต้องการให้การปรับค่าของมวลของอนุภาคฮิกซ์ในแบบจำลองดังกล่าวมีค่าไม่ละเอียดเกินกว่าที่ควรจะเป็นระบุว่าระดับพลังงานข้างต้นจะต้องมีค่าต่ำกว่า 2 TeV เนื่องจากระดับพลังงานระหว่าง 1-10 TeV นี้จะอยู่ในช่วงที่เครื่องเร่งอนุภาค LHC (Large Hadron Collider) ที่ CERN จะสามารถเข้าถึงได้ทำให้แบบจำลองฮิกซ์ขนาดย่อมซึ่งสามารถแก้ปัญหาข้างต้นเป็นได้ที่น่าสนใจ แต่เนื่องจากการที่แบบจำลองนี้เป็นแบบจำลองใหม่ทำให้ยังมีความขาดแคลนงานในลักษณะบทปฏิทัศน์อยู่

วิทยานิพนธ์ฉบับนี้ได้ถูกจัดทำขึ้นเพื่อเชื่อมโยงความรู้พื้นฐานระหว่างแบบจำลองมาตรฐานและแบบจำลองฮิกซ์ขนาดย่อมรวมทั้งแบบจำลองที่เกี่ยวข้องและมีความสำคัญในการศึกษาหรือวิจัยฟิสิกส์ในส่วนขยายของแบบจำลองมาตรฐานได้แก่เรื่อง การแตกหักเชิงพลวัตของสมมาตร (Dynamical Symmetry Breaking) การปรับแนวของสุญญากาศ (Vacuum Alignment) การปรากฏของสมมาตรในลักษณะไม่เชิงเส้น (Non-Linear Realisations of a Symmetry) และทฤษฎีการรวมครั้งใหญ่ (Grand Unification Theory) ในส่วนของแบบจำลองฮิกซ์ขนาดย่อมมีการศึกษาแบบจำลองฮิกซ์ขนาดย่อมที่สุด (Littlest Higgs) ในส่วนของทั้งการสร้างและการทดสอบแบบจำลอง โดยพบว่าแบบจำลองฮิกซ์ขนาดย่อมที่สุดซึ่งไม่ได้ถูกปรับปรุงไม่สามารถแก้ไขปัญหาปัญหาลำดับชั้นขนาดย่อมได้อย่างสมบูรณ์



ภาควิชา..ฟิสิกส์.............................................ลายมือชื่อนิสิต.....ภาวิน อิทธิสมัย.............................................
สาขาวิชา..ฟิสิกส์...........................................ลายมือชื่ออาจารย์ที่ปรึกษา.............................................
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The Little Hierarchy problem refers to the conflicts between the results from precision electroweak experiments, which require new physics beyond the standard model above 10 TeV scale, and the condition that the Higgs particle in the standard model is naturally light without extreme fine-tunings on some parameters, which, on the other hand, requires new physics below 2 TeV. Recently, a new class of models called Little Higgs was proposed to solve the problem while only few review articles were available.

This thesis provides rudiments of the Little Higgs and relevant topics that are useful for understanding physics beyond the standard model; including the hierarchy problem, dynamical symmetry breaking, vacuum alignment, non-linear realisations of a symmetry, and grand unification theory. Detail inspections on most economical model known as the Littlest Higgs along with some phenomenological signatures allowing distinction from other models which are testable in next generation particle colliders, such as the Large Hadron Collider (LHC) at CERN, are considered. It was found that this Littlest Higgs does not fully eliminate the little hierarchy problem.



Department.........Physics......... Student's signature..............................

Field of study......Physics......... Advisor's signature..............................

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# CHAPTER I

# INTRODUCTION

## 1.1 Introduction

The standard model of particle physics, especially the electroweak sector, pioneered by[1] Glashow, Weinberg, Salam, 't Hooft, and Veltman, is one of the most successful theories ever proposed. Its predictive power has been confirmed time after time by experiments. However, there is a part of the theory which is still far from complete. The main engine of the electroweak sector relies on the philosophy of spontaneous symmetry breaking which is explained by the so-called Brout-Englert-Higgs mechanism (more commonly called just as the Higgs mechanism). The BEH mechanism in the Glashow-Weinberg-Salam (GWS) theory is operated by an elementary scalar particle called the Higgs. The role of this Higgs is more or less indispensable as, roughly speaking, masses of all other particles depend on it (e.g., the couplings are even proportional to the Higgs' mass). Besides, the good high-energy behaviours of the standard model rely on the existence of the (fundamental) Higgs or any particle having similar properties. This is where the problem shows up. Every particle predicted or required by the GWS theory has been found except the Higgs. As a result, its mass remains an unknown parameter, rendering the electroweak symmetry breaking sector the poorly-understood part of the model.

At first sight, having one particle left undiscovered seems not to be a problem as physicists have been facing with the problem of the missing pieces for a long time (the neutrinos, the charm, and the top may be good examples). However, the Higgs, being a fundamental scalar, is unlike others. While physicists have come up with various reasons (e.g., symmetries) for predicting and explaining the "light" mass of gauge bosons and fermions, the standard model itself cannot explain anything about the mass of the Higgs except that it must be as large as the mass scale where the model is expected to work (the cut-off) if loop corrections are taken in to account. So the next question is, phenomenologically, how large should the cutoff be? How heavy should the Higgs be?

[1]For citations on well-known articles of the standard model, we refer the readers to other review articles or even textbooks (see the references).

There are several evidences making physicists never thought of the standard model as the holy grail of particle physics. The clearest one is that it does not explain anything about gravity except that the standard model works as long as gravity is negligible. This implies the standard model must have a dead end at the Planck scale (where gravity becomes important), no matter what. In other words, it must be regarded as a low-energy effective theory of some "more complete" theory. Then, it would have been a disaster if the standard model is completely satisfactory in all other aspects as it would mean that the Higgs is as heavy as the Planck scale ($10^{19}$ GeV). This contradicts our common sense as we know that all other members that were found up to now never weight more than 200 GeVs. Fortunately the standard model is not that complete. Interactions in the standard model is well-described by a product gauge group $SU(3)_{\text{colour}} \times SU(2)_{\text{weak}} \times U(1)_{\text{hypercharge}}$ which is phenomenologically useful but theoretically unsatisfactory. As can be expected by an optimist, an analysis on how the coupling strength of each interaction changes with energy revealed that all the three couplings come very close to each other at energies around $10^{14} - 10^{15}$ GeV, showing the sign of "Grand Unification". A prototype of a grand unification theory was proposed during the 70's by Georgi and Glashow [8], which, again, is operated with the arguments of spontaneous symmetry breaking relying on fundamental scalar particles. So it seems that now there are two cut-off scale; one is at $10^{15}$ GeV where the standard model ceases to work and the grand unification theory comes in, the other is at the Planck scale $10^{19}$ GeV. Then the properties of fundamental scalars tell us that the masses of both "Higgses" should be equal to their corresponding scales. That is what a theory can tell us and this is where experiments come in.

The discovery of the particles in the standard model, especially the electroweak gauge bosons, as well as the advances in the precision electroweak tests, tell us that the mass of the Higgs (of the standard model) should be only a few hundreds GeV's. Due to the fact that the quantum corrections to the mass of the Higgs are quadratically sensitive to the cutoff, this means extreme adjustments are required in order to make a collection of large numbers turns out to be a small one. This is known as the fine-tuning or naturalness problem. In other words, the Higgs is light without reasons (from the model). To make the Higgs naturally light, there must be new physics somewhere below $1 - 2$ TeV (i.e., small cut-off). However, the results from the precision electroweak tests tell us that the new physics will not show up *below* $5 - 10$ TeV (i.e., large cut-off), which certainly not go along with the naturalness arguments. Altogether the problem is usually referred to as the *little hierarchy problem* or the *LEP*[2] *paradox* [1]. An understanding of the electroweak symmetry breaking cannot be regarded as complete without solving these problems.

Several attempts have been made to deal with the fine-tuning problem. Some rely on the cancellation mechanism between the dangerous diagrams from heavy (yet unobserved) particles and the standard model maintained by a symmetry, such as *supersymmetry.* Some do not use an elementary scalar in the model at all. *Technicolour* models (see Farhi and Susskind [9] or Kaul [10] for reviews), based on a scaled-up quantum chromodynamics with symmetry breaking mechanism analogous to the BCS theory in superconductivity, fall into the latter category. In addition to these models, there were some attempts, first pioneered by Georgi and Pais [11], to realise the Higgs as a *pseudo Goldstone boson*, a Goldstone boson that becomes massive via explicit global symmetry breaking interactions. More serious attempts along this line were made by Georgi, Kaplan and others [12, 13, 14, 15, 16] during the mid 80's. Nevertheless, their models still suffered from the naturalness (and even the little hierarchy) problem. As a consequence it looks as if supersymmetry (including, supersymmetric grand unification and superstring theories) is the most promising candidate. Still, it is *not* the *only* solution.

When particle accelerators have been constantly developed, the opportunities of direct, along with precise, studies on physics of the electroweak symmetry breaking were broaden. Then there comes the Large Hadron Collider at CERN, scheduled to begin its action in 2008 or so, where TeV physics will be probed directly for the first time. Many models were proposed with the hope that they can be somehow tested in the LHC.

In 2001, the Georgi's models were resurrected with the inspirations from the extra-dimension physics (deconstruction) by Arkani-Hamed *et al.* [17, 18, 19] and Hill *et al.* [20] in a class of models called *Little Higgs.* The crucial ideas of the Little Higgs include the one called *collective symmetry breaking*, which plays the similar role as supersymmetry. The first realistic model designed dedicatory for being the theory beyond the standard model was proposed by Arkani-Hamed *et al.* in a minimal model called the *Littlest Higgs* [21], which is based on a product gauge group. The mass productions of the Little Higgs models began after that. The other type of the Little Higgs model

[2] The LEP (Large Electron and Positron) collider is the particle collider, operated during the 90's, at CERN.

which is based on a simple group was later proposed by Schmaltz and (the other) Kaplan in 2004 [22]. Since the Little Higgs models were constructed from the framework of the non-linear sigma model, there are many rooms available for modifications. Even though the littlest Higgs, which is the most economical model, suffered from various phenomenological constraints, many "modified" versions were proposed including the one so-called *Little Higgs with T-Parity*, by Cheng and Low [23, 24], which succeeded in avoiding conflicts with electroweak precision tests. A work in combination with supersymmetry is even possible (see, for example, Csáki *et al.* [25] or Berezhiani *et al.* [26]). Most importantly, like supersymmetry and some other models, the particle spectrum of the Little Higgs models contain various particles within the reach of the LHC. This means it is highly possible to perform even some direct search and see which model suits best for being recognised as physics beyond the standard model.

## 1.2 About the Thesis

### 1.2.1 Objectives of the Thesis

We have seen the reason for studying the Little Higgs. Now let us move to the reason for making this thesis. Rather than being a detail analysis of various Little Higgs models, this thesis should be considered as a "prelude" to the Little Higgs. This is because though there are some reviews of the models available, most of the topics in Little Higgs physics are not readily accessible for readers with just the basic knowledge on the rudiments of the standard model. In many papers, some important detail of calculations in Little Higgs were left to be desired. In addition, while there are very large numbers of nice reviews on the foundation of the standard model, only some numbers of resources that are designed to fill the gap (especially for the route to the Little Higgs) are available. So we try to make the bridge that helps to provide smoother transitions from the standard model to the some theories beyond it, not just the Little Higgs. This thesis is focused on the non-supersymmetric path as there are various articles on supersymmetry.

Still, it is not possible to cover all the important topics along the way to the Little Higgs, or to develop the story from scratches in the precise manners. Therefore, we will try to gather the models and tools that we already have on hand and study some important aspects. Once we are familiar with those tools, we can get into the Little Higgs arena with the strategy: we will "translate" what we do not yet know into the form or the system that we know how to deal with. We will focus on some detail of calculations when it is necessary. So this is a downside of the work; namely, we will gather as many ideas as possible while keeping some important insights, hence making some ideas not "probed" to their deepest level. We will use heuristic arguments on the thoughts that are related to the common ones and will go into some calculations for the topics presented in few literature.

### 1.2.2 What the Thesis Does Not Offer

There are some important topics that should have been included but are rather too "big" to fit into the thesis in a systematic manner. The left-out topics include (with examples of nice articles in parentheses): effective field theory ([27, 28, 29]), generalisation of non-linear realisation of symmetry ([30, 31]), detail calculations on loop corrections and beta functions[3], and precision electroweak measurements ([32]). The author tried to "dilute" some of the topics above and injected them into some related topics from time to time. Nevertheless, the interested readers are advised to consults the suggested articles and the references therein.

On the Little Higgs model itself, there are many topics that are not included in the thesis. For example, there is only one Little Higgs model, namely the Littlest Higgs, being studied here for the reason that it is the most economical model available. Though it will be shown later that while the model fails to survive the constraints from precision electroweak measurements, it does not mean that it is not useful anymore. Besides it is the model that have been studied most, comparing to other variations of the Little Higgs. This is simply due to the reason that the Littlest Higgs can be easily extended or tuned to cope with specific problem, and that it shares many things in common with the its modified version. For a comparison between the two famous Little Higgs models; namely the *Littlest Higgs* and the *Little Higgs from a simple group*, we advise the reader to consult the review papers by Schmaltz and Perelstein mentioned earlier, as well as the comparison on the phenomenological point of view made by Han *et al.* [33].

[3]See standard textbooks that provide the background field method for calculations of the beta functions.

### 1.2.3 Other Review Articles

As of mid 2007 there are few review articles providing various aspects of the Little Higgs models. Those that were aware of by the author will be listed below. Many of these reviews may provide some deeper insights of the Little models than those presented this thesis. So they are highly recommended to readers who are interested in as most of them are available on the `arXiv`. The author would also like to apologise to authors of articles that were not recognised below.

The first short, but illustrative, article providing an introduction to the model was given by Schmaltz in 2003 [34]. There he presented the *Simplest Little Higgs* which is pioneered by himself and Kaplan [35], [22]. Later, Schmaltz also provided two review articles (more detail); one in [36] with Tucker-Smith, and the other as a lecture note in the TASI 2004[4] [37]. For the Littlest Higgs, there is a nice review just published by Perelstein in 2007 in [38]. He emphasised on the *Littlest Higgs* model (which is our main topic of the thesis) and some on the theory space model ("Moose" type) as well as the Simplest Little Higgs, together with their phenomenology. There is also a very nice paper on the phenomenology of the Littlest Higgs model by Han *et al.* [3] which we use as one of the main articles.

There are also few master's theses related to Little Higgs models. Two are from the theory group at NIKHEF including, "The Hierarchy Problem in the Standard Model and Little Higgs Theories" by Maarten Brak (University of Utrecht) in 2004, and "Extensions of the Standard Model and their influence on single-top" by Erik Lascaris (University of Twente) in 2006. The Little Higgs model mentioned by these two theses is the Littlest Higgs model. The other master's thesis is "Little Higgs Models: Effective Gauge Theories Stabilizing the Electro-Weak Scale Employing Collective Symmetry Breaking" by Jos Postma (University of Groningen) in late 2006, which focuses on the the Simplest Little Higgs.

## 1.3 Organisation of the Thesis

The structure of the thesis is organised as follows. In chapter II, we will recall some fundamental concepts of the standard model. We focus on the idea of

[4] The lecture is provided in the school proceeding and is available online at `particle.physics.ucdavis.edu/workshops/TASI04/`.

spontaneous symmetry breaking and will study its application in the Glashow-Weinberg-Salam model. Though we will try to use the bottom-up strategy as much as possible, the treatments in this chapter is rather standard. Therefore, a casual reader may want to skip with some quick glances into the section 2.2 which we study the linear sigma model, the main tool for studying spontaneous symmetry breaking in this thesis.

Then in chapter III we will study how the loop corrections (quantum effects) affect the status of the standard model. To some detail, we learn how to deal with the loop corrections using a technique provided by Coleman and Weinberg which will be used to explain how electroweak symmetry breaks in Little Higgs models. There we will also point out various limitations on the mass of the Higgs, from both theoretical and experimental points of view, including the precision electroweak tests, which convince us that the Higgs should be light. At the end of the chapter, we summarise some of the shortcomings of the standard model, especially the little hierarchy problem, which will tell us the basic requirements of physics beyond the standard model.

In chapter IV, we gather various techniques that are used in many theories beyond the standard model. The main objective of the chapter is to provide the building blocks not only of the Little Higgs, but also of some aspects of the physics beyond the standard model in non-supersymmetric direction. It is best to have a quick glance at the appendix B where we present a short discussion on the minimal version of the grand unification theory, the $SU(5)$. This example should provide simple, but complex enough, situations that let us learn how to deal with particles appearing in various representations of the theory. We begin the chapter by bringing up another way to implement the BEH (Higgs) mechanism, but without requiring the existence of any fundamental scalars; i.e., dynamical symmetry breaking. This is the rudiments of a class of theories called technicolour. The same section will provide important viewpoints on the alignment of vacuum, explicit symmetry breaking, and the pseudo Goldstone bosons. After that, we study the formulation of the non-linear realisations of a symmetry which serve as a crucial tool for dealing with low-energy degrees of freedom of a theory. One of the resulting models called a non-linear sigma model will be used extensively in the chapter of Little Higgs. Later in that chapter, we present the simplest version of the Georgi-Kaplan model, which utilises what we have studied in the chapter and serves as a prototype of the Little Higgs.

In chapter V, we will introduce the Little Higgs models, which bring up the way out of the problem mentioned in chapter III by utilising what we have developed in IV. We will focus on the *Littlest Higgs* model. There we present detail calculations on most of its topics, leaving some lengthly calculations as outlines. After discussing the model building part, we will very briefly discuss some of the important findings from the phenomenological side of the model.

The conclusions of the thesis are given in chapter VI.

In the appendices, we provides several useful and interesting materials. Some rudiments on group theories can be found in appendix A. In the same appendix we also present an alternative way to evaluate the effective potential. A review of some aspects of the $SU(5)$ grand unification theory, which contains useful ideas of representations of a group and the hierarchy problem, is provided in the appendix B. Finally in the appendix C we list important mathematical formulae.

## 1.4 Conventions and Notations

These are essential notations that will be used throughout this thesis. Still, not all of the notations are standard and might be changed slightly from section to section depending on the contexts.

- The expression $A \simeq B = C$ means $A \simeq B$ and $B = C$.
- We use $\simeq$ for an approximation and $\sim$ for a very rough approximation.
- Electric charge of a particle is given by $Qe$ where $e$ is positive.
- The index $i$ on quarks and leptons fields or doublets ($L^i_L$, $Q^i_L$, $e^i_R$, $u^i_R$, etc...) denote *families*.
- $Q$ (sometimes appears with subscripts or superscripts) usually means an *electric charge* while $Q_L$ means the left-handed *quark doublet*.
- The superscript $c$ denotes *charge conjugation* unless in some special cases it means the *charm quark* (i.e., the charm doublet $Q^c_L$).
- A scalar field (both real and complex) is denoted as $\phi$. A scalar multiplet will be denoted as $\Phi$, $\Sigma$ or $\hat{\Phi}$, $\hat{\Sigma}$ depending on the context.
- The physical Higgs particle is denoted as $h$ while the complex fields living in a doublet or so will be denoted with superscripts as $h^0$, $h^+$, $h^-$, etc.

# CHAPTER II

# ASPECTS OF ELECTROWEAK PHYSICS

This chapter will give a brief review of the electroweak sector of the standard model. The presentations will be deliberately "non-rigorous" as we will try to build up various ideas as naturally as possible (some hand-waving arguments appear from time to time due to lack of spaces). The development of the models without gauge symmetries will be outlined in section 2.1. Implications from that section will underline the importance of a gauge theory. So mathematical aspects of spontaneous symmetry breaking, which are crucial for many gauge theories, will be discussed in section 2.2 and the results will be used frequently in this thesis. Many of the outcomes from the first two sections will be gathered up in the section 2.3 where we present the Glashow-Weinberg-Salam model for the electroweak interaction. Though we try to introduce the topics in natural ways, the contents are rather standard. So a casual reader may want to skip this chapter with a glance at the sections on the linear sigma model in 2.2.2 and the custodial symmetry $SU(2)$ in section 2.3.5.

## 2.1 Weak Interaction Before Gauge Theories: In Plain English

In this section we give an overview of some aspects of the developments of the standard model. Since this kind of information are widely available, we will not go into detail (especially, the mathematical ones). Please have a look at other review literature; for example, by Quigg [39], Aitchison and Hey [40], and Morii *et al.* [41], and the references therein for further information.

The first model capable of describing weak interactions (the beta decay) is based on an analogy from electromagnetism, resulting in a current-current interaction (Fermi's model) where the currents involved are of "charge-changing" type. The advantage of the model is that it allows some crucial phenomenological features; namely, the maximal parity violation (i.e., weak interaction treats left- and right-handed particles differently), and universality of couplings (all fundamental fermions participate in weak interaction with equal coupling strengths). It was found that hadrons experience weak interaction in a slightly different manner from leptons. This is not surprising as now we know that they are considered as composite particles. At the "fundamental" level, the model show similar family structures of leptons and quarks, which will be another crucial foundation of the standard model.

The locality of the interaction in Fermi's model brings up the problem of unitarity and renormalisability. Problems come from the fact that the coupling constant in the model is dimensional, which leads to the fact that a cross section of a particular process grows too rapidly with energy, rendering the theory useless in the sense of a perturbation theory at some energy scales. In the case that we want to deal with physics at these "deadly" energy scales (which are reachable by current experiments), we can no longer ignore the problem by consider the Fermi's model as an effective low-energy theory. It is also clear that dimensionality of the couplings ruins the renormalisation programme of weak interaction.

The problems mentioned above can be solved by further following an analogy from electromagnetism. Still, the picture of a gauge theory is not readily available for the weak interaction as the gauge fields cannot be introduced in a naive way. This is because the range of weak interaction is very limited (massless gauge fields are not applicable) and there were only charged-changing currents in the model (group theoretically incomplete since there are only two force mediators in an "$SU(2)$-like" theory). Still, the intermediate vector boson picture, being its "higher" energy limit, is helpful and consistent with the Fermi's theory. The "matching" between the two models allows a prediction of the mass of the vector bosons in terms of the Fermi coupling constant (naive dimensional analysis works as well). The mass of the gauge boson then acts as the energy scale of the theory. Its huge mass of $\mathcal{O}(100\,\mathrm{GeV})$ explains why Fermi's model works well - the "working arena" of Fermi's model lies at energies far below the mass of the vector boson, rendering the picture of contact interactions viable. Nevertheless, without a gauge theory as the main engine, the theory still suffers the problem of renormalisability and unitarity, only slightly less severe. The source of the problems lies in the longitudinal components of the vector boson and the trick for removing that component is applicable only to gauge theories.

An important step towards the standard model has been made when the existence of the neutral (i.e., charge-conserving) weak current and its corresponding (intermediate) vector boson were aware of, or at least anticipated. The need for the neutral vector boson can be explained from the theory side as it helps "soften" several processes involving the charged-currents. In addition, the neutral current gives us a clue that renormalisability might be possible. However, a particular adjustment by hand (for example, to adjust the couplings between the charged and neutral vector bosons and fermions so that severely diverging diagrams vanish) is needed and is guaranteed to work only at the energy scale where the adjustment was made. If just kicking the neutral vector boson into the game is not convincing enough, a simple analysis on the operator having the form of a conserved charge in a gauge theory also shows us that neutral weak boson must exist in order to close (i.e., complete) the group theoretical structure.

Weak interaction on the hadronic sector is more complicated. Neutron, protons, and other hadronic states enter weak interactions with coupling constants, especially the axial-vector part, that seem to be different from that of leptons. This problem, as mention a few paragraphs ago, subdued with the introduction of quarks as an internal structure. During the time when the family structure of the leptons were not very convincing and only 3 quarks seemed to be required, the asymmetry between leptons (strange quark did not have its partner) left strangeness-changing charged-current processes poorly explained. It was Cabibbo who suggested a hypothesis that quarks entering weak interactions are not the quarks in their mass eigenstates. The picture allowed a pair up between the $s$ and $u$, allowing the $s$ to live (in a small "room") in the $u - d$ family, merging with the $d$ as the Cabibbo's down quark.

Solving one problem of the process involving charged current brings up another problem when the neutral current is taken into account; namely, the process dealing with strangeness-changing neutral current is not heavily suppressed in the way people in the labs have seen. The problem of "non-diagonal" interaction of the neutral current is solved in an easy-to-expect, yet elegant, manner by making an analogy, under weak interaction processes, between quarks and leptons. The missing piece (the charm quark) was proposed as a partner of the quark orthogonal to the Cabibbo's down, completing the family structure, solving the above mentioned problem. The idea of families we have on hand can be easily (and successfully) extended when one member of the family was found.

We cannot leave this section without emphasising that to keep ourselves in the main courses that will eventually lead to the Little Higgs model, we have solely outlined *very small* fractions of what are important and interesting landmarks in the art and science of theoretical and experimental particle physics during the "pre-electroweak" era. It is by no means expected that this short outline alone will convince the readers to believe in what were written above and those who are interested are strongly encouraged to go spend some times with various review articles available out there.

## 2.2 Hidden Symmetries

Now as we are convinced that electromagnetic and weak interactions can be unified by the use of a gauge theory, we face an immediate difficulty right away: while we know that the gauge fields mediating weak interaction are massive, an explicit breaking of gauge symmetry by introducing mass terms for the gauge fields is not preferable. The gauge fields must be massless or else the corresponding gauge symmetry is violated. However, it is essential to underline that the reason that the mass terms are not introduced by hand is not because they spoil the beauty of gauge symmetry, but because they make the theory non-renormalisable (and hence loses its predictive power). Still, this is fine if we restrict ourselves to low energy phenomena. However, in order to keep up with the current accelerators and to construct a gauge theory that describes a unification between electromagnetic and weak interactions and so on, we must find a way to let the gauge fields be massive without spoiling gauge symmetries. When it is possible, a renormalisable theory is much more preferable.

The way that seems to work so far is commonly known as the BEH or the Higgs mechanism (or the Brout-Englert-Guralnik-Hagen-Kibble-Higgs-Anderson mechanism; see [42], [43], [44, 45]. [46]). It is the interplay between the spontaneously breaking of global symmetry and the gauge symmetry of electroweak interaction. In short, the BEH mechanism tells us how the gauge fields become massive by interacting with the Goldstone bosons[1].

In general, the mechanism requires that the Lagrangian under consideration contains a sector *preserving a gauge symmetry* and the other sector *generating spontaneous breaking of a global symmetry*

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{SB}} \,. \tag{2.1}$$

The existence of the Lagrangian that describes *global* continuous symmetry $G$ broken spontaneously to its subgroup (say $H$) is required so as to break the *gauge* symmetry of the $\mathcal{L}_{gauge}$. In other words, we can say that the BEH

[1]Despite its name, the Higgs particle is another story from the BEH (Higgs) mechanism. As we shall see, the BEH mechanism operates with or without the Higgs particle.

mechanism always occur when a gauged continuous symmetry is spontaneously broken. Notice that, it is not necessary that the global symmetry $G$ be the same group as the gauge group (which is the $SU(2) \times U(1)$ electroweak in most of our considerations). However, for phenomenological purpose on electroweak symmetry breaking, $G$ must be big enough to contain the electroweak gauge group and $H$ must be at least as big as the electromagnetic gauge group (which is the exact, unbroken, one).

The general idea of spontaneous breaking of a global symmetry will be explored in the next section, followed by its application in a model called the sigma model. The second part, which is the main ingredient of the BEH mechanism, that is played by the gauge sector will be discussed after that. Related discussions in the following sections can be found in Chanowitz [47], Pokorski [48], Georgi [49], and Cheng and Li [50].

### 2.2.1 Formalism

Here we will discuss about spontaneous symmetry breaking (SSB) in quite a general way. Let us begin with the fact that in some systems, the symmetry that is used to describe physical laws is not realised in its original form. In those system it is usually found that invariance of the ground state is not necessary the same as the invariance of the Hamiltonian (or the Lagrangian). Now suppose that we have a degenerate set of ground states[2], say $|0\rangle$, which will lead to different physical states in the context of quantum field theory (think of a system of ferromagnets below the Curie temperature). In addition, let us say that the Hamiltonian is constructed from an object called $\phi$. Then we see that it is possible to find a symmetry transformation $\{U(g)|g \in G\colon\ \phi \to \phi'\}$ that leaves the Hamiltonian (or the Lagrangian) invariant

$$U H_0 U^\dagger = H_0 \qquad (2.2)$$

but acts on a ground state in a non-trivial way

$$U|0\rangle = |0'\rangle \neq |0\rangle \,. \qquad (2.3)$$

[2] In this section, the Hamiltonian formalism will be used.

This implies that $U$ does not connect states (which is constructed out of the ground state by the fields $\phi_i$) that form an irreducible representation of $G$:

$$U|1\rangle \neq |2\rangle \tag{2.4}$$

which means $\langle 1|H_0|1\rangle \neq \langle 2|H_0|2\rangle$. In other words, the degeneracies of the energy eigenstates (spectrum), which are supposed to show up according to the symmetry of $H_0$, disappear. This also means that there exists a particular field $\phi_i$ whose vacuum expectation value transforms (non-trivially) under the operation $G$; i.e.,

$$\langle 0|U^\dagger \phi_i U|0\rangle \neq \langle 0|\phi_i|0\rangle \tag{2.5}$$

which suggests that

$$\langle 0|\phi_i|0\rangle = v_i \neq 0 \longrightarrow \text{SSB}\,. \tag{2.6}$$

Which element of the $\phi$ will be given non-zero vacuum expectation value is a question that has to be handled with care. Suppose we have associated the correspondence between physical particles and the fields in the Hamiltonian, as well as fixed the interpretation of the symmetry generators right from the start. Though each of the ground state in the space of the degenerate vacua results in equal vacuum energy, phenomenological observations will restrict only a specific set of the field $\phi_i$ to have non-zero vacuum expectation value. Otherwise we get a "wrong" ground state with the same mathematical relations between the generators but different meanings.

The other way we can go is that we start with a specific pattern of symmetry breaking. This completely defines a complete set of degenerate ground states, which we demand to be equivalent if they result in the same pattern of symmetry breaking. The choice of a ground state then becomes purely a matter of *convention*. After picking one up, we have to find its matching set of broken and unbroken generators. Then we have to re-associate the field in question with their new (and correct) physical interpretations. The resulting Lagrangian or Hamiltonian may be in an unfamiliar form which, however, can be recovered into the usual form by performing a $G$ transformation on the fields. One of the clear example is the linear sigma model that we shall soon meet in section 2.2.2.

To sum up, SSB occurs when the global symmetry (corresponding to the subgroup $H$ of $G$) of the ground state is not the same as the global symmetry ($G$) of the interaction of the Lagrangian or the Hamiltonian of the system. We may also view this result as follows. For a spontaneously broken symmetry system consisting of particles corresponding to the $\phi$, the interactions between these particles are arranged such that it is preferable to fill the vacuum with them instead of leaving it empty. The vacuum expectation value plays the role of the order parameter which signals whether to break or not to break the symmetry. It should be emphasised that the order parameter is not necessary a fundamental scalar. A composite object like a fermion condensate $\phi \sim \overline{\psi}\psi$ is allowed as well. The latter case will be examined in more detail in section 4.1.

For the case of continuous symmetries, we express the group elements in terms of the symmetry generators (or charge) $Q_a$; i.e., $U = \mathrm{e}^{\varepsilon^a Q^a}$. Then we have

$$\phi_i \longrightarrow \mathrm{e}^{\varepsilon^a Q^a}\phi_i \mathrm{e}^{-\varepsilon^b Q^b} = e^{-\mathrm{i}\varepsilon^a Q^a_{ij}}\phi_j \tag{2.7}$$

where $Q^a_{ij}$ is the matrix representation of the group, obeying the same algebra as those of $Q^a$'s. So we say that when the charge does not annihilate the vacuum

$$Q_a|0\rangle \neq 0\,, \tag{2.8}$$

(or $Q_a v_i \neq 0$) then SSB occurs and this particular $Q_a$ is called a *broken generator*. Supposing that a ground state occurs when the fields takes a value that is denoted collectively by $\phi_0$; In other words, when there exists a subgroup $H \subset G$ having elements $h \in H$ that leave the ground state invariant $h\phi_0 = \phi_0$, we say that $G$ is spontaneously broken to $H$. In this case, the number of broken generators are $\dim(G) - \dim(H)$. Notice that the elements corresponding to the broken generators do not form a group (clearly, they do not have an identity). However, they do form a *coset* $G/H$, which is basically a set whose elements themselves are sets of the $G$ group elements[3]. It is an equivalence class defined to contain all of the elements of $G$ related by a multiplication by an element in $H$.

In principle, we do not know which direction (choice) of the subgroup $H$ will be when spontaneous symmetry occurs. Suppose we have $x \in G/H$, by construction this $x$ will not leave the ground state invariant:

$$x|0\rangle \neq |0\rangle\,. \tag{2.9}$$

[3]The left coset (of g) is a set defined by $gH = \{gh|h \in H\}$ in one element of $G/H$.

However, the pattern of symmetry breaking and the structure of the group have not changed under this action. This means that we can define a vacuum

$$|\Omega\rangle = x|0\rangle \tag{2.10}$$

which in turns defines another subgroup $H'$ of $G$. The point is that, without other interactions; it is obvious that the new subgroup $H'$ is equivalent to the old one, the $H$. Interesting situations will show up when we turn on other interactions which has a preference in a specific *orientation* of the subgroup $H$ and the energy of the vacuum will depend on its orientation. We will come back to this case in section 4.1.2.

Now we will jump directly to the Goldstone theorem, which can be found in many literature nowadays. It says that each broken generator of a continuous symmetry $G$ leads to a spectrum of $\mathcal{L}_{SB}$ corresponding to a massless (its energy vanishes in the limit of zero momentum) spin-zero particle, whose state denoted[4] by $|\pi\rangle$, that can connect to the vacuum by the field operator $\phi$ or the current $J_0$; i.e.,

$$\langle\pi|\phi(0)|0\rangle \neq 0\,, \qquad \langle 0|J_0(0)|\pi\rangle \neq 0\,. \tag{2.11}$$

This massless particle is called a *Goldstone boson*. In general, a Goldstone boson can be created or destroyed by a symmetry current associated with the broken generator $Q_a$. This also implies that the number of the Goldstone bosons equals the dimension of $G/H$ and does not depends on what representations the fields belong to. The Lorentz invariance tells us the matrix element between the Goldstone boson and the vacuum state can be parametrised as

$$\langle 0|J^{\mu a}(x)|\pi_k(p)\rangle = -\mathrm{i}p^{\mu}F^{a}_{\ k}e^{-\mathrm{i}p\cdot x} \tag{2.12}$$

where $F^a_{\ k}$ is a constant matrix. In many cases, when the currents are the ones that correspond to an irreducible representation of the broken generators, the matrix $F^a_{\ k}$ can always be diagonalised; i.e., $F^a_{\ k} = F\delta_{ak}$. Upon taking $\mu = 0$ in (2.12) and the assumption (2.11) which says that the conserved charge corresponding to the broken generator does not annihilate the vacuum, we find that $F$ is non-zero. Then the conservation of the current $J^{\mu a}$ implies

$$\langle 0|\partial_{\mu}J^{\mu a}(x)|\pi_k(p)\rangle = m_{\pi}^2 F^{a}_{\ k}e^{-\mathrm{i}p\cdot x} = 0 \tag{2.13}$$

[4]We usually use the $\pi$ for the reason that *pion* is the usual suspect for being a Goldstone boson. See later sections.

which reproduces the Goldstone theorem; i.e., $m_\pi^2 = 0$.

Now we will turn to the case where the SSB is generated by *elementary scalars*. Let us consider a set of real scalar fields with the corresponding potential that is invariant under a particular global symmetry group $G$ (which maybe $O(N)$, $SU(N)$, etc.). This means

$$(T^a\phi)_i \frac{\partial V}{\partial \phi_i} = 0 \tag{2.14}$$

where $T^a$ is a generator (not necessarily corresponding to an irreducible representation) of $G$ and $(T^a\phi)_i = T^a_{ij}\phi_j$. Spontaneous global symmetry breaking from $G$ to its subgroup $H$ occurs when the structure of the potential of $\phi_i$ leads to the non-zero vacuum expectation value

$$\langle 0|\phi_i|0\rangle = v_i\,, \tag{2.15}$$

with

$$(T^a v)_i \begin{cases} = 0 & \text{for } a = 1, ..., n_H \\ \neq 0 & \text{for } a = n_H + 1, ..., n_G \end{cases} \tag{2.16}$$

Note that it is not necessary to restrict the vacuum to a particular $m^{th}$ direction like $\langle 0|\phi_i|0\rangle = v_i = \delta_{im} v$ (though this is mostly the case when we deal with SSB). Using (2.14), the vacuum configuration (2.15) implies

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_k}(T^a\phi)_i = 0\,, \tag{2.17}$$

that the $(T^a\phi)_i$ for $n_H + 1, ..., n_G$ are the Goldstone bosons with the relations

$$\langle 0|J^{\mu a}(x)|\phi_k(p)\rangle = -\mathrm{i}p^\mu (T^a v)_k e^{-\mathrm{i}p\cdot x}\,. \tag{2.18}$$

Let us call the $T^a$ for $a = n_H + 1, ..., n_G$ a broken generator $\tilde{T}^e$ with indices renamed so that $e = 1, ..., (n_G - n_H)$. Then the $\phi_i$ can be parametrised such that the Goldstone bosons lie along the direction of the broken generators,

$$\phi_i = e^{\mathrm{i}\tilde{T}^e_{ij}\chi^e(x)/v}(v_j + \rho_j(x))\,, \tag{2.19}$$

which means they are obtained from a ground state by a symmetry transformation with spacetime-dependent parameter. Here $\chi_i(x)$ and $\rho_i(x)$ are orthogonal. When the vacuum is aligned along the $m^{th}$ direction we get

$\phi = e^{i\tilde{T}^e_{ij}\chi^e(x)/v}\delta_{jm}(v_j + \rho_j(x))$ which can be arranged so that

$$\phi = e^{i\tilde{T}^e\chi^e(x)/v}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ v+\rho \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{n_G-1} \\ v+\rho \end{pmatrix} + \text{higher orders}\,. \qquad (2.20)$$

For the special case where the potential has the well-known"Mexican hat" form, we see that $\chi$ is a Goldstone field in the valley direction and $\rho$ is a massive field in the radial direction. Observe that the symmetry $G$ can be "realised" in terms of the transformation of the fields $\rho \to \rho$ while the $\chi^a$ transforms in a complicated way. To linear order, we have $\chi^a \to \chi^a + \text{constant}$, which clearly *protects* the $\chi$'s from being massive. The transformation of the Goldstone bosons does not concern us at the moment since they will eventually be traded with the massive gauge bosons (see section 2.2.3). We will come back to this story in the section 4.2.

Before we move on, it is very important to stress that spontaneous symmetry breaking is a solely theoretical concepts and the mechanism itself has nothing to do with experiments. The quantity directly related to SSB such as the vacuum expectation value of a scalar field can only be "touched" indirectly only within a particular theoretical scheme; i.e., it has no direct connection with experiments.

### 2.2.2 The Linear Sigma Model

We will illustrate the idea of spontaneous symmetry breaking (SSB) by an example of the linear sigma model. A popular example of the model is the $\lambda\phi^4$ theory with the mass term $\mu^2\phi^2$. However, in this section we will used another realisation of the scalar field which can be served as a simple prescription for dealing with pions. Though the model was originally made for describing chiral symmetry breaking in the strong interaction regime, the idea will be useful when we consider other symmetry breaking phenomena including the Little Higgs.

Pions are like no other hadrons. We know that hadrons consist of 2 or 3 quarks, 2 for the case of pions. So it is tempting to guess that the mass of a pion should be somewhere around $2 \times 1\,\text{GeV}/3 \sim 700\,\text{MeV}$ where the $1\,\text{GeV}$ is the mass of a proton. However, it turns out that the mass of a pion is approximately $135 - 140\,\text{MeV}$ while other hadrons, like the $\rho$, are heavier than $\sim 800\,\text{MeV}$.

Consequently, we do not have much choice but to construct an almost massless particle out of the massive ones by regarding the pions as (probably composite) Goldstone bosons. This is why we need to use spontaneous symmetry breaking. This is where the sigma model comes in.

Now let us turn to the linear sigma model. Consider a system consisting of massless (zero bare mass) nucleon doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$ and a massless spin-0 field $\Sigma$ consisting of a pion triplet $\boldsymbol{\pi}$ and its company, a scalar meson $\sigma$. Since the nucleon fields are massless, the theory does not possess only the $SU(2)$ isospin symmetry but also a larger symmetry group which is the $SU(2)_L \times SU(2)_R$ chiral symmetry defined by

$$\delta N_L = \mathrm{i}\varepsilon_L^a T^a N_L\,, \qquad \delta N_R = \mathrm{i}\varepsilon_R^a T^a N_R\,. \tag{2.21}$$

Both can be grouped together in the more useful form

$$\delta N = \mathrm{i}(\varepsilon^a - \gamma_5 \varepsilon_5^a) T^a N\,, \tag{2.22}$$

where $\varepsilon^a = (\varepsilon_R^a + \varepsilon_L^a)/2$ and $\varepsilon_5^a = (\varepsilon_R^a - \varepsilon_L^a)/2$. As usual, we can reverse the logic and say that this chiral symmetry "protects" the nucleon from being massive.

What is often referred to as the *sigma model* is actually the scalar ($\Sigma$) part that contains the potential of $\Sigma$ which is arranged so as to generate spontaneous symmetry breaking. It is a renormalisable field theory[5]. This $\Sigma$ system interacts with fermions via Yukawa interaction. The Lagrangian of the whole system $\mathcal{L}$ is therefore

$$\begin{aligned} \mathcal{L} &= \mathrm{i}\bar{N}\partial\!\!\!/ N - g\bar{N}\Sigma N + \mathcal{L}(\Sigma) \\ &= \mathrm{i}\bar{N}_R \partial\!\!\!/ N_R + \mathrm{i}\bar{N}_L \partial\!\!\!/ N_L - g\bar{N}_L \Sigma N_R - g\bar{N}_R \Sigma^\dagger N_L + \mathcal{L}(\Sigma)\,, \end{aligned} \tag{2.23}$$

which is invariant under the $SU(2)_L \times SU(2)_R$ symmetry if the $\Sigma$ transforms in as follows:

$$\Sigma \longrightarrow L\Sigma R^\dagger \tag{2.24}$$

for $N_L \to LN_L$ and $N_R \to RN_R$. In other words, $\Sigma$ transforms as the $(2,2)$ representation of the $SU(2) \times SU(2)$. There are many forms possible for the $\Sigma$ and we will use more than one of them in this thesis. The symmetry that is

[5]This linear sigma model stands on its own as a theory for describing spontaneous breaking of a chiral symmetry. Unlike the *non-linear* version, it is *not* to be considered as a low-energy effective theory of QCD or so.

manifested depends on the form of the field $\Sigma$. Here we pick out one that allows us to conveniently work with isospin symmetry. The $\Sigma$ can be decomposed into the triplet $\boldsymbol{\pi}$ and the singlet $\sigma$

$$\Sigma = \sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi} \tag{2.25}$$

where its elements transform under the $SU(2)$ isospin transformation $N \rightarrow e^{\mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\tau}/2}N$ as

$$\boldsymbol{\pi} \longrightarrow \boldsymbol{\pi} + \boldsymbol{\epsilon}\times\boldsymbol{\pi}\,, \qquad \sigma \longrightarrow \sigma \tag{2.26}$$

and under the axial transformation $N \rightarrow e^{\mathrm{i}\boldsymbol{\epsilon}_A\cdot\boldsymbol{\tau}\gamma^5/2}N$ as

$$\boldsymbol{\pi} \longrightarrow \boldsymbol{\pi} + \boldsymbol{\epsilon}_A\sigma\,, \qquad \sigma \longrightarrow \sigma - \boldsymbol{\epsilon}_A\cdot\boldsymbol{\pi}\,, \tag{2.27}$$

or in short (notice the different meanings between $\varepsilon^a$ and $\varepsilon^{abc}$)

$$\sigma \quad\longrightarrow\quad \sigma + \varepsilon_A^a\pi^a \tag{2.28}$$

$$\pi^a \quad\longrightarrow\quad \pi^a - \varepsilon^{abc}\varepsilon^b\pi^c - \varepsilon_A^a\sigma\,. \tag{2.29}$$

Then it is found that the conserved vector and axial vector currents are

$$\mathbf{J}_\mu^V \;=\; \bar{N}\gamma_\mu\frac{\boldsymbol{\tau}}{2}N + \boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi} \tag{2.30}$$

$$\mathbf{J}_\mu^A \;=\; \bar{N}\gamma_\mu\gamma^5\frac{\boldsymbol{\tau}}{2}N - (\boldsymbol{\pi}\partial_\mu\sigma - \sigma\partial_\mu\boldsymbol{\pi}) \tag{2.31}$$

respectively. After this it is easy to evaluate the conserved charges (from the $\mu = 0$ component). With these conserved currents on hand, we can reverse the argument below (2.22); i.e., the non-conservation of the currents can be used as a measure of the nucleon masses[6].

Next we will construct the potential for the $\Sigma$. The invariant object is constructed from the field $\Sigma$ via $\mathrm{Tr}\Sigma^\dagger\Sigma$. By nature of spontaneous symmetry breaking, we expect that there is a particular energy scale such that the $SU(2)\times SU(2)$ symmetry manifests at high energy but is hidden at energy below this particular energy scale. The Lagrangian for $\Sigma$ that leads to SSB is given by

$$\begin{aligned}\mathcal{L}(\Sigma) &= \frac{1}{2}\mathrm{Tr}(\partial^\mu\Sigma^\dagger\partial_\mu\Sigma) - \frac{\lambda}{4}\left[(\mathrm{Tr}\Sigma^\dagger\Sigma)^2 - F_\pi^2\right]^2 \\ &= \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} - \frac{\lambda}{4}\left[(\sigma^2 + \boldsymbol{\pi}^2) - F_\pi^2\right]^2\,;\end{aligned} \tag{2.32}$$

[6] It is better to use this statement when we refer to quark masses since we can talk about their *bare masses* in a more proper way.

i.e., the form of the potential leads to a minimum at

$$\sigma^2 + \boldsymbol{\pi}^2 = F_\pi^2 , \tag{2.33}$$

rather than at 0. Later we shall see that $F_\pi$ is the amplitude for a chiral current to create a Goldstone boson out of the vacuum, hence its name the *pion decay constant*. All "directions" of the ground state, determined by which fields ($\boldsymbol{\pi}$, or $\sigma$) are given non-zero vacuum expectation values, are *equivalent* as they satisfy (2.33), and are connectable via $SU(2)_L \times SU(2)_R$ transformations on $\Sigma$. Once a specific vacuum is "chosen", the symmetry is spontaneously broken down to $SU(2)$. But which $SU(2)$? Fortunately, the Lagrangian (2.32) tells us that all of the available vacua have the same energy and we are at liberty to pick out the desired, phenomenologically correct, one. Recall that axial symmetry implies the existence of a particle with mass similar to that of a neutron, but with opposite parity. Such the particle has never been found. So it is expected that the axial symmetry is broken, not the isospin (recall, $m_u \approx m_d$). In other words, chiral symmetry $SU(2)_L \times SU(2)_R$ should break down to $SU(2)_{L+R}$. Consequently, the vacuum must be an isospin singlet but transforms (being non singlet) under a chiral transformation; i.e.,

$$Q_V^a|0\rangle = 0 , \qquad Q_A^a|0\rangle \neq 0 , \tag{2.34}$$

where $\mathbf{Q}_V$ and $\mathbf{Q}_A$ are constructed from (2.30) and (2.31). In other words, only the isospin "charge" disappears into the vacuum. So the vacuum is said to be "filled" with the isoscalar meson $\sigma$ alone; i.e.,

$$\langle 0|\boldsymbol{\pi}|0\rangle = 0 , \qquad \langle 0|\sigma|0\rangle = F_\pi . \tag{2.35}$$

Now let us consider the state built from the vacuum by defining the shifted (physical) field

$$\sigma' = \sigma - F_\pi \tag{2.36}$$

which has zero vacuum expectation value (VEV). This leads to

$$\begin{aligned} \mathcal{L} \;=\; & \mathrm{i}\bar{N}\not{\partial}N + \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} \\ & -gF_\pi\bar{N}N - \frac{1}{2}(2\lambda F_\pi^2)\sigma'^2 \\ & -g\sigma'\bar{N}N + \mathrm{i}g\bar{N}\boldsymbol{\pi}\cdot\boldsymbol{\tau}\gamma^5 N - \frac{\lambda}{4}\left(\sigma'^2 + \boldsymbol{\pi}^2\right)^2 - \lambda F_\pi(\sigma'^3 + \sigma'\boldsymbol{\pi}^2) , \end{aligned} \tag{2.37}$$

where it is clear that the trace of the $SU(2)\times SU(2)$ symmetry is hidden from the particle spectrum. The $SU(2)\times SU(2)$ symmetric term $(\sigma^2+\boldsymbol{\pi}^2)$ now describes interactions between the $\sigma$ and the $\boldsymbol{\pi}$. In the Lagrangian (2.37) we see a nucleon with mass $m_N = gF_\pi$ interacting with a massive scalar field $\sigma$ having mass $m_\sigma = \sqrt{2\lambda}F_\pi$ and a triplet of massless fields $\boldsymbol{\pi}$. Here the fields $\boldsymbol{\pi}$ (the pions) are the Goldstone bosons corresponding to the broken generators $\mathbf{Q}_5$. Now the axial vector current (2.31) becomes

$$\mathbf{J}^A_\mu = \bar{N}\gamma_\mu\gamma^5\frac{\boldsymbol{\tau}}{2}N - \boldsymbol{\pi}\partial_\mu\sigma' + \sigma'\partial_\mu\boldsymbol{\pi} + F_\pi\partial_\mu\boldsymbol{\pi}\,. \tag{2.38}$$

Notice that the term linear in the Goldstone boson fields ($\boldsymbol{\pi}$) leads to the transition between the Goldstones and the vacuum via the axial current:

$$\langle 0|J^{A\mu}_a(x)|\pi_b(p)\rangle = -\mathrm{i}p^\mu F_\pi\delta_{ab}e^{-\mathrm{i}p\cdot x}\,, \tag{2.39}$$

where $a$ corresponds to the broken generators. For further references, we note that

$$\begin{aligned}
\langle 0|J^a_{V\mu}(x)J^b_{V\mu}(0)|0\rangle &\propto \delta^{ab} \\
\langle 0|J^a_{A\mu}(x)J^b_{A\mu}(0)|0\rangle &\propto \delta^{ab} \\
\langle 0|J^a_{V\mu}(x)J^b_{A\mu}(0)|0\rangle &= 0\,.
\end{aligned} \tag{2.40}$$

What we have also learnt here is that since the nucleon is massless in the absence of (chiral) symmetry breaking, its mass should also lie within the order of the symmetry breaking scale. In this sense we regard the $\Lambda \simeq m_\sigma$ as the mass scale of the theory where "new physics" shows up. Later we shall see that this new physics may be weakly interacting theory with spontaneous symmetry breaking (like what we are currently doing) or strongly interacting one[7]. Since the scale $F_\pi$ is completely arbitrary, it can be set to the value that we find appropriate. At first sight it seems that if we assume that $F_\pi$ (and hence the $m_\sigma$) be very high, $\infty$ for example, we can describe the low energy physics involving only the Goldstone boson fields by decoupling the $\sigma$ and the nucleons $N$'s from the Lagrangian (2.37). This also means setting the quartic coupling $\lambda$ to infinity. The problem is that this idea does not work. While the removal of the field $N$ is perfectly allowable, the naive removal of the $\sigma$ destroys the

[7] This case happens when we set $\lambda$ very large and hence the Landau pole stays close to the mass scale $m_\sigma$. So we have to introduce some cut-off before the theory we have on hand becomes unreliable. See section 3.2.1.

$SU(2) \times SU(2)$ symmetry altogether. Even worse, if we try to keep working while pretending not to worry too much about the symmetry, the effect of the $\sigma$ is still there in the $\sigma\pi\pi$ and the $\pi\pi\pi\pi$ couplings. This means a process like $\pi\pi \to \pi\pi$ still has an infinite contribution from the coupling, and hence cannot be neglected. A nice way to deal with the low energy physics in terms of the Goldstone boson fields while still being $SU(2)\times SU(2)$ invariant will be discussed in section 4.2.

#### 2.2.2.1 Remark 1: Another Representation

Let go back to the vacuum condition (2.33). It also reminds us the similarity between the structure of $SU(2) \times SU(2)$ and $SO(4)$. In the real representation, isospin generators become, for example,

$$T_V^1 = \left(\begin{array}{cc|cc} & & & \\ & & -\mathrm{i} & \\ \hline & \mathrm{i} & & \\ & & & \end{array}\right) \qquad T_V^2 = \ldots \tag{2.41}$$

Other generators are displayed in appendix C.3.0.3. The generators for the axial transformation must involve a change made to the $\sigma$; i.e., the $4^{th}$ component,

$$T_A^a = \left(\begin{array}{cc|cc} & & & \mathrm{i} \\ & & & \\ \hline & & & \\ -\mathrm{i} & & & \end{array}\right) \qquad T_A^2 = \ldots \tag{2.42}$$

Then the $SU(2) \times SU(2)$ quadruplet can be written as

$$\Phi = \begin{pmatrix} \pi^1 \\ \pi^2 \\ \pi^3 \\ \sigma \end{pmatrix} . \tag{2.43}$$

Note that this quadruplet (of real fields) still transform as a $(2, 2)$ under the chiral group. For the case that the symmetry is broken to $SU(2)_V$, this field can be parametrised as (see (2.20)),

$$\Phi = \exp\left\{ \frac{-\mathrm{i}}{F} \left( \begin{array}{ccc:c} & & & \mathrm{i}\chi_1 \\ & & & \mathrm{i}\chi_2 \\ \hdashline & & & \mathrm{i}\chi_3 \\ -\mathrm{i}\chi_1 & -\mathrm{i}\chi_2 & -\mathrm{i}\chi_3 & \end{array} \right) \right\} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{\pi^2+\sigma^2} \end{pmatrix} . \tag{2.44}$$

Observe that this "transfers" the degrees of freedom of the Goldstone bosons to the parameters of the $SO(4)$ rotation.

#### 2.2.2.2 Remark 2: Different Choices of the Vacuum

Another thing to remark is what we have mentioned in the section 2.2.1 (page 13). It can be shown that a different choice of vacuum; say, $\langle \pi^1 \rangle = \langle \pi^2 \rangle = \langle \sigma \rangle = 0$ and $\langle \pi^3 \rangle = F \neq 0$ also leads to the same physical content where the $\pi^1$, $\pi^2$, $\sigma$ are the Goldstone bosons. The mass terms for the nucleon will look like

$$\mathrm{i} g F_\pi \bar{N} \gamma^5 \tau^3 N \tag{2.45}$$

which can be reverted to the usual mass term $\bar{N}'_L N'_R + \bar{N}'_R N'_L$ by a rotation on $N_L$ by $-\mathrm{i}\tau^3$ while leaving $N_R$ not transformed.

#### 2.2.2.3 Remark 3: The Background Field Point of View

In this remark, we review a realisation of the symmetry breaking that makes the claim "vacuum filled with a scalar field" more transparent. Let us say that the minimum of the potential in (2.32) occurs at[8] $\sigma = \sigma_c = F_\pi$. This means after the symmetry is broken, the state of a particle is described by an excitation of the field $\sigma$ near the classical field $\sigma_c$. This excitation can be described by changing the variable

$$\sigma \longrightarrow \sigma + \sigma_0 \, . \tag{2.46}$$

Just notice that $\sigma_0$ is considered as a constant background field (no kinetic term) and is not necessarily the same as $\sigma_c$. Then quadratic-field part of (2.32)

[8] For simplicity, we will deliberately be less rigorous and focus on the field $\sigma$ alone with the assumption that the reader will remember that we always have the $\pi$ in the system.

becomes

$$\mathcal{L}(\sigma+\sigma_0)_\sigma \supset +\frac{\lambda}{2}\sigma^2 F_\pi^2 - \frac{3}{2}\lambda\sigma_0^2\sigma^2$$
$$= -\frac{\lambda}{2}(3\sigma_0^2 - F_\pi^2)\sigma^2 \,. \quad (2.47)$$

In addition, we also have various interaction terms between $\sigma$, $\pi$ and the background $\sigma_0$. In general when $\sigma_0 \neq 0$ we see a $\sigma$ field propagating with effective mass

$$m_\sigma^2 = \lambda(3\sigma_0^2 - F_\pi^2)\,, \quad (2.48)$$

assuming that $3\sigma_0^2 > F_\pi^2$. Only when $\sigma_0 = \sigma_c = F_\pi$ will the effective mass $m_\sigma$ become

$$m_\sigma^2 = 2\lambda F_\pi^2 > 0\,. \quad (2.49)$$

Clearly, this field description allows us to view the current situation as follow. There is a particle (field quanta) corresponding to the field $\sigma$ with $m_\sigma^2 = 2\lambda F_\pi^2$ propagating in the vacuum filled with the constant background field $\sigma_0$.

### 2.2.3 Gauge Theory with SSB: Conventional Aspects

In this section we consider the important part of the BEH mechanism: when the (spontaneously) broken generator of $\mathcal{L}_{SB}$ "coincides" with the generator of the $\mathcal{L}_{gauge}$. In other words, it is the case when the Goldstone bosons (from spontaneous breaking of a global symmetry) are coupled with the gauge fields of $\mathcal{L}_{gauge}$. *Universally of the coupling strength* tells us that the gauge bosons couple with universal strength to all quanta carrying the charge of the gauge group. Then it is almost transparent that in the minimal model with one vacuum expectation value, the masses of the gauge bosons are determined by the gauge coupling constant.

Let the scalar fields $\phi_i$ (corresponding to the global symmetry $G$) couple with the gauge fields of a (local) gauge group $G_{\text{gauge}}$. When $G_{\text{gauge}} = G$, we say that the global symmetry of the system is promoted into the local (gauge invariant) one. The coupling between these fields and the gauge fields $W_\mu^a$ is displayed in the kinetic term

$$\begin{aligned}\frac{1}{2}\left|\partial_\mu\phi_i - \mathrm{i}gW^a_\mu(T^a\phi)_i\right|^2 &= \frac{1}{2}(\partial_\mu\phi_i)^2 - \mathrm{i}gW^a_\mu\left[\partial_\mu\phi_i(T^a\phi)_i\right] \\ &\quad + \frac{1}{2}g^2W^a_\mu W^{b\mu}(T^a\phi)_i(T^b\phi)_i\,. \end{aligned} \tag{2.50}$$

which clearly reflects the universally property. Recall that in the $\phi$ sector, we have

$$\phi_i = e^{\mathrm{i}\tilde{T}^e_{ij}\chi^e(x)/v}(v_j + \rho_j(x))\,, \tag{2.51}$$

where the $\chi$'s were used to parametrise the Goldstone boson for the global symmetry case. Gauge symmetry claims that not all of the particle represented by the fields displayed above (the gauge fields $W$ and the $\phi$) are physical and a gauge transformation $W_\mu \to W'_\mu$ removes all the $\chi$ dependence (i.e., by setting the gauge transformation parameter $\theta(x) = -T^c\chi^c(x)/v$). Actually nothing is missing because the gauge fields, which is equal in number to the missing Goldstone bosons, in the system are massive with their masses described by

$$M^2_{cd} = \frac{g^2}{2}(v^\dagger v)T^cW'^c_\mu T^dW'^{d\mu}\,. \tag{2.52}$$

In other words, the Goldstone bosons become the longitudinal degrees of freedom of the gauge fields. For this reason these $\chi$ are called the "would-be" Goldstone bosons. Literature may prefer to say that the Goldstone boson was "eaten" and become the longitudinal polarisation of the gauge field. Remember that mass terms for charged and neutral particles are different in most standard conventions.

To illustrate how this works, let us consider the Abelian $U(1)$ case,

$$\mathcal{L} = |\partial_\mu\phi - \mathrm{i}eA_\mu\phi|^2 - \lambda\left(\phi^*\phi + \frac{\mu^2}{2\lambda}\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{2.53}$$

where the global symmetry is spontaneously broken when $m^2 < 0$. The vacuum expectation value (for $\phi$) $v = \langle 0|\phi|0\rangle = (-\mu^2/\lambda)^{1/2}$ and the parametrisation

$$\phi(x) = \rho(x)e^{\mathrm{i}\chi(x)T/v}\begin{pmatrix}0\\1\end{pmatrix} \tag{2.54}$$

where $T = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$ bring us the field

$$\phi'(x) = \phi(x) - \langle 0|\phi|0\rangle = (\rho(x) - v)e^{\mathrm{i}\chi(x)T/v} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{2.55}$$

that has zero vacuum expectation value. The Goldstone boson ($\chi$) degree of freedom can be removed by choosing an appropriate $U(1)$ gauge choice called the *unitary gauge*, say $\theta(x) = -\chi(x)$, where only *physical* particles show up in the Lagrangian. Gauge symmetry is said to be broken spontaneously (or hidden) and we get

$$\phi'_g(x) = \phi_g(x) - \langle 0|\phi_g|0\rangle = (\rho(x) - v)e^{\mathrm{i}\chi(x)T/v} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.56}$$

$$A^\mu_g(x) = A^\mu - \frac{1}{ev}\partial^\mu \chi(x) \tag{2.57}$$

which leads to the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F_{g\mu\nu}F^{\mu\nu}_g + \frac{1}{2}\partial_\mu \rho' \partial^\mu \rho' + \frac{1}{2}e^2 v^2 A^\mu A_\mu - \frac{1}{4}\lambda \rho'^4 \\ & + e^2 v \rho' A^\mu A_\mu + \frac{1}{2}e^2 \rho'^2 A^\mu A_\mu - \lambda v^2 \rho'^2 - \lambda v \rho'^3 , \end{aligned} \tag{2.58}$$

where the gauge field becomes massive with mass determined by the couplings, $m_A = ev$, as expected. Nevertheless, gauge invariance is not spoiled but is hidden; i.e., (2.58) is still gauge invariant. The price we have to pay is the existence of the uninvited guest, the massive scalar field (the $\rho'$ here). When the BEH mechanism is implemented in the standard model, the missing particle is commonly known as the Higgs.

However, since the massive gauge field is generally not friendly with renormalisation, we must find another gauge where renormalisability is manifest. Remember that the theory was renormalisable, at least, before the symmetry had been hidden. Here, let us parametrise $\phi = (\varphi + \mathrm{i}\chi)/\sqrt{2}$. Denoting the physical fields as $\varphi' = \varphi_1 - v$ and $\chi' = \chi$, the Lagrangian then becomes

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2a}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu\varphi')^2 + \frac{1}{2}(\partial_\mu\chi')^2 \\ & +\frac{1}{2}e^2v^2A^\mu A_\mu - evA^\mu\partial_\mu\chi \\ & -\frac{\lambda}{4}(\varphi^2+\chi^2)^2 - \lambda v(\phi^2+\chi^2)\phi - e^2vA^\mu A_\mu\varphi \\ & -\frac{1}{2}e^2A^\mu A_\mu\varphi^2 - \frac{1}{2}e^2A^\mu A_\mu\chi^2 + eA_\mu\chi\partial^\mu\varphi - eA_\mu\varphi\partial^\mu\chi\,, \end{aligned} \tag{2.59}$$

where the second term is the gauge fixing term. Let us further work this out in more detail. When first line of (2.59) is taken as a free Lagrangian, we easily read off the free propagators in the Landau gauge ($a \to 0$)

$$A_\mu : \quad -\frac{\mathrm{i}}{p^2}(g^{\mu\nu} - p^\mu p^\nu/p^2) \equiv -\mathrm{i}D_0^{\mu\nu} \tag{2.60}$$

$$\varphi : \quad \frac{\mathrm{i}}{p^2 - 2\lambda v^2} \tag{2.61}$$

$$\chi : \quad \frac{\mathrm{i}}{p^2}\,. \tag{2.62}$$

Next recall that since the scalar field carries the quantum number of the symmetry current, the Goldstone boson (the $\chi$) can couple, via the gradient coupling, with the gauge boson according to the term $-evA^\mu\partial_\mu\phi_2'$ with the coupling constant $ev$. The mass term can also be viewed as a vertex. Both are displayed in fig.2.1. We can also arrive at the similar pictures using the fact


**Figure 2.1:** Contributions from the scalar field

that the Goldstone bosons have non-vanishing couplings to the electromagnetic current which couples to the gauge field $A^\mu$:

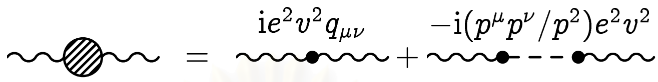
$$\langle 0|J^\mu(0)|\pi(p)\rangle = \mathrm{i}p^\mu F_\pi. \tag{2.63}$$

This also yields the same coupling $\mathrm{i}F_\pi p^\mu$ where we can identify $F_\pi = v$.

To calculate the propagator for $A^\mu$ we first evaluate $\Pi(p^2)$ defined from the vacuum polarisation tensor

$$\mathrm{i}\Pi^{\mu\nu}(p^2) = e^2 \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}px} \langle 0|T J^\mu(x) J^\nu(0)|0\rangle = \mathrm{i}(g^{\mu\nu}p^2 - p^{\mu\nu})\Pi(p^2) \tag{2.64}$$

which, to the lowest order, we find




$$\begin{aligned} &= \mathrm{i}e^2v^2 g^{\mu\nu} - \frac{\mathrm{i}}{p^2}p^\mu p^\nu e^2 v^2 \\ &= \mathrm{i}e^2v^2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \end{aligned} \tag{2.65}$$

which means $\Pi(p^2) = e^2v^2/p^2$. Notice that this vacuum polarisation tensor is transverse; namely,

$$p^\mu \Pi_{\mu\nu} = 0\,. \tag{2.66}$$

Moreover, since the $\Pi(p^2)$ is singular when $p \to 0$, it cancels exactly the massless pole $p^2 = 0$ of the gauge boson propagator given in (2.60). This can be seen by recalling that the $A$ propagator, constructed from a geometric series, is given by

$$\begin{aligned} D^{\mu\nu} &= (D_0 + D_0\Pi D_0 + \ldots)^{\mu\nu} \\ &= -\frac{\mathrm{i}}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left[1 + \frac{e^2v^2}{p^2} + \ldots\right] \\ &= -\mathrm{i}\frac{g^{\mu\nu} - p^\mu p^\nu/p^2}{p^2 - m_A^2} \qquad (2.67) \\ &= -\mathrm{i}\frac{D_0^{\mu\nu}}{(1 - \Pi(p^2))}\,. \qquad (2.68) \end{aligned}$$

Put differently, the $A$ acquires mass through its pole (of its propagator) at $m_A = ev$. It is important to emphasise that the Goldstone boson does not appear as an external particle (because it is not the physical one). So it maybe useful to find the other gauge that removes the coupling $evA_\mu\partial^\mu\chi$, which is found to be accomplished by the gauge fixing

$$-\frac{1}{2}(\partial_\mu A^\mu - \xi ev\chi)^2,. \tag{2.69}$$

This gauge, which was shown by 't Hooft to preserve renormalisability [51], is known as the $R_\xi$ gauge ("R" stands for renormalisable). With this gauge fixing, the first two lines of (2.59) becomes

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu\varphi')^2 + \frac{1}{2}(\partial_\mu\chi')^2 + \frac{1}{2}m_A^2 A^\mu A_\mu - \frac{1}{2}\xi m_A^2 \chi'^2\,, \tag{2.70}$$

leading, when being regarded as a free Lagrangian, to the propagator for the gauge boson

$$\begin{aligned} A_\mu : \quad & -\frac{\mathrm{i}}{p^2 - e^2v^2}\left[g^{\mu\nu} - \frac{(1-\xi)p^\mu p^\nu}{p^2 - \xi e^2 v^2}\right] \\ & = -\mathrm{i}\frac{g^{\mu\nu} - p^\mu p^\nu/m_A^2}{p^2 - m_A^2} - \mathrm{i}\frac{p^\mu p^\nu/m_A^2}{p^2 - \xi m_A^2} \end{aligned} \tag{2.71}$$

which can be easily seen to reduce to the unitary gauge (massive gauge field) in the $\xi \to \infty$ limit. The unphysical pole in the propagator of $A_\mu$ can be shown to cancel exactly with the other from the propagator for $\chi'$. The Landau gauge and the 't Hooft-Feynman gauge are recovered in the $\xi = 0$ and $\xi = 1$ limits respectively.

The result in this discussion can be easily extended to the non-Abelian case. The expression for the mass of the gauge boson (2.52) can be rewritten (with some obvious generalisations) in this way

$$\frac{g^2}{2}(T^c v^\dagger)_i (T^d v)_i W'^c_\mu W'^{d\mu} \tag{2.72}$$

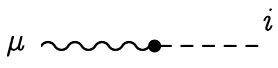
which gives us the *mass matrix*

$$m_{ab}^2 = g^2 (T^a v)_i (T^b v)_i\,. \tag{2.73}$$

Moreover, the interaction between the gauge fields $W$ and the Goldstone bosons is then generalised to

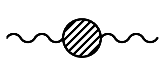
$$-\mathrm{i}gW_\mu^a \partial^\mu \phi_i (T^a v)_i \tag{2.74}$$

which leads to the transition amplitude




$$= \quad gp^\mu (T^a v)_i \tag{2.75}$$

and eventually to the $W$ propagator



$$= \quad \mathrm{i}m_{ab}^2 \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\,. \tag{2.76}$$

When there are more than one coupling in the theory there will be more amplitudes of the form like (2.75) corresponding to different couplings between different kinds of the Goldstone bosons and the gauge fields.

## 2.3 Gauge Theory for Electroweak Interaction

From now on we will work with the assumption in mind that interactions are explained by local symmetries. When gravity is excluded, the "local symmetries" then reduces to "gauge symmetries" and we are will be ready to proceed with the tools on hand. We begin in 2.3.1, where we will argue heuristically to find the suitable gauge group. Then we associate particles their electroweak quantum numbers, grouping them into doublet-singlet structure. The physical current like the electromagnetic current will be constructed and expressed in terms of currents corresponding to the electroweak gauge group. These currents will interact with the gauge fields. The detail of the latter will be shown in section 2.3.2. All those sections alone will be, to some degrees, useless because the concepts of electric charge and so on will not make any sense unless the symmetry breaking occurs. So in section 2.3.3 we present the usual strategy of the spontaneous symmetry breaking; namely the BEH (Higgs) mechanism triggered by a fundamental scalar. After that we will show, in section 2.3.4, how fermions, whose masses protected by chiral symmetry, receive masses by interacting with the Higgs particle. Finally the idea of custodial symmetry $SU(2)$ will be introduced in 2.3.5.

### 2.3.1 The Gauge Group for Electroweak Interaction

In this section we will start with a "given" structure of the fermions and their interactions with the bosons that are known so far according to experimental observations of weak and electromagnetic processes. We shall seek for the appropriate gauge group in order to realise the theory as a gauge theory. Then we can study some properties of the corresponding gauge fields and imagine how they should look like *after* spontaneous symmetry occurs, pretending that we do not care (for the moment) that they are massless.

Weak interaction can be explained by the prescription of interactions that is rather similar to the electromagnetic one; namely by introducing 3 (spin-1) bosons and let them couple with fermions. Since two of the bosons are charged, the interaction between these bosons and a photon is inevitable. Therefore, it is very tantalising to look for a theory that "unifies" electromagnetic and weak interactions into the electroweak one. The main task is simply to look for a suitable gauge group. By "suitable" we mean that the group must not only be large enough to contain at least 4 gauge bosons and allow us to fit all known fermions into some of its representation, but the group should also require minimal introduction of new particles. The $SU(2)$ seems to be a good candidate but is, however, not good enough (see next paragraph). Therefore, we will look for an extension of this group.

We will list the fields that the sought-for theory is supposed to handle. The three lepton families are

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L , \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L , \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L , \quad e_R^- , \quad \mu_R^- , \quad \tau_R^- , \tag{2.77}$$

and are now regarded as doublets and singlets (not just a collection of fields). The right-handed electron, muon, and tau are also needed since they participate in electromagnetic interaction. They are expected to appear as (electroweak) singlets because charged-changing current weak interaction does not "touch" right-handed particles or left-handed anti-particles. The right-handed neutrino fields are not needed because they do not participate in weak interaction. The left-handed doublet $L_L$ and right-handed singlet $l_R$ are given explicitly by

$$L_L^l = \frac{1-\gamma^5}{2}\begin{pmatrix}\nu_l\\ l\end{pmatrix} , \qquad l_R = \frac{1+\gamma^5}{2} l , \tag{2.78}$$

and similarly

$$\bar{L}_L^l = \begin{pmatrix}\bar{\nu}_l & \bar{l}\end{pmatrix}\frac{1+\gamma^5}{2} , \qquad \bar{l}_R = \bar{l}\left(\frac{1-\gamma^5}{2}\right) , \tag{2.79}$$

Quarks appear in quite a similar structure except that *all* of them have right-handed partners

$$\begin{pmatrix}u\\ d'\end{pmatrix}_L , \quad \begin{pmatrix}c\\ s'\end{pmatrix}_L , \quad \begin{pmatrix}t\\ b'\end{pmatrix}_L , \quad u_R , \quad d'_R , \quad c_R , \quad s'_R , \quad t_R , \quad b'_R . \tag{2.80}$$

Notice that the arrangement of fermions in doublets and singlets in this way automatically leads to parity violation in weak interactions. However, it is important to remember that this does not explain "why" parity is violated. It just describes "how". Also remember that members of the multiplet are *exactly* the same before electroweak symmetry is broken. In addition, their masses are protected by chiral symmetry. Only after the spontaneous symmetry breaking occurs will the definitions of masses and charges be technically reasonable.

A unification with electromagnetism means that we must incorporate the electric charge $Q$ into the group as a symmetry generator. Mathematically, this immediately rules out the $SU(2)$ group because the generator of $SU(2)$ must be traceless ($T_3$ is not qualified for being electric charge: neutrinos are neutral while electrons have charge $-1$). Phenomenologically, the group $SU(2)$ is also rejected as we have seen that there *is* a neutral gauge boson for weak interaction. In other words, $SU(2)$ can provide only 3 gauge bosons while 4 are required. Then the next candidate that is just big enough to allow the room for 4 gauge bosons, which is the one that nature seems to chose, is the group $SU(2) \times U(1)$.

The $U(1)$ in $SU(2) \times U(1)$ does not correspond directly to pure electromagnetism (i.e., its charge cannot be the electromagnetic one) and there is no physical reason for it to be so. Actually the $U(1)$ charge *cannot* be $Q$ since $Q$ has a preferred direction in the weak isospin space as it distinguishes $e$ from $\nu_e$ and so on. In other words, $Q$ is not a "constant of motion". What we need is a new quantum number corresponding to an operator which commutes with the $SU(2)$ generators $T_a$. Let us call it the weak hypercharge $Y$. The connection between the $U(1)$ (hypercharge) or its gauge boson, which we will call $B^\mu$, and electromagnetism or $A^\mu$ (photon) can be identified only with the use of further physical arguments. So it is common to denote the gauge group for electroweak interaction as $SU(2)_L \times U(1)_Y$.

The form of $Y$ can be found by observing that for each lepton or quark family, the electric charge $Q$ can be decomposed into two parts. The first is the right-handed part called $Q_{\text{Right}}$ which obviously commutes with $T_a$. The second is the left-handed part[9] $Q_{\text{Left}}$ that can be further broken down into

$$Q_{\text{Left}} = T_3 - \frac{1}{2}\int \mathrm{d}^3x L_l^\dagger L_l \tag{2.81}$$

for the lepton family, and

$$Q_{\text{Left}} = T_3 + \frac{1}{6}\int \mathrm{d}^3x Q_q^\dagger Q_q\,. \tag{2.82}$$

for the quark family. Here the indices $l$ and $q$ stand for lepton or quark families

[9] It should be clear from the context that these do not mean the left- or right-handed quark doublets.

(*not summed* here). Since the second part of $Q_{\text{Left}}$ commutes with $T_a$, we then see that $Y \propto Q - T_3$. So we set a *convention*

$$Y = 2(Q - T_3)\,. \tag{2.83}$$

Let us denote the $SU(2)_L$ coupling as $g$ and the $U(1)_Y$ coupling as $g'$. Before proceeding further, let us notice that though we can start with $g' = g$, there is no symmetry that relates these two couplings (e.g., we cannot write $Y$ in terms of a commutator of the $SU(2)$ generators). This also means they are renormalised differently and tend to differ from one another as we watch them "run". Once we have the definition (2.83) and the fermions structures (2.77) and (2.80), we can assign the hypercharge to all fermions as shown in Table 2.1. The assignment

**Table 2.1:** Hypercharge assignments for fermions.

| fermions | $T$ | $T_3$ | $Q$ | $Y$ |
|---|---|---|---|---|
| $\nu_e, \nu_\mu, \nu_\tau$ | 1/2 | 1/2 | 0 | -1 |
| $e_L, \mu_L, \tau_L$ | 1/2 | -1/2 | -1 | -1 |
| $e_R, \mu_R, \tau_R$ | 0 | 0 | -1 | -2 |
| $u_L, c_L, t_L$ | 1/2 | 1/2 | 2/3 | 1/3 |
| $d'_L, s'_L, b'_L$ | 1/2 | -1/2 | -1/3 | 1/3 |
| $u_R, c_R, t_R$ | 0 | 0 | 2/3 | 4/3 |
| $d'_R, s'_R, b'_R$ | 0 | 0 | -1/3 | -2/3 |

of hypercharges leads to, for example, the electron's electromagnetic current

$$\begin{aligned} J^{(e)\mu}_{\text{em}} &= -\bar{e}\gamma^\mu e = -\bar{e}_L\gamma^\mu e_L - \bar{e}_R\gamma^\mu e_R \\ &= -\bar{L}^e_L\gamma^\mu \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} L^e_L - \bar{e}_R\gamma^\mu e_R \\ &= -\bar{L}^e_L\gamma^\mu \left(\frac{1}{2} - T_3\right) L^e_L - \bar{e}_R\gamma^\mu e_R\,, \end{aligned} \tag{2.84}$$

and the charge changing current

$$J^{(e)\mu}_{-} = \bar{e}\gamma^\mu(1-\gamma^5)e = 2\bar{L}^e_L\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} L^e_L = 2\bar{L}^e_L\gamma^\mu T^- L^e_L \tag{2.85}$$

$$J^{(e)\mu}_{+} = 2\bar{L}^e_L\gamma^\mu T^+ L^e_L \tag{2.86}$$

where $T^{\pm} = T^1 \pm \mathrm{i}T^2$. So it is now clear how these currents are related to the $SU(2)$ and $U(1)$ structures. In addition, notice that (2.83) allows us to "define"

$$J^{\mu}_{em} = J^{\mu}_3 + \frac{1}{2} J^{\mu}_Y \,, \tag{2.87}$$

where

$$J^{\mu}_a = \bar{L}^l_L \gamma^{\mu} T^a L^l_L \,. \tag{2.88}$$

The definition of the hypercharge also implies that the gauge interaction term contains

$$gT^a W^a_\mu + \frac{1}{2} g' Y B_\mu = g(T^1 W^1_\mu + T^2 W^2_\mu) + T^3 (g W^3_\mu - g' B_\mu) + g' Q B_\mu \,, \tag{2.89}$$

where the factor 1/2 is a convention (recall that $T^a = \tau^a/2$). By inspecting at one of the neutral (charge-preserving) part, we see that the term $(gW^3 - g'B)$ must corresponds to the $Z$ due to its coupling with $T^3$. Moreover, since $A$ is orthogonal and linearly independent of $Z$ we then have the relations

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gW^3_\mu - g'B_\mu) = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W \tag{2.90}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gB_\mu + g'W^3_\mu) = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W \,, \tag{2.91}$$

where the $\theta_W$ from

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \,, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \tag{2.92}$$

is defined as the *weak mixing angle* or the *Weinberg angle*. After using (2.90) for $Z$ and inverting the expression (2.90) and (2.91) for $B$ and $W^3$, we put them into (2.89) and obtain

$$gT^a W^a_\mu + \frac{1}{2} g' Y B_\mu = \frac{g}{\sqrt{2}}(T^+ W^+_\mu + T^- W^-_\mu) + \frac{g}{2\cos\theta_W}(2T^3 - 2Q\sin^2\theta_W) Z_\mu + eQA_\mu \,, \tag{2.93}$$

where $W^{\pm}_\mu = (W^1_\mu \mp \mathrm{i}W^2_\mu)/\sqrt{2}$, and

$$e = g\sin\theta_W = g'\cos\theta_W \tag{2.94}$$

which is defined by inspecting the coupling between $A^\mu$ and Q. Notice that the factor 2 in the $Z^\mu$ term in (2.93) is a convention - to make it parallel with (2.87). Observe that $Z$ has non-zero coupling with neutrinos as expected.

### 2.3.2 Interactions Between the $SU(2)\times U(1)$ Gauge Fields and Fermions

Now we (should) have convinced ourselves how the gauge $SU(2)_L \times U_Y(1)$ group has something to do with the physical fields $(W^{\pm}, Z, A)$ for the (low-energy) effective weak interaction and the electromagnetic one. The next task is then to work out explicitly their interactions with matter fields. This can be accomplished in the familiar way; i.e., by introducing the gauge-current coupling. Here we will concentrate on the leptonic part of the theory. The hadronic (quark) part will look somewhat similar. By inspecting (2.89) we find that it is reasonable to write the neutral current[10] as

$$J^{\mu}_{NC} = 2J^{\mu}_3 - 2\sin^2\theta_W J^{\mu}_{\text{em}}\,, \tag{2.95}$$

so that

$$\begin{aligned}\mathcal{L}_{int} &= gJ^{a\mu}W^a_\mu + \frac{g'}{2}J^\mu_Y B_\mu \\ &= \frac{g}{2\sqrt{2}}(J^{+\mu}W^+_\mu + J^{-\mu}W^-_\mu) + \frac{g}{2\cos\theta_W}J^\mu_{NC}Z_\mu + eJ^\mu_{\text{em}}A_\mu\,. \end{aligned} \tag{2.96}$$

Observe that the neutral current for electron (family) can be written out explicitly as

$$\begin{aligned} J^{(e)\mu}_{NC} &= 2\bar{L}^e_L\gamma^\mu T^3 L^e_L - 2\sin^2\theta_W\left[\bar{L}^e_L\gamma^\mu\left(\frac{1}{2}-T^3\right)L^e_L - \bar{e}_R\gamma^\mu e_R\right] \\ &= \bar{L}^e_L\gamma^\mu\begin{pmatrix}1 & 0\\ 0 & -\cos 2\theta_W\end{pmatrix}L^e_L + 2\sin^2\theta_W\bar{e}_R\gamma^\mu e_R \\ &= \frac{1}{2}\bar{\nu}\gamma^\mu(1-\gamma^5)\nu + \bar{e}\gamma^\mu\left[\left(2\sin^2\theta_W - \frac{1}{2}\right)+\frac{1}{2}\gamma^5\right]e \\ &= \bar{\nu}\gamma^\mu(C^\nu_V - C^\nu_A\gamma^5)\nu + \bar{e}\gamma^\mu\left(C^e_V - C^e_A\gamma^5\right)e\,, \end{aligned} \tag{2.97}$$

where

$$C^\nu_V = \frac{1}{2}\,, \qquad C^\nu_A = \frac{1}{2}\,, \qquad C^e_V = -\frac{1}{2} + 2\sin^2\theta_W\,, \qquad C^e_A = -\frac{1}{2}\,. \tag{2.98}$$

The universality and the fermion family replication tell us that the superscripts on each lepton family are actually overkill; i.e., all neutrinos have $C_A = C_V = \frac{1}{2}$ and all $e$, $\mu$, and $\tau$ share the same $C_V$'s and $C_A$'s. The total contribution from

[10] In some literatures, the "neutral current" include the electromagnetic current which is also a charge-preserving (neutral) type.

a number of left-handed lepton doublets $L_L^i = \begin{pmatrix} \nu_i \\ l_i \end{pmatrix}_L$ is then

$$J^{\mu}_{NC} = \frac{1}{2}\sum_{\nu_i} \bar{\nu}_i\gamma^{\mu}(1-\gamma^5)\nu_i + \sum_{l_i} \bar{l}_i\gamma^{\mu}(C_V - C_A\gamma^5)l_i \tag{2.99}$$

together with

$$C_V = T^3 - 2Q\sin^2\theta_W\,, \qquad C_A = T^3\,, \tag{2.100}$$

In some cases, it is useful to introduce the "chiral coupling"

$$C_L = C_V + C_A = 2(T^3 - Q\sin^2\theta_W) \tag{2.101}$$

$$C_R = C_V - C_A = -2Q\sin^2\theta_W\,. \tag{2.102}$$

Then the leptonic neutral current part of Lagrangian can be written as

$$\mathcal{L}_Z = \frac{g}{2\cos\theta_W}\sum_i \left\{\overline{\nu}_i\gamma^{\mu}(1-\gamma^5)\nu_i + \bar{l}_i\left[C_L\gamma^{\mu}(1-\gamma^5) + C_R\gamma^{\mu}(1+\gamma^5)\right]l_i\right\}\,. \tag{2.103}$$

This means the neutral current of the $SU(2)\times U(1)$ theory can be expressed in terms of the variables that are frequently considered as

$$\begin{aligned}\mathcal{L}_{\rm NC} = & -\sum_i e\bar{l}_i\gamma^{\mu}{}_iA_{\mu} + \frac{1}{2}\left(\frac{M_W^2G_F\sqrt{2}}{\cos^2\theta_W}\right)^{1/2}\sum_i \overline{\nu}_i\gamma^{\mu}(1-\gamma^5)\nu_iZ_{\mu} \\ & + \frac{1}{2}\left(\frac{M_W^2G_F\sqrt{2}}{\cos^2\theta_W}\right)^{1/2}\sum_i \bar{l}_i\Big[(2\sin^2\theta_W - 1)\gamma^{\mu}(1-\gamma^5) \\ & + 2\sin^2\theta_W\gamma^{\mu}(1+\gamma^5)\Big]l_iZ_{\mu}\,. \end{aligned} \tag{2.104}$$

Similar terms, with slight modifications of the chiral couplings, are applicable when quarks are brought in.

It is important to emphasise that what we have done so far is mostly to convince ourselves that the $SU(2)_L \times U(1)_Y$ gauge fields are related to the physical fields[11]. In other words, we have used a bottom-up approach. We still have not used one of the essential features of gauge theory, namely the coupling between the gauge fields and the particle fields via the gauge covariant derivative. This is considered as a top-down approach where what we have found so far can

[11] Though these fields must be massless at this stage, we know that this can be resolved by the BEH (Higgs) mechanism.

be summarised in the Lagrangian

$$\begin{aligned}
\mathcal{L} &= \sum_i \mathrm{i}\bar{\Psi}_{Li}\gamma^\mu D_\mu \Psi_{Li} + \sum_i \mathrm{i}\bar{\Psi}_{Ri}\gamma^\mu D_\mu \Psi_{Ri} \\
&\quad -\frac{1}{2}\mathrm{Tr}\left\{F^a_{\mu\nu}T^a F^{\mu\nu a}T^a\right\} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \qquad (2.105)\\
&= \sum_l \mathrm{i}\bar{L}^i_L\gamma^\mu D_\mu L^i_L + \sum_l \mathrm{i}\bar{l}^i_R\gamma^\mu D_\mu l^i_R \\
&\quad + \sum_i \mathrm{i}\bar{Q}^i_L\gamma^\mu D_\mu Q^i_L + \sum_i \mathrm{i}\bar{u}^i_R\gamma^\mu D_\mu u^i_R + \sum_i \mathrm{i}\bar{d}^i_R\gamma^\mu D_\mu d^i_R \\
&\quad -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\,, \qquad (2.106)
\end{aligned}$$

where $\Psi$ stands for all fermions and the $D_\mu$ is the covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i}gT^aW^a_\mu - \mathrm{i}\frac{g'}{2}YB_\mu\,, \qquad (2.107)$$

or, using (2.96),

$$D_\mu = \partial_\mu - \mathrm{i}\frac{g}{\sqrt{2}}\left(T^+W^+_\mu + T^-W^-_\mu\right) - \mathrm{i}\frac{g}{2\cos\theta_W}\left(2T^3 - 2\sin^2\theta_W Q\right)Z_\mu - \mathrm{i}eQA_\mu \qquad (2.108)$$

and

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon_{abc}W^b_\mu W^c_\nu\,, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \qquad (2.109)$$

The Lagrangian (2.106) contains all the fields corresponding to all of the elementary particles except the gluons of the strong interaction. The particle content of the standard model (including quarks and gluons) can also be summed up according to the way they transforms under the $SU(3)_C \times SU(2)_L \times U(1)_Y$:

$$\begin{aligned}
&(\mathbf{3},\mathbf{2},1/3)_L\,,(\mathbf{3},\mathbf{1},4/3)_R\,,(\mathbf{3},\mathbf{1},-2/3)_R\,,(\mathbf{1},\mathbf{2},-1)_L\,,(\mathbf{1},\mathbf{1},-2)_R\,,\\
&(\mathbf{8},\mathbf{1},0)\,,(\mathbf{1},\mathbf{3},0)\,,(\mathbf{1},\mathbf{1},0) \qquad (2.110)
\end{aligned}$$

Though these structures may not look very satisfactory at first, they tell us that all the particles cannot have bare masses. We say that $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry protects the bosons from being massive and the chiral symmetry does the same job with the fermions. This is not very satisfactory since nature has no massless particles but photon and the gluons. Many ways of generating masses are proposed and one of them is the BEH mechanism.

### 2.3.3 Electroweak Symmetry Breaking

It is widely believed that the mechanism is the promising one, maybe due to its simplicity, its analogy with superconductivity, or its minimal requirements of new particles (just one). In this section we will consider the BEH mechanism in the "minimal" sense; i.e., only one Higgs doublet will be needed. Before we start, let us notice the important fact: when particles get their mass via the BEH mechanism, their mass must be proportional to the vacuum expectation value of a scalar field. This means that their mass must not be too different from each other and must lie somewhere below the energy scale of the theory (the vacuum expectation value). However, we shall see in 3.4.2.2 that this argument is true except for the Higgs itself (or any fundamental scalar particle).

The ingredient for the BEH mechanism to be discussed in the conventional standard model is a scalar field with non-zero vacuum expectation value. As we have said earlier, the reason why it exists is rather *ad hoc*. In other words, the fundamental scalar field must exist, "otherwise the symmetry will not break". For the pattern of symmetry breaking, the simplest choice is to argue that the global symmetry is exactly the same as the gauge symmetry in question. Since the upper bound of the photon mass is very low, it is then assumed that the $U(1)_{\mathrm{em}}$ symmetry is exact is the symmetry of the vacuum and hence

$$SU(2) \times U(1) \longrightarrow U(1)\,, \tag{2.111}$$

which will be eventually identified with the $SU(2)_L \times U(1)_Y \to U(1)_{em}$. One of the Lagrangians that can do the job is

$$\mathcal{L}(\Phi) = \partial_\mu \Phi^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2\,, \tag{2.112}$$

with $\mu^2 < 0$. The pattern (2.111) requires that all the generators of the $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$ are "broken"; i.e.,

$$T^a \langle \Phi \rangle_0 \neq 0\,, \qquad \text{and} \qquad Y \langle \Phi \rangle_0 \neq 0 \tag{2.113}$$

while

$$Q \langle \Phi \rangle_0 = \left( T^3 + \frac{Y}{2} \right) \langle \Phi \rangle_0 = 0\,. \tag{2.114}$$

Observe that we can write the $T^3$-$Y$ combination of the broken generator as $T^3 - \frac{Y}{2}$ which will later be associated with the $Z$. So the simplest form of the

$\Phi$ that can interact with the $SU(2)$ gauge fields is then a doublet[12], often called the *Higgs doublet* $H$,

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} \tag{2.115}$$

with $Y = +1$. To ensure that the vacuum does not carry electric charges, the vacuum can be *aligned*[13] such that

$$\langle H \rangle_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} , \tag{2.116}$$

with $v^2 = -\mu^2/\lambda$. How about the triplet? In case we want to include the "Higgs" triplet, there are more freedom on assigning the charges of the fields. For example, we can have a triplet consisting of $\Phi^{++}, \Phi^+, \Phi^0$, or $\Phi^+, \Phi^0, \Phi^-$, etc. In the first case we have

$$\Phi = \begin{pmatrix} \Phi^{++} \\ \Phi^+ \\ \Phi^0 \end{pmatrix} \tag{2.117}$$

together with the associated isospin

$$T^3(\Phi) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} . \tag{2.118}$$

This triplet will have the hypercharge assignment $Y = 2$ (from $Q = T^3 + Y/2$). The isospin generators $T^a$ for the triplet version are given in the appendix C.3.

Following the section 2.2.3, we parametrise the Higgs doublet as

$$H = \exp\left\{ \mathrm{i}\frac{T^a \chi^a}{v} \right\} \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} , \tag{2.119}$$

where $\chi^a$'s and $h$ now have zero vacuum expectation values. The field $h$ corresponds to the so-called Higgs particle. After that we couple the system defined by the Lagrangian (2.112) with the electroweak gauge fields. In this state we may say that the $SU(2) \times U(1)$ is promoted to a local symmetry. Then the gauge fixing $\theta^a = \chi^a/v$ clearly removes the unphysical degrees of freedom,

[12] Notice that we have introduce the specific symbol $H$ for the Higgs doublet, rather than the $\Phi$. The symbol $\Phi$ will be used for other purposes in later chapters.

[13] For more detail on vacuum alignment, see section 4.1.

leaving

$$H = \frac{v+h}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.120}$$

Therefore, the Gauge invariant Lagrangian reduces to a simple (but less elegant) form

$$\begin{aligned} \mathcal{L} &= \left| \left( \partial_\mu - \mathrm{i} g T^a W^a_\mu - \mathrm{i} g' \frac{Y}{2} B \right) \frac{(v+h)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 \\ &\quad - \frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 \\ &= \left| \left( \partial_\mu - \mathrm{i} \frac{g}{\sqrt{2}} \left( T^+ W^+_\mu + T^- W^-_\mu \right) \right. \right. \\ &\qquad \left. \left. - \mathrm{i} \frac{g}{2\cos\theta_W} \left( 2T^3 - 2\sin^2\theta_W Q \right) Z_\mu - \mathrm{i} e Q A_\mu \right) \frac{(v+h)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 \\ &\quad - \frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 \end{aligned} \tag{2.121}$$

or

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{g^2 v^2}{4} W^+_\mu W^{-\mu} + \frac{1}{2}\frac{g^2 v^2}{4\cos^2\theta_W} Z_\mu Z^\mu - \frac{1}{2}(-2\mu^2) h^2 \\ &\quad + \frac{g^2 v}{2} h W^+_\mu W^{-\mu} + \frac{g^2 v}{4\cos^2\theta_W} Z_\mu Z^\mu + \frac{\mu^2}{v} h^3 \\ &\quad + \frac{g^2}{4} h^2 W^+_\mu W^{-\mu} + \frac{g^2}{8\cos^2\theta_W} h^2 Z_\mu Z^\mu + \frac{\mu^2}{4v^2} h^4 . \end{aligned} \tag{2.122}$$

By observing the quadratic terms listed above, we see that the $SU(2)$ gauge boson are massive and identify

$$M_W^2 = \frac{g^2 v^2}{4} , \tag{2.123}$$

as well as

$$M_Z^2 = \frac{g^2 v^2}{4\cos^2\theta_W} = \frac{M_W^2}{\cos^2\theta_W} = \frac{g^2 + g'^2}{4} v^2 . \tag{2.124}$$

Notice that the mass terms are defined a bit differently; i.e., $M_W^2 W^+_\mu W^{-\mu}$ and $\frac{1}{2} M_Z^2 Z^\mu Z_\mu$ . We see that the quadratic term $A^\mu A_\mu$ is absent and the photon is massless. However, the price we have to pay in a gauge theory with SSB is the introduction of the Higgs particle. Its mass is given by

$$M_h^2 = -2\mu^2 = 2\lambda v^2 . \tag{2.125}$$

Without quantum effects (see the section 3.1.5), this value can be anything in principle, including a value very close to zero. It cannot be predicted within the framework of the standard model so that it ($M_h$) has to be taken from experiments. Unfortunately, no one has ever seen the Higgs so far and we only know that $v^2 \approx (246\,\text{GeV})^2$. Nevertheless, we can still learn something about the mass of the Higgs. Recall that what we have done so far relies on the validity of perturbation technique. So it is required that $\lambda < 1$ which means

$$M_h^2 < (350\text{GeV})^2 \tag{2.126}$$

or else perturbation theory will break down.

Still, the masses of the $W$'s and $Z$ can be evaluated from the low energy phenomenology. Using the "matching" relation between the electroweak coupling and the Fermi constant

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}, \tag{2.127}$$

(2.123) and the value of the Fermi constant obtained from experiment, we find the vacuum expectation value parameter

$$v = \left(\sqrt{2}G_{\text{F}}\right)^{-1/2} \simeq 246\text{GeV} \tag{2.128}$$

or the vacuum expectation value of the scalar field $\langle H\rangle_0 \simeq 175\text{GeV}$. Then this leads to the tree-level predictions

$$M_W \simeq 78\,\text{GeV}\,, \qquad \text{and} \qquad M_Z \simeq 89\,\text{GeV}\,, \tag{2.129}$$

where the latter one (that predicts the mass of the $Z$) is one of the prominent features of the model.

### 2.3.4 Fermion Masses

In this section we will have a quick glance on how quarks become massive within the conventional BEH mechanism. The results of the previous section allow us to safely say that masses of the gauge fields in the spontaneously broken gauge theory are the consequences of the interactions between the corresponding particles and the Higgs doublet. So it is expected that fermions (leptons and quarks, to be specific) can be "dragged" in a more or less similar way. The problem is that the couplings between scalars and fermions do not originate from the gauge interactions, so we do not really know what "forms" of interactions to put in. The gauge symmetry of the Lagrangian only tells us what cannot be put in but does not tell us what what to look for. Thus, the usual strategy applies: start with the simplest one that is renormalisable, then leave the remaining works to experimentalists. In other words, the mass of the fermions must be thought of as free parameters in the context of the standard model and only experiments will tell what their values are. Since there are many kinds of fermions in the theory, their coupling strengths may differ by many orders of magnitude - from $\sim 10^{-6}$ for an electron to $\sim 1$ for a top quark. Again, the standard model do not have explanations for this.

The simplest gauge invariant form of scalar-fermion interaction is the "$h\psi\psi$" Yukawa interaction

$$\begin{aligned} \mathcal{L}_Y &\supset -y_{f'}\left[\bar{f}'_R(H^\dagger F_L) + (\bar{F}_L H)f'_R\right] = -\frac{y_f v}{\sqrt{2}}\left[\bar{f}'_R f'_L + \bar{f}'_L f'_R\right] \\ &= -\frac{y_{f'} v}{\sqrt{2}}\bar{f}' f' - \frac{y_f}{\sqrt{2}}\bar{f}' f' h\,, \end{aligned} \tag{2.130}$$

where

$$F_L = \begin{pmatrix} f \\ f' \end{pmatrix}_L \tag{2.131}$$

This readily tells us that the mass of the $T_3 = \frac{1}{2}$ member of the $SU(2)$ fermion doublet is

$$m_{f'} = \frac{y_{f'} v}{\sqrt{2}}\,. \tag{2.132}$$

This also says that the coupling between the Higgs and the fermions is proportional to the fermion's mass. For example, we have

$$y_e = \frac{\sqrt{2}m_e}{v} \approx 2 \times 10^{-6}\,. \tag{2.133}$$

Observe that while this kind of mass term is fine for the leptons as the neutrinos are automatically massless, it is not sufficient to provide any of the $T_3 = -\frac{1}{2}$ quarks. Thus we have to "flip" the Higgs doublet in a specific way; i.e., we introduce the conjugate of $H$:

$$\tilde{H} = \mathrm{i}\tau^2 H^* = \begin{pmatrix} h^{0*} \\ -h^- \end{pmatrix}\,, \tag{2.134}$$

which is also an $SU(2)$ doublet, with $Y = -1$. It is clear that this will lead to the masses of the up-type quarks when the scalar field receives the vacuum expectation value. So the general Yukawa interaction between the scalar and fermions in the $SU(2)\times U(1)$ theory is supposed to be of the form

$$\mathcal{L}_Y = -y_l^i(\bar{L}_L^i H)l_R^i - y_d^i(\bar{Q}_L^i H)d_R^i - y_u^i(\bar{Q}_L^i \tilde{H})u_R^i + \text{h.c.}\,. \tag{2.135}$$

However, this is not entirely correct. There is no *a priori* reason why quarks must pair up within their own family member, which is defined by weak interaction, and interact with the Higgs. So the following couplings are also possible

$$\begin{aligned}
&(\bar{Q}_L^u\tilde{H})u_R\,,\quad (\bar{Q}_L^u H)d_R\,,\quad (\bar{Q}_L^u H)s_R\,,\quad (\bar{Q}_L^u H)b_R\\
&(\bar{Q}_L^c H)d_R\,,\quad (\bar{Q}_L^c\tilde{H})c_R\,,\quad (\bar{Q}_L^c H)s_R\,,\quad (\bar{Q}_L^c H)b_R\\
&(\bar{Q}_L^t H)d_R\,,\quad (\bar{Q}_L^t H)s_R\,,\quad (\bar{Q}_L^t H)b_R\,,\quad (\bar{Q}_L^t\tilde{H})t_R\\
&\quad +\text{h.c.}
\end{aligned} \tag{2.136}$$

Each of them requires its own coupling constant (not all independent) which must be chosen so as to give the correct quark masses; i.e., the quark fields $u, d, c, s, t$, and $b$ are the mass eigenstates not the electroweak (gauge) eigenstates.

### 2.3.5 Custodial Symmetry $SU(2)$

What we have done so far was to assume the existence of the complex scalar doublet to break the global (and hence the gauge) symmetry $SU(2)\times U(1)\to U(1)$. In this section we shall see that this $SU(2)\times U(1)$ is not the largest symmetry the Higgs system can have and study the consequences.

Let us start with the Higgs system alone, neglecting all the gauge symmetries. This system is described by the Lagrangian (2.112), rewritten here,

$$\mathcal{L}(H) = \partial_\mu H^\dagger \partial^\mu H - \mu^2 H^\dagger H - \lambda(H^\dagger H)^2\,. \tag{2.137}$$

Also recall that we have the conjugate given by (2.134)

$$\tilde{H} = \mathrm{i}\tau^2 H^* = \begin{pmatrix} h^{0*} \\ -h^- \end{pmatrix}\,. \tag{2.138}$$

The point is that we can treat both $H$ and $\tilde{H}$ on equal footing by introducing a matrix

$$\Sigma \equiv \sqrt{2}\left(\tilde{H}, H\right) = \sqrt{2}\begin{pmatrix} h^{0*} & h^+ \\ -h^- & h^0 \end{pmatrix} \tag{2.139}$$

which satisfies

$$\Sigma^\dagger \Sigma = 2H^\dagger H \mathbb{I} = 2\mathbb{I} \det \Sigma \tag{2.140}$$

and is pseudo-real (see the appendix A.2) via

$$\Sigma^\dagger = \tau^2 \Sigma \tau^2 \,. \tag{2.141}$$

Then the Lagrangian (2.137) becomes

$$\mathcal{L}(\Sigma) = \frac{1}{4}\mathrm{Tr}(\partial^\mu \Sigma^\dagger \partial_\mu \Sigma) + \frac{\mu^2}{4}\mathrm{Tr}(\Sigma^\dagger \Sigma) - \frac{\lambda}{16}\left[\mathrm{Tr}(\Sigma^\dagger \Sigma)\right]^2 \,. \tag{2.142}$$

Clearly the Lagrangian has an $SU(2)_L \times SU(2)_R$ *global* symmetry which will be spontaneously broken to the $SU(2)$ when

$$\langle 0|\Sigma|0\rangle = v\mathbb{1}. \tag{2.143}$$

Neither the left- or the right-handed transformations leaves the vacuum expectation value invariant

$$L\langle 0|\Sigma|0\rangle \neq \langle 0|\Sigma|0\rangle \,, \qquad R\langle 0|\Sigma|0\rangle \neq \langle 0|\Sigma|0\rangle \tag{2.144}$$

but their combinations $R^\dagger = L$ does; i.e.,

$$L\langle 0|\Sigma|0\rangle \neq \langle 0|\Sigma|0\rangle L^\dagger = L\langle 0|\Sigma|0\rangle \neq \langle 0|\Sigma|0\rangle \,. \tag{2.145}$$

Consequently the Lagrangian (2.142) is equivalent to the linear sigma model Lagrangian introduced in (2.32), except for some irrelevant differences in the definitions of the couplings. Observe that now we have a set of degenerate vacua parametrised by an $SU(2)$ transformation. So we arrive at an interesting result: when all the electroweak interactions are switched off, the Higgs Lagrangian alone has a global symmetry that is *larger* than that is required by the electroweak symmetry of the standard model. This extra symmetry on the (pure) Higgs sector is known[14] as a *custodial symmetry* $SU(2)$ for a reason that we shall see in this section and in the section 4.1.1.

If we follow the usual strategy of the sigma model and gauge all the global $SU(2)_L \times SU(2)_R$ we would have seen that out of the 6 gauge bosons (one triplet for the $SU(2)_L$ and the other for the $SU(2)_R$), only 3 would have become massive

---

[14]Some literature refer to the global $SU(2)_R$ as the custodial symmetry, while some prefer to mention the whole $SU(2)_L \times SU(2)_R$.

while others remain massless. However, we know that nature did not chose this path and only the $SU(2)_L$ and $U(1) \subset SU(2)_R$ are gauged. Here, 4 gauge bosons are massless. To see things more clearly, observe that a local $SU(2)_L \times U(1)_Y$ transformation on $H(x)$

$$H \longrightarrow \mathrm{e}^{-\mathrm{i}g\theta^a(x)T^a}\mathrm{e}^{-\mathrm{i}g'\alpha(x)Y}H \tag{2.146}$$

is transferred to the $\Sigma(x)$ field as

$$\Sigma \longrightarrow \mathrm{e}^{-\mathrm{i}g\theta^a(x)T^a}\Sigma\mathrm{e}^{\mathrm{i}g'\alpha(x)YT^3} \tag{2.147}$$

where $T^a = \tau^a/2$. In addition, notice that the $T^3$ is attached at the $U(1)$ part due to the different hypercharges of the two Higgs doublets. Then the introduction of electroweak symmetry *explicitly* breaks the global $SU(2)_L \times SU(2)_R$. The dangerous part comes from the hypercharge coupling as the $U(1)_Y$ is planted in the $SU(2)_R$ part, as a subgroup, of the transformation (2.147). This equation tells us that to gauge the sigma model (2.142), we introduce the $SU(2)_L \times U(1)_Y$ covariant derivative on the $\Sigma(x)$

$$D_\mu\Sigma = \partial_\mu\Sigma + \mathrm{i}gW^a_\mu T^a\Sigma - \mathrm{i}g'Y\Sigma B_\mu T^3\,. \tag{2.148}$$

Consequently, the gauge invariant Lagrangian is

$$\mathcal{L}(\Sigma) = \frac{1}{4}\mathrm{Tr}(D^\mu\Sigma^\dagger D_\mu\Sigma) + \frac{\mu^2}{4}\mathrm{Tr}(\Sigma^\dagger\Sigma) - \frac{\lambda}{16}\left[\mathrm{Tr}(\Sigma^\dagger\Sigma)\right]^2\,. \tag{2.149}$$

In this way we can reconstruct the standard model using the $\Sigma$ field instead of the Higgs doublet and follow the usual strategies. Observe that the Yukawa couplings which are allowed by the electroweak gauge symmetry explicitly break the global $SU(2)_L \times SU(2)_R$ as well.

As we have said earlier, not all the electroweak interaction breaks the custodial symmetry, only the $U(1)_Y$ does. When the $g'$ is turned off, we have $\cos\theta_W = 1$, and the covariant derivative reduces to

$$D_\mu\Sigma = \partial_\mu\Sigma + \mathrm{i}gW^a_\mu T^a\Sigma\,. \tag{2.150}$$

This adds the *global* $SU(2)_R$

$$\Sigma \longrightarrow \Sigma R^\dagger \tag{2.151}$$

back to the Lagrangian, recovering the $SU(2)_L \times SU(2)_R$ global symmetry. Since $W^a_\mu$ is an $SU(2)_R$ singlet, it is also a $SU(2)_{L+R}$ triplet. Then the action of covariant derivative on the vacuum is

$$D_\mu \Sigma_0 = \mathrm{i} g W^a_\mu T^a \Sigma_0 = \mathrm{i}\frac{gv}{2}\begin{pmatrix} W^3_\mu & W^1_\mu - \mathrm{i}W^2_\mu \\ W^1_\mu + \mathrm{i}W^2_\mu & -W^3_\mu \end{pmatrix} \tag{2.152}$$

leading to the mass term

$$\frac{1}{4}\mathrm{Tr}(D^\mu \Sigma_0^\dagger D_\mu \Sigma_0) = \frac{g^2 v^2}{4} W^{a\mu} W^a_\mu \tag{2.153}$$

with $M_{W^\pm} = M_{W^3}$, as expected. In this aspect, we can view the equality between the masses of the charged and neutral bosons

$$\frac{M_W^{\pm 2}}{M_{W^3}^2} = 1 \tag{2.154}$$

as a result of the global symmetry $SU(2)_L \times SU(2)_R$. Note that when the electromagnetic interaction is absent, the weak isospin is strictly valid and the $Z$ mass would be the same as the $W$ mass.

We can go further by bringing back the hypercharge which results in

$$\begin{aligned} D_\mu \Sigma_0 &= \mathrm{i} g W^a_\mu T^a \Sigma_0 - \mathrm{i} g' \Sigma_0 B_\mu Y T^3 = \mathrm{i}\frac{gv}{2}\begin{pmatrix} W^3_\mu - \frac{g'}{g}B_\mu & W^1_\mu - \mathrm{i}W^2_\mu \\ W^1_\mu + \mathrm{i}W^2_\mu & -\left(W^3_\mu - \frac{g'}{g}B_\mu\right) \end{pmatrix} \\ &= \mathrm{i}\frac{gv}{2}\begin{pmatrix} \frac{Z_\mu}{\cos\theta_W} & W^1_\mu - \mathrm{i}W^2_\mu \\ W^1_\mu + \mathrm{i}W^2_\mu & -\frac{Z_\mu}{\cos\theta_W} \end{pmatrix} . \end{aligned} \tag{2.155}$$

In other words, the ‘hypercharge” interactions corresponding to the neutral and electromagnetic ones introduce the difference between the mass of the charged bosons and the neutral boson $Z^\mu$

$$\frac{M_W^2}{M_Z^2} = \cos^2\theta_W . \tag{2.156}$$

This brings us to an important quantity - the *rho parameter*

$$\rho \equiv \frac{M_W^2}{\cos^2\theta_W M_Z^2} = 1 , \tag{2.157}$$

which is respected at tree-level while the value obtained in the lab deviates very slightly. Hence, it can be said that the global $SU(2)_L \times SU(2)_R$ symmetry protects the relation (2.157) between the masses of the electroweak gauge bosons;

and hence its name the custodial symmetry[15]

The other benefit the custodial symmetry provides comes from, again, the very tenuous (but calculable) deviation from the tree-level prediction $\rho \approx 0.99 \pm 0.01$. The other essential feature of the $\rho$-parameter is that it is sensitive to the additional Higgs fields. To see this, let us consider a system consisting of the Higgs fields (scalar) $h_n$ in various representations $R_n$ of $SU(2)$, each comes with a $T^2$ eigenvalue $T_n(T_n+1)$. By restricting ourselves to neutral vacuum expectation values, we can write

$$D_\mu v_n \to \frac{g}{2}\left[T^+W^+_\mu + T^-W^-_\mu + gT^3W^3_\mu - g'T^3B_\mu\right] v_n \tag{2.158}$$

which we can easily evaluate $\sum_n (D_\mu v_n)^\dagger D^\mu v_n$. Then the mass of the gauge fields are written in terms of the eigenvalues of these $T^2$ and $T^3_n$ of $T^3$

$$\begin{aligned} M_W^2 &= \frac{1}{2}g^2 \sum \left[T_n(T_n+1) - (T^3_n)^2\right] v_n^2 \\ M_Z^2 &= \left(g^2 + g'^2\right) \sum (T^3_n)^2 v_n^2 \end{aligned} \tag{2.159}$$

where $v_n$ is the vacuum expectation value of the *neutral* scalar field in the $n^{\text{th}}$ representation. Notice that these formulae make sense only when all the Higgs are properly aligned (see section 4.1.2) so that the $SU(2)$ part of the electroweak symmetry is broken (not the $U(1)$). Then the $\rho$-parameter becomes

$$\rho = \frac{\sum\left[T_n(T_n+1) - (T^3_n)^2\right] v_n^2}{2\sum (T^3_n)^2 v_n^2}\,. \tag{2.160}$$

This relation put a strong constraint on the form of the Higgs; in other words, only the Higgs doublets are welcome. For future references we write

$$M_W^2 = \left[\frac{g^2}{g^2+g'^2}\frac{\sum[T_n(T_n+1) - (T^3_n)^2]v_n^2/2}{\sum T_3^2 v_n^2}\right] M_Z^2\,. \tag{2.161}$$

The moral of the story is that, as we shall see in section 4.1.1, there are many ways to incorporate the custodial symmetry to the theory, either with or without the fundamental scalar. What is important is the breaking global symmetry which also implies the Goldstone bosons which will eventually be the

[15]The name custodial symmetry is fairly generic and may apply to other symmetries that protect some particles from getting large mass. Hence, in general, we may refer the custodial symmetry to a symmetry protecting the small value of a parameter from receiving large radiative corrections (the vanishing of the parameter leads to a symmetry forbidding radiative corrections from inducing non zero value of that parameter). An example of this aspect of the custodial symmetry is the chiral symmetry protecting the fermion mass presented in section 3.4.2.2.

longitudinal component of the gauge bosons. In other words, these Goldstone bosons were "seen", though indirectly, in the labs. The excellent agreement of the predicted $\rho$ parameter and those measured in experiments convinces us that at least the philosophy of spontaneous symmetry breaking should have something to do with nature, one way or another.

# CHAPTER III

# QUANTUM EFFECTS IN THE STANDARD MODEL

In this chapter we will study techniques for dealing with the quantum effects in the standard model. Once we are accustomed with the idea we can further analyse the problem they bring in. In the section 3.1, we will start by studying how the radiative corrections (from particles running in loops) have influences on the potential of the system which can eventually result in symmetry breaking. Then in section 3.2, we will analyse some of the bounds on the mass of the Higgs from the theory side. A brief review on the experimental constraints will be presented in the section 3.3. After gathering things up, we will study some shortcomings of the standard model that are relevant to the Little Higgs in 3.4.

## 3.1 Coleman-Weinberg Mechanism

The goal of this section is to discuss how to find the true vacuum of the system when spontaneous symmetry breaking occurs with quantum effects taken into account. Basically, we will study a quantum field in the presence of a classical external source (not necessary scalars) where the object that plays an important role is the *effective action* and the *effective potential*. We shall eventually see that spontaneous symmetry breaking can occur, due to radiative corrections, even when we do not start with the "Mexican hat-like" potential.

A number of nice articles on the Coleman-Weinberg mechanism are availables and will be our main references. They include, the original paper by Coleman and (Erick) Weinberg themselves [52] (summarised in Pokorski [48], Huang [53], and Cheng and Li [50]), papers by Sher [54] and Brandenberger [55], as well as books by Rivers [56], Srednicki [57] and (Steven) Weinberg [58].

### 3.1.1 The Effective Action

Consider a Lagrangian describing an interaction between a scalar field $\phi$ with an external source $J$. Recall that the generating functional Z[J] for the full Green's function constructed from a vacuum-to-vacuum transition amplitude, is related to the connected one, the W[J], by

$$
\begin{aligned}
Z[J] &= \langle 0|0\rangle = \int \mathcal{D}\phi \exp\left\{ \mathrm{i}S[\phi] + \mathrm{i}\int \mathrm{d}^4x J(x)\phi(x) \right\} \\
&= \sum_0^\infty \frac{1}{N!}(\mathrm{i}W[J])^N = e^{\mathrm{i}W[J]} \,, \qquad (3.1)
\end{aligned}
$$

where $S[\phi] = \int \mathrm{d}^4x \mathcal{L}(\phi)$ and $N$ is the number of connected components. Here the $W[J]$ is expressed in terms of the connected Green's functions (sum of all connected diagrams with $n$ external fields):

$$
W[J] = \sum \frac{1}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n G^{(n)}_{\mathrm{conn}}(x_1, \ldots, x_n) J(x_1) \ldots J(x_n) \,. \qquad (3.2)
$$

Then the *classical field* $\phi_c(x)$ is defined as a vacuum expectation value of $\phi(x)$ in the presence of the source

$$
\phi_c(x) = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} = \frac{\delta W[J]}{\delta J(x)} \qquad (3.3)
$$

where the second equality tells us that it can be written in terms of the connected diagrams. Then let us define the (quantum) effective action, which is a functional of the expectation value of the field in the presence of the source, as a Legendre transformation ("dual") of $W[J]$

$$
\Gamma[\phi_c] = W[J] - \int \mathrm{d}^4x J(x)\phi_c(x) \,, \qquad (3.4)
$$

where $J(x)$ here is the current obtained, in terms of $\phi_c(x)$, from (3.3). This description easily leads to the "dual equation of motion" of (3.3)

$$
\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{\phi_c} = -J(x) \,, \qquad (3.5)
$$

where quantum effects (loop corrections) are already included. The formula helps us find the (external) field $\phi(x)$ in the absence of the source as it is the one that makes $\Gamma$ extremum. This explains why $\Gamma$ is called the effective "action". Observe, that if we assume that the vacuum expectation value of the field $\phi(x)$ is zero when the external field is turned off; i.e.,

$$
\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{J\equiv 0} = 0 \,, \qquad (3.6)
$$

then the extremum condition is stated as

$$\left.\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)}\right|_{\phi_c\equiv 0} = 0\,. \tag{3.7}$$

Also notice that, by construction, $W[J]$ is also a Legendre transformation of $\Gamma[\phi]$, namely

$$W[J] = \Gamma[\phi_c] - \int \mathrm{d}^4x\phi_c(x)J(x)\,,. \tag{3.8}$$

To illustrate the role of the $\Gamma$ as the effective action and make things a bit less abstract let us see what we will get if we use it instead of the usual action $S[\phi]$; i.e., consider

$$\exp\{\mathrm{i}W_\Gamma[J,\hbar]\} \equiv \int \mathcal{D}\phi \exp\left\{\frac{\mathrm{i}}{\hbar}\left[\Gamma[\phi] + \int \mathrm{d}^4xJ(x)\phi(x)\right]\right\} \tag{3.9}$$

where the dimensionless parameter called $\hbar$ is introduced. Next recall that for every (connected) diagram, the number of unfixed internal momenta is equal to the number of loops and that the overall factor of $\hbar$ is equal to $\hbar^{L-1}$. This means the $L$-loop term in $W_\Gamma[J,\hbar]$ carries a factor $\hbar^{L-1}$. Thus, $W_\Gamma[J,\hbar]$ can be expressed as a power series of $\hbar$ as

$$W_\Gamma[J,\hbar] = \sum_{L=0}^{\infty} \hbar^{L-1}W_\Gamma^{(L)}[J] \tag{3.10}$$

which is dominated by tree-level ($L=0$) diagrams in the $\hbar \to 0$ limit. To isolate the tree-level contributions in (3.10) we use the method of stationary phase (also known as the steepest descent, or saddle-point approximation) to evaluate the integral (3.9). Observe that the classical field, by definition, extremises the combination

$$\Gamma[\phi] + \int \mathrm{d}^4xJ(x)\phi(x) \tag{3.11}$$

and hence leads to

$$\exp\{\mathrm{i}W_\Gamma[J,\hbar]\} \propto \exp\left\{\frac{\mathrm{i}}{\hbar}\left[\Gamma[\phi_c] + \int \mathrm{d}^4xJ(x)\phi_c(x)\right]\right\}\,, \tag{3.12}$$

giving, to order $\hbar^{-1}$,

$$W_\Gamma^{(0)}[J] = \Gamma[\phi_c] + \int \mathrm{d}^4xJ(x)\phi_c(x) = W[J]\,. \tag{3.13}$$

In other words, the tree-level diagrams generated by the effective action $\Gamma[\phi]$ gives the *complete* connected diagrams and hence a complete description for scattering amplitude in the original theory that is described by the action $S[\phi]$.

Now, let us leave the formalism for a moment and see how the effective action is evaluated. First we try to work out the $W[J]$ in the path integral $Z[J] = e^{\mathrm{i}W[J]/\hbar}$ by using the saddle-point approximation where the saddle "point" $\phi_s$ is obtained from

$$\left.\frac{\delta\left[\int \mathrm{d}^4y(\mathcal{L} + J(y)\phi(y))\right]}{\delta\phi(x)}\right|_{\phi_s} = 0\,. \tag{3.14}$$

In the simple example where $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V(\phi)$ we find

$$\Box^2\phi_s + V'(\phi_s) = J(x)\,. \tag{3.15}$$

Note that the $\phi_s$ depends on the structure of the potential and hence need not be unique. Next, we write $\phi = \phi_s + \sqrt{\hbar}\bar{\phi}$, expand the exponential of $Z[J]$ about the saddle point, and then, to the lowest order in $\hbar$, we obtain

$$Z[J] \approx e^{\frac{\mathrm{i}}{\hbar}[S[\phi_s]+\int J\phi_s]}\int \mathcal{D}\phi' \exp\left\{\mathrm{i}\int \mathrm{d}^4x\frac{1}{2}\left[(\partial\bar{\phi})^2 - V''(\phi_s)\bar{\phi}^2\right]\right\}\,. \tag{3.16}$$

As the second integral is now in the Gaussian form, we use the formula

$$\int \mathcal{D}\phi \exp\left\{-\frac{1}{2}\int \mathrm{d}^4x\phi(x)K\phi(x)\right\} = (\det K)^{-1/2} = (\exp\{\mathrm{Tr}\ln K\})^{-1/2} \tag{3.17}$$

and arrive at

$$Z[J] = exp\left\{\frac{\mathrm{i}}{\hbar}\left[S[\phi_s] + \int J\phi_s\right] - \frac{1}{2}\mathrm{Tr}\ln[\Box^2 + V''(\phi_s)]\right\}\,, \tag{3.18}$$

which means, to order $\mathcal{O}(\hbar)$,

$$W[J] = \frac{\mathrm{i}}{\hbar}\left[S[\phi_s] + \int J\phi_s\right] + \frac{\mathrm{i}\hbar}{2}\mathrm{Tr}\ln[\Box^2 + V''(\phi_s)]\,. \tag{3.19}$$

By observing that $\phi_c = \delta W[J]/\delta J = \phi_s + \mathcal{O}(\hbar)$ allows us to replace $\phi_s$ by $\phi_c$ within errors of $\mathcal{O}(\hbar^2)$. Then we finally arrive at the Legendre transformation (the effective action)

$$\Gamma[\phi_c] = S[\phi_c] + \frac{\mathrm{i}\hbar}{2}\mathrm{Tr}\ln[\Box^2 + V''(\phi_c)] + \mathcal{O}(\hbar^2)\,. \tag{3.20}$$

The techniques used here can be easily generalised to the case when fermions and gauge fields are taken into account.

Now let us see what we can do after we have the effective action on hand. The effective action can be expanded in the following ways: in powers of $\phi_c$

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n \Gamma^{(n)}(x_1, \dots, x_n)\phi_c(x_1)\dots\phi_c(x_n)\,, \tag{3.21}$$

or in position space, with the local term defined as the effective potential $V_{\text{eff}}(\phi_c)$,

$$\Gamma[\phi_c] = \int \mathrm{d}^4x \left[-V_{\text{eff}}(\phi_c) + \frac{1}{2}Z(\phi_c)(\partial_\mu\phi_c)^2 + \dots\right] \tag{3.22}$$

where $Z(\phi_c)$ is the wave function renormalisation, or in powers of momentum

$$\Gamma[\phi_c] = \int \mathrm{d}^4x \sum_n \frac{1}{n!} \left\{\tilde{\Gamma}^{(n)}(0,\dots,0)[\phi_c(x)]^n + \dots\right\}\,. \tag{3.23}$$

Note that $\Gamma^{(n)}(x_1,\dots,x_n)$ is the proper vertex function (or one-particle irreducible, 1PI, Green's function) and $\tilde{\Gamma}^{(n)}$ is its momentum space representation. These set of expansions tell us that $V_{\text{eff}}$ is effective in the sense that its $n^{\text{th}}$ derivative is the sum of all one-particle irreducible diagram for the *original* field with $n$ vanishing external momenta; that is to say,

$$V_{\text{eff}}(\phi_c) = -\sum_{n=2}^{\infty} \frac{1}{n!}\tilde{\Gamma}^{(n)}(0,\dots,0)[\phi_c(x)]^n\,. \tag{3.24}$$

Observe that the summation starts from $n = 2$ since the tadpole diagrams can be safely neglected[1] because it is momentum dependent and can be subtracted by a particular renormalisation for the mass counterterm anyway.

Using the effective action from (3.20) with the assumption that $\phi_c$ is a constant when the source is turned off, and using

$$\mathrm{Tr}\ln[\Box^2 + V''(\phi_c)] = \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \ln[-k^2 + V''(\phi_c)]\,, \tag{3.25}$$

we find that

$$V_{\text{eff}}(\phi_c) = V(\phi_c) - \frac{\mathrm{i}\hbar}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \ln\left[\frac{-k^2 + V''(\phi_c)}{k^2}\right] + \mathcal{O}(\hbar^2)\,, \tag{3.26}$$

where we have shifted the potential by a constant $\propto -\int \mathrm{d}^4k \ln k^2$.

It is important to note that the reason that we use loop expansion is because we want to deal with the expansion in terms of a parameter that

[1] Tadpoles do not disappear automatically since there is no symmetry argument that forbids the existence of the tadpole here (unlike in QED).

multiplies the Lagrangian so that the results are not affected by the shifts of fields, e.g., when symmetry is broken, or by the different way of partitioning the Lagrangian into free and interaction parts. Moreover, this expansion parameter can even be set to 1 without ruining the convergence of the result since we can always rescale the fields in the Lagrangian to absorb the change. This is unlike the situations in the usual perturbation theories where we expand the object of interests in terms of the parameter (e.g, the coupling constant) that multiplies a particular *part* of the Lagrangian.

### 3.1.2 Spontaneous Symmetry Breaking and Effective Potential

Radiative corrections are usually thought of as being small; i.e., we treat them as perturbations. The point is that: despite of their size, if they induce terms related to spontaneous symmetry breaking the effect will be enormous in the sense that we may come up with a totally different theory. So now we will discuss how the effective action method works in the case when symmetry breaks spontaneously; that is, when the vacuum expectation value of the external field does not vanish and has a constant value $v$. Instead of (3.6), we now set

$$\left.\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\right|_{J\equiv 0} = v\,, \tag{3.27}$$

so that

$$\left.\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)}\right|_{\phi_c = v} = 0\,. \tag{3.28}$$

Then it is found that the effective action generates the 1PI vertex function for the shifted field $\bar{\phi} = \phi - v$.

The case we will take as an example is a self-interacting, $\lambda\phi^4$, massless scalar field theory where the un-renormalised Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 \tag{3.29}$$

which clearly shows no signs[2] of spontaneous symmetry breaking at tree level. The generalisation to $\phi^n$ case is possible but not necessary since a theory having

[2] As we shall see in section 3.4.2.2, the choice of vanishing mass parameter for the scalar field, though leading to interesting phenomena, is just as unnatural as others because there is no symmetry protecting the mass of the scalar.

$\phi^n$ for $n > 4$ is not renormalisable. The (normalised Lagrangian) is then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}Z(\partial_\mu\phi)^2 - \frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4. \tag{3.30}$$

where the terms with coefficients $A$, $B$, and $C$ are counterterms introduced so as to absorb the cut-off dependence. Observe that since we did not assume the reflection symmetry $\phi \to -\phi$. Mass renormalisation counterterm must also be introduced as we did not explicitly impose any reason to prevent it. Since the structure of $\phi^4$ does not allow any diagram with odd number of external lines, we only have to consider the diagrams shown in (3.1) which gives a contribution

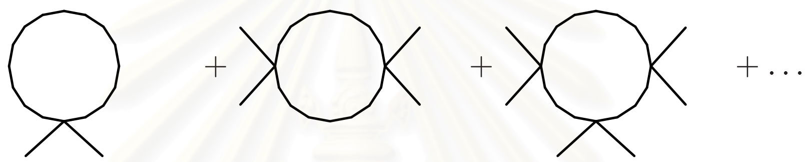


**Figure 3.1:** Loop diagrams for a $\phi^4$ theory.

$$\tilde{\Gamma}^{(2n)}(0,\ldots,0) = \mathrm{i}\frac{(2n)!}{2^n 2n}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[(-\mathrm{i}\lambda)\frac{\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\right]^n, \tag{3.31}$$

where the factor preceding the integral is a symmetry factor which is introduced to avoid over-counting contributions from diagrams having the same structure. Notice that $1/2^n$ factor is due to Bose-Einstein statistics of the field. To one loop, this results in the effective potential

$$\begin{aligned} V_{\text{eff}}(\phi_c) &= \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\sum_n \frac{1}{2n}\left[\frac{(\lambda/2)\phi_c^2}{k^2+\mathrm{i}\varepsilon}\right]^n \\ &= \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\phi_c^2}{2k^2}\right) \\ &= \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \frac{\lambda\Lambda^2}{64\pi^2}\phi_c^2 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\frac{\lambda\phi_c^2}{2\Lambda^2} - \frac{1}{2}\right) \end{aligned} \tag{3.32}$$

where we have performed a Wick rotation to the momentum Euclidean space, integrated using a large momentum cut-off $k_E^2 = \Lambda^2$, and discarded terms that go to zero in the large $\Lambda$ limit. The equation (3.32) also shows that the corrections to $\phi^2$ and $\phi^4$ terms are quadratically, and logarithmically divergent respectively.

Next, we will try to absorb the cut-off parameter by imposing some renormalisation conditions that fix the values of the counterterms. As usual, the renormalised mass squared of the field is defined as the value of the inverse propagator at zero momentum; i.e., $\tilde{\Gamma}^2(0) = \mu^2$. Hence we have the condition

$$\left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\phi^2}\right|_{\phi=0} = \mu^2\,, \tag{3.33}$$

which can be used to determine the counterterm $B$. However, if we move on to the renormalisation condition for the coupling constant using the four point function

$$\tilde{\Gamma}^{(4)}(0) = \lambda\,; \tag{3.34}$$

i.e.,

$$\left.\frac{\mathrm{d}^4 V_{\text{eff}}}{\mathrm{d}\phi^4}\right|_{\phi=0} = \lambda\,, \tag{3.35}$$

we will face the problem of the infrared singularity from the logarithm right away. However, this singularity is just an artifact resulted from a particular renormalisation choice, namely the fact that the usual subtraction is performed at finite value of momenta characterised by a mass scale. So we introduce the alternative condition at an arbitrary mass scale $M$:

$$\left.\frac{\mathrm{d}^4 V_{\text{eff}}}{\mathrm{d}\phi^4}\right|_{\phi=M} = \lambda(M)\,, \tag{3.36}$$

and set the wave function renormalisation counterterm $Z(M) = 1$ instead of the usual one ($Z(0) = 1$). The condition (3.36) yields $C = -\frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2} - \frac{25}{6}\right)$ and hence leading to the effective potential

$$V_{\text{eff}}(\phi_c) = \frac{\lambda(M)}{4!}\phi_c^4 + \frac{\lambda^2(M)}{256\pi^2}\phi_c^4\left(\ln\frac{\phi_c^2}{M^2} - \frac{25}{6}\right) \tag{3.37}$$

which seems to shift the minimum away from the origin to the location satisfying

$$\lambda(M)\ln\frac{\langle\phi\rangle^2}{M^2} = -\frac{32}{3}\pi^2 + \mathcal{O}(\lambda)\,. \tag{3.38}$$

However, since the term on the left hand side of (3.38) is large and negative for small $\phi$, the quantum correction will be larger than the tree level part. The two terns in (3.38) are of order $\lambda$ and $\lambda^2$ respectively and hence are not comparable in the sense of perturbation theory. Though the approximation is not very reliable, we see that the quantum correction must have something to do with spontaneous symmetry breaking.

### 3.1.3 Massless Scalar Electrodynamics

In the previous chapter, we have seen that the one-loop approximation lead to the effective potential with corrections lying outside the territory of validity of perturbation theory. The problem is actually not from the formalism itself, but from the fact that there is only one coupling in the theory which is clearly not enough when we want to consider an interplay between the classical term, $\mathcal{O}(\lambda)$, and the one-loop term, $\mathcal{O}(\lambda^2)$. Thus we guess that the effective potential technique may be able to give a "perturbatively correct" result if we introduce another independent coupling (hence another interaction) to the theory so as to "fix" the loop-correction and also prevent it from getting too large.

In this section we consider the Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |\partial_\mu\phi - \mathrm{i}eA_\mu\phi|^2 - \frac{\lambda}{3!}(\phi^*\phi)^2 + \text{counter terms} \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi_1 - eA_\mu\phi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_2 + eA_\mu\phi_1\right)^2 \\ &\quad -\frac{\lambda}{4!}(\phi_1^2+\phi_2^2)^2 + \text{counter terms} \end{aligned} \tag{3.39}$$

which describes a system of complex scalar field coupled with photon. It is clear that the theory has a $U(1)$ gauge symmetry and hence the photon is exactly massless (at tree level). Notice that we have decomposed the complex scalar field into two real fields $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$. Let us see what diagrams will contribute. First notice that the gauge coupling introduces the trilinear coupling $\phi A^\mu \partial\phi$ which generates the diagrams like that is shown in Fig.3.2. However, the

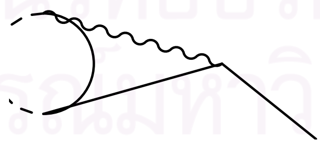


**Figure 3.2:** A loop generated from a trilinear coupling in (3.39). The dashes refer to other parts in the diagram.

vanishing of external momenta tells us that the momentum for the scalar is the same as that of the gauge boson. When we choose to work in the Landau gauge, the contributions from this kind of diagrams vanishe due to the vanishing contraction between the momentum of the scalar field and the gauge boson propagator. Consequently, we are left with diagrams having the same structure as those considered in the previous section which are shown in Fig.3.3.

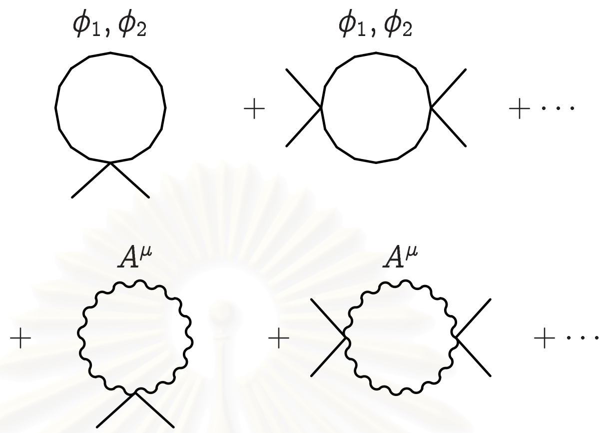



**Figure 3.3:** Loop diagrams that contribute to $V_{\text{eff}}(\phi_c)$ in scalar electrodynamics

Loop calculations can be simplified by noticing that variable entering the effective potential is only the combination $(\phi_1^2 + \phi_2^2)$ which, therefore, allowing us to switch off the external field $\phi_2$ and consider only the $\phi_1$. It is then clear that the contributions from the these loop diagrams have the same structure as the scalar loop that we dealt with earlier. The symmetry factors (the number preceding the integral in (3.31) are different for each kind of loop, however. For example, a loop having $\phi_2$ running inside is associated with the factor $-\frac{2\lambda}{4!}\phi_1^2\phi_2^2$. The result is found to be (See Coleman and Weinberg [52])

$$V_{\text{eff}}(\phi_c) = \frac{\lambda(M)}{4!}\phi_c^4 + \frac{1}{64\pi^2}\left[\frac{5\lambda^2(M)}{18} + 3e^4\right]\phi_c^4\left(\ln\frac{\phi_c^2}{M^2} - \frac{25}{6}\right), \qquad (3.40)$$

where the factor 3 comes from the trace of the gauge propagator (the $g^{\mu\nu} - k^\mu k^\nu/k^2$ part). Even though $\lambda$ and $e^4$ emerge from different order of loop calculations (zero-loop for $\lambda$ and one-loop for $e^4$), they are independent which means they are comparable. Thus, there is nothing preventing us from assuming that $\lambda \ll e^2 \ll 1$ such that $\lambda$ is of order $e^4$. If this is the case[3], then the term

[3]In fact, Coleman and Weinberg had shown in [52] that if this is not the case, we can always make it such by changing the renormalisation scale. In other words, SSB occurs for any arbitrary small parameters $\lambda$ and $e$.

containing $\lambda^2$ can be neglected and (3.40) becomes

$$V_{\text{eff}}(\phi_c) = \phi_c^4 \left[ \frac{\lambda(M)}{4!} + \frac{3e^4(M)}{64\pi^2} \left( \ln \frac{\phi_c^2}{M^2} - \frac{25}{6} \right) \right] , \tag{3.41}$$

which yields

$$V'_{\text{eff}} = \phi_c^3 \left[ \frac{\lambda(M)}{6} + \frac{3e^4(M)}{16\pi^2} \left( \ln \frac{\phi_c^2}{M^2} - \frac{11}{3} \right) \right] . \tag{3.42}$$

So the effective potential is minimum for non-zero $\langle 0|\phi|0\rangle$ when

$$\langle 0|\phi|0\rangle = M \exp \left[ \frac{11}{6} - \frac{4\pi^2 \lambda(M)}{9e^4(M)} \right] . \tag{3.43}$$

When the renormalisation scale $M$ is adjusted so that $\langle 0|\phi|0\rangle = M$ we arrive at the result

$$\lambda = \frac{33}{8\pi^2} e^4 , \tag{3.44}$$

which is interesting if we remember that we started with independent couplings $\lambda$ and $e$. Though these parameters are related in (3.44), the number of independent parameters is not reduced to 1. This is because the dimensionless parameter $\lambda$ has been *transmuted* into the dimensional one (the $\langle 0|\phi|0\rangle$). Nevertheless, the $\lambda$ can still be taken as the free parameter of the theory. It is then clear that the $U(1)$ symmetry is spontaneously broken if the effective potential is written in this form

$$V_{\text{eff}}(\phi_c) = \frac{3e^4}{64\pi^2} \phi_c^4 \left( \ln \frac{\phi_c^2}{\langle\phi\rangle^2} - \frac{1}{2} \right) , \tag{3.45}$$

where $\langle\phi\rangle \equiv \langle 0|\phi|0\rangle$ . So there exists a (would-be) Goldstone boson corresponding to the direction of the broken generator. Then this Goldstone will be eaten by the (photon or photon-like) gauge field via the gauge interaction (like the usual BEH mechanism). The gauge field then becomes massive with

$$m_V^2 = e^2 \langle\phi\rangle^2 . \tag{3.46}$$

In addition, the resulting effective potential (3.45) tells us right away that the scalar particle develops mass from quantum effects which is proportional to the strength of the coupling between itself and the gauge field, namely

$$m_S^2 = \left. \frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\phi_c^2} \right|_{\langle\phi\rangle} = \frac{3e^4}{8\pi^2} \langle\phi\rangle^2 . \tag{3.47}$$

It is also pointed out in Coleman and Weinberg's paper [52] that the high-order corrections are small enough so that they will not turn the origin of the potential back into the absolute minimum.

In addition we note that the relation

$$e^4 = \frac{m_V^4}{v^4} \tag{3.48}$$

can be generalised to the case where there are many gauge bosons as

$$e^4 \to \frac{\sum_V m_V^4}{v^4} \tag{3.49}$$

where now the $m_V^4$ is fourth power of the gauge boson *mass matrix* in the background $\Phi_c$ (recall that in the zero loop $M^2_{g,ab} \sim g^2 v^2 T^a T^b$).

As a final note there is another method where we can evaluate the contribution from the tadpole alone (with some modifications). The method is interesting and can be used to provide a quick check as well. It is presented in the appendix A.4.

### 3.1.4 Extension to Non-Abelian Cases

Now we want to determine the contributions to the effective potential when there are (non-Abelian) gauge bosons and fermions in the system. Since scalar fields come in a multiplet (let us call it $\Phi_c$), we will assume for simplicity that only one scalar field gets a vacuum expectation value (i.e., we will assume $\Phi_c \sim \mathrm{Re}H^0 = h$ when we deal with the standard model). Also notice that, because the internal lines in the loop carry indices, we have to consider the transition between indices which is archived by introducing the matrix element

$$m^2_{s,ab}(\Phi_c) = \left.\frac{\partial^2 V_0}{\partial \phi_a \partial \phi_b}\right|_{\Phi_c}. \tag{3.50}$$

This is a generalisation to the mass matrix which contributes to each vertex. This will reduce to the Goldstone boson mass matrix when the $\Phi_c$ takes the classical vacuum expectation value (see the relevant formula for the Higgs mass in (2.48)). So they generate diagrams that can be thought of as generalisations of those shown in Fig.3.1. We have to take every possible arrangement of the vertices and internal lines into account. This results in

$$\sum_{\text{all } a_i} m^2_{s,a_1 a_2} m^2_{s,a_2 a_3} \cdots m^2_{s,a_n a_1} = \mathrm{Tr}[m_s^2]^n \tag{3.51}$$

for each diagram with a *fixed* number of external lines, say $n$. Calculations are very similar to those in (3.32) and the resulting one-loop effective potential is[4]

$$V_s^{(1)}(\Phi_c) = \frac{1}{64\pi^2}\mathrm{Tr}\left\{[m_s^2(\Phi_c)]^2 \ln \frac{m_s^2(\Phi_c)}{M^2}\right\}, \tag{3.52}$$

at a particular renormalisation scale $M$. Then it is clear that in a basis which make $m_s^2$ diagonal, the one-loop contribution becomes

$$V_s^{(1)}(\Phi_c) = \frac{1}{64\pi^2}\sum_\alpha m_{s,\alpha}^4(\Phi_c) \ln \frac{m_{s,\alpha}^2(\Phi_c)}{M^2}, \tag{3.53}$$

where $m_{s,\alpha}^2$ are the eigenvalues of the mass matrix.

Next, consider the contributions from non-Abelian gauge fields. Again, the form of the effective potential will depend on the gauge choice while the resulting physical quantities evaluated from it will not. Thus we will (again) work in the Landau gauge. Similar to the previous case of scalar fields, we define the "mass matrix"

$$m_{g,ab}^2(\Phi_c) = g^a g^b \mathrm{Tr}\left\{\Phi_c^\dagger T^a T^b \Phi_c\right\}, \tag{3.54}$$

(no summation), which contributes to every vertex. At tree-level, we recover $M_W^2 \sim g^2v^2/4$, for example. The rest of the calculations require nothing new and the one-loop effective potential from the non-Abelian gauge loop, in gauge eigenstates (diagonal $m_g$), is

$$V_g^{(1)}(\Phi_c) = \frac{3}{64\pi^2}\mathrm{Tr}\left\{[m_g^2(\Phi_c)]^2 \ln \frac{m_g^2(\Phi_c)}{M^2}\right\}, \tag{3.55}$$

where the factor 3, again, comes from the gauge boson propagator.

Now let us turn to the fermion loop. The part of the Lagrangian that contributes to vertices in loop diagrams is

$$-\bar{\Psi}_a \left\{A(\Phi_c) + \mathrm{i}\gamma_5 B(\Phi_c)\right\}_{ab}\Psi_b \equiv -\bar{\Psi}_a m_{f,ab}\Psi_b, \tag{3.56}$$

where $m_{f,ab} = \{A(\Phi_c) + \mathrm{i}\gamma_5 B(\Phi_c)\}_{ab}$ is the fermion mass matrix (it reduces to the fermion mass after the scalars received a vacuum expectation value, according to the first non-vanishing order). For $n$ internal massless fermion

[4]The subscript "eff" is omitted when its meaning is clear from the context.

lines, we find that the loop contributes

$$\cdots m_f \frac{1}{\not k} m_f \frac{1}{\not k} \cdots m_f \frac{1}{\not k} m_f \frac{1}{\not k} \cdots \longrightarrow \text{trace term}\,, \tag{3.57}$$

where all indices are suppressed. As a trace of odd number of the gamma matrices vanish, only loops with even number of internal fermions survive. Thus the neighbouring fermion propagators can be grouped in a pair $m_f^2 \frac{1}{k^2}$; i.e., $n$ pairs of this term for $n$ even external lines. Also notice that, a fermion loop puts up a factor $-1$ (Fermi-Dirac statistics from reordering the fields in the loop). Now we can proceed with the calculations analogous to the previous cases and obtain

$$V_f^{(1)} = -\frac{1}{64\pi^2} \mathrm{Tr} \left\{ \left[ m_f m_f^\dagger(\Phi_c) \right]^2 \ln \frac{m_f m_f^\dagger(\Phi_c)}{M^2} \right\} \tag{3.58}$$

where the trace also runs over the Dirac indices[5]. In the simplest case where there is only one Yukawa coupling $y = \sqrt{2} m_f / v$, the fermion contributes[6]

$$V_f^{(1)} = -\frac{N}{64\pi^2} y^4 \Phi_c^4 \ln \frac{\Phi_c^2}{M^2} \tag{3.59}$$

The factor $N$ equals 4 for Dirac fermions (from a trace of the Dirac matrix), and 2 for Weyl or Majorana fermions.

Consequently, the one-loop contribution to the effective potential is

$$\begin{aligned} V_{\text{eff}}^{(1)} &= \frac{1}{64\pi^2} \mathrm{Tr} \left\{ [m_s^2(\Phi_c)]^2 \ln \frac{m_s^2(\Phi_c)}{M^2} \right\} \\ &\quad - \frac{1}{64\pi^2} \mathrm{Tr} \left\{ \left[ m_f m_f^\dagger(\Phi_c) \right]^2 \ln \frac{m_f m_f^\dagger(\Phi_c)}{M^2} \right\} \\ &\quad + \frac{3}{64\pi^2} \mathrm{Tr} \left\{ [m_g^2(\Phi_c)]^2 \ln \frac{m_g^2(\Phi_c)}{M^2} \right\}, \end{aligned} \tag{3.60}$$

which can be reduced to a more compact version[7]

$$V_{\text{eff}}^{(1)} = \frac{1}{64\pi^2} \left\{ \lambda^2 + 3 \sum g_i^4 - N \sum y_i^4 \right\} \Phi_c^4 \ln \frac{\Phi_c^2}{M^2}\,. \tag{3.61}$$

[5]The arrows of fermion lines forbid reflection symmetry (i.e., the loops is oriented) but the trace on the Dirac indices kills the odd terms so we have to sum over only the even terms.

[6]Notice that we are working with a colourless fermion. The right-hand side of (3.59) will be multiplied by $N_c$ for a theory with $N_c$ colours.

[7]This simplified form has to be used with care as we did not take the factors like colours into account.

Observe that in a theory with more than one couplings, it is possible that one-loop diagrams dominate the tree-level diagrams without requiring large values of coupling constants (and hence does not sacrifice perturbation theory).

Before we leave this section, let us note that by recalling (3.26), the general formula for the logarithmically and quadratically divergent parts of the one-loop correction to the effective potential can be written as

$$\begin{aligned} V_{1-\text{loop}} &= \frac{1}{64\pi^2}\int \mathrm{d}^4k \mathrm{STr}\ln(k^2+m^2(\Phi_c)) \\ &= \frac{1}{64\pi^2}\mathrm{Str}\left[[m^2(\Phi_c)]^2\ln(m^2(\Phi_c))\right] + \frac{\Lambda^2}{64\pi^2}\mathrm{STr}m^2(\Phi_c) + \ldots \end{aligned} \quad (3.62)$$

where the supertrace is defined with $\mathrm{STr} = \mathrm{Tr}(-1)^{\mathrm{F}}$ where $F = 1$ for a loop containing fermions and zero otherwise.

### 3.1.5 Coleman-Weinberg Potential for the Electroweak Theory

In this section we will study the application of the Coleman-Weinberg technique in the standard electroweak theory. Let us recall that the interactions that contribute to the (one-loop) effective potential are

$$\begin{aligned} &-\lambda(H^\dagger H)^2 + \left(\frac{g}{2}\tau^a W^a_\mu + \frac{g'}{2}B_\mu\right)^2 H^\dagger H \\ &-y^i_l(\bar{L}^i_L H)l^i_R - y^i_d(\bar{Q}^i_L H)d^i_R - y^i_u(\bar{Q}^i_L \tilde{H})u^i_R + \text{h.c.}. \end{aligned} \quad (3.63)$$

The earlier sections tell us that the contributions from scalar, gauge, and fermion loops are proportional to $\lambda^2, g^4$, and $y^4$ respectively. Now we have 2 charged $W$ bosons and 1 neutral $Z$. For the quarks, we can neglect all but the top, which is the heaviest quark. The mass matrix in this (broken) basis $W$, $Z$ is already diagonal where

$$\begin{aligned} M^2_{W^\pm} &= g^2v^2/4 \\ M^2_Z &= (g^2+g'^2)v^2/4 \end{aligned} \quad (3.64)$$

which means we will get an extra factor $\frac{1}{16}$ in the effective potential (from $(m^2_g)^2$). There will be an additional factor 3 for the top quark (comparing to the previous section) since we have to include all the *three colours*. The effective potential (to 1-loop) in the general form is then

$$V_{\text{eff}} = -\frac{\mu^2}{2}\Phi_c^2 + \frac{\lambda}{4}\Phi_c^4 + \frac{3}{64\pi^2}\left\{\frac{1}{16}\left[2g^4 + \left(g^2 + g'^2\right)^2\right] - 4y_t^4\right\}\Phi_c^4 \ln\frac{\Phi_c^2}{M^2} + \frac{1}{64\pi^2}\left[M_h^2(\Phi_c)\right]^2 \ln\frac{M_h^2(\Phi_c)}{M^2} + \frac{3}{64\pi^2}\left[M_{GB}^2(\Phi_c)\right]^2 \ln\frac{M_{GB}^2(\Phi_c)}{M^2} \quad (3.65)$$

where

$$M_h^2(\Phi_c) = (\mu^2 + 3\lambda\Phi_c^2) \quad (3.66)$$

$$M_{GB}^2(\Phi_c) = (\mu^2 + \lambda\Phi_c^2) \quad (3.67)$$

are the masses of the Higgs and the 3 Goldstone bosons in the background $\Phi_c$ (again, see the relevant formula for the Higgs mass in (2.48)). These Goldstone bosons will not show up in the unitary gauge. To get the more familiar form, we take $\Phi_c^2 = v^2 = -\mu^2/\lambda$. In that case, we usually write the one-loop contributions as

$$V_{\text{eff},g-f}^{(1)} = \frac{1}{64\pi^2 v^4}\left\{3\left[2M_W^4 + M_Z^4\right] + M_h^4 - 12m_t^4\right\}\Phi_c^4 \ln\frac{\Phi_c^2}{M^2} \quad (3.68)$$

where $M_Z$, $M_Z$, $M_h$ and $m_t$ are now the masses of the physical fields. Observe that fermion makes a large negative contribution (at large $\Phi_c$).

Let us consider the case when the mass of the Higgs (and hence the $\lambda$) is assumed to be small. This will eventually help us identify the lower bound of the Higgs mass itself. In this case the $\lambda^2$ term in (3.61) can be neglected and the major fermion contributions come from the top quark. The bound on the mass of the Higgs is then based on the condition that the electroweak symmetry breaking vacuum is absolutely stable against radiative corrections; i.e.,

$$V_{\text{eff}}(\langle H\rangle) < V(0) \quad (3.69)$$

where $V(0) = 0$. (Steven) Weinberg [59] and Linde [60] were the first two to pioneer the work in this scheme. The bound on the Higgs corresponding to the gauge bosons loop corrections was found to be

$$M_h^2 \geq \frac{3\sqrt{2}G_F}{16\pi^2}(2M_W^4 + M_Z^4) = \frac{3\alpha^2(2 + \sec^4\theta_W)}{16\sqrt{2}G_F \sin^4\theta_W}. \quad (3.70)$$

At that time (1976), the top quark had not yet been found and hence some major contributions were missing. The relation above yields

$$M_h > \mathcal{O}(10\text{GeV}) \tag{3.71}$$

which is already in the range probed by experiments nowadays. This kind of lower bound is superseded by the vacuum stability bound evaluated from the framework of the running of the Higgs quartic coupling which we will consider in section 3.2.2. Even more, this bound is also superseded by the the direct search at LEP ([61]) suggesting that $M_h \gtrsim 114\,\text{GeV}$ (95% C.L.).

## 3.2 Bounds on the Mass of the Higgs

There are many ways to understand the bound of the Higgs within the framework of the standard model; i.e., as a limit of its "correctness". We will consider only the theoretical ones. This section should somehow convince the reader that the Higgs should be light.

### 3.2.1 Landau Pole and Triviality Bound

Now we will use simple arguments to claim that the Higgs cannot be arbitrarily heavy. Let us go back to the scalar theory with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}M_h^2 h^2 - \frac{\lambda}{4}h^4\,. \tag{3.72}$$

It is quite legitimate to use this simplified version and compare it with the interaction term in the Higgs potential (2.122). This agrees with the large-$\lambda$ approximation in the renormalisation group equation in (3.86). The quartic coupling scales with energy as[8]

$$\lambda(Q) = \frac{\lambda(M_h)}{1 - \frac{3}{2\pi^2}\lambda(M_h)\ln\frac{Q}{M_h}}\,. \tag{3.73}$$

The coupling increases with energy ($Q$) and will eventually hit a pole, blowing up at a particular energy scale regardless of how small the $\lambda(M_h)$ is. The pole is known as the *Landau pole*;

$$Q_\infty = \Lambda_{\text{Landau}} = M_h e^{2\pi^2/3\lambda(M_h)}\,. \tag{3.74}$$

[8]The variable $Q$ means any energy scale in general, while $\Lambda$ usually indicates the cut-off of the theory.

$\Lambda_{\text{Landau}}$ decreases as we increase $M_h$ and eventually they will meet at some point. (2.125) tells us that for a fixed $\Lambda_{\text{Landau}}$ we can obtain the upper bound on the $M_h$; i.e.,

$$M_h^2 = 2\lambda v^2 < \frac{4\pi^2 v^2}{3\ln\frac{\Lambda_{\text{Landau}}}{M_h}}\,. \tag{3.75}$$

Still, we will further assume, hand-wavingly, the hierarchy

$$M_h < \Lambda_{\text{Landau}}/2 \tag{3.76}$$

since we do not expect to do anything with energy close to $\Lambda_{\text{Landau}}$ anyway. Thus,

$$M_h^2 < (1070\,\text{GeV})^2\,. \tag{3.77}$$

Observe that the choice (3.76) puts the limit of the $\lambda$ too far ($\lambda < 6$). This is fine if we do not care whether the perturbative approach works or not - just think of the TeV limit as the ultimate one. To get a reasonable constraint on the quartic coupling, let us use $M_h < \Lambda_{\text{Landau}}/100$ (which gives $\lambda < 1$). This one yields $M_h < 415\,\text{GeV}$.

The consideration above is a crude one as it relies heavily on the choice (3.76) and the arbitrariness of the Higgs mass itself. Who knows whether the Higgs mass scale be much lighter than the Landau pole scale $\Lambda_{\text{Landau}}$ or just about the same. Moreover, it is not expected that the standard model is valid at arbitrarily high energy scale. So we will, as usual, regard the standard model as a low-energy effective theory and introduce the $\Lambda$ as the point where the simple model stops to be valid and new physics enters. First, we observe that we can input the known parameters at electroweak scale ($M_W$) into (3.74)

$$\Lambda_{\text{Landau}} = M_W \exp\left\{\frac{4\pi^2\sin^2\theta_W}{3\alpha}\frac{M_W^2}{M_h^2}\right\} \approx M_W \exp\left\{(370)\frac{M_W^2}{M_h^2}\right\} \tag{3.78}$$

where we have used $\alpha(M_W) \approx \frac{1}{128}$. Suppose we expect that the Landau pole lies beyond the grand unification scale (see B.4) $10^{15}$GeV we find

$$M_h \lesssim 3M_W \lesssim 240\text{GeV}\,, \tag{3.79}$$

which is a very rough approximation. However, if the new physics is expected to enter so soon; like $\Lambda = 10^3\,\text{GeV}$, the Landau pole will be lowered and the upper bound on $M_h$ is relaxed to

$$M_h \lesssim 600\,\text{GeV}\,. \tag{3.80}$$

Given a value of $M_h$, the upper limit $\Lambda$ of the theory (the cut-off) can be evaluated as well. Observe that (3.73) can be written as

$$\frac{1}{\lambda(M_h)} = \frac{1}{\lambda(Q)} + \frac{3}{2\pi^2}\ln\frac{Q}{M_h}\,, \tag{3.81}$$

which implies that

$$\frac{1}{\lambda(M_h)} \geq \frac{3}{2\pi^2}\ln\frac{\Lambda}{M_h} \tag{3.82}$$

In other words, this forces an upper bound of the $\lambda$; i.e.,

$$\lambda(M_h) \leq \frac{2\pi^2}{3\ln\frac{\Lambda}{M_h}}\,. \tag{3.83}$$

The equation is exactly the same as (3.75). However, here we see that the quartic coupling is sensitive to the limit which we "claim" that the theory is valid. If we set $\Lambda \to \infty$, the coupling decreases to zero and this scalar sector will become a free theory; which is said to be *trivial.* The bound of the $\Lambda$ depends on the Higgs mass as

$$\Lambda \leq M_h \mathrm{e}^{4\pi^2 v^2/3M_h^2}\,. \tag{3.84}$$

For the Higgs mass in the range 120 GeV to 200 GeV, the $\Lambda$ ranges from $10^{17}$ GeV down to $10^8$ GeV, which is very broad. This is why it is said that the Higgs mass and the "cut-off" scale are very sensitive to each other.

Notice that what we have done so far was based on the assumption that the fermions and the gauge bosons are negligible up to the scale $\Lambda$. Moreover, we also assumed that higher orders terms in $\lambda$ in the renormalisation group equation are negligible. Still, their effects are not so negligible. To see this, first observe that the factor $\frac{9}{4\pi^2}$ on the right-hand side of (3.81) is the coefficient of the beta function, satisfying

$$\frac{\partial\lambda(Q)}{\partial\ln Q} = \frac{3}{2\pi^2}\lambda^2(Q)\,. \tag{3.85}$$

A generalisation to the standard model case is obtained by the renormalisation group equation for the quartic coupling

$$\begin{aligned}\frac{\partial\lambda}{\partial\ln Q} &= \frac{1}{(4\pi)^2}\Big[24\lambda^2 + 12\lambda\left(y_t^2 - (9g^2 + 3g'^2)\right) \\ &\qquad -6y_t^4 + \frac{3}{8}(2g^4 + (g^2+g'^2)^2)\Big]\,, \end{aligned} \tag{3.86}$$

together with similar couplings for the other couplings. The first thing we see is that the running of the coupling is slowed down by the interaction of the scalar field with the quarks and the gauge bosons. If the effects from self-interactions of the quarks and the gauge fields are negligible, (3.86) reduces to

$$\frac{\partial \lambda}{\partial \ln Q} = \frac{12\lambda}{(4\pi)^2}\left[2\lambda + \left(y_t^2 - (9g^2 + 3g'^2)\right)\right] \equiv \frac{24\lambda}{(4\pi)^2}\left[\lambda - \lambda_c\right] , \tag{3.87}$$

where $\lambda_c \equiv \frac{1}{2}(9g^2+3g'^2) - \frac{1}{2}y_t^2$. Thus, if we insist on working with a perturbative method, we have to impose the minimum requirement that the quartic coupling not explode at a particular scale, which requires that

$$\lambda(Q) < \lambda_c(Q) , \tag{3.88}$$

leading to the upper bound of the mass of the Higgs. Unfortunately, the evaluation of the bound requires the knowledge of the running of the Yukawa and the gauge couplings as well. Then the mass of the Higgs can be plotted as a function of the mass of the top quark. The analysis by Beg *et al.* [62] suggests that for $M_t \approx 170$GeV the Higgs is bounded to $m_h \approx 175$GeV .

### 3.2.2 Vacuum Stability, A Lower Bound

In the case when the Higgs is light, (3.86) becomes

$$\frac{\partial \lambda}{\partial \ln Q} = \frac{1}{(4\pi)^2}\left[ - 6y_t^4 + \frac{3}{8}(2g^4 + (g^2 + g'^2)^2)\right] . \tag{3.89}$$

When the energy scale under consideration is low enough that the strong interaction becomes very strong, (3.89) is further trimmed down to

$$\frac{\partial \lambda}{\partial \ln Q} = -\frac{1}{(4\pi)^2} 6y_t^4 \tag{3.90}$$

which simply results in

$$\lambda(Q) = \lambda(v) - \frac{3y_t^4(Q)}{8\pi^2} \ln \frac{Q}{v} . \tag{3.91}$$

Clearly, top quarks can drive the quartic coupling down to negative values, preventing the breaking of symmetry which is the main ingredient of the standard model. Therefore, we must impose the constraint

$$\lambda(Q) > 0 \tag{3.92}$$

to avoid that. In this case, the running of the top's Yukawa coupling is, to the lowest order, not affected by other couplings. So its renormalisation group equation is in a simple form and the numerical results can be obtained easily. The Yukawa coupling "runs" as

$$\frac{\partial y_t}{\partial \ln Q} = \frac{1}{(4\pi)^2}\frac{9}{2}y_t^3 + \cdots \,, \tag{3.93}$$

which leads to

$$y_t^2(Q) = \frac{y_t^2(v)}{1 - \frac{9}{(4\pi)^2}y_t^2(v)\ln\frac{Q}{v}}\,, \tag{3.94}$$

and

$$\lambda(Q) = \lambda(v) - \frac{\frac{3}{8\pi^2}y_t^4(v)\ln\frac{Q}{v}}{1 - \frac{9}{(4\pi)^2}y_t^2(v)\ln\frac{Q}{v}}\,. \tag{3.95}$$

The condition (3.92) then translates into

$$2\lambda(v)v^2 = M_h^2 > \frac{\frac{3v^2}{4\pi^2}y_t^4(v)\ln\frac{Q}{v}}{1 - \frac{9}{(4\pi)^2}y_t^2(v)\ln\frac{Q}{v}} = \frac{\frac{3}{v^2\pi^2}m_t^4\ln\frac{Q}{v}}{1 - \frac{9}{8v^2\pi^2}m_t^2\ln\frac{Q}{v}}\,. \tag{3.96}$$

This means at the scale $Q = \Lambda$, there must be new physics showing up or the vacuum stability is destroyed. In this way, we see that the lower bound of the mass of the Higgs can be written as a function of the cut-off and the mass of the top quark. At present, this bound is also superseded by the direct search of the Higgs at LEP (see section 3.3).

### 3.2.3 Tree Level Unitarity

For the case of the Fermi's current-current, the intermediate vector boson theories, or theories based on the perturbation technique, unitarity bound plays a very important role. In general, the scattering cross section may depend on some parameter related with the $C.M.$ energy which can be expanded in terms of partial waves. The unitary bound is then basically a requirement that the contributions from the tree level to the first partial wave expansion (s-waves) of scattering amplitudes not exceed the unitary bound (think of an expansion with coefficients greater than 1; i.e., a scattering with probability greater than unity). In this section, we will briefly outline about the effects of the tree-level unitary condition to the bound on the Higgs mass and the scale of new physics.

For example, recall that in Fermi's theory we have $\sigma \sim G_F E^2_{\mathrm{C.M.}}$, where

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s^2}|\mathcal{M}|^2. \tag{3.97}$$

Then the expansion of the scattering amplitude in terms of the partial waves amplitudes

$$\mathcal{M}(\theta) = \beta \sum_J (2J+1) P_J(s,\theta) a_J(s) \tag{3.98}$$

($\beta$ is a numerical value of order $\mathcal{O}(1)$, and $s = E^2_{CM}$) tells us that

$$s < \frac{\beta'}{G_F}, \tag{3.99}$$

otherwise partial-wave unitarity is not respected by tree diagrams. In other words, new physics (theory) must shows up at a scale

$$\Lambda_F < \sqrt{\frac{\beta'}{G_F}} \tag{3.100}$$

in order to modify Fermi's theory. It was found (see, for example, Chanowitz [47])that $\Lambda_F \approx 1\,\mathrm{TeV}$. Still, this does note necessary means that we must wait till we reach the $\Lambda_F$ to see new physics and the new physics actually surfaces at $\mathcal{O}(100\,\mathrm{GeV}) \sim M_W$; i.e., at the electroweak scale.

Similar situations happen in electroweak theory where the process under consideration is $W^+ + W^- \to Z + Z$ and

$$\frac{s}{16\pi v^2} < 1 \tag{3.101}$$

or, in terms of the cut-off,

$$\Lambda_{\mathrm{SB}} < \sqrt{16\pi v^2} \approx 2\,\mathrm{TeV} \tag{3.102}$$

By observing the previous case where $M_W \approx \frac{\Lambda_F}{10}$, we can make an analogy by introducing a mass scale of the "usual suspect" that may break the electroweak symmetry; i.e., the Higgs,

$$M_h \approx \frac{\Lambda_{\mathrm{SB}}}{10}\,. \tag{3.103}$$

This may be a very rough guess of the mass of the Higgs. Still, the unitarity bound can provide something more useful. Using (minimal) the standard model the contribution to the amplitude of $W^+ + W^- \to Z + Z$ from the gauge sector can be written as (taking the $\rho$-parameter to be 1)

$$\mathcal{M}_g \approx \frac{g^2}{4M_W^2} s = \frac{s}{v^2} \tag{3.104}$$

and the Higgs exchange (s-chanel) adds

$$\mathcal{M}_h \approx -\frac{s}{v^2}\frac{s}{s - M_h^2} \,. \tag{3.105}$$

(see Kolda and Murayama [5]). That is,

$$\mathcal{M} \approx \mathcal{M}_g \approx \frac{s}{v^2} - \frac{s}{v^2}\frac{s}{s - M_h^2} \,. \tag{3.106}$$

Now we can compare between the cases of having and not having the Higgs (to be more precise, the latter should be $M_h^2 >> s$). With the Higgs around, the amplitude at $s >> M_h^2$ becomes

$$\mathcal{M} = -\frac{g^2}{4M_W^2} m_h^2 \tag{3.107}$$

which, at least, does not grow with $s = E_{\mathrm{CM}}^2$ and hence guarantees the good behaviour at high energy. It was found that the unitarity bound gives ([5])

$$M_h < 780\,\mathrm{GeV} \,. \tag{3.108}$$

Notice that this is bound on unitarity, *not* the strict bound on the mass of the Higgs; i.e., the Higgs can be heavier than the value specified in (3.108) and the perturbation theory is not valid.

On the other hand, in the absence of the Higgs (or very heavy Higgs), the amplitude grows in a way similar to the Fermi's case and is bound with the "new physics" scale at $\sim 2\,\mathrm{TeV}$ (recall (3.102)), which, is no longer reliable due to large higher order corrections around that scale (unitarity is violated). This is why it is usually expected that there will be new physics at $\Lambda_N < 2\,\mathrm{TeV}$.

The moral of the story is that though unitarity bound does not act directly on the mass of the Higgs, it strongly suggests that either the Higgs or a Higgs-like particle exists or there is new physics at a TeV scale.

## 3.3 Some Experimental Related Bounds

In this section we will briefly review two important clues from experiments (both direct and indirect search of the Higgs). One can be related to the bounds on the mass of the Higgs and the other can guide us where the scale of the new physics is. Since the topic of electroweak precision test is a very big and sophisticated one, to keep us within the scope of the thesis, the content of this section will be far from self-contained and logically consistent. What we are trying to do is to outline a few of the crucial findings and to point out where to look for further information.

### 3.3.1 Bounds on the Higgs From Precision Electroweak Tests

The (minimal) standard model is considered to be one of the most successfully tested so far. Many parameters can be measured with extremely high degrees of accuracy. So there are many ways to get the bounds of the mass of the missing Higgs with clues from experiments. Here we will mention some of them.

First there is the Higgs direct production where we expect the Higgs to reveal itself in the final state of the collision events. This may be the best way to identify the particle if we can find one. However, the problem is that if we still do not have large enough energy to produce one, we cannot conclude that it does not exists. We can only mention about the *excluded region*. One of the processes that looks promising is the Higgsstrahlung initiated from colliding leptons, mainly electrons and positrons, as shown in Fig. 3.4 Unfortunately,

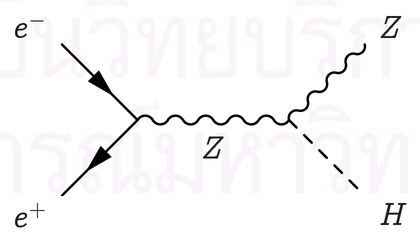



**Figure 3.4:** Higgsstrahlung $e^+ + e^- \to Z + h$.

despite its name, the Higgs has not been found yet. Still, this kind of experiments gives a lower-bound of the mass of the Higgs. It was found at LEP (CERN) that

([63])

$$M_h \gtrsim 114.4\,\mathrm{GeV}\,, \tag{3.109}$$

at 95% C.L..

Apart from the direct searches, *precision measurements* can serve as another crucial tools, thanks to the success of the standard model itself. The strategy is simple, we focus on parameters that can be measured to some "fine" detail (say, better than one percent). Take them as fix inputs and put them into relevant results from loop corrections. This will provide "best fit" values for other parameters. These *resulting* best fit parameters can then be compared with not only the parameters observed directly, but also with those predicted from theory. On the one hand, the deviation (if any) of the latter will suggest the need of the new physics. On the other hand, the excellent agreements (usually valid up to a specific energy scale) will tell us that new physics is not welcome.

The tricks mentioned above will not apply directly in the case of the mass of the Higgs as the resulting best fit parameter (since it has not been found and its mass is not predictable within the framework of the standard model), but we can use it for some other purposes (see later in this section).

Now one can study how the results from a model deviate from these electroweak precision parameters. After takening some uncertainties from both theory (e.e., the need of higher-loop calculations) and experiments (including some input parameters like the mass of the top quark) into account, the *global fits* (to all electroweak data) of these data can be studied. Most of the cases, the fundamental parameters

$$g,\, g',\, \lambda_H,\, \mu_H^2,\, y_i \tag{3.110}$$

(where $\lambda_H$ and $\mu_H^2$ are given by (2.125)) are traded by a combination of parameters that are directly (or easily) accessible by experiments and are denoted collectively as $\{\mathbb{P}\}$:

$$\{\mathbb{P}\} \equiv \left\{\alpha = \frac{e^2}{4\pi}, G_F, M_Z, M_h, m_i\right\} \tag{3.111}$$

where the $m_i$'s are fermion masses. For example,

$$\sin^2 2\theta_W = \frac{2\sqrt{2}\pi\alpha}{G_F M_Z^2}\,. \tag{3.112}$$

Usually, $\alpha$, $G_F$, and fermion masses $m_i$ (except the top) are taken as input parameters.

The case we are interested in is the mass of the Higgs. The best fits result is (Langacker [64])

$$M_h = 86^{+49}_{-32}\,\mathrm{GeV} \tag{3.113}$$

which lies mostly inside the excluded region by LEP as shown above. Still, naively, we see that this shows some room within the region 114 GeV − 140 GeV . Usually the best fits for the Higgs is shown in the $\Delta\chi^2$ vs $M_h$ plot where $\Delta\chi^2 = \chi^2 - \chi^2_{\rm min}$ in the Fig. 3.5. In the figure theoretical prediction with uncertainties (usually due to loop corrections that are related with top quarks) is plotted as a blue-band. The solid line represents the global fits.

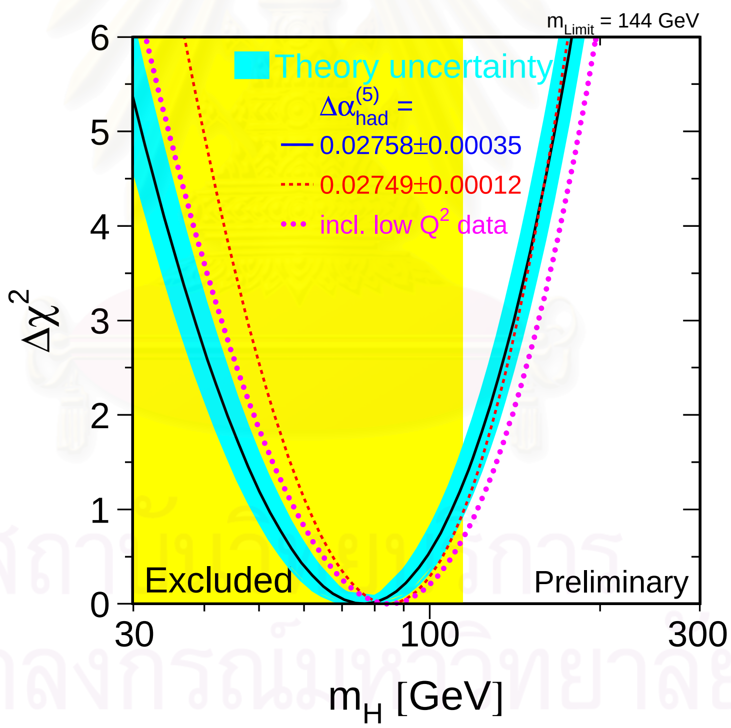



**Figure 3.5:** A sample $\Delta\chi^2$ vs. $M_h$ plot (from LEP EWWG '07 [4]).

In addition, the success of the standard model can be represented by the plot of the "pulls" which are defined by the differences (of the observables) between the values predicted by theory and the measured ones, divided by the error from the theory side. The plot is shown in the Fig. 3.6. It does not only show that the standard model fits very well with the results (from experiments) but also that the results do not favour any additional particles of new physics[9], at least, within $\sim 5$ TeV.

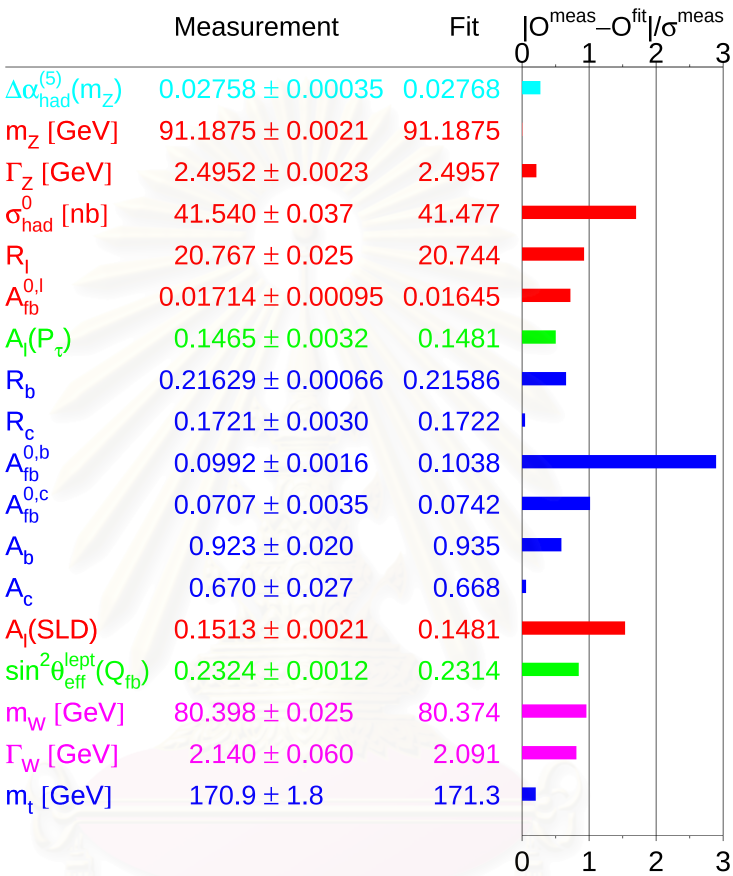





**Figure 3.6:** The list of "pulls" of various electroweak parameters (from LEP EWWG '07 [4]).

Finally, we can also summarise theoretical (triviality and vacuum stability) and experimental (precision electroweak) bounds in the figure 3.7 taken from the paper by Kolda and Murayama [5]. There the shaded block region (on the left hand side) labelled "Standard Model" which is bounded from below by direct search (see above) and the precision electroweak "best fits" from the standard model.

[9] See the next section.

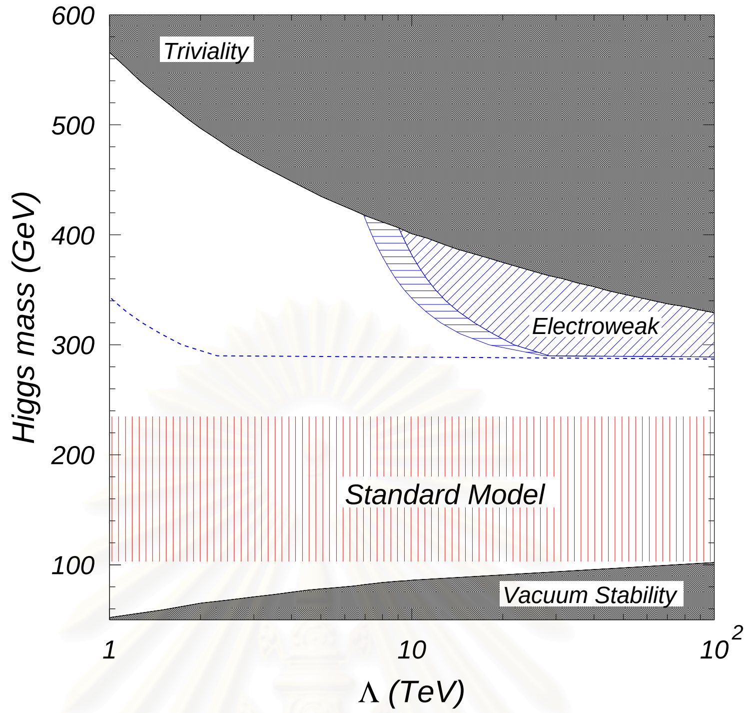



**Figure 3.7:** Bounds of the Higgs mass, including triviality, vacuum stability, and precision electroweak tests [5]. The region above the dash lines is disfavoured by naturalness arguments (see section 3.4.2).

### 3.3.2 Implications on New Physics

In this section we will roughly consider the influences from electroweak precision tests on theories that are considered "beyond the standard model". Now we will regard the standard model as a low-energy effective theory of some higher-energy, more complete theory. Effective field theory is another big subject in physics that requires much more careful treatments than what we are doing in this section. The reader is encouraged to consult the following literatures by Georgi [27] and those by his students: Kaplan [28] and Manohar [29].

What we are considering is more or less analogous to what physicists have done half a century ago where the *Fermi current-current model*

$$\frac{G_F}{\sqrt{2}}\bar{\psi}\gamma_\mu(1-\gamma^5)\psi\bar{\psi}\gamma^\mu(1-\gamma^5)\psi \tag{3.114}$$

can be thought of as a low-energy description of the *Intermediate Vector Boson* theory described by

$$\mathcal{L}_W = -\frac{1}{4}\left(\partial_\mu W^\dagger_\nu - \partial_\nu W^\dagger_\mu\right)(\partial^\mu W^\nu - \partial^\nu W^\mu) + M_W^2 W^\dagger_\mu W^\mu + g(J^{\mu\dagger} W^+_\mu + J^\mu W^-_\mu) \quad (3.115)$$

having the vector boson's propagator

$$-\,\mathrm{i}\frac{(g^{\mu\nu} - k^\mu k^\nu / M_W^2)}{k^2 - M_W^2}\,. \quad (3.116)$$

With this propagator, the passage to low-energy limit ($E << M_W$) is transparent, and $G_{\mathrm{F}} \sim \frac{g^2}{M_W^2}$. We trade the non-local interactions of the high-energy theory (i.e., the intermediate vector boson theory or the standard model) with the local interactions of the low-energy theory (Fermi's). To see this, just consider the expansion

$$\frac{g^2}{k^2 - M_W^2} = -g^2\frac{1}{M_W^2} + g^2\frac{k^2}{M_W^4} + \ldots\,. \quad (3.117)$$

The Fermi's Lagrangian (3.114) is constructed solely from the ingredients available at low energy while the effects of the heavy vector bosons are encoded in the "coefficient" $G_F$. The value of $G_F$ can be obtained from the low-energy side only via experiments.

Now we will proceed in the step fairly similar to what we did above. To consider the standard model as a low-energy effective theory, we have to include the effects of all possible dimensional operators into the low-energy effective Lagrangian. They will be denoted as $\mathcal{O}^{(4+p)_i}$ where $(4+p) \geq 5$ is their dimensions. Lower dimensional operators can be "absorbed" into operators of the standard model (in $\mathcal{L}_{SM}$). Each operator is constructed solely from the building blocks of the standard model where the effects of the heavy particles beyond the standard model are encoded in its coefficient. Experiments is the only way to determine the values of these coefficients if we do not yet know exactly what the high-energy theory is[10]. The effect of these operators on the predictions by the standard model will be suppressed by the mass scale ($\Lambda$) preceding them. We write the low-energy effective Lagrangian as (see Barbieri

[10]See the section B on the $SU(5)$ Grand Unification in the next chapter. There we can "calculate" the value of the Weinberg angle from the higher-energy theory.

and Strumia [2], and Han and Skiba [65])

$$\mathcal{L}_{\text{eff}}(E < \Lambda) = \mathcal{L}_{SM} + \sum_{i,p} \frac{c_i}{\Lambda^p} \mathcal{O}_i^{(4+p)} \tag{3.118}$$

where $\Lambda$ is the scale where the new physics is expected to show up. Then the strategy is we throw in all possible forms of operators that are relevant to a specific problem (here we need operators that are stringently constrained by electroweak precision tests, otherwise we cannot evaluate the bound $\Lambda$) and do not violate some particular symmetry (Lorentz symmetry, for example). The dimension five operators include the cross-family interaction of the leptons

$$\frac{\lambda_{ij}}{\Lambda}(L_L^i H^\dagger)(L_L^j H^\dagger) + \text{h.c.} \tag{3.119}$$

where $i, j$ are family indices, which violate lepton numbers, generate masses of neutrinos when the Higgs receives a vacuum expectation value. Due to the smallness, of neutrino masses, this operator is not a good candidate for the problem we have on hand.

It turns out that operators that are useful for the electroweak precision tests are of dimension 6 which include various forms of four-fermion interactions

$$\mathcal{O}_{Ld} \sim (\bar{L}_L \gamma^\mu L_L)(\bar{d}\gamma_\mu d)\,, \tag{3.120}$$

Higgs-gauge intersections

$$\mathcal{O}_{WB} \sim (H^\dagger \sigma^a H) W^a_{\mu\nu} B^{\mu\nu}\,, \tag{3.121}$$

($W^{\mu\nu}$, $B^{\mu\nu}$ are the field strengths), or the higher order Higgs interaction

$$\mathcal{O}_H \sim (H^\dagger D_\mu H)^2\,. \tag{3.122}$$

Some of the "induced" operators will violate different kinds of symmetries; for example, baryon number, lepton number, or the custodial symmetry. For the operators that break the essential symmetries of the standard model, such as the CP, the corresponding lower-bound of the new physics must be very high, say $\sim 100\,\text{TeV}$. These operators will not be taken into account in this context of the precision tests. We will focus on operators that respect some symmetries of the standard model which we can watch the effects of them in the experiments. Suppose we consider a particular operator that generates flavour changing neutral current (which is well-maintained by the standard model) that are not observed in the laboratory. We can then evaluate the lower energy bound $\Lambda$ of that operator so that the new physics correspond to it is pushed up to higher energy, making the effects from this particular operator "automatically" unobservable at low energies.

We can go on with other operators in a similar manners. In general one usually assume $c_i \sim \mathcal{O}(1)$ (while they can be either positive or negative). Once we choose the appropriate operators, the rich indirect experiment results (precision tests) will do the rest of the job, translating the information into the lower bound of the new physics corresponding to each operator. The more agreements between the standard model and experiments, the higher the energy of the new physics. Nevertheless, this is not the whole story. The Higgs mass is still an unknown of the standard model. So the evaluations of the lower bounds must include variation of the mass of the Higgs. Some of the examples of the operators are taken from the papers by Babieri and Strumia [1, 2] and are shown the table 3.1. What we can conclude is that new physics is not welcome by precision electroweak test, at least below $5 - 7\,\text{TeV}$. In one shot, the table 3.1 also tells us that if the Higgs exists, it should be light (actually, if it does not exist, the considerations we have done in this section will cease to make sense). The lower-bound obtained in this way put a very stringent constraint on every kinds of physics beyond the standard model. It is the requirement that every theory proposed must find its way to avoid spoiling the bound.

| Dimensions six operators | $M_h = 115\text{GeV}$ | $M_h = 300\text{GeV}$ |
|---|---|---|
| $\mathcal{O}_{WB} = (H^\dagger \tau^a H) W^a_{\mu\nu} B_{\mu\nu}$ | 9.7 | 7.5 |
| $\mathcal{O}_{H} = \lvert H^\dagger D_\mu H \rvert^2$ | 4.6 | – |
| $\mathcal{O}_{LL} = \frac{1}{2}(\bar{L}\gamma_\mu \tau^a L)^2$ | 7.9 | – |
| $\mathcal{O}'_{HL} = i(H^\dagger D_\mu \tau^a H)(\bar{L}\gamma_\mu \tau^a L)$ | 8.4 | 7.5 |
| $\mathcal{O}'_{HQ} = i(H^\dagger D_\mu \tau^a H)(\bar{Q}\gamma_\mu \tau^a Q)$ | 6.6 | – |
| $\mathcal{O}_{HL} = i(H^\dagger D_\mu H)(\bar{L}\gamma_\mu L)$ | 7.3 | – |
| $\mathcal{O}_{HQ} = i(H^\dagger D_\mu H)(\bar{Q}\gamma_\mu Q)$ | 5.8 | – |
| $\mathcal{O}_{He} = i(H^\dagger D_\mu H)(\bar{e}\gamma_\mu e)$ | 8.2 | – |
| $\mathcal{O}_{Hu} = i(H^\dagger D_\mu H)(\bar{u}\gamma_\mu u)$ | 2.4 | – |
| $\mathcal{O}_{Hd} = i(H^\dagger D_\mu H)(\bar{d}\gamma_\mu d)$ | 2.1 | – |

**Table 3.1:** The lower-bound of energy scale of new physics (in the unit of TeV) evaluated from various dimension six operators, related to the mass of the Higgs. This table shows the 99% C.L. bounds where the blanks mean no fits are possible. The table is from [1, 2].

## 3.4 Shortcomings of the Standard Model

Though the standard model is considered as one of the most successful theory of particle physics, its success is not unlimited. There are still problems or frustrations that the model leaves to us. In this section we will consider some of those problems; both from the general structure of the model itself and from the its fundamental scalar (Higgs) sector.

### 3.4.1 The General Problem with the Model Itself

#### 3.4.1.1 The Standard Model is Only Partially Unified

It is often said that the Glashow-Weinberg-Salam theory is a unification theory of weak and electromagnetic interactions. However, it is still quite frustrating to think of any theory governed by a product group as a unified one, especially due to the fact that the coupling constants in the $SU(2)_L \times U(1)_Y$ theory are related only through experiments. Therefore, the quest of a more "unified" theory is not out of questions at all. A quick review of the $SU(5)$ grand unification theory, which solve this "problem", is presented in the appendix B.

#### 3.4.1.2 The Family Problem

It is important to note that the way we put the quarks in a particular family (e.g., the up and down quarks) is not completely unique. This is because $u$ is actually related, by the $SU(2)$, to the $d$, $s$, and $b$ instead of just with the $d$. Similar arguments apply to the $c$ and $t$ families as well. Moreover, the way quarks and leptons are put in to a family is also not natural. The reason we pair the $e^-$, $\mu^-$, and $\tau^-$ lepton families with the $u$, $c$, and $t$ quarks family respectively is partly due to their masses (which also affects the order they were discovered - light particles were discovered first, of course). However, there is nothing to guarantee that this must be so because in the standard model, there is no mechanism describing the transition between quarks and leptons.

The problem described above is fairy related to the problem of number of families in the standard model. Though the way particles are grouped into a family is not unique, these particles nicely organised themselves into three familes. Despite of phenomenological considerations, the standard model does not have any explanation for this.

## 3.4.2 Problems from The Higgs Sector

### 3.4.2.1 Electroweak Symmetry is Broken "Because it has to"

Spontaneous symmetry breaking is a very important feature of the standard model. Actually the theory dealing with electroweak symmetry should find its way to incorporate the BEH (Higgs) mechanism, otherwise the gauge bosons will be massless. However, the Glashow-Weinberg-Salam model utilises the BEH mechanism with the help of a fundamental scalar (the Higgs). As the title said, the scalar sector of the standard model does not provide any explanation why the electroweak symmetry is broken. We have to assume the "hyper-Mexican hat" potential as the starting point or else the symmetry will not be broken. Clearly this picture lacks any dynamical process that might occurs around the symmetry breaking scale. Of course, this problem alone will not doom the whole theory. However, a better mechanism with dynamical explanations for incorporating the BEH mechanism, like the BCS theory in superconductor, would be nicer.

### 3.4.2.2 Quadratic Divergences: Part 1. Hierarchy Problem

There are many experimental evidences, or theoretical constraints, telling us that the Higgs should be light. We have seen some of the latter kind of arguments in section 3.2. However, it turns out that theories we have on hand tend to introduce infinities into this parameter in various ways via "quantum effects", i.e., loop corrections. Renormalisation programmes tell us that these can be somehow controlled by adjusting free parameters in the theory. So now we will consider the question whether these adjustments are natural and see whether the Higgs is "happy" to be kept light when quantum effects are included. The example we used in this section is actually a toy model. Still, the idea is readily applicable to the standard model.

First we will start with the particle that are said to be "naturally light". They are the gauge fields and the fermions. We will work out the fermion self-energy and see why they are said to be so. To simplify the calculations and keep focusing on the physics, we will use the Lagrangian for a fermion interacting with a *massive* scalar field:

$$\mathcal{L} = \mathrm{i}\overline{\psi}\partial\!\!\!/\psi - \partial_\mu\phi^*\partial^\mu\phi - m_s^2\phi^*\phi - \lambda_s(\phi^*\phi)^2 - y_f\overline{\psi}\psi\phi + h.c.\,. \tag{3.123}$$

Notice that the "*h.c.*" actually contains only the Yukawa coupling term. In fact, this simplification is somewhat legitimate because the top quark is the heaviest fundamental particle we that have found. The field $\phi$ is parametrised by $\phi = (h+v)/\sqrt{2}$ and the fermion gets its mass from the tree level: $m_f = y_f v/\sqrt{2}$. The scalar field contributes to the fermion self-energy via the diagram shown in fig. 3.8. We find the self-energy

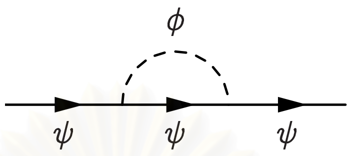



**Figure 3.8:** Fermion self-energy diagram contributed from a scalar field.

$$-\,\mathrm{i}\Sigma_{f,s}(p) = \left(\frac{-\mathrm{i}y_f}{\sqrt{2}}\right)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}\mathrm{Tr}(\not{k}+m_f)}{[k^2-m_f^2]}\frac{\mathrm{i}}{[(k-p)^2-m_s^2]}\,. \tag{3.124}$$

Recalling that the renormalised mass is defined as $m_f^R = m_f + \delta m_f$, we find

$$\delta m_f = \Sigma_f(p=m_f) = \mathrm{i}\frac{y_f^2}{32\pi^4}\int_0^1 \mathrm{d}x \int \mathrm{d}^4k' \frac{m_f(1+x)}{[k'^2 - m_f^2x^2 - m_s^2(1-x)]^2} \tag{3.125}$$

where we have used the Feynman parameter for parametrising the integral (see the appendix C.2). The calculation is straightforward. The term that remains there when the cut-off becomes large is

$$\delta m_{f,s} = -\frac{3y_f^2 m_f}{64\pi^2}\ln\left(\frac{\Lambda^2}{m_f^2}\right). \tag{3.126}$$

This is not beyond our expectation since we know that the chiral symmetry protecting the mass of the fermion is broken when the fermion interacts with the scalar field; i.e., when Yukawa coupling (the mass term) of the fermion is introduced. The chiral symmetry is restored when this parameter vanishes. This means that the correction term $\delta m_f$ should be proportional to the term that breaks the symmetry - which is the $m_f$. Consequently, a fermion is said to have *multiplicative renormalisations* which means it is natural to be light. In other words, the chiral symmetry $U(1)_L \times U(1)_R$ plays the role of the *custodial symmetry* protecting the fermion's mass.

Now let us turn to the self-energy of the scalar field which, in this simplified theory, only comes from the interaction with fermion. We write

$$(m_s^R)^2 = m_s^2 + \delta m_s^2 \,. \tag{3.127}$$

The corresponding diagram shown in fig. 3.9 results in

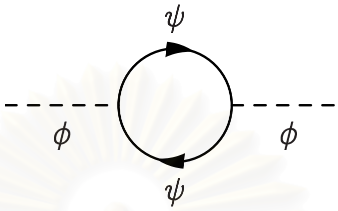



**Figure 3.9:** Scalar self-energy diagram contributed from a fermion.

$$-\,\mathrm{i}\Sigma_{s,f}(p^2) = (-1)\left(\frac{-\mathrm{i}y_f}{\sqrt{2}}\right)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}^2\mathrm{Tr}\left\{(\not{k}+m_f)((\not{k}-\not{p})+m_f)\right\}}{[k^2-m_f^2][(k-p)^2-m_f^2]} \tag{3.128}$$

Then we find that the mass of the scalar field (or the Higgs) diverges:

$$\begin{aligned}
\delta m_{s,f}^2 &= \Sigma_s(p^2 = m_s^2) \\
&= -\frac{y_f^2}{8\pi^2}\Bigg[\Lambda^2 + \frac{1}{2}(m_s^2 - 6m_f)^2 \ln\left(\frac{\Lambda^2}{m_f^2}\right) \\
&\quad + \frac{1}{2}(4m_f^2 - m_s^2)\left\{1 + \int_0^1 \mathrm{d}x \ln\left(1 - \frac{m_s^2}{m_f^2}x(1-x)\right)\right\} \\
&\quad + \mathcal{O}\left(\frac{1}{\Lambda^2}\right)\Bigg] .
\end{aligned} \tag{3.129}$$

Situations are different this time. Considering the quadratic divergent piece

$$\delta m_{s,f}^2 = -\frac{y_f^2}{8\pi^2}\Lambda^2 \,, \tag{3.130}$$

we see that is no symmetry recovered when the (bare) mass of the scalar reduces to zero due to the fact that the correction is not proportional to $m_s$. The "physical" mass of the scalar becomes

$$(m_s^R)^2 \simeq -\frac{y_f^2}{8\pi^2}\Lambda^2 \,, \qquad \text{when } m_s = 0 \,, \tag{3.131}$$

which also illustrates that it is not protected by any symmetry. The scalar (Higgs) seems to prefer to be as heavy as the largest mass scale of the theory (the cut-off). The severity of the problem then depends on the value of the

$m_s^R$ that experimental results expect it to be, and also on the limit $\Lambda$ that the theory is expected to be valid. We have seen in the section 3.2 that the mass of the Higgs should not be too heavy (comparing with electroweak gauge bosons). Thus, taking $m_s^R = 100$GeV, the rough approximation of the Higgs mass, and $\Lambda = 10^{19}$GeV, the Planck scale, we see that the correction $\delta m_s^2$ is huge and has to be balanced by $m_s^2$ with amazing precision. To see this, let us write (3.131) in terms of dimensionless parameters:

$$\mu^2 \equiv \frac{m_s^2}{\Lambda^2} = \mu_R^2 + \frac{y_f^2}{8\pi^2}\,, \tag{3.132}$$

where $\mu_R^2 \equiv (m_s^R)^2/\Lambda^2$. Using the values of $m_s$ and $\Lambda$ given above, together with $y_f \sim 1$, we find

$$\mu^2 \sim y_f^2 \frac{(1+10^{-32})}{100}\,. \tag{3.133}$$

In other words, the tree level parameter $\mu^2$ must be adjusted to the $32^{nd} - 34^{th}$ decimal places. If such the adjustment is not satisfied the mass will come out to be of order $\Lambda = 10^{19}$GeV again. Even if it is so, higher order corrections are very likely to violate it and hence infinite re-adjustments are required. So we say that the Higgs mass is quadratically unstable against the quantum corrections. Too many "coincidences" are required to make it light. This is the *fine-tuning problem.*

In the appendix B, we show that the situation does not gets much better if we take the standard model as an effective theory of some *unified* theory at a particular energy scale, say $10^{15}$ GeV. We show that there will be at least two fundamental scalars to do the job of breaking the symmetry spontaneously. That means we should expect two fundamental scalars with masses of order $10^{15}$ GeV and $10^{19}$ GeV which are the cut-off of the standard model and the unified theory. However, we know that one of the scalar, says the Higgs of the standard model, should be as light as $10^2$ GeV. Then there must be some unnatural separation between the two mass scales and some extreme fine-tunings must be done to get things right.

The lack of naturalness and the requirement of fine-tuning is commonly referred to as the *Hierarchy problem.* In some literatures, it is also known as the "*big*" or the "*full*" hierarchy problem when the effective theory is expected to valid to the grand unification or the Planck scales. One of the reasons that this kind of problem is not currently considered as the defect of the theory is because the extreme sensitiveness of the mass of the Higgs to the cut-off did not manifest itself in the region that is already "probed" (indirectly) by experiments. This is clear by setting $\Lambda = 1\,\text{TeV}$ in (3.132). However, the need for fine-tuning reappears as soon as we extent the cut-off to 10TeV (recall the electroweak precision tests and see below) whose verification is within the reach of the LHC.

By the way, it is important to emphasise that the hierarchy problem is not the problem of mathematical inconsistencies of the model but is more or less a kind of phenomenological frustrations.

#### 3.4.2.3 Quadratic Divergences: Part 2. Implications on New Physics

The Higgs of the standard model has three different kinds of couplings. We just have discussed the most severe quadratic divergent contribution due to top quarks. Now we will include the loop corrections to the scalar (the Higgs) from gauge bosons and the Higgs itself and perform some rough calculations. By rough we mean we focus on only the contributions to quadratic divergent diagrams. Then we can regard the "loop-particles" as massless since we will focus on high energy limits of the theory.

Let us begin with electroweak gauge fields (2 charged $W$'s and 1 neutral $Z$) with the diagram providing major contribution shown in fig. 3.10. For simplicity, we will assume that $M_W \simeq M_Z$.

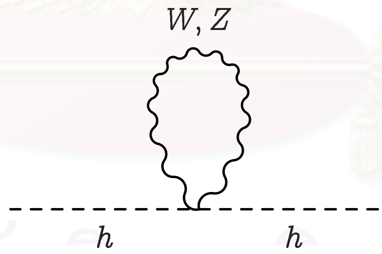



**Figure 3.10:** Higgs self-energy diagram contributed from gauge bosons.

For *each* gauge field, the contribution is (Landau gauge)

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}(\mathrm{i}g^{\mu\nu})\frac{g^2}{4}\frac{-\mathrm{i}(g_{\mu\nu} - k^\mu k^\nu/k^2)}{k^2} = \frac{1}{4}(3)g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2} = \frac{3}{64\pi^2}g^2\Lambda^2 \qquad (3.134)$$

Observe the factor 3 from the trace of the propagator of the gauge boson (see also (3.40). Altogether, the three electroweak gauge bosons add to the mass of the Higgs

$$\delta M_{h,g}^2 = \frac{9}{64\pi^2} g^2 \Lambda^2 \,. \tag{3.135}$$

Out of all the loop contributions from quarks, that from the top quark is the largest. As we might mentioned from time to time, this is not surprising due to its large Yukawa coupling with the Higgs (or its large mass). We have already evaluated the contribution from fermion field to a scalar in the previous section (see (3.129). For the case at hand, we just need the factor 3 for quarks have three colours. Thus

$$\delta M_{h,t}^2 = -\frac{3}{8\pi^2} y_t^2 \Lambda^2 \,. \tag{3.136}$$

Finally we have the quartic coupling of Higgs as shown in the fig.3.11.

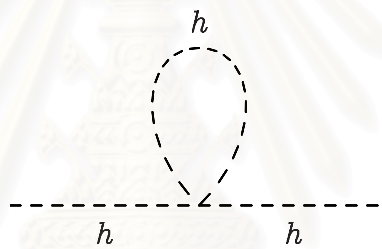



**Figure 3.11:** Higgs self-energy from its quartic coupling.

In this case we have

$$\delta M_{h,h}^2 = \frac{\mathrm{i}\lambda}{4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{k^2} 4 = \frac{4}{64\pi^2} \lambda \Lambda^2 \,. \tag{3.137}$$

where 4 is a symmetry factor.

We now collect the corrections to the Higgs mass together and write

$$\begin{aligned}
(M_h^{\mathrm{R}})^2 &= M_{h(\mathrm{tree})}^2 + \delta M_{h,g}^2 + \delta M_{h,h} + \delta M_{h,t}^2 \\
&= M_{h(\mathrm{tree})}^2 + \left(9g^2 + 4\lambda - 24 y_t^2\right) \frac{\Lambda^2}{64\pi^2} \\
&= M_{h(\mathrm{tree})}^2 + \left(9g^2 + 4\lambda - 24 y_t^2\right) \frac{\Lambda^2}{160}
\end{aligned} \tag{3.138}$$

which is just a more complex version of (3.131). Thus, the analysis is very similar. Since there are no relations between the Higgs couplings, natural cancellations are hopeless. As in the previous case, the Higgs mass is obtained *naturally* only when the cut-off is at $\sim 1\,\text{TeV}$ which is the approximate limit of current accelerators, while fine-tuning is still required if we raise the cut-off to $\sim 10\,\text{TeV}$ of next generation accelerators. When $\Lambda \sim 10\,\text{TeV}$ we get the approximation

$$\delta M_{h,g}^2 = \frac{9}{64\pi^2} g^2 \Lambda^2 \simeq \frac{9}{160} \frac{M_W^2}{v^2} \Lambda^2 \sim (800\text{GeV})^2 \tag{3.139}$$

$$\delta M_{h,h}^2 = \frac{4}{64\pi^2} \lambda \Lambda^2 \simeq \frac{1}{160} \frac{M_h^2}{v^2} \Lambda^2 \sim (450\,\text{GeV})^2 \tag{3.140}$$

$$\delta M_{h,t}^2 = -\frac{3}{8\pi^2} y_t^2 \Lambda^2 \simeq -\frac{3}{40} \frac{m_t^2}{v^2} \Lambda^2 \sim -(2\,\text{TeV})^2 \tag{3.141}$$

where we have used $M_h \sim 200\,\text{GeV}$. The illustration of the situation can be plot in a chart shown in figure 3.12presented by Schmaltz [34].

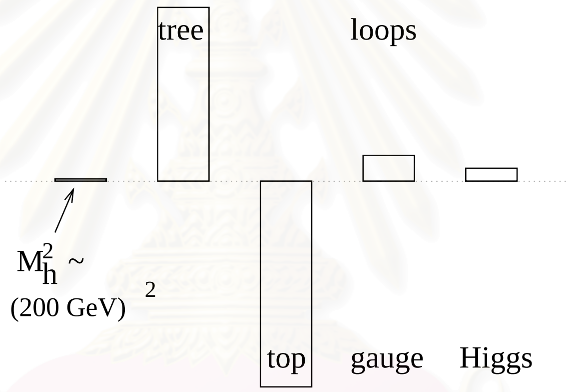



**Figure 3.12:** An illustration of how bad the situation of the naturalness of the standard model is, when $\Lambda \sim 10\,\text{GeV}$.

In the previous section, we have argued that there must be new physics somewhere within this scale. Now, we can turn the argument around and see at what scales will we find new physics if we require only a few amount of fine-tuning. In other words, we will start by arguing that the loop correction should not be much larger than the tree-level. Then the degree of naturalness (of the mass of the Higgs) is evaluated by considering (see Casas *et al.* [66, 67] for a more complete analysis)

$$\mathcal{F} \equiv \left| \frac{\delta(M_h^2)}{M_h^2} \right| . \tag{3.142}$$

The mass of the Higgs would be most natural if we require $\mathcal{F} << 1$. However, this would imply $\Lambda < 1\,\text{TeV}$ which has already been ruled out by precision electroweak measurements. So, let us say we want to fine-tune no more than 1 part in 10 (i.e., less than 10%), which means

$$\left|\frac{\delta(M_h^2)}{M_h}\right| \lesssim 10\,. \tag{3.143}$$

The (3.143) then tells us that $\Lambda_{\mathrm{SM}} < 2-3\,\mathrm{TeV}$ (due to the large top quark loop contribution) if we let the mass of the Higgs varies between $115-200\,\mathrm{GeV}$.

Remember that we have hierarchy of particles: gauge bosons, quarks, and the Higgs. Each has liberty to live on its own. This means, providing a particular acceptable amount of fine-tuning, each kind of particle can "request" new physics at different energy scale, depending on how severe it contributes to the quadratic divergent loop correction to the mass of the Higgs. As we have seen, as the top quark contributes most, it requires (see, for example Schmaltz [36])

$$\Lambda_{\mathrm{top}} \lesssim 2\,\mathrm{TeV} \tag{3.144}$$

to keep the fine-tuning better than 10%. For the gauge bosons and the Higgs we have

$$\Lambda_{\mathrm{gauge}} \lesssim 5\,\mathrm{TeV} \tag{3.145}$$

$$\Lambda_{\mathrm{Higgs}} \lesssim 10\,\mathrm{TeV}\,. \tag{3.146}$$

With the limit of the current accelerators around 1 TeV, it is still fine that we did not see any signal of new physics. When we say new physics we mean that there must be some other heavy particles (so that have not been seen already) that produce diagrams to cancel the dangerous diagrams from the standard model particles.

There are at least two ways to resolve the naturalness problem. One is to remove the problem right from the start; i.e., remove the Higgs or any fundamental scalar. The idea is every scalar (the Goldstone boson; for example) in the theory is regarded as a composite particle. In fact, this is not a new idea. It was used (and proved "correct") in the physics of superconductor through the BCS theory. The breaking of symmetry is dynamical and is managed by composite scalar fermion condensates. Theories developed with this scheme in mind are categorised in the class of *technicolour*. We shall investigate some important features of *dynamical symmetry breaking* and technicolour in the section 4.1.

The other way out of the problem is using symmetry, as we may have already mentioned from time to time. Symmetry can organise cancellations between dangerous quadratically divergent loop diagrams. *Supersymmetry* and *Little Higgs* fall into the examples of this kind. Actually, we may have seen similar cancellations when we introduce the $SU(2)$ electroweak gauge symmetry to ensure the cancellations between dangerous diagrams from $W$ and $Z$. In supersymmetry, the cancellations occur between loops from particle having opposite spin statistics (particles and superparticles) related via supersymmetry. In Little Higgs, as we shall soon see, there is a global symmetry playing the role of supersymmetry, to manage cancellations between particles of the same statistics.

Up to now we have been assuming that we do not believe in a perfect coincidence so that there should be very few percents fine-tunings, and that there must be new particles showing up at around 2 TeV to cancel the unpleasant part. Unfortunately, life is not that easy. The effects of the new particles cannot be turned on and off as we wishes. They will leave some footprints within the precision electroweak measurements or processes taken place at energy scale not far below the cut-off $\Lambda$. Even though we can enforce the perfect cancellations between the quadratically divergent diagrams by introducing some symmetries, there will always be the logarithmic divergent diagrams left behind. That also put some constraint on the cut-off as well.

Still, this is not the whole story we can learn from the naturalness (fine-tuning) arguments. A careful reader might have noticed that the *maximum* limit of the standard model with better than 10% fine-tuning (say $2-3\,\text{TeV}$) lies way below the *minimum* energy scale where new physics will appear (around $5-7\,\text{TeV}$) according to precision electroweak tests. If we take the precision tests as our first priority and set $\Lambda_{\text{top}}$ we will eventually end up at arond $2-3\%$ fine-tuning. which is hardly acceptable. In addition, even if the fine-tuning fails very slightly[11] and the Higgs turns out to be just “a bit” heavier, the other bound from the precision tests saying that the Higgs should be $\sim 100-200\,\text{GeV}$ is not satisfied. This problem is known as the *little hierarchy problem* or the *LEP paradox*. Unless we take the “desperate” solution; i.e., ignore the fine-tuning and accept the world as is, we have to be careful when introduce new physics. So the new physics predicts results which are deviated only slightly from those predicted by the standard model (which agrees well with precision electroweak tests).

[11]This is not likely to occur though. If the fine-tuning mechanism fails, for any reason, the mass of the Higgs will be driven to the cut-off scale.

# CHAPTER IV

# PRELUDE TO THE LITTLE HIGGS

The purpose of this chapter is to provide not only the basic ingredients required to understand the Little Higgs, but also many crucial ideas and thoughts that should be useful for studying physics beyond the standard model in many directions. In section 4.1, we consider another approach to the spontaneous symmetry breaking which does not require the fundamental scalar. It serves us as another way to implement the BEH mechanism, as well as gives us some insights on the problem of vacuum alignment. As a by-product, it provides a nice way to avoid the naturalness problem and provides the basis of understanding the high-energy limit of the Little Higgs (the UV completion). We will show some simple (but complex enough to be illustrative) model to point out how the structure of the vacuum affects the pattern of gauge symmetry breaking. After that we present methods to deal with low-energy effective theory in terms of the non-linear sigma model in section 4.2. Finally, in section 4.3, we put the last three main ideas together in a model that is considered as a prototype of the Little Higgs.

## 4.1 Dynamical Symmetry Breaking

In this section we will investigate a gauge theory with spontaneous symmetry breaking from different point of view than what we have done in previous chapters. Let us recall that what is important in the spontaneous breaking of a global symmetry is the symmetry group $G$ and $H$ of the system (and the representations used), not the existence of an elementary scalar. Spontaneous symmetry breaking without an elementary scalar, usually dubbed *dynamical symmetry breaking*, is our main topic here. In the first section (4.1.1) we will see what happens to the electroweak sector if the Higgs does not exist. A more general treatment will be found in section 4.1.2, where we include the discussions of vacuum alignments. Formalisms of a situation where the global symmetry is broken by weak interaction are discussed in section 4.1.2.1, together with their applications in section 4.1.3.

### 4.1.1 A Simple Case

Consider a system of massless quarks $u_L$, $u_R$ and $d_L$, $d_R$ living in a flavour doublet and a colour triplet. Now let us say there is no fundamental scalars included in this system. In this way, when the electroweak gauge couplings $g$ and $g'$ are turned off we see that in addition to the gauge symmetry $G_{\text{strong}}$ of the strong interaction, the theory will have a chiral symmetry $G = SU(2)_L \times SU(2)_R$, one acting on the doublet $(u_R, d_R)$ while the other acting on $(u_L, d_L)$. This global chiral symmetry is sometimes dubbed "accidental symmetry" of the electroweak Lagrangian for the reason that should be obvious. We shall see that due to the nature strong interaction (colour), naturalness, hierarchy, and triviality problems are solved (or prevented) in one shot.

Next, suppose that the system is arranged in a way that the quark-antiquark condensate is easy to produce (i.e., the $q\bar{q}$ is the lowest energy state) and tend to stay with each other with strong interaction as a "glue". For a space filled with infinite amount of these pairs to act as a vacuum, each pair must have zero momentum and zero angular momentum. This immediately tells us that helicity cannot be zero for each pair. In other words, chiral symmetry is not respected by the vacuum. The chiral symmetry $SU(2)_L \times SU(2)_R$ will break down spontaneously to $SU(2)_{L+R}$ vector isospin subgroup. The breaking is said to be triggered by a "*composite*" scalar consisting of a quark bilinear having non-zero vacuum expectation value

$$\langle \bar{u}_L u_R \rangle = \langle \bar{d}_L d_R \rangle \neq 0 \,, \tag{4.1}$$

which clearly provides links between right-handed quarks with the left-handed antiquarks. The configuration of the condensate maybe written as follows:

$$\begin{pmatrix} \langle \bar{u}_L u_R \rangle & \langle \bar{u}_L d_R \rangle \\ \langle \bar{d}_L u_R \rangle & \langle \bar{d}_L d_R \rangle \end{pmatrix} \propto (\Lambda_{\chi SB})^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{4.2}$$

where the $\Lambda_{\chi SB}$ is the energy scale where the breaking is expected to occur. For strong interaction described by QCD, this scale lies around 1 GeV and should not be confused with the confinement scale $\Lambda_{\text{QCD}}$ which is somewhere around 1 GeV as well. Also note the pion decay constant $F_\pi \sim 100\,\text{MeV}$.

Before we move on, let us note the crucial fact that when electroweak interactions are turned off, strong interaction alone does not distinguish between up and down quarks. This means a configuration

$$\begin{pmatrix} \langle \bar{u}_L u_R \rangle & \langle \bar{u}_L d_R \rangle \\ \langle \bar{d}_L u_R \rangle & \langle \bar{d}_L d_R \rangle \end{pmatrix} \propto (\Lambda_{\chi SB})^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{4.3}$$

is equally probable and is reachable by a unitary transformation. However, as we shall see later, this is not a preferred configuration when we turn electroweak interaction on.

Now, recall that regardless whether there is a fundamental scalar in the system or not, we always have the Goldstone theorem on hand when a global symmetry is spontaneously broken. In this case, we have 3 Goldstone bosons (the pions) corresponding to the broken axial currents (see the section 2.2.2). Even in a case without a fundamental scalar like this one we are considering, we should not write $\mathcal{L}_{SB} = 0$ though the interactions between the Goldstone bosons and other particles many not manifest themselves in the Lagrangian. Consequently, in the sense of the $\mathcal{L}_{SB}$ given by (2.1), we find that we should write

$$\mathcal{L}_{QCD} \leftrightarrow \mathcal{L}_{SB} , \tag{4.4}$$

where the role of the fundamental scalar was replaced by the fermion condensate. Being the Goldstone bosons, we see that masses of these pions are protected to all order. Their derivative interactions are characterised by a scale $F_\pi$ which is typically of $\mathcal{O}(100\,\mathrm{MeV})$. Other states are at $\Lambda_{\chi SB} >> F_\pi$. Empirically, $\Lambda_{\chi SB} \sim \mathcal{O}(1\mathrm{GeV})$.

Notice that by saying that a pion is a "composite" particle, we mean at energies higher than $\Lambda_{\chi SB}$ the particle has quark-antiquark substructure, bound by strong interaction. However, at energies far below $\Lambda_{\chi SB}$ it looks point-like; i.e., like other elementary particles. Then we can use the effective Lagrangian to explain its behaviour (see later sections).

Now we will proceed to the second stage: when the electroweak interaction is switched on and coupled with our system, and is treated as a perturbation. The latter means the weak couplings are weak at the energy scale where the strong interaction (the binding force) becomes strong[1]. Strictly speaking, turning on a gauge interaction means we introduce a weakly interacting

[1] In principle, the "electroweak interaction" here cannot be the same as the one we are familiar with. The gauge fields in this case are very light. As we shall see this symmetry breaking mechanism based on $F_\pi \sim \mathcal{O}(100\mathrm{MeV})$ will result in the electroweak like interaction with $M_W, M_Z \sim \mathcal{O}(100\mathrm{MeV})$ not $\mathcal{O}(100\mathrm{GeV})$.

gauge group $G_W \subset G$ such that[2] when $G$ is spontaneously broken to $H$, $G_W$ is also broken to its subgroup, say $H_W$, controlled by the intersection between $G_W$ and $H$. In the case we are dealing with, it is clear that the global chiral symmetry group must be broken *explicitly* to a subgroup preserving the electroweak interaction. This case is interesting because the currents in electroweak interactions are handed ones. So they can "communicate" with the condensate the presence of symmetry breaking. Then the resulting interaction will give mass to the gauge bosons. The following considerations will be rather similar with those in the section 2.2.3. So let us just fill what are missing.

The parts of the Lagrangian containing $SU(2) \times U(1)$ currents that are capable of producing the pions are written explicitly as

$$\begin{aligned} & g W_\mu^a \overline{\psi} \gamma^\mu \left(\frac{1-\gamma^5}{2}\right) \frac{\tau^a}{2} \psi + g' B_\mu \overline{\psi} \gamma^\mu \left(\frac{1+\gamma^5}{2}\right) \frac{\tau^3}{2} \psi \\ & = g J^{\mu a} W_\mu^a + g' J_Y^\mu B_\mu \end{aligned} \tag{4.5}$$

These currents produce the pions according to

$$\langle 0 | J^{\mu a}_{\text{gauge}}(0) | \pi^b(p) \rangle = \mathrm{i} p^\mu F_\pi \delta^{ab} \,, \tag{4.6}$$

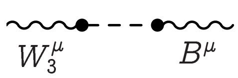
However, in this case there are two gauge fields coupled with the neutral pion with the couplings given in fig.4.1. Up to this point, we can proceed in the same


**Figure 4.1:** Gauge bosons and pions couplings

way as what we did in the section 2.2.3. The diagrams in Fig.4.1 contribute to the $\Pi(p^2)$ in the vacuum polarisation tensor as follows:

$$\Pi_{WW}(p^2) = \frac{g^2 F_\pi^2}{p^2} \,, \quad \Pi_{BB}(p^2) = \frac{g'^2 F_\pi^2}{p^2} \,, \quad \Pi_{WB}(p^2) = \frac{g g' F_\pi^2}{p^2} \,. \tag{4.7}$$

The last term is coming from




$$= \quad (\mathrm{i} g p^\mu F_\pi) \frac{\mathrm{i}}{p^2} (-\mathrm{i} g' p^\mu F_\pi) \,. \tag{4.8}$$

[2] It is not necessary that $G_W$ and $H_W$ be the $SU(2)_L \times U(1)_Y$ and $U(1)_{em}$, respectively. However, these are usually the cases.

In this way, the mass terms for the gauge fields can be read off easily. They can be organised into a mass matrix

$$M^2 = F_{\pi^2} \begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & gg' \\ & & gg' & g'^2 \end{pmatrix} \tag{4.9}$$

with rows and columns labelled by the states $W_\mu^1, W_\mu^2, W_\mu^3, B_\mu$. The $2 \times 2$ matrix in the lower-right corner, corresponding to the $W_\mu^3$ and $B_\mu$, can be diagonalised into

$$F_\pi^2 \begin{pmatrix} g^2 + g'^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.10}$$

with the eigenstates being the linear combinations (2.90) and (2.91)

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gW_\mu^3 - g'B_\mu) = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W \tag{4.11}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gB_\mu + g'W_\mu^3) = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W \,. \tag{4.12}$$

The eigenvalues of the diagonalised mass matrix are therefore

$$M_W^2 = g^2 F_\pi^2 \,, \quad M_Z^2 = (g^2 + g'^2) F_\pi^2 \,, \quad M_A^2 = 0 \,, \tag{4.13}$$

which implies

$$\frac{M_W^2}{M_Z^2 \cos\theta_W^2} = \rho = 1 \,. \tag{4.14}$$

What is interesting is that for any strongly interacting gauge theory with a chiral symmetry breaking from $G \supset SU(2)_L \times U(1)_Y$ to $H \supset U(1)_{em}$, in such a way that $SU(2)_L \times U(1)_Y \not\subset H$, will always break the electroweak interaction down to electromagnetism. Phenomenological constraints further requires that $H$ must also contain an $SU(2)$ group to ensure that the relation (4.14) is satisfied at tree levels. We say that the $SU(2)$ protects such a relation and hence its is the $SU(2)$ *custodial symmetry* (see section 2.3.5).

Though the relation (4.14) as well as the global symmetry that protects it is the same as those found in the section 2.3.5, we tried not to claim right from the beginning that they are the same. This is because we do not really know what is the mechanism behind the electroweak symmetry breaking. Nor do we know if the (standard model) Higgs doublet exists. But, as we have seen, various theoretical arguments and their results together with some verifications from the laboratories convinced us that the electroweak symmetry must be broken. So there must be three would-be Goldstone bosons produced and later eaten by the $W$ and $Z$ gauge fields. This assures, as we have mentioned from time to time, that the Higgs doublet is not necessary and the important part is the symmetry breaking pattern. The constraint (4.14) tells us that the custodial symmetry is more or less vital for any model of electroweak symmetry breaking we use. Once the custodial symmetry is there, it protects the $\rho$ from receiving large radiative corrections.

The results we have here show some nice features of the dynamical symmetry breaking model in the sense that the inputs (parameters) of the theory are minimal and the outputs are quite a lot. We started with massless fermions forming condensates which broke the chiral isospin symmetry down to the vector isospin symmetry. Then it was this strong interaction that broke the $SU(2) \times U(1)$ down to $U(1)_{em}$ and gave masses to the gauge bosons. Still, the numerical results are not satisfactory; for example, taking $F_\pi \sim 100\mathrm{MeV}$, we find that the mass of the gauge bosons

$$M_W \sim 30\,\mathrm{MeV}\,. \tag{4.15}$$

are about 2500 times lower than the experimental values. This particular problem was a motivation for the introduction of a stronger colour-like interaction called *technicolour* or *hypercolour*. Besides, observe that this simple mechanism cannot give mass to fermions. Yukawa interaction is not an option since the theory does not have fundamental scalar. This problem can be solved in a modified theory generically called the extended technicolour. We will not pursue that topic here. Still it is worth emphasising that despite all the good features and all the defects that the families of technicolour theories may have, these kinds of theories do not have the Higgs (nor the Higgs doublet). A particle having quantum numbers similar to the Higgs may exist but merely not qualifies, phenomenologically, for being so. At least it does not have the Higgs feature such as its couplings with other particles are proportional to their masses. Actually, this is fairly obvious because it is "by construction". We tried to avoid it right from the beginning and it did not come back to us.

To sum up, we have tried to see what happens if the Higgs does not exist. We saw that electroweak symmetry still breaks down to that for electromagnetism and the weak gauge bosons are massive, though they are even lighter than the pions, due to their interactions with the condensate.

### 4.1.2 Explicit Symmetry Breaking and Vacuum Alignment

In a system where spontaneous symmetry breaking occurs, we have seen that in many cases there exist infinite possibilities of equivalent ground states (vacua). The previous section provided one of the examples. An illustrative example maybe the breaking of the global symmetry $SO(3)/SO(2)$ of a ferromagnetic system. Though we know that the symmetry is broken down to $SO(2)$, we cannot tell *which* $SO(2)$ subgroup is left unbroken. An $SO(3)$ transformation will rotate a particular vacuum to others which results in the new unbroken subgroup (which are equivalent). Nevertheless, things change when we apply an external "disturbance" (in this case it is the external magnetic field) to the system which breaks the $SO(3)$ symmetry *explicitly*. In other words, the explicit perturbation may transform as a 3-vector in $SO(3)$ and will force a specific $SO(2)$ rotation that leaves it intact. Therefore, the $SO(3)$ symmetry is not exact even before the spontaneous symmetry breaking occurs. So, this specific $SO(2)$ (second) vacuum may not be the same as the choice picked up by the spontaneous breaking mechanism. The interesting point is that instead of having the $O(1)$ as a survival exact symmetry that satisfies both the two vacuum alignments, the system has the $SO(2)$ as an exact symmetry. This means that the vacuum of the spontaneous symmetry breaking tends to "align" itself with the explicit symmetry breaking interaction. Some of the aspects of the *vacuum alignment* will be reviewed in this section.

In this section, we will deal with explicit symmetry breaking in general sense. Then we focus on the explicit breaking caused by gauge interactions. Finally we will briefly review some of the examples of the vacuum alignment problems.

First let us recall the invariant condition (2.14) for a potential $V_0$ leading to spontaneously broken symmetry

$$\frac{\partial V_0}{\partial \phi_i}(T^a\phi)_i = 0\,. \tag{4.16}$$

Suppose we introduce a small explicit symmetry breaking perturbation $V_1(\phi)$ so that the new potential becomes $V = V_0 + V_1$, we will find that the vacuum changes to $\phi = \phi_0 + \phi_1$ for a small $\phi_1$. This new vacuum satisfies

$$\left.\frac{\partial V(\phi)}{\partial \phi_i}\right|_{\phi=\phi_0+\phi_1} = 0 \tag{4.17}$$

and is no longer degenerate. Expanding this equation, the stationary condition becomes

$$0 = \left.\frac{\partial V_0(\phi)}{\partial \phi_i}\right|_{\phi_0} + \left.\frac{\partial^2 V_0(\phi)}{\partial \phi_j \partial \phi_i}\right|_{\phi_0} \phi_{1j} + \left.\frac{\partial V_1(\phi)}{\partial \phi_i}\right|_{\phi_0} + \left.\frac{\partial^2 V_1(\phi)}{\partial \phi_j \partial \phi_i}\right|_{\phi_0} \phi_{1j} + \ldots . \tag{4.18}$$

After a multiplication with $T^a_{ik}\phi_{0k} = (T^a\phi_0)_i$ and the application of a slightly-modified version of (2.17), we find, to the first order

$$(T^a\phi_0)_i \left.\frac{\partial V_1(\phi)}{\partial \phi_i}\right|_{\phi_0} = 0 . \tag{4.19}$$

As we have seen in the section A.4 on the effective potential, the above equation technically tells us that the Goldstone bosons have no tadpoles (to first order in $V_1$). This equation also tells us that it is the symmetry breaking terms in the Hamiltonian that control the alignment of the true vacuum. Hence this condition is known as the *vacuum alignment condition.*

Group theoretical techniques allow us to find a condition to identify whether the vacuum under consideration is the "true" one; i.e., whether it has minimum energy. Consider a spontaneous breaking of global symmetry group from $G$, with $T^a$ $(a = 1, \ldots, n_G)$ as generators, down to $H$, with $Y^i$ $(i = 1, \ldots, n_H)$ as generators. The broken generators are denoted by $X^z$ $(z = 1, \ldots, n_G - n_H)$. So the set of degenerate vacua is described by

$$|\Omega(\Theta)\rangle = e^{\mathrm{i}\Theta^z X^z}|0\rangle . \tag{4.20}$$

If $G$ is an exact symmetry (but hidden), then the orientation of the $H$ in $G$ is arbitrary. Now suppose we introduce an explicit breaking part, which introduces a second vacuum alignment, into the system. By treating the small symmetry breaking term in the Hamiltonian $\mathcal{H}'$ as a perturbation, we find that the vacua are no longer degenerate and are shifted by

$$\Delta E(\Theta) = \langle\Omega|\mathcal{H}'|\Omega\rangle = \langle 0|e^{-\mathrm{i}\Theta^z X^z}\mathcal{H}' e^{\mathrm{i}\Theta^z X^z}|0\rangle . \tag{4.21}$$

In general, we do not know which vacuum aligns with $\mathcal{H}'$. Look at the vacua defined by (4.21). One of them, say the $|0\rangle$, should correspond to the true vacuum such that the $\Delta E(\Theta)$ be extremal; i.e.,

$$\frac{\partial}{\partial \Theta^z}\Delta E(\Theta)\Big|_{\Theta=0} = \mathrm{i}\langle 0|[X^z, \mathcal{H}'(0)]|0\rangle = 0 . \tag{4.22}$$

Furthermore, the contribution from $\mathcal{H}'$ must lead to an upward curvature of the effective potential:

$$\frac{\partial^2}{\partial\Theta^x\partial\Theta^z}\Delta E(\Theta)\Big|_{\Theta=0} = -\langle 0|[X^z,[X^x,\mathcal{H}'(0)]]|0\rangle \geq 0\,. \tag{4.23}$$

These two conditions are also known as the Dashen's conditions. The latter condition tells us that the Goldstone boson will acquire a mass matrix

$$m^2_{xz} = C\frac{\partial^2}{\partial\Theta^x\partial\Theta^z}\Delta E(\Theta)\Big|_{\Theta=0} = -C\langle 0|[X^z,[X^x,\mathcal{H}'(0)]]|0\rangle \geq 0\,, \tag{4.24}$$

where $C$ is a constant that can be calculated using a technique, that requires some knowledge on strong interaction, like the current algebra method (see Dashen [68] or the review paper by Pagels [69] and the references therein) where it was found that $C = \frac{1}{F_\pi^2}$. The Goldstone bosons that become massive, due to*approximate* symmetry (spontaneously broken from $G$ to $H$ but explicitly broken by $\mathcal{H}'$), are referred to as *pseudo Goldstone bosons* (sometimes called *pGB*, *pNGB*, or just "*pseudos*").

Instead of looking for a vacuum $|0\rangle$ that corresponds to minimal energy by the method prescribed above, there is another way to view the situation on hand. Let us say that we are given a fixed vacuum $|0\rangle$. Then the problem is to find a "$G$-rotated" perturbation $\mathcal{H}'(\Theta) = U(\Theta)\mathcal{H}'(0)U^\dagger(\Theta)$ such that the $H$-invariant vacuum has minimal energy. The conditions that $\mathcal{H}'(\Theta)$ must satisfy are then (4.22), and (4.23). In practice, it may turn out that the masses of the Goldstone bosons given by (4.24) are, or has a potential to be, negative. This signifies that we had picked up either the wrong vacuum, or the wrong $\mathcal{H}'(\Theta)$, or we have to find some conditions (on several parameters) that the particular vacuum we have chosen is the true one. A $G$ transformation on the perturbation can also brings us, indirectly, the "preferred" vacuum.

#### 4.1.2.1 Global Symmetry Explicitly Broken by Electroweak Interaction

Now, recall that gauge interactions can also *explicitly* break the global symmetry even though the global symmetry is a result of *another* gauge symmetry. As an example, consider a system of *massless colourless technifermions* which behave likes $u$ and $d$ quarks[3]. Chiral symmetry then breaks dynamically due to technicolour interaction. The reason technifermions are chosen instead of the

[3] Sometimes we will refer to the technifermions just as fermions when it is clear from the context.

quarks is because we can assume that the influences of the weak interaction on the dynamics of the strong interaction acting on colourless technifermions are moderately suppressed so that the pattern of symmetry breaking of the global (chiral) flavour symmetry is not altered. This is because the technicolour interactions are assumed to have energy scale much greater than the usual colour ones. We will refer to the technifermions as $\mathcal{U}$, $\mathcal{D}$, $\mathcal{C}$, etc.... Global symmetry is explicitly broken by the Yukawa interaction as well if the theory admits the existence of fundamental scalars.

The gauge fields are introduced by gauging a subgroup of $G$, called $G_W$; i.e., promoting the $G_W$ to a local one. Though we are interested in cases where $SU(2) \times U(1) \subset G_W$, many other possibilities are available. As indicated before, it is not necessary that $G_W$ be the same as the global group $G$. So there may be an intersection between $H$ and $G_W$. The so called the alignment of $H$ relative to $G_W$, is not fixed but is determined dynamically such that the energy of the specific vacuum is minimised. Another way of mentioning the vacuum alignment problem is that "what subgroup $H_W$ of $G_W$ is left unbroken by strong dynamics?"

Particles in the spectrum are classified as follows. The gauge fields corresponding to the overlapped section (of $G_W$ and $H$) will be massless because there are no Goldstone bosons to feed to them. Moreover, the Goldstone bosons corresponding to the intersection between the broken generators and the gauge generators will be exactly massless and will eventually get eaten by interacting with the corresponding gauge fields.

Still, some of the massless Goldstone bosons will make it to the physical spectrum while being massless if they are protected by another (non-gauged) subgroup of $G$ having all of its generators commute with $G_W$. So, by construction, this subgroup cannot be defined beforehand and may share its generators with those corresponding to the unbroken symmetry. Electroweak interaction can be responsible for generating such the group because it introduces the doublet-singlet structure to the group which may or may not result in the additional group depending on whether it treats each family differently. The largest possible (maximal) subgroup of this kind will be referred to as $S$. Consequently, the appearance of (weak) gauge interactions *explicitly* break $G$ to $G_W \times S$. Finally, there remain the Goldstone bosons whose masses are not protected by either $G_W$ or $S$. These Goldstones will interact with the gauge field and become massive. They are the pseudo-Goldstone bosons.

Now we will let some of the $G$ generators coupled with the gauge fields belonging to the group $G_W$ of weak interactions. In this section, it is helpful to use different indices for $G$-currents and $G_W$-currents. The latter will be accompanied by first few Greek indices $\alpha$, $\beta$, .... With the use of the suitable representation, the gauge couplings can be recasted into the form

$$\mathcal{L}' = \sum_{\alpha} g^{\alpha} A_{\mu}^{\alpha} J^{\alpha\mu}, . \tag{4.25}$$

Though the $G_W$-currents do not "know" anything about the global symmetry and its breaking, $G_W$ must be a subgroup of $G$. Hence, the $G_W$-current must be a linear combination of some of the $G$-currents

$$g^{\alpha} J_{\mu}^{\alpha} = \sum_{a} g^{\alpha a} J_{\mu}^{a} \tag{4.26}$$

corresponding to the mixing between the unbroken and broken $G$ generators $Y^i$'s and $X^z$'s respectively. The $g^{\alpha a}$ is provided to manipulate the linear combinations of the couplings. Expressing things in terms of the $G$-currents will bring in some convenience since, as we shall see, we have to consider a $G$-transformation of an $\mathcal{H}'$ constructed from the $G_W$-currents $J^{\alpha}$. Notice that, under a global $G$-transformation $J^a$ transforms as an adjoint representation $R(\Theta)$ of $G$

$$g^{\alpha} U^{\dagger} J^{\alpha\mu} U = \sum_{a} g^{\alpha a} U^{\dagger} J_{\mu}^{a} U = \sum_{ac} g^{\alpha a} R^{ac}(\Theta) J_{\mu}^{c} . \tag{4.27}$$

In addition, we can decompose the generators of $G_W$ into the unbroken and the broken $G$-generators, denoted collectively by,

$$\Lambda = \Lambda_Y + \Lambda_X , \tag{4.28}$$

which, obviously, depends on a particular choice of vacuum. This also means $J_{\mu}^{\alpha}$ can be projected into the subspace determined by the $Y^i$ and $X^z$ such that the resulting $J$ corresponding to the unbroken and broken subgroups do not mix:

$$\begin{aligned} g^{\alpha} J_{\mu}^{\alpha} &= g^{\alpha} U^{\dagger} J_{\mu}^{\alpha} U\Big|_{Y} + g^{\alpha} U^{\dagger} J_{\mu}^{\alpha} U\Big|_{X} = \sum_{ac} g^{\alpha c} R^{ac}(\Theta) J_{\mu}^{c}\Big|_{Y} + \sum_{ac} g^{\alpha c} R^{ac}(\Theta) J_{\mu}^{c}\Big|_{X} \\ &\equiv {}^{U}J_{\mu}^{\alpha_Y} + {}^{U}J_{\mu}^{\alpha X} . \end{aligned} \tag{4.29}$$

Observe that this way of projecting into $Y^i$ and $X^z$ clearly depends on the relative orientation of $H$ and $G_W$. We also define the ${}^{U}J_{\mu}^{\alpha}$'s in such a way that they include the gauge couplings corresponding to the 2 parts of the current given in (4.29). For convenience, $\Lambda$'s are then defined so that they contain the

$g$'s. These projections are useful because we know how the $G$ generators act on a vacuum.

Now we will arrange perturbations $\mathcal{H}'$ due to weak gauge fields in more, group theoretically, useful forms so we can get better versions of the Dashen's conditions. The exchanges of the gauge fields contribute, to leading order in $g$,

$$\mathcal{H}'(0) = -\frac{1}{2}\sum_{\alpha\beta} g^{\alpha} g^{\beta} \int \mathrm{d}^4x D^{\mu\nu}_{0\alpha\beta}(x) T\left\{J^{\alpha}_{\mu}(x) J^{\beta}_{\nu}(0)\right\} \tag{4.30}$$

where $D^{\mu\nu}_{0\alpha\beta}(x)$ is the free gauge boson propagator

$$D^{\alpha\beta}_{0\mu\nu}(x) = \mathrm{i}\langle 0|T A^{\alpha}_{\mu}(x) A^{\beta}_{\nu}(0)|0\rangle \equiv \delta^{\alpha\beta} D_{0\mu\nu}\,. \tag{4.31}$$

This leads to the energy shift of a vacuum state $|\Omega\rangle$

$$\Delta E(\Theta) = -\frac{1}{2}\sum_{\alpha}(g^{\alpha})^2 \int \mathrm{d}^4x D^{\mu\nu}_{0}(x)\langle\Omega|T J^{\alpha}_{\mu}(x) J^{\alpha}_{\nu}(0)|\Omega\rangle \tag{4.32}$$

In general, the shifted energy depends on the alignment of the vacuum. Picking a specific vacuum $|0\rangle$ we find that the vacuum $|\Omega\rangle$ is related to others by an element in $G$ by $|\Omega\rangle = U(g)|0\rangle$. So,

$$\begin{aligned}(g^{\alpha})^2\langle\Omega|T J^{\alpha}_{\mu}(x) J^{\alpha}_{\nu}(0)|\Omega\rangle &= (g^{\alpha})^2\langle 0|T U^{\dagger} J^{\alpha}_{\mu}(x) U U^{\dagger} J^{\alpha}_{\nu}(0) U|0\rangle \\ &\equiv \langle 0|T\,{}^{U}\!J^{\alpha}_{\mu}(x)\,{}^{U}\!J^{\alpha}_{\nu}(0)|0\rangle\,. \end{aligned} \tag{4.33}$$

Only H-invariant quantities can contribute to the expression in (4.33) because the vacuum is invariant under transformations belonging to the subgroup $H$. To see what we have on hand, recall the assumption

$$\langle 0|T J^{Y}_{\mu}(x) J^{X}_{\nu}(0)|0\rangle = 0 \tag{4.34}$$

or

$$\mathrm{Tr} X^{z} Y^{i} = 0 \tag{4.35}$$

which brings us to

$$\mathrm{Tr}\left(Y^{i}[Y^{j}, X^{z}]\right) = \mathrm{Tr}\left([Y^{i}, Y^{j}]X^{z}\right) = \mathrm{i}f^{ijk}\mathrm{Tr}\left(Y^{k}X^{z}\right) = 0 \tag{4.36}$$

and hence $[Y, X] \sim X$. Then, the further assumption that the symmetry breaking respects parity in the sense that $PY^{i}P^{-1} = +Y^{i}$ and $PX^{z}P^{-1} = -X^{z}$, reduces the general relation

$$[X^y, X^z] = \mathrm{i} f^{yzi} Y^i + \mathrm{i} f^{yzw} X^w \tag{4.37}$$

down to

$$[X^y, X^z] = \mathrm{i} f^{yzi} Y^i \,, \tag{4.38}$$

which completely specifies the symmetric space $(G/H)$. Then it can be shown that (see Peskin [70]) the product of two unbroken currents are proportional to only one invariant, the $\delta^{ij}$,

$$\begin{aligned} \langle 0|T J^i_\mu(x) J^j_\nu(0)|0\rangle &= \delta^{ij} \langle 0|T J^Y_\mu(x) J^Y_\nu(0)|0\rangle \\ &= \mathrm{Tr}\left\{ Y^i Y^j \right\} \langle 0|T J^Y_\mu(x) J^Y_\nu(0)|0\rangle \end{aligned} \tag{4.39}$$

Here, $J^Y$ denotes any single generator corresponding to $Y^i$ and we do not sum over them. Similarly,

$$\langle 0|T J^x_\mu(x) J^z_\nu(0)|0\rangle = \mathrm{Tr}\left\{ X^x X^z \right\} \langle 0|T J^X_\mu(x) J^X_\nu(0)|0\rangle \,, \tag{4.40}$$

and

$$\langle 0|T J^Y_\mu(x) J^X_\nu(0)|0\rangle = 0 \tag{4.41}$$

allows us to write

$$\begin{aligned} \langle 0|T^U J^\alpha_\mu(x) {}^U J^\alpha_\nu(0)|0\rangle &= \langle 0|T^U J^{\alpha_Y}_\mu(x) {}^U J^{\alpha_Y}_\nu(0)|0\rangle + \langle 0|T^U J^{\alpha_X}_\mu(x) {}^U J^{\alpha_X}_\nu(0)|0\rangle \\ &= \mathrm{Tr}\left\{ (U^\dagger \Lambda^\alpha U)_Y (U^\dagger \Lambda^\alpha U)_Y \right\} \langle 0|T J^Y_\mu(x) J^Y_\nu(0)|0\rangle \\ &\quad + \mathrm{Tr}\left\{ (U^\dagger \Lambda^\alpha U)_X (U^\dagger \Lambda^\alpha U)_X \right\} \langle 0|T J^X_\mu(x) J^X_\nu(0)|0\rangle \\ &= \mathrm{Tr}\left\{ (U^\dagger \Lambda^\alpha U)_Y \left[ U^\dagger \Lambda^\alpha U \right] \right\} \langle 0|T J^Y_\mu(x) J^Y_\nu(0)|0\rangle \\ &\quad + \mathrm{Tr}\left\{ (U^\dagger \Lambda^\alpha U)_X \left[ U^\dagger \Lambda^\alpha U \right] \right\} \langle 0|T J^X_\mu(x) J^X_\nu(0)|0\rangle \\ &= \mathrm{Tr}\left\{ \left[ U^\dagger \Lambda^\alpha U \right] \left[ U^\dagger \Lambda^\alpha U \right] \right\} \langle 0|T J^Y_\mu(x) J^Y_\nu(0)|0\rangle \\ &\quad + \mathrm{Tr}\left\{ (U^\dagger \Lambda^\alpha U)_X (U^\dagger \Lambda^\alpha U)_X \right\} \\ &\qquad \langle 0|T \left\{ J^X_\mu(x) J^X_\nu(0) - J^Y_\mu(x) J^Y_\nu(0) \right\} |0\rangle \,, \end{aligned} \tag{4.42}$$

where $(U^\dagger \Lambda^\alpha U)_Y$ is a linear combination of the $G$-generators corresponding to the current ${}^U J^{\alpha Y}_\mu$. Clearly we cannot naively "undock" the $U$ from an expression such as

$$\mathrm{Tr}\left\{ (U^\dagger \Lambda^\alpha U)_X (U^\dagger \Lambda^\alpha U)_X \right\} = \mathrm{Tr}\left\{ {}^U\Lambda^\alpha_X {}^U\Lambda^\alpha_X \right\} \neq \mathrm{Tr}\left\{ \Lambda^\alpha_X \Lambda^\alpha_X \right\} \,, \tag{4.43}$$

where ${}^{U}\Lambda^{\alpha}_{X} \equiv (U^{\dagger}\Lambda U)_{X}$, because the mixing is implicitly described by (4.30). The example of the above mixing and the dependence on the vacuum alignment will be given below. Later, we will see the use of expressing the $G_W$-currents in terms of the unbroken and broken $G$-currents in the form displayed in the last line of (4.42).

It may be illustrative to see things in more details: for the case $SU(N) \times SU(N) \to SU(N)$ the $G$-currents are

$$
\begin{aligned}
J^{a}_{L\mu} &= \bar{\Psi}_{L}\gamma_{\mu}T^{a}\Psi_{L} = \bar{\Psi}\gamma_{\mu}T^{a}\frac{(1-\gamma^{5})}{2}\Psi && (4.44)\\
J^{a}_{R\mu} &= \bar{\Psi}_{R}\gamma_{\mu}T^{a}\Psi_{R} = \bar{\Psi}\gamma_{\mu}T^{a}\frac{(1+\gamma^{5})}{2}\Psi\,, && (4.45)
\end{aligned}
$$

which can be arranged into vector and axial currents in the usual way:

$$
J^{a}_{V\mu} = \bar{\Psi}\gamma_{\mu}T^{a}\Psi\,, \qquad J^{a}_{A\mu} = \bar{\Psi}\gamma_{\mu}T^{a}\gamma^{5}\Psi\,. \tag{4.46}
$$

Then the $G_W$ currents and their transformations are

$$
\begin{aligned}
g^{\alpha}J^{\alpha}_{L\mu} &= \bar{\Psi}_{L}\gamma_{\mu}g^{\alpha}\theta^{\alpha}_{L}\Psi_{L} \to \bar{\Psi}_{L}\gamma_{\mu}L^{\dagger}g^{\alpha}\theta^{\alpha}_{L}L\Psi_{L}\,,\\
g^{\alpha}J^{\alpha}_{R\mu} &= \bar{\Psi}_{R}\gamma_{\mu}g^{\alpha}\theta^{\alpha}_{R}\Psi_{R} \to \bar{\Psi}_{R}\gamma_{\mu}R^{\dagger}g^{\alpha}\theta^{\alpha}_{R}R\Psi_{R}
\end{aligned} \tag{4.47}
$$

where $\theta^{\alpha}_{L,R}$ are generators for the $G_W$ (which are actually a linear combination of the $T^{a}$'s), and the $U_L$ and $U_R$ are the $SU(N)$ matrices. In (4.33) we then expect the following terms, due to $J^{\alpha} = J^{\alpha}_{L} + J^{\alpha}_{R}$,

$$
\begin{aligned}
\langle 0|{}^{U}J^{\alpha}_{\mu}{}^{U}J^{\alpha}_{\nu}|0\rangle &\propto \mathrm{Tr}\left(L^{\dagger}\theta^{\alpha}_{L}LL^{\dagger}\theta^{\alpha}_{L}L\right) + \mathrm{Tr}\left(R^{\dagger}\theta^{\alpha}_{R}RR^{\dagger}\theta^{\alpha}_{R}R\right)\\
&\quad + \mathrm{Tr}\left(L^{\dagger}\theta^{\alpha}_{L}LR^{\dagger}\theta^{\alpha}_{R}R\right)\\
&= \mathrm{Tr}\left(\theta^{\alpha}_{L}\theta^{\alpha}_{L}\right) + \mathrm{Tr}\left(\theta^{\alpha}_{R}\theta^{\alpha}_{R}\right) + \mathrm{Tr}\left(\theta^{\alpha}_{L}U_{N}\theta^{\alpha}_{R}U^{\dagger}_{N}\right)
\end{aligned} \tag{4.48}
$$

where $U_N = LR^{\dagger}$ is an $SU(N)$ matrix, which clearly exhibits dependence on the orientation. In terms of the generators these currents, the projections are given by

$$
\begin{aligned}
\Lambda^{\alpha}_{Y} &= g^{\alpha}\sum_{i}Y^{i}\mathrm{Tr}\left\{Y^{i}\theta^{\alpha}\right\} && (4.49)\\
\Lambda^{\alpha}_{X} &= g^{\alpha}\sum_{z}X^{z}\mathrm{Tr}\left\{X^{z}\theta^{\alpha}\right\}\,. && (4.50)
\end{aligned}
$$

So the $G_W$ current can be decomposed to

$$\begin{aligned} g^\alpha J^\alpha_\mu &= g^\alpha \bar\Psi\gamma_\mu\theta^\alpha\Psi = \bar\Psi\gamma_\mu\Lambda^\alpha_Y\Psi + \bar\Psi\gamma_\mu\Lambda^\alpha_X\Psi \\ &= g^\alpha\bar\Psi\gamma_\mu\sum_i Y^i \mathrm{Tr}\left\{Y^i\theta^\alpha\right\}\Psi + g^\alpha\bar\Psi\gamma_\mu\sum_z X^z\mathrm{Tr}\left\{X^z\theta^\alpha\right\}\Psi\,. \end{aligned} \quad (4.51)$$

Now let us come back to (4.42), where we have factored out the term

$$\mathrm{Tr}\left\{U^\dagger\Lambda U U^\dagger \Lambda U\right\}\langle 0|T J^Y_\mu(x) J^Y_\nu(0)|0\rangle \quad (4.52)$$

which does not depend on the alignment $\Theta$ of the vacuum. The generators of $G$ as "seen" by the vacuum is the same no matter which direction it points to. Consequently, the energy shift of $|\Omega\rangle$ in (4.30) is

$$\begin{aligned} \Delta E(\Theta) &= E_0 + \frac{1}{2}\sum_\alpha \mathrm{Tr}\left\{{}^U\Lambda^\alpha_X\,{}^U\Lambda^\alpha_X\right\} \\ &\qquad \int \mathrm{d}^4x D_0^{\mu\nu}(x)\langle 0|T\left\{J^Y_\mu(x)J^Y_\nu(0) - J^X_\mu(x)J^X_\nu(0)\right\}|0\rangle. \end{aligned} \quad (4.53)$$

The unbroken generators of $G_W$ raise the value of $\Delta E$ while the broken generators try to lower it. It can be argued that the integral is positive (see Preskill [71], or Peskin [70]). This is quite natural since the lightest particle created by the broken current $J^X_\mu$ should be heavier than the lightest particle (massive due to explicit symmetry breaking) created by the unbroken one, the $J^Y_\mu$. The examples are the axial and the vector mesons, respectively. Therefore, to find the preferred vacuum orientation, we have to find the configuration which minimises

$$\sum_\alpha \mathrm{Tr}\left\{{}^U\Lambda^\alpha_X\,{}^U\Lambda^\alpha_X\right\}\,, \quad (4.54)$$

which is the projections of the generators (corresponding to the currents) on the subspace of possible equivalent representations of $Y^i$ or $X^z$. We may say that the vacuum prefers the direction that results in the minimal number of broken generators of $G_W$ in the projection. To see the meanings of what we have on hand, recall that the interaction between the Goldstone bosons and the gauge fields is given by

$$\langle 0|J^\alpha_\mu(0)|\pi^x(p)\rangle = \mathrm{i}p_\mu\delta^{\alpha x}F_\pi\,. \quad (4.55)$$

Observe that the arguments preceding the equations (4.39) and (4.40) leads, in the low energy limit, to

$$\begin{aligned}\langle 0|T J_\mu^\alpha(p) J_\nu^\beta(-p)|0\rangle &\approx \sum_z \langle 0|J_\mu^\alpha|\pi^z(p)\rangle \frac{\mathrm{i}}{p^2}\langle \pi^z(-p)|J_\nu^\beta|0\rangle \\ &= \sum_z \left[\mathrm{i}p_\mu F_\pi \delta^{\alpha z}\right]\frac{\mathrm{i}}{p^2}\left[-\mathrm{i}p_\mu F_\pi \delta^{\beta z}\right] \\ &= \sum_z \mathrm{i}\frac{p_\mu p_\nu}{p^2} F_\pi^2 \mathrm{Tr}\left\{\Lambda^\alpha X^z\right\}\mathrm{Tr}\left\{\Lambda^\beta X^z\right\} \\ &= \mathrm{i}\frac{p_\mu p_\nu}{p^2} F_\pi^2 \mathrm{Tr}\left\{\Lambda_X^\alpha \Lambda_X^\beta\right\}\,. \end{aligned} \tag{4.56}$$

On other vacua $|\Omega\rangle$, we find

$$\langle \Omega|T J_\mu^\alpha(p) J_\nu^\beta(-p)|\Omega\rangle \approx \mathrm{i}\frac{p_\mu p_\nu}{p^2} F_\pi^2 \mathrm{Tr}\left\{{}^U\Lambda_X^\alpha\, {}^U\Lambda_X^\beta\right\}\,, \tag{4.57}$$

which allows us to read off the mass term for the (weak) gauge fields

$$M_{\alpha\beta}^2 = F_\pi^2 \mathrm{Tr}\left\{{}^U\Lambda_X^\alpha\, {}^U\Lambda_X^\beta\right\} \propto g^\alpha g^\beta F_\pi^2 \mathrm{Tr}\left\{X^\alpha X^\beta\right\}\,, \tag{4.58}$$

which can be compared with the results found in section 4.1.1. This means

$$\Delta E(\Theta) \propto \sum_{\alpha\beta} M_{\alpha\beta}^2 = \mathrm{Tr} M^2\,. \tag{4.59}$$

The interpretation agrees with that below (4.54); namely, the $G_W$ and $H$ prefer to line up in such a way, determined by the fermion condensate alignment, that the masses of the $G_W$ gauge fields be minimal. Hence electroweak symmetry is broken as little as possible. In other words, as a crucial result, the largest possible subgroup $G_W$ (i.e., largest overlap between $G_W$ and $H$) will survive.

At this stage, we can convert (4.57) into a more useful form. Using the vacua defined by (4.20), the Dashen's condition (4.22) becomes

$$\begin{aligned} 0 = \frac{\partial}{\partial \Theta^z}\sum_\alpha \mathrm{Tr}\left\{(U^\dagger \Lambda U)_X^2\right\}\Big|_{U=1} &= 2\mathrm{i}\sum_\alpha \mathrm{Tr}\left\{([X^z, \Lambda^\alpha])_X \Lambda_X^\alpha\right\} \\ &= 2\mathrm{i}\sum_\alpha \mathrm{Tr}\left\{X^z[\Lambda_Y^\alpha, \Lambda_X^\alpha]\right\} \end{aligned} \tag{4.60}$$

where the orthogonality $\mathrm{Tr}\{XY\} = 0$ and $[X, X] = \mathrm{i}Y$ were used in the last step. This tells us that the vacuum is stationary when $\mathrm{Tr}\left\{X^z[\Lambda_Y^\alpha, \Lambda_X^\alpha]\right\} = 0$ for all gauge generators $\Lambda^\alpha$. The result is useful since it is entirely group theoretic;

i.e, we do not have to mess with complicated strong dynamics. Next, consider

$$
\begin{aligned}
&\frac{\partial^2}{\partial\Theta^x\partial\Theta^z}\sum_\alpha \mathrm{Tr}\left\{(U^\dagger\Lambda^\alpha U)_X^2\right\}\Big|_{U=1} \\
&= -2\sum_\alpha\left(\mathrm{Tr}\left\{[X^x,[X^z,\Lambda_X^\alpha]]\,\Lambda_X^\alpha\right\}+\mathrm{Tr}\left\{([X^x,\Lambda^\alpha])_X([X^z,\Lambda^\alpha])_X\right\}\right) \\
&= -2\sum_\alpha\left(\mathrm{Tr}\left\{[X^x,[X^z,\Lambda_X^\alpha]]\,\Lambda_X^\alpha\right\}+\mathrm{Tr}\left\{[X^x,\Lambda_Y^\alpha][X^z,\Lambda_Y^\alpha]\right\}\right) \\
&= 2\sum_\alpha\left(\mathrm{Tr}\left\{[\Lambda_Y^\alpha,[\Lambda_Y^\alpha,X^x]]\,X^z\right\}-\mathrm{Tr}\left\{[\Lambda_X^\alpha,[\Lambda_X^\alpha,X^x]]\,X^z\right\}\right) \qquad (4.61)
\end{aligned}
$$

Then the Goldstone mass matrix from the second Dashen's condition (4.24) becomes

$$
\begin{aligned}
m_{xz}^2 \;=\; &\frac{1}{F_\pi^2}\sum_\alpha\left(\mathrm{Tr}\left\{[\Lambda_Y^\alpha,[\Lambda_Y^\alpha,X^x]]\,X^z\right\}-\mathrm{Tr}\left\{[\Lambda_X^\alpha,[\Lambda_X^\alpha,X^x]]\,X^z\right\}\right) \\
&\int \mathrm{d}^4x D_0^{\mu\nu}(x)\langle 0|T\left\{J_\mu^Y(x)J_\nu^Y(0)-J_\mu^X(x)J_\nu^X(0)\right\}|0\rangle \qquad (4.62)
\end{aligned}
$$

which can be written in a less precise form as a combination of the masses due to the broken and unbroken gauge generators

$$
m_{\mathrm{NGB}}^2 \propto m_{\Lambda_Y}^2 - m_{\Lambda_X}^2 \,. \qquad (4.63)
$$

Observe that the latter tends to destabilise the vacuum. This mass term will control the resulting subgroup of the gauge group.

### 4.1.3 Examples of a Symmetry Broken Explicitly by Weak Gauge Interactions

In this section we will study the applications of (4.62) to the symmetry breaking[4] $SU(2N)\times SU(2N)\to SU(2N)$. We consider the case where the electromagnetic or weak gauge groups are included in the $SU(2N)\times SU(2N)$ of $G$.

Consider a system of $2N$ left-handed and $2N$ right-handed massless fermions transforming under the (same) *complex* representation of a *strong* interaction gauge group. The system will have a global flavour symmetry namely $SU(2N)_L\times SU(2N)_R$. Then the condensate

$$
\langle 0|\bar\psi_L^i\psi_R^j|0\rangle = \Lambda^3\delta^{ij}\,, \qquad (4.64)
$$

[4]The number $N$ may be regarded as the number of the left-handed doublets when weak interaction is taken into account.

with $i,j$ being flavour indices and[5] $\Lambda \propto F_\pi$, breaks the chiral group down to $SU(2)_{L+R}$ (see section 4.1.1). At this stage, we have equivalent sets of vacua that are related by a $SU(2N)_L \times SU(2N)_R$ (unitary) transformation $L\langle\bar{\psi}^i_L\psi^j_R\rangle R^\dagger = \delta^{ij}\Lambda^3 LR^\dagger$ where $LR^\dagger$ is a unitary unimodular matrix which can be parametrised by the pion fields (the Goldstones) $\pi^a(x)$; i.e., $\exp\left\{\frac{\pi^a(x)\tau^a}{F_\pi}\right\}$.

#### 4.1.3.1 Masses of the Pions Due To Quark Masses and Electromagnetism

Now let us take the strong interaction to be the usual colour interaction for a moment and regard the $\mathcal{U}$, $\mathcal{D}$ as the usual quarks $u$, $d$. Let the condensate that breaks the chiral symmetry and keeps vector isospin be

$$\langle\bar{u}_L u_R\rangle = \langle\bar{u}_L u_R\rangle \neq 0\,. \tag{4.65}$$

We will consider the coupling of the Goldstone bosons produced above with the electromagnetic field. The $T^3$ generator of the global symmetry (isospin) is coupled with the electromagnetic field via $Q = T^3 + Y/2$ which has a preference in the $\tau^3$ direction, and therefore explicitly breaks the chiral symmetry. In the simplest case where $N = 1$ we know that these Goldstones are the pions. The perturbation from electromagnetic interaction is

$$\mathcal{H}'_\gamma(0) = -\frac{1}{2}e^2\int \mathrm{d}^4x D_0^{\mu\nu}(x) T\left\{J_\mu(x)J_\nu(0)\right\} \tag{4.66}$$

where $J^\mu \equiv \bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d$ is the electromagnetic current. Since the pattern of symmetry breaking is $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$ (in our current notation we have $Y^i = T^i = \tau^i/2$ as generators of $H$), we see that $\Lambda^\alpha_Y = eT^3$. In addition, the unbroken isospin leads to $\Lambda^\alpha_X = 0$ hence the Dashen's stationary condition is satisfied. So the estimation of the pseudo-Goldstone boson is easy in this case: to lowest order, we expect that the contribution from one photon exchange does not give any mass to the neutral pion. Since the $X^z_A$ (chiral) charges are electrically neutral and hence commute with $J^\mu_{em}$,

$$(m^2_{\pi^0})^\gamma = 0\,, \tag{4.67}$$

while the charged ones receive

$$(m^2_{\pi^\pm})^\gamma = e^2 M^2_\gamma\,. \tag{4.68}$$

[5] $F_\pi$ is either of order MeV or GeV, depending on the interactions being considered (colour or technicolour).

The strong-dynamics part is embedded in

$$M_\gamma^2 = \frac{1}{F_\pi^2}\int \mathrm{d}^4x D_0^{\mu\nu}(x)\langle 0|T\left\{J_{V\mu}(x)J_{V\nu}(0) - J_{A\mu}(x)J_{A\nu}(0)\right\}|0\rangle. \qquad (4.69)$$

where the unbroken and broken currents are just the vector and axial vector currents, respectively. As expected, the correction is proportional to the parameter corresponding to the term that breaks the symmetry ($m_\pi^2 \sim e^2 M_\gamma^2$). Notice that we have used the superscript $\gamma$ in (4.68) to note that this mass correction comes from photon exchange only. Explicit symmetry breaking perturbation from the quark masses, which affects both charged and neutral pions, is neglected. We can even cheat a little by taking the $M_\gamma$ measured from experiments to determine the effects due to electromagnetic interaction. In order to do so we have to bring back the contribution from non-zero quark masses which also explicitly breaks the chiral symmetry.

Next we will workout the contribution to pion masses according to non-zero $m_u, m_d, \ldots$. This is sort of unnecessary if we insist on finding out the value of $M_\gamma^2$ from experiments. Still this example may give some idea of vacuum alignment. Let us note that the quark mass perturbation can be written as

$$\begin{aligned}\mathcal{H}'_q &= m_u\bar{u}u + m_d\bar{d}d = \frac{1}{2}(m_u+m_d)(\bar{u}u+\bar{d}d) + \frac{1}{2}(m_u-m_d)(\bar{u}u-\bar{d}d)\\ &= \frac{1}{2}(m_u+m_d)\bar{\Psi}\Psi + \frac{1}{2}(m_u-m_d)\bar{\Psi}\tau^3\Psi\,, \end{aligned} \qquad (4.70)$$

which means this explicit breaking term also prefers the breaking of axial isospin symmetry while leaving the (vector) isospin there: the term containing $\bar{\Psi}\Psi$ is invariant under the isospin transformation but not under the chiral one. The effect of the explicit isospin breaking $\tau^3$ operator (recall the coupling with electromagnetic field) from the second part is suppressed by the small value of $m_u - m_d$ (comparing to $m_u + m_d$). The point is that, without the mass terms $m_u$ and $m_d$ which keeps the vacuum in the $\bar{\Psi}\Psi$ direction, other possibilities related by an $SU(2)\times SU(2)$ transformation lying on a linear combination between

$$\Phi_V^0 \sim \bar{\Psi}\Psi\,, \qquad \text{and} \quad \Phi_V^a \sim \bar{\Psi}\gamma^5\tau^a\Psi \qquad (4.71)$$

are equally possible. It was the mass terms (4.70) that align the vacuum in this example. Gauge and Yukawa interactions can also do similar jobs. For further reference, we note that in general the $\mathcal{H}'_q$ can be written as a linear combination of the commuting generators of the group in question (including an identity);

i.e.,

$$\mathcal{H}'_q = c_0 u^0 + c_3 u^3 \tag{4.72}$$

where $u^0$ is proportional to a unit $2 \times 2$ matrix and $u^a = \bar{\Psi}\tau^a\Psi$. An unbroken isospin clearly implies $c_3 = 0$ so that $m_u = m_d = m$.

To workout the pion masses due to non-zero quark masses using (4.24), we need to evaluate

$$\int\int \mathrm{d}^3x \mathrm{d}^3y \langle 0| \left[ J^z_{A0}(\mathrm{x}), \left[ J^x_{A0}(\mathbf{0}), m\bar{\Psi}(y)\Psi(y) \right] \right] |0\rangle \tag{4.73}$$

where $J^x_{A0} = \bar{\Psi}(x)\gamma^0\gamma^5\frac{\tau^x}{2}\Psi(x)$. First we recall that

$$\{\Psi_\alpha(\mathrm{x}), \Psi_\beta(\mathrm{y})\} = 0 = \left\{\Psi^\dagger_\alpha(\mathrm{x}), \Psi^\dagger_\beta(\mathrm{y})\right\} \tag{4.74}$$

$$\left\{\Psi^\dagger_\alpha(\mathrm{x}), \Psi_\beta(\mathrm{y})\right\} = \delta_{\alpha\beta}\delta^3(\mathrm{x} - \mathrm{y}) . \tag{4.75}$$

Then

$$\begin{aligned}
&\left[\bar{\Psi}(0)\frac{\tau^x}{2}\gamma^0\gamma^5\Psi(0), \bar{\Psi}(\mathrm{y})\Psi(\mathrm{y})\right] \\
&= \left[\Psi(0)^\dagger_\alpha\Psi(0)_\beta, \Psi(y)_\gamma\Psi(y)_\delta\right] (\tau^x\gamma^5/2)_{\alpha\beta}\gamma^0_{\gamma\delta} \\
&= \Big(\Psi^\dagger_\alpha\{\Psi_\beta, \Psi^\dagger_\gamma\}\Psi_\delta - \Psi^\dagger_\gamma\{\Psi^\dagger_\alpha, \Psi_\delta\}\Psi_\beta \\
&\quad + \{\Psi^\dagger_\alpha, \Psi^\dagger_\gamma\}\Psi_\delta\Psi_\beta - \Psi^\dagger_\alpha\Psi^\dagger_\gamma\{\Psi_\delta, \Psi_\beta\}\Big)(\frac{\tau^x}{2}\gamma^5)_{\alpha\beta}\gamma^0_{\gamma\delta} \\
&= \left(\Psi^\dagger_\alpha\Psi^\dagger_\gamma\Psi_\delta\Psi_\beta \times 0 + \left[\Psi^\dagger_\alpha\Psi_\beta, \Psi^\dagger_\gamma\Psi_\delta\right]\right)(\frac{\tau^x}{2}\gamma^5)_{\alpha\beta}\gamma^0_{\gamma\delta} \\
&= \left(\Psi^\dagger_\alpha\Psi_\delta(\frac{\tau^x}{2}\gamma^5\gamma^0)_{\alpha\delta} - \Psi^\dagger_\gamma\Psi_\beta(\frac{\gamma^0\tau^x}{2}\gamma^5)_{\gamma\beta}\right)\delta^3(\mathrm{y}) \\
&= 2\left(\Psi^\dagger_\alpha\Psi_\delta(\frac{\tau^x}{2}\gamma^5\gamma^0)_{\alpha\delta}\right)\delta^3(\mathrm{y}) ,
\end{aligned} \tag{4.76}$$

which leads to

$$\int \mathrm{d}^3y \left[J^x_{A0}(\mathbf{0}), m\bar{\Psi}(y)\Psi(y)\right] = m\Psi^\dagger_\alpha\Psi_\delta(\frac{\tau^x}{2}\gamma^5\gamma^0)_{\alpha\delta} . \tag{4.77}$$

Next,

$$\begin{aligned}
\int \mathrm{d}^3x \left[J^z_{A0}(\mathrm{x}), m\Psi^\dagger_\alpha\Psi_\delta(\frac{\tau^x}{2}\gamma^5\gamma^0)_{\alpha\delta}\right] \\
&= \int \mathrm{d}^3x \left[\bar{\Psi}(\mathrm{x})\gamma^0\gamma^5\frac{\tau^z}{2}\Psi(\mathrm{x}), m\Psi^\dagger_\alpha\Psi_\delta(\frac{\tau^x}{2}\gamma^5\gamma^0)_{\alpha\delta}\right] \\
&= \int \mathrm{d}^3x \left[\Psi^\dagger_\gamma\Psi_\beta, m\Psi^\dagger_\alpha\Psi_\delta\right](\gamma^0\gamma^5\frac{\tau^z}{2})_{\gamma\beta}(\frac{\tau^x}{2}\gamma^5\gamma^0)_{\alpha\delta} \\
&= m\int \mathrm{d}^3x\Psi^\dagger_\alpha\Psi_\delta(\frac{\tau^z}{2}\gamma^5\gamma^5\gamma^0\tau^x - \gamma^5\gamma^0\tau^x\frac{\tau^z}{2}\gamma^5)_{\alpha\delta}\delta^3(\mathrm{x}) \\
&= m\Psi^\dagger_\alpha\Psi_\delta(\gamma^0\frac{1}{2}\{\tau^z,\tau^x\})_{\alpha\delta} \\
&= m\bar{\Psi}\Psi\delta^{zx}\,. \qquad (4.78)
\end{aligned}$$

which brings us to

$$\langle 0|\left[X^z_A,[X^x_A,\mathcal{H}']\right]|0\rangle = m\langle 0|\bar{u}u + \bar{d}d|0\rangle\delta^{zx}\,. \qquad (4.79)$$

Therefore the masses of the pions due to non-zero quark masses are

$$m^2_\pi = -\frac{1}{F^2_\pi}m\langle 0|\bar{u}u + \bar{d}d|0\rangle = -\frac{2}{F^2_\pi}m\langle 0|\bar{u}u|0\rangle \equiv Cm\,, \qquad (4.80)$$

where $C = -\frac{2}{F^2_\pi}\langle 0|\bar{u}u|0\rangle$. It is important to emphasise that all the pions, charged or neutral, received the same amount of masses.

Finally, the mass of the charged pion is

$$m^2_{\pi^+} = m^2_\pi + (m^2_{\pi^+})^\gamma = m^2_\pi + e^2M^2_\gamma\,. \qquad (4.81)$$

Since the corrections due to the two explicit breaking sources are of the same order, we find

$$m_{\pi^+} - m_{\pi^0} \approx \frac{(m^2_{\pi^+})^\gamma}{2m_{\pi^0}}\,. \qquad (4.82)$$

Recap: under the influence of electromagnetism, the charged condensates like

$$\langle\bar{u}_L d_R\rangle = \langle\bar{d}_L u_L\rangle \neq 0\,, \qquad (4.83)$$

which were "equivalent" to the neutral condensates (4.65) when the electromagnetism was not introduced, will now repel other condensates. This means they tend to stay further from one another (comparing to the case when electromagnetism is absent) and are more difficult to produce. Hence they cannot be the true vacuum. The true vacuum will aligned itself to the neutral condensate direction. We can also say that the electromagnetic interaction (i.e., weak gauge interaction) tries to make the condensate neutral so that the vacuum energy is lowest, and the binding is highest. Consequently electromagnetic symmetry is not broken by the condensate. In other words, we started with the correct vacuum (4.65). In the next section we shall see that when this is not the case, electromagnetic gauge symmetry can be broken and photons can be massive.

#### 4.1.3.2 Explicit Symmetry Breaking by Electroweak Interaction

Next, by letting the Goldstone bosons of the spontaneous symmetry breaking coupled with the electroweak gauge fields, we find, as in the section 4.1.1, that this introduces left-handed doublet and right-handed singlet structures. Let $q$ be the mean electric charge of the *doublet*. We find the weak coupling for the case $N = 1$ is

$$\mathcal{L}' = g \sum_{\alpha} \bar{\psi}_L \gamma^\mu T^\alpha \psi_L W^\alpha_\mu + g' \left( \bar{\psi}_R \gamma^\mu T^3 \psi_R + q \bar{\psi} \gamma^\mu \psi \right) B_\mu \, . \tag{4.84}$$

with the corresponding generators written as

$$\theta^\alpha \ni \left( \begin{array}{c:c} T^\alpha & \\ \hdashline & 0 \end{array} \right) , \quad \left( \begin{array}{c:c} q & \\ \hdashline & q + T^3 \end{array} \right) , \tag{4.85}$$

based on a $(\psi_L, \psi_R)$ basis[6], where each box is a $2 \times 2$ matrix. In this case, we still stick with the complex representation of the technifermions and hence the usual pattern of symmetry breaking, $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$, can be studied. The "original" vacuum condensate $\langle 0 | \bar{\psi}^i_L \psi^j_R | 0 \rangle = \Lambda^3 \delta^{ij}$, which can be represented by

$$\Lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.86}$$

based on a row $(\bar{\mathcal{U}}_L \, \bar{\mathcal{D}}_L)$ and a column $(\mathcal{U}_R \, \mathcal{D}_R)$. With this condensate, broken and unbroken generators of $G$ can be defined. Then the $G_W$ generators in (4.85) can be easily "partitioned" into unbroken and broken parts of $G$ (vector isospin

[6] We use different notations from Peskin [70] who uses $(\psi_L, \psi^c_L)$.

and axial isospin)

$$\Lambda^\alpha = \left[\frac{g}{2}\left(\begin{array}{c|c} T^\alpha & \\ \hline & T^\alpha \end{array}\right) + \frac{g'}{2}\left(\begin{array}{c|c} 2q+T^3 & \\ \hline & 2q+T^3 \end{array}\right)\right] + \left[-\frac{g}{2}\left(\begin{array}{c|c} -T^\alpha & \\ \hline & T^\alpha \end{array}\right) + \frac{g'}{2}\left(\begin{array}{c|c} -T^3 & \\ \hline & T^3 \end{array}\right)\right], \quad (4.87)$$

respectively. Observe that (4.86) can be recovered from a rotated one $(L\bar{\psi}^i_L \psi^j_R R^\dagger)$; for example, in the basis of (4.86),

$$\Lambda^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.88)$$

(which comes in the wrong, electrically charged, pairs $\bar{\mathcal{U}}_L \mathcal{D}_R = \bar{\mathcal{D}}_L \mathcal{U}_R = \Lambda^3$) by using a transformation of the weak gauge sector. This means there is no alignment problem as we can always "choose" a vacuum condensate that leads to phenomenological acceptable outcomes; namely, the one that is electrically neutral. As all the vacua are equivalent under the electroweak symmetry $SU(2)\times U(1)$, there is no chance for this group to evade breaking. So the three Goldstone bosons from the broken axial isospin symmetry, though remain exactly massless with the protection of the $SU(2)_L$ weak gauge symmetry, were all eaten up by the weak gauge bosons. The masses of these gauge bosons can be evaluated and are equal to those found in section 4.1.

More interesting cases are those having $N > 1$. Let us stick with $N = 2$, by introducing another *similar* copy of the $\mathcal{UD}$ doublet with the same charge assignments. Most of the arguments used here are fairly heuristic, which are sufficient to provide us the rudiments of a model that will eventually become the Little Higgs. As usual, the pattern of global symmetry breaking is $SU(4)_L \times SU(4)_R \to SU(4)_{L+R}$. One of the direction of the condensate that leads to this breaking has the usual diagonal form

$$\Lambda^3 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad (4.89)$$

based on a row $(\bar{\mathcal{U}}_L \bar{\mathcal{D}}_L \bar{\mathcal{C}}_L \bar{\mathcal{S}}_L)$ and a column $(\mathcal{U}_R \mathcal{D}_R \mathcal{C}_R \mathcal{S}_R)$. The broken and unbroken generators of $G$ can now be properly defined. Then let us write the "fermion multiplet" to be acted on by weak interaction as

$$\Psi = (\mathcal{U}_L,\, \mathcal{D}_L,\, \mathcal{C}_L,\, \mathcal{S}_L, \mathcal{U}_R,\, \mathcal{D}_R,\, \mathcal{C}_R,\, \mathcal{S}_R)\,. \tag{4.90}$$

By using a $2 \times 2$ matrix that acts on the space of the two doublets (the family space) we see that the gauge couplings, written in terms of the $8 \times 8$ (matrix) generators, are

$$\mathcal{L}' = g \sum_a \bar{\Psi}\gamma^\mu \left(\begin{array}{c:c} T^\alpha \otimes \mathbb{1} & \\ \hdashline & 0 \end{array}\right) \Psi W_\mu^\alpha + g'\bar{\Psi}\gamma^\mu \left(\begin{array}{c:c} q & \\ \hdashline & q + T^3 \otimes \mathbb{1} \end{array}\right) \Psi B_\mu\,. \tag{4.91}$$

The $\mathbb{1}$ in $T^\alpha \otimes \mathbb{1}$ acts on the space between the two doublets. In this case we see the additional symmetry group $S$. Electroweak interaction leaves us with two copies of the left-handed doublets $(\mathcal{U}_L,\, \mathcal{D}_L)$ and $(\mathcal{U}_R,\, \mathcal{D}_R)$, as well as two similar up-type fermions $\mathcal{U}_R$, $\mathcal{C}_R$ and two similar down-type fermions $\mathcal{D}_R$, $\mathcal{S}_R$. Since the members of the three set come with equal $U(1)$ charge assignments, we then have the additional global symmetry,

$$S = SU(2)_L \times SU(2)_R \times SU(2)_R\,, \tag{4.92}$$

where the $SU(2)_{L+R}$ remains unbroken when the spontaneous symmetry breaking of $G = SU(4)_L \times SU(4)_R$ due to the condensate occurs. The breaking of the $S$ section alone produces $2 \times 3 = 6$ neutral Goldstone bosons (since the global symmetry $S$ "links" members with similar charges). Consequently, out of 15 Goldstone bosons produced, 3 representing the generators $T^a \otimes \mathbb{1}$ will be eaten by the electroweak gauge fields. The other 6 of them will survive massless to a physical spectrum. The remaining 6 Goldstone bosons may receive masses by interacting with the gauge bosons.

To work out the Goldstone bosons' masses, we seek for terms that are not invariant under the alignment of vacuum in (??). We note, similar to (4.87), that

$$\begin{aligned} \Lambda^\alpha \;=\; & \left[\frac{g}{2}\left(\begin{array}{c:c} T^\alpha \otimes \mathbb{1} & \\ \hdashline & T^\alpha \otimes \mathbb{1} \end{array}\right) + \frac{g'}{2}\left(\begin{array}{c:c} 2q + T^3 \otimes \mathbb{1} & \\ \hdashline & 2q + T^3 \otimes \mathbb{1} \end{array}\right)\right] \\ & + \left[-\frac{g}{2}\left(\begin{array}{c:c} -T^\alpha \otimes \mathbb{1} & \\ \hdashline & T^\alpha \otimes \mathbb{1} \end{array}\right) + \frac{g'}{2}\left(\begin{array}{c:c} -T^3 \otimes \mathbb{1} & \\ \hdashline & T^3 \otimes \mathbb{1} \end{array}\right)\right], \end{aligned} \tag{4.93}$$

where the second term corresponds to the broken generator. An inspection of the form of the above generators reveals that the $SU(2)$ couplings do not contribute to the Goldstone boson's mass (notice the signs of the upper left and the lower right blocks and see the Dashen's formula (4.62)). The relations in (4.48), saying that only the mixing $\langle J_L J_R \rangle$ contribute to the $\Theta$-dependent part of the potential, also confirm our inspection.

The unpleasant results of having Goldstone bosons in the physical spectrum can be avoided by fixing the source of the problem; namely, the symmetry that links between the family. One of the possible ways to do so is to associate the different $U(1)$ charges of the electroweak gauge group as follows. For

$$\begin{pmatrix} \mathcal{U}_L \\ \mathcal{D}_L \end{pmatrix}_L \qquad Y = \Delta \tag{4.94}$$

and for

$$\begin{pmatrix} \mathcal{C}_L \\ \mathcal{S}_L \end{pmatrix}_L \qquad Y = -\Delta\,, \tag{4.95}$$

as well as

$$\begin{aligned} Y(\mathcal{U}_R) &= \Delta + \frac{1}{2} \qquad & Y(\mathcal{D}_R) &= \Delta - \frac{1}{2} \\ Y(\mathcal{C}_R) &= -\Delta + \frac{1}{2} \qquad & Y(\mathcal{S}_R) &= -\Delta - \frac{1}{2} \end{aligned} \tag{4.96}$$

which completely prevent the existence of additional group $S$. The $U(1)$ exchanges are now expected to contribute to the vacuum energy. On the basis (4.90) the $U(1)$ interaction becomes

$$\begin{aligned} & g'\bar{\Psi}\gamma^{\mu}\left[\left(\begin{array}{c:c} 0 & \\ \hdashline & T^3 \otimes \mathbb{1} \end{array}\right) + \Delta \left(\begin{array}{c:c} \mathbb{1} \otimes T^3 & \\ \hdashline & \mathbb{1} \otimes T^3 \end{array}\right)\right] \Psi B_{\mu} \\ = \; & \frac{g'}{2}\bar{\Psi}\gamma^{\mu} \left(\begin{array}{cccc:cccc} \Delta &&&&&&& \\ & \Delta &&&&&& \\ && -\Delta &&&&& \\ &&& -\Delta &&&& \\ \hdashline &&&& \Delta + 1 &&& \\ &&&&& \Delta - 1 && \\ &&&&&& -\Delta + 1 & \\ &&&&&&& -\Delta - 1 \end{array}\right) \Psi B_{\mu} \end{aligned} \tag{4.97}$$

With the usual "diagonal" vacuum condensate (4.89), we can determine the unbroken and broken generators.

Next, we will quote the results found by Preskill [71] and Peskin [70] that: in addition to the 7 diagonal (in the space of the two doublets) broken generators consisting of

$$T^a \otimes \mathbb{1},\ \mathbb{1} \otimes T^3 \quad \text{and} \quad T^a \otimes T^3\,, \tag{4.98}$$

there are 8 of the broken generators coming out in the forms that allow mixing between quarks of different families like $\bar{U}C$ which appearing in the form

$$\left(\begin{array}{cc:cc} & F & & \\ F^* & & & \\ \hdashline & & & -F \\ & & -F^* & \end{array}\right) \tag{4.99}$$

where $F$ are $2 \times 2$ matrices having 1 *or* i in one of their elements (while other elements are zero). For example, the $\bar{U}C$ mixing comes from

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ .$$

Any vacuum rotated by $SU(4)_L \times SU(4)_R$ transformations should be equivalent when $G_W$ interactions are absent. However, with $G_W$ interactions turned on, the set of degenerate vacua

$$L\bar{\psi}^i_L \psi^j_R R^\dagger = \Lambda^3 L \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} R^\dagger = \Lambda^3 L R^\dagger = \Lambda^3 \Sigma \tag{4.100}$$

are not all equivalent but it was found that there are only two configurations of the condensate that are the stationary points; namely the (4.89) and

$$\Sigma' = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}\ . \tag{4.101}$$

This latter vacuum condensate is invariant under the different $SU(4)$ transformations. To see which one of them will be the minimum of the true vacuum we have to find the mass of the pseudo-Goldstone bosons. A quick estimate clearly yields,

$$m^2 \propto g'^2 \Delta^2 M^2 \tag{4.102}$$

because turning off $g'$ restores a larger symmetry. However, a full calculation by Preskill [71] and Peskin [70] shows that there some of them, which are

$$\begin{aligned} m^2_{\bar{\mathcal{U}}\mathcal{S}} &\propto g'^2 \Delta(\Delta+1) M^2\,, \\ m^2_{\bar{\mathcal{D}}\mathcal{C}} &\propto g'^2 \Delta(\Delta-1) M^2\,, \end{aligned} \tag{4.103}$$

that have the potential to be negative when $|\Delta| < 1$. When it is so, we know that the vacuum in (4.89) is not stable and the preferred one, with the influences from the $B^\mu$ interaction, will be (4.101). To see this notice that when $|\Delta| > 1$ the form of the generator (4.97) is "almost" $SU(4)_{L+R}$ symmetric (especially when $|\Delta| >> 1$). We have to preform a rotation on the $U(1)$ $G_W$ generator and partition the resulting generator into an unbroken and a broken part with respect to the configuration of the condensate. The rotated generators must satisfy

$$\sum_\alpha \mathrm{Tr}\left\{{}^U\Lambda^\alpha_X {}^U\Lambda^\alpha_X\right\} \quad min, \quad \text{or} \sum_\alpha \mathrm{Tr}\left\{{}^U\Lambda^\alpha_Y {}^U\Lambda^\alpha_Y\right\} \quad max\,, \tag{4.104}$$

which is the case for the condensate (4.89) when $|\Delta| > 1$. Hence we also expect that the rotated current preserve as much as possible the $SU(4)_{L+R}$ in the way that the trace $\mathrm{Tr}\left\{{}^U\Lambda_Y {}^U\Lambda_Y\right\}$ be maximum. This can be accomplished by a rotation on the lower-right block of the $U(1)$ $G_W$ interaction matrix in (4.97) into

$$U\Lambda^\alpha_{\mathrm{lower-right}}U^\dagger \to \frac{g'}{2}\begin{pmatrix} \Delta+1 & & & \\ & -\Delta+1 & & \\ & & \Delta-1 & \\ & & & -\Delta-1 \end{pmatrix} \tag{4.105}$$

(notices the two terms in the middle). Doing so is just equivalent to a switching between $\mathcal{D}_R$ and $\mathcal{C}_R$ and therefore the form of the condensate (4.101). That condensate leads to the pairings

$$\langle\bar{\mathcal{U}}_L\mathcal{U}_R\rangle = \langle\bar{\mathcal{D}}_L\mathcal{C}_R\rangle = \langle\bar{\mathcal{C}}_L\mathcal{D}_R\rangle = \langle\bar{\mathcal{S}}_L\mathcal{S}_R\rangle \neq 0 \tag{4.106}$$

which have energy lower than (4.89) when $\Delta < 1$. The existences of charged condensates in (4.106) signify the breaking of $U(1)_{em}$.

### 4.1.4 Another Pattern of Symmetry Breaking: Real Representation

Now we will explore another possible pattern of symmetry breaking that is relevant to Little Higgs models. The group of interest corresponds to fermions in real representation, $\mathbf{r}$ of the strong interaction group with the property that the product of $\mathbf{r}$ with itself contains a singlet and hence the fermion condensate appears in a symmetric form.

Before we go to the real representation, let us first recall the convention where all quarks are left-handed (transform under $\mathbf{N}$ and $\bar{\mathbf{N}}$). Then for a Dirac spinor $\psi$ we have $\bar{\psi}\psi = \varepsilon^{\alpha\beta}\psi^{\alpha}_{\mathbf{N}}\psi^{\beta}_{\bar{\mathbf{N}}}$ where $\alpha, \beta$ and $\varepsilon^{\alpha\beta}$ are two-component spinor indices and the 2-index Levi-Civita (antisymmetric) tensor, respectively. The important feature we require is that the condensate must be invariant under a strong interaction gauge symmetry $G_{\text{strong}}$ (i.e., $G_{\text{strong}}$ is not broken) that binds the condensate together. Then the form of the condensate also depends on how the fermions transform under this $G_{\text{strong}}$ and is characterised by a $G_{\text{strong}}$-invariant tensor $\lambda^{rs}$. Denoting $i$, $j$, *etc*... as flavour indices, the operator to form a condensate becomes[7]

$$\langle 0|\psi^{\alpha}_{i}\varepsilon_{\alpha\beta}\psi^{\beta}_{j}|0\rangle \tag{4.107}$$

The $r, s$ are the indices for the representation of the strong symmetry group and $\lambda^{rs}$ is a $G_{\text{strong}}$ invariant tensor.

For $2N$ multiplets[8] of left-handed fermions with $N$ in the *complex* representation $\mathbf{r}$ while the other $N$ in the conjugate representation $\bar{\mathbf{r}}$, we see that the condensate is constructed from

$$\Psi^{(\bar{\mathbf{r}})\alpha}_{i}\varepsilon_{\alpha\beta}\Psi^{(\mathbf{r})\beta i} \tag{4.108}$$

[7] If this form was used in the previous sections, we would have obtained the form

$$\theta^{\alpha} = \begin{pmatrix} \theta^{\alpha}_{L} & \\ & -\theta^{\alpha *}_{R} \end{pmatrix}$$

(with the extra minus) for the generators of the $G_W$.

[8] Odd numbers are equally possible. The $2N$ notations are used, just for this moment, for compatibility with the previous discussions.

where the strong interaction indices have been suppressed. When the left-right notation is recovered (4.108) becomes the usual form

$$\langle \overline{\psi}_L \psi_R \rangle \,. \tag{4.109}$$

Let us turn to the case of $2N$ multiplets of fermions transforming under a real representation $\mathbf{r}$, of $G_{\text{strong}}$ such that a symmetric product of this real representation with itself contains a singlet (or we cannot construct a condensate). Therefore we cannot distinguish between left and right handed fermions. So, unlike the previous case, the condensate is now a fermion-fermion type (not fermion-anti-fermion). What follows is that the $\Sigma_{rs}$ is also symmetric due to the Fermi statistics between the indices of $\lambda_{(rs)}$ and the $\varepsilon^{\alpha\beta}\Psi_r^\alpha\Psi_s^\beta$. The $G_{\text{strong}}$-invariant form becomes

$$\Psi_r^{(\mathrm{r})\alpha i}\varepsilon_{\alpha\beta}\Psi_s^{(\mathrm{r})\beta i}\lambda_{(rs)} \,. \tag{4.110}$$

This leads to a flavour symmetry $G = SU(2N)$. Then the maximal subgroup of $G$ leaving the symmetric tensor $\Sigma_{ij}$ invariant, being the vacuum expectation value of (4.110), is $H = O(2N)$.

## 4.2 Non-Linear Realisation of a Symmetry

In this section we will introduce the concept or the "realisation of a symmetry", which is different from the representation of a group, and study some of its formal properties. The considerations will be useful when we are dealing with the problem where the "complete" high-energy theory is still unknown, or next-to-impossible to calculate, and we have to work with the effective low-energy ones. A good example of a model to be modified in this aspect is the sigma model concerning the pions. The hefty mass gap between the pions and *all* other hadronic states in QCD suggests the possibility of describing the low-energy hadronic physics in terms of the effective field theory where the only kind of strongly interacting particles are the pion fields (Goldstone bosons). In fact, this is necessary since we do not know which particle to associate with the $\sigma$. We will follow up with what we have discussed on the linear realisation of the chiral $SU(2)$ symmetry in the section 2.2.2. There, we have seen that the effective Lagrangian must share all the symmetry properties with that of the high energy theory. However, it turned out that the linear version[9] is not

[9] Recall that "linear" means the broken symmetry is realised via linear transformations.

very desirable when we want to concentrate on light particles only (pions as the Goldstone bosons). It did not provide any rationale to cast off contributions from higher order diagrams. This is where the non-linear realisation comes in. It helps describe the low energy arena of the theory in terms of light particles only while still preserves the symmetry of the high energy theory.

We will start with a brief discussion on some formal aspects of the non-linear realisations in section 4.2.1, then followed by their application: the non-linear sigma model $[SU(N)]^2/SU(N)$ in section 4.2.2. We will follow the discussions by Scherer [72] and Weinberg [58]. Readers who seek for more formal, and more general, treatments may want to look up the papers by Coleman *et al.* in [30] and [31].

### 4.2.1 Formal Aspects

Let us consider an $n$-component vector called $\Phi(x)$ with $\phi^a(x)$'s as its components. This enables us to define a vector space $\mathcal{M}$,

$$\mathcal{M} \equiv \Big\{ \Phi : \mathbb{M}^4 \to \mathbb{R}^n \Big\} \tag{4.111}$$

where, as usual, $\phi^a : \mathbb{M}^4 \to \mathbb{R}$. Then we can define an operation of a particular group $G \ni g$ on $\mathcal{M}$ by considering a map $\varphi$ which associates $(g, \mathcal{M}) \in G \times \mathcal{M}$ with an element $\varphi(g, \Phi) \in \mathcal{M}$. This operation requires that $\varphi$ has an identity $\mathbb{1}$

$$\varphi(\mathbb{1}, \Phi) = \Phi\,, \qquad \forall \Phi \in \mathcal{M} \tag{4.112}$$

and the mapping preserves the group structure (homomorphism)[10]

$$\varphi(g_1 g_2, \Phi) = \varphi\big(g_1, \varphi(g_2, \Phi)\big) \tag{4.113}$$

for $g_i \in G$. If $\varphi$ satisfies an "optional" condition $\varphi(g, \lambda\Phi) = \lambda\varphi(g, \Phi)$, the mapping is said to be linear and $\varphi$ will form a *representation* of $G$. Then recall that in a theory with spontaneous symmetry breaking from $G$ to $H$, the ground state under consideration is invariant under a subgroup $H$. So the configuration of $\Phi$, say $\Phi = 0$, can be associated a with the ground state. Therefore it is required that $\varphi(h, 0) = 0$, where $h_1 h_2 \in H$ if $h_1 \in H$ and $h_2 \in H$. In addition we obviously require $h^{-1} \in H$ for $h \in H$. Then it can be shown (see Scherer [72]) that we can set up an isomorphism (a bijective homomorphism) between

[10] Some literatures prefer the notation $\varphi(g_1 \star g_2) = \varphi(g_1) \circ \varphi(g_2))$ where $\star$ and $\circ$ are operations on $G$ and on $M$ respectively.

the left-coset and the Goldstone bosons. This requires that when picking up any element of a coset $gH$, its action on the "origin" $\Phi = 0$ (our vacuum) is the same as that done by $g$; i.e.,

$$\varphi(gh, 0) = \varphi(g, \varphi(h, 0)) = \varphi(g, 0)\,, \tag{4.114}$$

which means that different vacuum states may be reached by a transformation in the coset space. It is then obvious, in this sense, that the coset $gH$ must depend on spacetime. In addition, the action of $g \in G$ on $\Phi = \varphi(\tilde{g}h, 0)$ satisfies

$$\varphi(g, \Phi) = \varphi(g\tilde{g}h, 0) = \Phi' \tag{4.115}$$

which means that the new left coset representing $\Phi'$ can be reached by multiplying the coset $\tilde{g}H$ by $g$; i.e., by a $G-$transformation. What we need is then an appropriate variables to parametrise the coset space $(G/H)$. What we have shown above then uniquely defines the transformation behaviour of $\Phi$ (which is our Goldstone bosons).

### 4.2.2 Non-Linear Sigma Model $SU(N) \times SU(N)/SU(N)$

In this section we will discuss an extension of the linear sigma model considered in section 2.2.2. Our goal is to find a way to deal with the low energy physics in terms of the Goldstone boson fields that respects the $SU(2) \times SU(2)$ symmetry and does not mix multiplets of the unbroken $SU(2)_V$ subgroup; namely the triplet with the singlet, under general $SU(2)\times SU(2)$ transformations. We expect that under the symmetry breaking scale, the sought for realisation allows the $\sigma$ to be frozen out while only the pions ($\pi$) transform. This realisation of the non-linear sigma model can also be served as a toy model for the Little Higgs.

#### 4.2.2.1 Matrix Representation of the Goldstone Bosons

To illustrate the differences between the linear and non-linear sigma models we will start with the matrix representation of the Goldstone boson (like what we did in section 2.2.2). Consider a case where the symmetry is broken from $G = SU(N) \times SU(N) = \{(L, R)|L \in SU(N), R \in SU(N)\}$ to its "vector" subgroup $H = \{(V, V)|V \in SU(N)\}$, where $g = (L, R)$ denotes any element of the group. The way $g$ acts on an object to be considered should be clear from its notation. Then the left coset of $\tilde{g} = (\tilde{L}, \tilde{R})$ is $\tilde{g}H = \{(\tilde{L}V, \tilde{R}V)|V \in SU(N)\}$. Here it is helpful to introduce a matrix $\Sigma = \tilde{L}\tilde{R}^\dagger$ satisfying

$$\tilde{g}H = (\tilde{L}V, \tilde{R}V) = (\tilde{L}\tilde{R}^\dagger\tilde{R}V, \tilde{R}V) = (\tilde{L}\tilde{R}^\dagger, 1)(\tilde{R}V, \tilde{R}V) = (\tilde{L}\tilde{R}^\dagger, 1)H\,, \quad (4.116)$$

where $(\tilde{R}V, \tilde{R}V) \in H$, valid for the subgroup elemets, is used in the last step. Moreover,

$$g\tilde{g}H = (L\tilde{L}\tilde{R}^\dagger, R)H = (L\tilde{L}\tilde{R}^\dagger R^\dagger R, R)H = \left(L(\tilde{L}\tilde{R}^\dagger)R^\dagger, 1\right)(R, R)H \quad (4.117)$$

tells us that $\Sigma$ transforms as

$$\Sigma \longrightarrow L\Sigma R^\dagger\,. \quad (4.118)$$

Here, the identification of $\Sigma$ with $\Phi$ (in the previous section) is also transparent.

In the case $G = SU(N) \times SU(N)$, we know that the vector space $\mathcal{M}$, introduced in the previous section, is defined by

$$\mathcal{M} \equiv \left\{\Phi : \mathbb{M}^4 \to \mathbb{R}^n\right\}\,. \quad (4.119)$$

For example, when $N = 2, 3$ we have $n = 3, 8$ respectively. Now, let us define another vector space formed by a set $\mathcal{H}_N$ of all traceless, Hermitian, $N \times N$ matrices. The elements of $\mathcal{M}$ are related to the elements of $\mathcal{H}_N$, called $\hat{\phi}$, by

$$\mathcal{M}_2 \equiv \left\{\hat{\phi} : \mathbb{M}^4 \to \mathcal{H}_N\right\}\,. \quad (4.120)$$

We have seen the $SU(2)$ example in the section on the linear sigma model:

$$\hat{\phi} = \sum_{a=1}^{3} \tau^a\phi^a = \begin{pmatrix} \phi^3 & \phi^1 - \mathrm{i}\phi^2 \\ \phi^1 + \mathrm{i}\phi^2 & -\phi^3 \end{pmatrix} \quad (4.121)$$

In this case, the realisation of $SU(2) \times SU(2)$ on $\mathcal{M}_2$ is still a linear one.

Next, let us turn to the case where the set we defined does not allow the formation of a vector space. Many possibilities are available. The simplest, yet useful, definition is the one that maps $\mathbb{M}^4$ to a set of $SU(N)$ matrices instead of the hermitian traceless ones (the $\mathcal{H}_N$). Obviously, the sum of $SU(N)$ matrices need not be of the $SU(N)$ type. Therefore, we define

$$\mathcal{M}_3 \equiv \left\{\hat{\phi} : \mathbb{M}^4 \to SU(N)|\Sigma = \exp\left\{\frac{\mathrm{i}\hat{\phi}}{F_\pi}\right\}, \hat{\phi} \in \mathcal{M}_2\right\}\,. \quad (4.122)$$

Then we set up the group action on $\mathcal{M}_3$:

$$\varphi : G \times \mathcal{M}_3 \longrightarrow \mathcal{M}_3 \tag{4.123}$$

where $\varphi\left[(L,R),\Sigma\right] \equiv L\Sigma R^{\dagger} \in \mathcal{M}_3$. The successive transformations $g_1 = (L_1, R_1)$ then $g_2 = (L_2, R_2)$ are given by

$$\varphi\left[g_1, \varphi[g_2, \Sigma]\right] = L_1 L_2 U R_2^{\dagger} R_1^{\dagger} = \varphi\left[g_1 g_2, \Sigma\right] \ , \tag{4.124}$$

which guarantee a realisation of $G$ on $\mathcal{M}_3$. The advantage of (4.122) is that we can easily expand the exponential in terms of momenta; hence allowing us to concentrate on the low-energy phenomena only. If the energy is low enough, only the Goldstone bosons can be produced as they are the lightest particles out there. In addition, as all that will be used in constructing the Lagrangian (for the $\Sigma$ or the $\hat{\phi}$) are symmetry arguments, the underlying dynamics does not concern us and the power of group theory will ensure that a theory based on a particular realisation of the symmetry is equivalent to any other realisations (see [30, 31]). This means we can choose to work with any one we find convenient (if we keep working at low-energy region).

Now, let us see how to use this formalism in the context of spontaneous symmetry breaking. First, consider the ground state $\Sigma_0 = \mathbb{1}$ which is invariant under the vector transformation $(V, V)$; i.e., $\varphi[(V,V),\Sigma_0] = \Sigma_0$, but is not so under the axial one; namely, $\varphi[(A, A^{\dagger}), \Sigma_0] = A^{\dagger} A^{\dagger} \neq \Sigma_0$. This is in contrast to what we have seen in the linear sigma model, where the fields $\sigma$ and $\pi^a$ transform linearly (c.f. (2.26-2.29)) under both transformations corresponding to broken and unbroken subgroups. The situation is different here. For the unbroken subgroup (pure vector transformations) we can make use of $VV^{\dagger} = 1$ which leads to

$$V\Sigma V^{\dagger} = V\left(1 + \mathrm{i}\frac{\hat{\phi}}{F_\pi} - \frac{\hat{\phi}^2}{2F_\pi^2} + \ldots\right)V^{\dagger} = 1 + \mathrm{i}\frac{V\hat{\phi}V^{\dagger}}{F_\pi} - \frac{V\hat{\phi}V^{\dagger}V\hat{\phi}V^{\dagger}}{2F_\pi^2} + \ldots \ . \tag{4.125}$$

Since the transformed matrix is still traceless, (4.125) brings us back to a linear *representation* on $\mathcal{M}_2$; i.e.,

$$\hat{\phi} \to V\hat{\phi}V^{\dagger} \in \mathcal{M}_2 \, . \tag{4.126}$$

On the other hand, for the transformation in the broken subgroup (pure chiral transformations), we have

$$A^\dagger \Sigma A^\dagger = A^\dagger \left(1 + \mathrm{i}\frac{\hat{\phi}}{F_\pi} - \frac{\hat{\phi}^2}{2F_\pi^2} + \ldots\right) A^\dagger = 1 + \mathrm{i}\frac{A^\dagger \hat{\phi} A^\dagger}{F_\pi} - \frac{A^\dagger \hat{\phi}\hat{\phi} A^\dagger}{2F_\pi^2} + \ldots . \quad (4.127)$$

Since we do not have the relation $A^\dagger A^\dagger = 1(wrong)$ to put into the slot between the $\hat{\phi}\hat{\phi}$ in the last term, the transformation of the $\hat{\phi}$ is highly non-linear. To see that, let us write $A^\dagger \equiv \mathrm{e}^{\mathrm{i}\Theta} = \mathrm{e}^{\mathrm{i}\lambda^a\theta^a}$ so that the above expression becomes

$$1 + \mathrm{i}\frac{\hat{\phi}'}{F_\pi} = (1 + \mathrm{i}\Theta + \ldots)\left(1 + \mathrm{i}\frac{\hat{\phi}}{F_\pi} + \ldots\right)(1 + \mathrm{i}\Theta + \ldots) . \quad (4.128)$$

The non-linear realisation allows us to consider only the low-momentum terms. We will keep working with the global $SU(3)_L \times SU(3)_R$ case. The form of $\Sigma$ is given in (A.38). First, a term without a derivative is proportional to a constant

$$\mathrm{Tr}\Sigma^\dagger\Sigma = \text{constant} . \quad (4.129)$$

Those containing $\mathrm{Tr}\Sigma$ or $\mathrm{Tr}\partial_\mu\Sigma\Sigma^\dagger$ also vanish:

$$\begin{aligned}
\mathrm{Tr}\Sigma &= \frac{\phi^a}{F_\pi}\mathrm{Tr}\lambda^a = 0 \\
\mathrm{Tr}\left[\partial_\mu\Sigma^\dagger\Sigma\right] &= \frac{1}{F_\pi}\mathrm{Tr}\left[\mathrm{i}(\partial_\mu\hat{\phi})\Sigma^\dagger\Sigma\right] = 0 ,
\end{aligned}$$

where we have used $\Sigma^\dagger\Sigma = 1$ in the second equation. In addition, the term $\mathrm{Tr}\left[(\partial_\mu\partial^\mu\Sigma^\dagger)\Sigma\right]$ can be transformed into $\mathrm{Tr}\partial^\mu\Sigma^\dagger\partial_\mu\Sigma$ without the need to worry about the total derivative term. Consequently, the kinetic term for $\Sigma$ is given by

$$\begin{aligned}
\frac{F_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) &= \frac{F_\pi^2}{4}\mathrm{Tr}\left(\mathrm{i}\frac{-\partial^\mu\hat{\phi}}{F_\pi}\mathrm{i}\frac{\partial_\mu\hat{\phi}}{F_\pi}\right) + \ldots \\
&= \frac{F_\pi^2}{4}\mathrm{Tr}\left(\lambda^a\partial^\mu\phi^a\lambda^b\partial_\mu\phi^b\right) + \ldots = \frac{1}{2}\partial^\mu\phi^a\partial_\mu\phi^a , \quad (4.130)
\end{aligned}$$

which is invariant for the case of global symmetry.

#### 4.2.2.2 Real-vector Representation of the Goldstone Bosons

It might be more illustrative to consider the real representation of the Goldstone bosons. This case should explain what we really meant by freezing the $\sigma$ field. First, recall that $SU(2) \times SU(2) \sim SO(4)$. In the simplest case without the nucleons[11], the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - \frac{\mu^2}{2}\Phi^T\Phi - \frac{\lambda}{4}(\Phi^T\Phi)^2\,, \tag{4.131}$$

with $\mu^2 < 0$ for the case under consideration. Note that $\Phi$ is a 4-component vector

$$\Phi^T = (\phi_1, \phi_2, \phi_3, \phi_4) \tag{4.132}$$

where $\phi$ is an isovector pseudoscalar field and $\phi_4$ is an isoscalar scalar field.

At a particular point $x$, the isovector (Goldstone boson) field can be set to zero by a suitable redefinition of vacuum; i.e., writing the field as a chiral rotation acting on the $\Phi_0^T = (0, 0, 0, \sigma_0)$. In other words,

$$\phi_i(x) = R_{i4}(x)\sigma_0(x)\,, \tag{4.133}$$

where $i = 1, \ldots, 4$. The $\sigma_0$ is given by the positive root of $\sigma_0^2 = \phi_i^2$ because the matrix $R(x)$ is orthogonal

$$R^T R = 1\,. \tag{4.134}$$

This condition also fixes one of the element in $R_{i4}$; i.e.,

$$\sum R_{i4}^2 = 1\,. \tag{4.135}$$

So the Lagrangian (4.131) is simplified to

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma_0 \partial^\mu \sigma_0 - \frac{1}{2}\sigma_0^2 \partial_\mu R_{i4} \partial^\mu R_{i4} - \frac{\mu^2}{2}\sigma_0^2 - \frac{\lambda}{4}\sigma_0^4\,. \tag{4.136}$$

The degrees of freedom described by $\phi_i$ are now transferred to the zero-vacuum expectation value field $\sigma_0 - \sqrt{-\mu^2/\lambda}$ and the parameters of the rotation matrix $R_{i4}$. Since only 3 parameters are required from $R_{i4}$, one of the (infinite) possibilities is to chose them as $R_{a4}$ where $a = 1, 2, 3$. We set up a map

$$\zeta_a \equiv \frac{\phi_a}{\phi_4 + \sigma_0} \tag{4.137}$$

together with

$$R_{a4} \equiv \frac{2\zeta_a}{1+\zeta^2} = \frac{\phi_a}{\sigma_0}\,. \tag{4.138}$$

[11] A more complete treatment, especially on the $SU(3)$ group can be found in Weinberg's book [58].

Here the $R_{44}$ is automatically given by (4.135);

$$R_{44} = \frac{1-\zeta^2}{1+\zeta^2} = \frac{\phi_4}{\sigma_0}\,. \tag{4.139}$$

Therefore,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma_0\partial^\mu\sigma_0 - 2\sigma_0^2\frac{\partial_\mu\zeta_a\partial^\mu\zeta_a}{(1+\zeta^2)^2} - \frac{\mu^2}{2}\sigma_0^2 - \frac{\lambda}{4}\sigma_0^4\,, \tag{4.140}$$

which is clear that the mass term for the $\zeta$ is absent. The minimum requirement that there are massless particles is now satisfied.

Now we can study the transformation of the Goldstone boson $\zeta$. Using the explicit forms of the generators given in (C.49) and (C.50), we find that the $\zeta$ and $\sigma_0$ transform as an isovector and an isoscalar, respectively, under the isospin (unbroken) transformation

$$\boldsymbol{\zeta} \longrightarrow \boldsymbol{\zeta} + \boldsymbol{\epsilon}\times\boldsymbol{\zeta}\,, \tag{4.141}$$

$$\sigma_0 \longrightarrow \sigma_0\,. \tag{4.142}$$

It is clear that the Lagrangian (4.136) is invariant under this transformation. In other words, *unbroken isospin symmetry is realised linearly on the fields* $\boldsymbol{\zeta}$ *and* $\sigma_0$. Next, we will find out how the $\boldsymbol{\zeta}$ transform under the *broken* symmetry. First, recall that in terms of the original field, the broken (axial) symmetry is linearly realised (as it should):

$$\boldsymbol{\phi} \longrightarrow \boldsymbol{\phi} + \boldsymbol{\epsilon}_5\sigma_0\,, \tag{4.143}$$

$$\sigma_0 \longrightarrow -\boldsymbol{\epsilon}_5\cdot\boldsymbol{\phi}\,. \tag{4.144}$$

Putting these into the (4.137), we find that the $\boldsymbol{\zeta}$ and $\sigma_0$ transform under the *broken* symmetry in a non-linear way

$$\boldsymbol{\zeta} \longrightarrow \boldsymbol{\zeta} + \frac{\boldsymbol{\epsilon}_5}{2}(1-\zeta^2) + \boldsymbol{\zeta}(\boldsymbol{\epsilon}_5\cdot\boldsymbol{\zeta}) \tag{4.145}$$

$$\sigma_0 \longrightarrow \sigma_0\,,. \tag{4.146}$$

Otherwise stated, the *broken axial symmetry is realised non-linearly on the fields* $\boldsymbol{\zeta}$ *and* $\sigma_0$. If we define the so-called covariant derivative of the pion field

$$\mathbf{D}_\mu = \frac{\partial_\mu\boldsymbol{\zeta}}{1+\zeta^2}\,, \tag{4.147}$$

we find that it transforms as

$$\mathbf{D}_\mu \longrightarrow \mathbf{D}_\mu + (\boldsymbol{\zeta} \times \boldsymbol{\epsilon}_5) \times \mathbf{D}_\mu \,, \tag{4.148}$$

leaving the Lagrangian invariant. Though the $\mathbf{D}_\mu$ transforms in a linear way under the broken subgroup transformation, the $\boldsymbol{\zeta}$ does not. (4.141-4.142) together with (4.145-4.146) define a non-linear realisation of the $SU(2) \times SU(2)$ group. The striking feature of the non-linear realisation is that the triplet and the singlet of the unbroken isospin subgroup do not mix, as shown in (4.146). The singlet $\sigma_0$ do not even transform. In other words, this means that the $\sigma_0$ plays no role in maintaining the invariance of the Lagrangian under the isospin and axial transformations. Consequently, the physical content of the theory can be changed in a legitimate way: we can toss away the $\sigma_0$ degree of freedom by arguing that it is very heavy, keeping its vacuum expectation value finite. In this way the physical properties of process involving the light pions as external particles will not change. Defining $F = 2\langle\sigma_0\rangle$, we find that the low-energy effective Lagrangian becomes

$$\mathcal{L} = \frac{F^2}{2}\mathbf{D}^\mu \cdot \mathbf{D}_\mu \,. \tag{4.149}$$

If we define the new pion field $\boldsymbol{\pi} \equiv F\boldsymbol{\zeta}$, we get the usual form of the so-called "non-linear sigma model"

$$\mathcal{L} = \frac{1}{2}\frac{\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}}{(1+\boldsymbol{\pi}^2/F^2)^2} = \frac{1}{2}\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} - \frac{\boldsymbol{\pi}^2}{F^2}\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} \,. \tag{4.150}$$

Moreover, we can now talk about how the $SU(2)_L \times SU(2)_R$ is realised on the pion field $\boldsymbol{\pi}$, with a few modifications on (4.141) and (4.145) without the need to mention about the singlet at all.

### 4.2.3 $SU(N)/SO(N)$ Non-Linear Sigma Model

In this last example of a class of non-linear sigma models, we will consider the $SU(N)/SO(N)$ type. This pattern of symmetry breaking is due to fermions transforming under real representation forming a condensate. First assume that the space spanned by $G/H$ is symmetric as mentioned in the appendix A.3. So let us arrange the $SU(N)$ generators in a form that is easy to work with. Then recall that for any group element $g \in G$ we can write

$$g = e^{\mathrm{i}t^a T^a} \,. \tag{4.151}$$

It is clear that $T$ is Hermitian (since $G$ is unitary) and traceless ($\det g = 1$). Then the symmetry properties of the complex matrix $T$ show up when we decompose $T$ into two real matrices: $T^a = A^a + \mathrm{i}B^a$. We have

$$A^{a\mathsf{T}} = A^a \,, \tag{4.152}$$

and

$$B^{a\mathsf{T}} = -B^a \,, \tag{4.153}$$

where the latter is antisymmetric. Consequently, we assign the $A^a$ and $\mathrm{i}B^a$ to broken $X^z$ and unbroken $Y^i$ generators, respectively. The results are just as we have advertised; namely, the unbroken subgroup can be spanned by an anti-symmetric matrices and hence being the $SO(N)$.

Now let us be more specific and consider the case when $N = 5$, which is relevant for all the models we consider in this thesis. There are many possibilities of symmetric condensates[12] transforming as a $\mathbf{15}$ of $SU(5)$. One of them is

$$\Sigma_0' = \mathbb{1}_{5\times 5} \,. \tag{4.154}$$

Still, the one that was commonly chosen in the Little Higgs model is ([21])

$$\Sigma_0 = \begin{pmatrix} & & \mathbb{1}_{2\times 2} \\ & 1 & \\ \mathbb{1}_{2\times 2} & & \end{pmatrix} = \begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 1 \\ & & 1 & & \\ 1 & 0 & & & \\ 0 & 1 & & & \end{pmatrix} . \tag{4.155}$$

This (4.155) can be connected to the $\Sigma_0' = \mathbb{1}$ by a rearrangement of the basis from the original basis $T'^a$ corresponding to $\Sigma_0'$ by an $SU(5)$ $U_0$ matrix via

$$T^a = U_0 T'^a U_0^\dagger \,, \tag{4.156}$$

[12] In the $SU(5)/SO(5)$ model by Georgi and his colleagues [14], [16], present during the mid 80's, they used different sets of vacua from from (4.155).

with $\Sigma_0 = U_0 {U_0}^{\mathsf{T}}$. The explicit form $U_0$ is not necessary but might help clearing things up. Consider

$$U_0 = \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & 0 & 0 & 1-\mathrm{i} & 0 \\ 0 & 1+\mathrm{i} & 0 & 0 & 1-\mathrm{i} \\ 0 & 0 & 2 & 0 & 0 \\ 1-\mathrm{i} & 0 & 0 & 1+\mathrm{i} & 0 \\ 0 & 1-\mathrm{i} & 0 & 0 & 1+\mathrm{i} \end{pmatrix}. \tag{4.157}$$

We find that

$$U_0 U_0^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.158}$$

as advertised. In addition, we can verify that $U_0^\dagger U_0 = \mathbb{1}_{5\times 5}$. To make sure that the $SU(5)$ generators enjoy the transformation by (4.156), let us write $T^a = U_0 \lambda^a U_0^\dagger$ where the $SU(5)$ generators $\lambda^a$ are given in section C.3.2. Then

$$\begin{aligned} [T^a, T^b] &= \left[U_0 \lambda^a U_0^\dagger, U_0 \lambda^b U_0^\dagger\right] = U_0 [\lambda^a, \lambda^b] U_0^\dagger \\ &= \mathrm{i} f^{abc} T^c \end{aligned} \tag{4.159}$$

indicates that the $SU(5)$ Lie algebra is satisfied. Up to this step, the explicit forms of $T^a$'s are not necessary. Then to see how $T^a$ acts on the vacuum $\Sigma_0$, consider

$$\begin{aligned} T^a \Sigma_0 &= \left(U_0 \lambda^a U_0^\dagger\right) U_0 {U_0}^{\mathsf{T}} = U_0 \lambda^a {U_0}^{\mathsf{T}} = U_0 \lambda^a U_0 \\ &= \pm (U_0 \lambda^a U_0)^{\mathsf{T}} \end{aligned} \tag{4.160}$$

where in the last step we consider the $(U_0 \lambda^a U_0)$ as one piece of a matrix and the plus and minus signs depend on whether the $(U_0 \lambda^a U_0)$ is symmetric or not. It is clear that the symmetry property of the whole piece depends on the $\lambda^a$. So we have found the condition for partitioning the $SU(5)$ generators into symmetric $(X^z)$ and antisymmetric $(Y^i)$ parts. For the 14 symmetric generators we have

$$\begin{aligned} X^z \Sigma_0 &= +(U_0 \lambda^z U_0)^{\mathsf{T}} = \left(U_0 \lambda^z U_0^\dagger U_0 U_0\right)^{\mathsf{T}} \\ &= +(X^z \Sigma_0)^{\mathsf{T}} = \Sigma_0 {X^z}^{\mathsf{T}}. \end{aligned} \tag{4.161}$$

Then the antisymmetric generators, corresponding to the $SO(5)$ unbroken subgroup, satisfy the minus version of (4.160); i.e.,

$$Y^i\Sigma_0 = -(U_0 Y^i U_0)^{\mathsf{T}} = -\Sigma_0 {Y^i}^{\mathsf{T}} . \tag{4.162}$$

There is also another way to arrive at the relations just found above. Observe that in the original basis the unbroken $SO(5)$ (antisymmetric) generators obey

$$Y'^i = -{Y'^i}^{\mathsf{T}} . \tag{4.163}$$

We find

$$\begin{aligned} 0 &= U_0 Y'^i {U_0}^{\mathsf{T}} + U_0 {Y'^i}^{\mathsf{T}} {U_0}^{\mathsf{T}} \\ &= U_0 Y'^i U_0^\dagger U_0 {U_0}^{\mathsf{T}} + U_0 {U_0}^{\mathsf{T}} {U_0^\dagger}^{\mathsf{T}} {Y'^i}^{\mathsf{T}} {U_0}^{\mathsf{T}} \\ &= Y^i \Sigma_0 + \Sigma_0 {Y^i}^{\mathsf{T}} . \end{aligned} \tag{4.164}$$

We can also work out this relation by starting with the $\Sigma_0$ right away. By requiring that the vacuum be invariant under $H$-transformation; i.e., $\exp\{\mathrm{i}\alpha^a Y^a\}\Sigma_0 \exp\{\mathrm{i}\alpha^a Y^a\}^{\mathsf{T}} = \Sigma_0$ we find, by expanding this expression,

$$\begin{aligned} \Sigma_0 &= \left(1 + \mathrm{i}\alpha^i Y^i\right) \Sigma_0 \left(1 - \mathrm{i}\alpha^i {Y^i}^{\mathsf{T}}\right) \\ &= \Sigma_0 + \mathrm{i}\alpha^i \left(Y^i \Sigma_0 + \Sigma_0 {Y^i}^{\mathsf{T}}\right) + \mathcal{O}(\alpha^2) \end{aligned} \tag{4.165}$$

which clearly leads to

$$Y^i \Sigma_0 + \Sigma_0 {Y^i}^{\mathsf{T}} = 0 . \tag{4.166}$$

The conditions (4.163) and (4.155) do not only guarantee that the subgroup of interested is $SO(5)$ but also provides a condition for finding the $Y^i$'s. Then it is transparent that broken generators satisfy the symmetric version

$$X^z \Sigma_0 - \Sigma_0 {X^z}^{\mathsf{T}} = 0 , \tag{4.167}$$

which follows from the condition in the $\Sigma_0' = \mathbb{1}$ basis

$$X'^z = {X'^z}^{\mathsf{T}} , \tag{4.168}$$

with the transformation (4.156). Note that there are 10 antisymmetric generators $Y^i$ of $SO(5)$ and 14 symmetric ones of $X^z \in SU(5)/SO(5)$. Knowing that these generators are Hermitian and traceless, we can use the conditions on the vacuum expectation value to find their exact forms.

Then the Goldstone bosons are parametrised along the $SU(5)/SO(5)$ (broken) direction by symmetric generators as

$$\Sigma(x) = \mathrm{e}^{\mathrm{i}\Pi^z(x)X^z/F_\pi}\Sigma_0\mathrm{e}^{\mathrm{i}\Pi^z(x)X^{z\mathsf{T}}/F_\pi}\,. \tag{4.169}$$

The relation (4.167) provides the way to move $\exp\{\mathrm{i}\Pi^z X^z\}^\mathsf{T}$ through the $\Sigma_0$ and rewrite the $\Sigma$ in the non-linear sigma model in a more useful form:

$$\Sigma(x) = \mathrm{e}^{2\mathrm{i}\Pi^z(x)X^z/F_\pi}\Sigma_0\,. \tag{4.170}$$

In this $SU(5)$ model, we expect that the Goldstone boson matrix looks fairly similar to that of the $SU(3)$ model given in (A.38). In addition, we "know in advance" that we are going to deal with electroweak interaction. So let us decompose the representations of the pions under the electroweak gauge group as follows:

$$\mathbf{1}_0 \oplus \mathbf{3}_0 \oplus \mathbf{2}_{\mathbf{1/2}} \oplus \mathbf{3}_{\mathbf{1}}\,. \tag{4.171}$$

With some proper normalisations the Goldstone boson matrix is

$$\Pi^z X^z = \begin{pmatrix} \frac{\omega^0}{2} - \frac{\eta}{\sqrt{20}} & \omega^+/\sqrt{2} & \pi^-/\sqrt{2} & \phi^{--} & \frac{\phi^-}{\sqrt{2}} \\ \omega^-/\sqrt{2} & -\frac{\omega^0}{2} - \frac{\eta}{\sqrt{20}} & (H^0 - \mathrm{i}\pi^0)/2 & \frac{\phi^-}{\sqrt{2}} & \phi^0 \\ \pi^+/\sqrt{2} & (H^0 + \mathrm{i}\pi^0)/2 & \sqrt{4/5}\eta & \pi^-/\sqrt{2} & (H^0 - \mathrm{i}\pi^0)/2 \\ \phi^{++} & \frac{\phi^+}{\sqrt{2}} & \pi^+/\sqrt{2} & \frac{\omega^0}{2} - \frac{\eta}{\sqrt{20}} & \omega^-/\sqrt{2} \\ \frac{\phi^+}{\sqrt{2}} & \phi^0 & (H^0 + \mathrm{i}\pi^0)/2 & \omega^+/\sqrt{2} & -\frac{\omega^0}{2} - \frac{\eta}{\sqrt{20}} \end{pmatrix}. \tag{4.172}$$

We can organise the members into sub blocks that have definite electroweak quantum numbers: a Hermitian traceless $2 \times 2$ matrix

$$\Omega = \begin{pmatrix} \frac{1}{2}\omega^0 & \frac{1}{\sqrt{2}}\omega^+ \\ \frac{1}{\sqrt{2}}\omega^- & -\frac{1}{2}\omega^0 \end{pmatrix} = \omega^a\tau^a\,, \tag{4.173}$$

and a $2 \times 2$ symmetric matrix

$$\Phi = \begin{pmatrix} \phi^{++} & \frac{1}{\sqrt{2}}\phi^+ \\ \frac{1}{\sqrt{2}}\phi^+ & \phi^0 \end{pmatrix}, \tag{4.174}$$

as well as a complex doublet

$$H^T = \begin{pmatrix} \frac{\pi^+}{\sqrt{2}} \\ \frac{H^0 + i\pi}{2} \end{pmatrix} . \tag{4.175}$$

In the next chapter we shall see that the $SU(2)$ triplet $\Omega$ will be associated with $SU(2) \times SU(2) \to SU(2)$ breaking. We write

$$\hat{\Pi}(x) = \begin{pmatrix} \Omega & H^\dagger & \Phi^\dagger \\ H & 0 & H^* \\ \Phi & H^\mathsf{T} & \Omega^\dagger \end{pmatrix} + \frac{\eta}{\sqrt{20}} \mathrm{diag}(1, 1, -4, 1, 1) , \tag{4.176}$$

where $\hat{\Pi} = \Pi^z X^z$. Then the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \frac{F_\pi^2}{4} \mathrm{Tr} \left\{ \partial_\mu \Sigma^\dagger \partial_\mu \Sigma \right\} , \tag{4.177}$$

which is constructed from non-trivial terms with lowest derivatives (momenta) possible. The factor 1/2 is introduced give the right factor to the kinetic term of the "would-be" Higgs due to the convention $\mathrm{Tr} T^a T^b = \delta^{ab}$. This factor can be absorbed at the cost of rescaling the $F$ and the fields inside the $\Sigma$. As usual for the non-linear sigma model, explicit mass terms for the Goldstone bosons are not allowed due to the non-linear realisation of the broken symmetry on the Goldstone boson fields. However, we know that gauge interactions break the global symmetry at tree level. When the gauge interactions are introduced via the covariant derivative such as

$$D^\mu \Sigma = \partial^\mu \Sigma + i g^\alpha W^{\alpha\mu} \left( T^\alpha \Sigma + \Sigma T^{\alpha \mathsf{T}} \right) , \tag{4.178}$$

they tend to align the orientation of the original vacuum and results in the Goldstone bosons acquiring masses from quantum effects. Their quadratic divergent contributions can be evaluated from the (Coleman-Weinberg) effective potential

$$V_{g,CW} \supset \frac{\Lambda^2}{(4\pi)^2} \frac{F^2}{4} \mathrm{Tr} \left[ (g^\alpha T^\alpha \Sigma + g^\alpha \Sigma T^{\alpha \mathsf{T}})(g^\beta \Sigma^\dagger T^\beta + g^\beta T^{\beta \mathsf{T}} \Sigma^\dagger) \right] . \tag{4.179}$$

### 4.2.4 $\Lambda_{\chi SB} = 4\pi F$

One important thing that should not be left unmentioned when using the non-linear sigma model (or most low-energy effective theories) is the suppression of the effects from terms of higher mass dimensions. In the context of effective theory, dimensional operators are allowed as we are not going to touch the high-energy region, regardless of the fact that they are non-renormalisable. So the strategy we usually take is to collect all possible operators, constrained only by symmetries of the theory, with some unknown coefficients. They will be suppressed by factors proportional to $\frac{E}{\Lambda}$ where $E$ is the energy scale of interest. In this way these operators will become important as we go up in energy. The coefficient of the operators will determine which one becomes strong before others.

The idea of this section is based on the justification of validity of the chiral perturbation expansion when quantum effects are taken into account. According to the previous paragraph, we further claim that for the perturbation philosophy to work, the correction terms must be smaller than the "more principal" terms. It is, however, not always the case without specific conditions on the cut off of the theory as loop corrections with appropriate dimensions are always included.

Also note that in a theory like the linear sigma model (see section 2.2.2) where $m_\sigma \simeq \lambda F_\pi$, we can have more than one mass scales. One is the scale $F_\pi$ (the decay constant) where the symmetry breaks, the other is the cut-off, or the new physics scale, $\Lambda \sim m_\sigma$. For example, we know that there is a difference of order $10 \approx 4\pi$ between the pion decay constant ($F_\pi \sim \mathcal{O}(100\text{MeV})$) and the cutoff of the chiral perturbation theory ($\Lambda_{\text{CutOff}} \sim \mathcal{O}(1\text{GeV})$). For each loop, we get the suppression factor

$$\frac{m_\sigma^2}{16\pi^2 F_\pi^2} \to \frac{\lambda}{16\pi^2} \,. \tag{4.180}$$

Then we see that the loop correction will become large when the above expression is equal to one. If we insist on working with a perturbation theory, we then have to work below the scale

$$\Lambda^2 \simeq m_\sigma^2 simeq (4\pi F_\pi)^2 \sim (1\,\text{GeV})^2 \tag{4.181}$$

where the $\lambda$ is not too large.

A similar conclusion can be drawn in the case of a system described by a chiral Lagrangian. The detail discussions are given in; for example, the papers by Manohar and Georgi [73], Luty [74], and Cohen *et al.* [75]. The main ideas are quite similar to those given above. The 4-pion vertex $\pi^4$ is extracted from the usual non-gauged chiral Lagrangian $\frac{1}{F^2}(\text{Tr}\partial\Sigma^\dagger\partial\Sigma)$. In this case, we have to

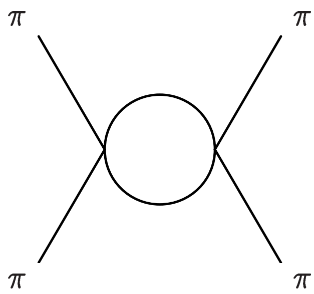



**Figure 4.2:** $\pi - \pi$ scattering in the lowest order in $p$.

take into account the one-loop contribution to the $\pi - \pi$ scattering (four powers of external momenta - and we can expect logarithmic divergence) shown in the figure 4.2 which comes with correction ofthe form

$$\frac{p^4}{F^4}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{k^2} \approx \frac{p^4}{F^4}\frac{1}{(4\pi)^2}\ln\frac{\Lambda^2_{\mathrm{CutOff}}}{\mu^2} \tag{4.182}$$

and compare them with the operators having higher mass dimensions (non-renormalisable) like

$$\frac{F^2}{\Lambda^2_{\chi SB}}\mathrm{Tr}\left(\partial\Sigma\partial\Sigma^\dagger\partial\Sigma\partial\Sigma^\dagger\right) \tag{4.183}$$

which can also produce similar contribution ($p^4$). We can also say that the interaction like (4.183) is radiatively generated by higher-order corrections to the tree level kinetic term.

The point is that once we change the renormalisation scale $\mu$ by order (of magnitude) 1, (4.182) will change by order

$$\frac{1}{(4\pi)^2}\frac{p^4}{F^4} \tag{4.184}$$

with the extra factor $(4\pi)^2$ which will eventually result in a new competition (at different subtraction scale; for example) between that loop and terms with $F^2/\Lambda^2_{\chi SB}$ by an order of magnitude $\frac{1}{(4\pi)^2}$. In other words, a condition like

$$\frac{F^2}{\Lambda^2_{\chi SB}} << \frac{1}{(4\pi)^2} \quad \text{(wrong)} \tag{4.185}$$

or

$$\Lambda_{\chi SB} >> 4\pi F \quad \text{(wrong)} \tag{4.186}$$

which we might want to assume in order to suppress the effects of the higher-dimension, non-renormalisable, operators such as that in (4.183) will not work in general. This is because the typical size of the effects from these higher dimensional operators is now of order $1/(4\pi)^2$; i.e.,

$$\frac{F^2}{\Lambda_{\chi SB}^2} \gtrsim \frac{1}{(4\pi)^2} \,. \tag{4.187}$$

Therefore, it is better to assume

$$\Lambda_{\text{CutOff}} \approx \Lambda_{\chi SB} \approx 4\pi F \tag{4.188}$$

which guarantees that the quantum corrections are of the same order of magnitude as the renormalised interaction terms and that higher dimension terms are always suppressed. At the scale $\Lambda = 4\pi F$, the non-renormalisable interactions, once suppressed by the scale $F$ in the Lagrangian, become strongly coupled. New physics is required there.

## 4.3 Vacuum Misalignment Caused by $SU(2) \times U(1)$ Breaking

This section illustrates an application of various ideas we have gathered so far; especially those on vacuum alignment and the non-linear sigma model. The model to be discussed here was proposed by Georgi and Kaplan during the mid 80s and was resurrected in 2001 with the name *Little Higgs*. We will follow some part of a series of papers by Kaplan *et al.* [12, 13], [15], outlining some of their important findings[13].

In a few words, in this class of models the Higgs (scalar) is a pseudo-Goldstone boson of a nonlinearly realised approximate global symmetry. Its mass will be protected against large radiative corrections. It is exactly massless if the symmetry is exact and the non-linear realisation trick tells us that they can have only *derivative interactions*. So the primary goal of the model is to incorporate a fundamental scalar, or whatever resembles the Higgs, into a system having global symmetry dynamically broken. Thus we expect that there are two sources of explicit symmetry breaking: one is the weak gauge

[13] Kaplan *et al.* even proposed an $SU(5)/SO(5)$ in [14] and [16], which can be considered as the "prototype" of the Littlest Higgs model presented in the next chapter. Nevertheless, we shall focus on the $SU(3) \times SU(3)/SU(3)$ model where calculations are simpler. The $SU(5)/SO(5)$ will be studied in detail in the next chapter.

symmetry that is assumed to leave electroweak symmetry unbroken, the other is the breaking of electroweak by other explicit perturbations. The first stage of symmetry breaking is *assumed* to happen at a moderately high scale by fermions, interacting via another kind of strong interaction forming condensates. We shall call the parameter characterising this scale $F$ or $F_\pi$ analogous to the pion decay constant. We shall call this interaction *ultracolour* ($UC$, for short) and call those fermions *ultrafermions*. Unlike what happens in technicolour theories, the ultrafermion condensate is aligned so that $SU(2) \times U(1)$ symmetry of the standard electroweak survives at this stage of symmetry breaking. In the language of 4.1.2, this means that there must be a room inside the unbroken subgroup for the $SU(2)_L \times U(1)_Y$ to live in. Our additional task is to find a pattern of symmetry breaking that allows the existence of a Goldstone boson transforming in the same way as the Higgs doublet.

The task of the electroweak symmetry breaking at lower scale ($M_W$) can be accomplished by introducing a perturbation that "turns" the ultracolour condensate's alignment away from its original $SU(2) \times U(1)$ preserving direction. A tiny deviation will result in a smaller scale of the electroweak symmetry breaking. The second stage of symmetry breaking is done by introducing a fundamental scalar which Yukawa interacts with both ultrafermions and the standard model fermions. This scalar will develop a vacuum expectation value when the ultrafermion condensate is rotated by the perturbation. The Higgs is a bound state between this fundamental scalar and a composite scalar (Goldstone boson).

Having a fundamental scalar in the theory provides at least one benefit: masses of the fermions can be generated via their Yukawa coupling with the (vacuum expectation value of the) scalar field. However, we see right away that this will spring up the hierarchy problem (and the need for fine-tuning) as the fundamental scalar has no symmetry to protect its mass from driving itself up to the highest mass scale of the theory. However, we shall keep dealing with the theory with the current knowledge that a special mechanism in Little Higgs model will eventually solve (or prevent) the problem. Another frustrating outcome we may have to face is that if the $\Lambda_{UC} = 4\pi F$, the global (approximate) symmetry breaking scale, is demanded to be too high the differences between this composite Higgs and the Higgs from the conventional standard model may not be easy to realise.

Now, let us begin with the spontaneous symmetry breaking at $\Lambda_{\rm UC}$, by a condensate of ultrafermions. These fermions are in a complex representation with

$$\Psi = \begin{pmatrix} \mathcal{U} \\ \mathcal{D} \\ \mathcal{S} \end{pmatrix} \tag{4.189}$$

where we clearly have a $SU(3)_L \times SU(3)_R$ in the flavour space. Then we assume the form of the condensate

$$\langle 0|\bar{\psi}^i_L \psi^j_R|0\rangle = \Lambda^3_{\mathrm{UC}} \Sigma^{ij}_0 \,, \tag{4.190}$$

instead of (4.64), where $\Sigma^{ij}_0$ is a unitary matrix. To deal with the pseudo-Goldstone degrees of freedom, we use the non-linear realisation where the bilinear (not the condensate) is,

$$\bar{\psi}^i_L \psi^j_R = \Lambda^3_{\mathrm{UC}} \Sigma^{ij} + \ldots \tag{4.191}$$

where $\langle\Sigma\rangle = \Sigma_0$. As usual, we have an octet of Goldstone bosons $\pi^a(x)$ which are introduced as fields parametrising the $G/H$ space:

$$\Sigma(x) = \mathrm{e}^{\mathrm{i}\pi^a(x)T^a/F_\pi} \Sigma_0 \mathrm{e}^{\mathrm{i}\pi^a(x)T^a/F_\pi} \tag{4.192}$$

where $\Sigma_0$ is a ("rotatable") vacuum configuration. At this stage, it might be helpful to recall the appendix A.2.1 that in this $SU(3)$ representation we have the "ultrameson" octet

$$\begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^s - & \bar{K^0} & -2\eta/\sqrt{6} \end{pmatrix} . \tag{4.193}$$

The key player here is the $K^0$ which eventually plays the role of the Higgs field.

Notice that the electroweak symmetry $SU(2) \times U(1)$ is not broken by the condensate of the form

$$\Sigma_0 = \mathrm{e}^{\mathrm{i}\theta T^8} \tag{4.194}$$

where $T^8 \equiv \lambda^8/2$ is the usual $SU(3)$ generator. This is clear as the $T^8$ behaves as an identity in the $SU(2)$ subspace and $[T^8, T^3] = 0$. If all the explicit symmetry breaking interactions are absent, the vacuum (4.194) is always obtainable by a $SU(3) \times SU(3)$ transformation. What we want to know is whether the true vacuum is still characterised by this condensate $\Sigma_0$ even after the electroweak interaction is turned on. If so the electroweak symmetry remains unbroken (when global symmetry is broken spontaneously).

The next step is to consider the *explicit* symmetry breaking terms; namely, the $SU(2) \times U(1)$ interaction, the Yukawa couplings between a fundamental scalar and fermions or ultrafermions which preserve $SU(2) \times U(1)$, and the $SU(2) \times U(1)$ invariant ultrafermion mass terms (which break the chiral flavour symmetry). Starting with the electroweak interaction, the free Lagrangian is

$$\mathcal{L}_{\text{eff}} \supset \frac{F_\pi^2}{4} \text{Tr}\left( D_\mu \Sigma D^\mu \Sigma^\dagger \right) \tag{4.195}$$

where $D_\mu$ is the usual covariant derivative

$$\begin{aligned} D_\mu \Sigma &= \partial_\mu \Sigma - \text{i} g^a A_\mu^a \left( Q^a \Sigma + \Sigma Q^{a^\dagger} \right) \\ &= \partial_\mu \Sigma - \text{i} g_L A_{L\mu}^a Q_L^a \Sigma + \text{i} g_R A_{R\mu}^a \Sigma Q_R^a \end{aligned} \tag{4.196}$$

acting on $(\mathbf{N}, \bar{\mathbf{N}})$, under which the $\Sigma$ transforms. The counterterms responsible for gauge bosons exchange, to $\mathcal{O}(Q^2)$ are required. Since the gauged $(Q_L^a, Q_R^a)$ transform as $(\text{Adj}, \mathbf{1})$ and $(\mathbf{1}, \text{Adj})$ respectively, it is found that the invariant object, to $\mathcal{O}(Q^2)$ is

$$\text{Tr}\left( Q_L^a \Sigma Q_R^a \Sigma^\dagger \right) . \tag{4.197}$$

The form of this effective potential can be thought of as an analogy from the $G_W = U(1)$ case (electromagnetism) with global symmetry breaking $SU(2)_L \times SU(2)_R / SU(2)_V$. We will follow exactly the strategy leading to (4.48) and (4.53). Denoting $q$ as a doublet (say of $u, d$ quarks), the $U(1)$ current is rotated by the $G$ rotation as

$$\begin{aligned} J_W^\mu &= -\bar{q}_L \gamma^\mu T^3 q_L + \bar{q}_R \gamma^\mu T^3 q_R + (G\,\text{singlet}) \\ &\rightarrow J_W^\mu(L, R) = -\bar{q}_L \gamma^\mu L^\dagger T^3 L q_L + \bar{q}_R \gamma^\mu R^\dagger T^3 R q_R + (G\,\text{singlet}) \end{aligned} \tag{4.198}$$

The potential from this $U(1)$ interaction

$$V_W \sim e^2 \int \text{d}^4 x D_0^{\mu\nu} \langle 0 | T(J_W^\mu(x) J_W^\nu(0)) | 0 \rangle \tag{4.199}$$

then becomes

$$\begin{aligned} V_W &\sim e^2 \text{Tr}(T^3 L^\dagger R T^3 R^\dagger L) \int \text{d}^4 x D_0^{\mu\nu} \langle 0 | T(J_{3L}^\mu(x) J_{3R}^\nu(0)) | 0 \rangle \\ &= e^2 \Delta_W \text{Tr}(T^3 \Sigma T^3 \Sigma) \end{aligned} \tag{4.200}$$

where $\Delta_W$ is assumed to be positive and $\Delta_W \sim \mathcal{O}(\Lambda^4)$ (see Dimopoulos and Preskill [76]).

Consequently, the effective Lagrangian becomes

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\text{Tr}\left(D_\mu \Sigma D^\mu \Sigma^\dagger\right) + (\Lambda_1)^4 \text{Tr}\left(Q^a \Sigma Q^a \Sigma^\dagger\right) + \dots\,, \tag{4.201}$$

where the $Q^a = Q^a_L = Q^a_R$ are the usual $SU(2) \times U(1)$ written in terms of $SU(3)$ generators

$$Q^a = \left\{gT^1, gT^2, gT^3, g'T^8/\sqrt{3}\right\}\,. \tag{4.202}$$

Next, we construct the effective potential in order to consider the vacuum alignment when the $SU(2) \times U(1)$ interaction is turned on. It was found by Kaplan and Georgi [12] that the vacuum is characterised $\Sigma_0 = \exp\{\text{i}\theta T^8\}$ which means it leaves $SU(2) \times U(1)$ in the unbroken subgroup (these generators commute). This is more or less like the case where the vacuum prefers the electrically neutral configuration ($\bar{u}_L u_R$ like) when the electromagnetic interaction is introduced as an explicit perturbation. In the present case, it "chooses" to behave as a $SU(2) \times U(1)$ singlet.

To see what happens with the ultrakaons let us consider

$$\Sigma = \exp\left\{\frac{2\text{i}}{F_\pi}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K^0/2 \\ 0 & K^0/2 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(K^0) & \text{i}s(K^0) \\ 0 & \text{i}s(K^0) & c(K^0) \end{pmatrix} \tag{4.203}$$

where $c(K^0) = \cos(K^0/F_\pi)$ and $s(K^0) = \sin(K^0/F_\pi)$. Then we can put this into the potential (4.201) and shall see that the $K^0$ get mass of order $\mathcal{O}(g^2(\Lambda_1)^2) \sim \mathcal{O}(g^2\Lambda^2)$.

It is now time to introduce a fundamental scalar (Higgs) to the theory. We expect that it will develop a vacuum expectation value at lower energy scale $\langle\phi\rangle = v << \Lambda$. Since this scalar communicates with the ultra-$K^0$ and the ultrafermions as well as the usual fermions via Yukawa coupling, we will use the doublet structure

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_4 + \text{i}\phi_5 \\ \phi_6 + \text{i}\phi_7 \end{pmatrix}\,. \tag{4.204}$$

The real fields are numbered from 4 to 7 as they will be coupled with the $T^4, \ldots, T^7$ generators; i.e.,

$$\mathcal{L}_{\text{Yukawa}} = \sum_{\alpha=4}^{7} \left( h\bar{\Psi}_R T^\alpha \Psi_L + y^* \bar{\Psi}_L T^\alpha \Psi_R \right) \phi_\alpha \tag{4.205}$$

Now observe that the chiral transformations (which also rotates the ultrafermion condensate)

$$\Psi_L \quad \rightarrow \quad \mathrm{e}^{\mathrm{i}\theta T^8} \Psi_L \tag{4.206}$$

$$\Psi_R \quad \rightarrow \quad \mathrm{e}^{-\mathrm{i}\theta T^8} \Psi_R \tag{4.207}$$

have an effect on $\bar{\Psi}_R T^\alpha \Psi_L$ for $\alpha = 4, \ldots, 7$ as

$$\bar{\Psi}_R T^\alpha \Psi_L \rightarrow \mathrm{e}^{-\mathrm{i}\theta/\sqrt{12}} \bar{\Psi}_R T^\alpha \Psi_L \, . \tag{4.208}$$

This shows how the Yukawa coupling depends on the orientation $\theta$

$$\mathcal{L}_{\text{Yukawa}}(\theta) = \sum_{\alpha=4}^{7} \left( y\mathrm{e}^{-\mathrm{i}\theta} \bar{\Psi}_R T^\alpha \Psi_L + y^* \mathrm{e}^{\mathrm{i}\theta} \bar{\Psi}_L T^\alpha \Psi_R \right) \phi_\alpha \tag{4.209}$$

The introduction of the Yukawa coupling further "tilts" the vacuum. With new explicit symmetry breaking perturbation introduced, we expect that the condensate will further re-align itself. Since,the ultrafermions have very large mass compared to the "ultrapion" scale $F$ or the strong dynamics scale $\Lambda$, it is then expected that they will not show themselves up in the low-energy effective Lagrangian. In the $\Sigma$ version (i.e., low energy effective Lagrangian), the Lagrangian (4.209) can be rewritten as

$$\mathcal{L}_{\text{Yukawa}\Sigma}(\theta) = (\Lambda_2)^3 \sum_{\alpha=4}^{7} \left( y\mathrm{e}^{-\mathrm{i}\theta} \mathrm{Tr} \left\{ T^\alpha \Sigma \right\} \phi_\alpha \right) \, . \tag{4.210}$$

As we have added a new low-energy effective interaction (the Yukawa), we also have to consider all possible interactions with the same mass dimensions (analogous to (4.200) and (4.201)). The resulting effective Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\; & \frac{F^2}{4}\mathrm{Tr}\left\{D_\mu\Sigma D^\mu\Sigma^\dagger\right\} + \frac{1}{2}\sum_{\alpha=4}^{7} D_\mu\phi_\alpha D^\mu\phi_\alpha \\ & -\frac{1}{2}m_\mu^2\sum_{\alpha=4}^{7}\phi_\alpha\phi_\alpha - \frac{\lambda}{4}\sum_{\alpha=4}^{7}(\phi_\alpha\phi_\alpha)^2 + (\Lambda_1)^4\mathrm{Tr}\left\{Q^a\Sigma Q^a\Sigma^\dagger\right\} \\ & +(\Lambda_2)^3\sum_{\alpha=4}^{7}\left(y\mathrm{e}^{-\mathrm{i}\theta}\mathrm{Tr}\left\{T^\alpha\Sigma\right\}\phi_\alpha\right) + (\Lambda_3)^4\sum_{\alpha=4}^{7}\left(y\mathrm{e}^{-\mathrm{i}\theta}\mathrm{Tr}\left\{T^\alpha\Sigma\right\}\right)^2 \\ & +(\Lambda_4)^4\sum_{\alpha=4}^{7}\left|y\mathrm{e}^{-\mathrm{i}\theta}\mathrm{Tr}\left\{T^\alpha\Sigma\right\}\right|^2 + (\Lambda_5)^4\sum_{\alpha=4}^{7}\left(y^2\mathrm{e}^{-2\mathrm{i}\theta}\mathrm{Tr}\left\{T^\alpha\Sigma T^\alpha\Sigma\right\}\right) \\ & +\text{h.c.}\,. \end{aligned} \tag{4.211}$$

Notice that the mass parameters $\Lambda_1, \ldots, \Lambda_5$ need not be equal since each operators can have its own scale which is determined by strong interactions (just like the $\Delta_W$ in (4.200) that depends on the spectral integral).

Now we will not follow the detail of their calculations but will quote the results right away. The strategy is to start with the vacuum (condensate) that preserves $SU(2)\times U(1)$ ($\mathbb{1}$ is possible) and see if we can get pseudo Goldstone bosons with positive masses at the end. It was found that the parameter which decides whether to break or not to break the $SU(2)\times U(1)$ is the $(\alpha_5)^4$ which can be positive or negative. When $(\alpha_5)^4 < 0$ the $\phi - K$ mixing will results in (see Kaplan *et al.* series of papers [15] [12]) the mass squared matrix [14]

$$M_{\phi K}^2 \sim \begin{pmatrix} M_\phi^2 & y\Lambda^2 \\ y\Lambda^2 & (g^2+y^2)\Lambda^2 \end{pmatrix}. \tag{4.212}$$

The situation at hand is interesting because if it happens that

$$M_\phi^2 < \frac{y^2}{g^2}\Lambda^2 \equiv M_{ct}^2 \tag{4.213}$$

then the determinant of the matrix $M_{\phi K}^2$ will be negative (recall the arguments leading to (4.103)). If that is the case, it was found that

$$\langle\phi_6\rangle = \frac{M_\phi^2}{\lambda}\left(1 - \frac{M_\phi^2}{M_{\text{ct}}^2}\right) \tag{4.214}$$

where $\langle\phi_6\rangle$ is expected to be of $\mathcal{O}(100\,\text{GeV})$. Also the vacuum is rotated to

$$\langle 0|\bar{\psi}_L^i\psi_R^j|0\rangle = \Lambda_{\text{UC}}^3\delta^{ij} + \mathrm{i}\varepsilon^6(T^6)^{ij}\,, \tag{4.215}$$

[14] We will come back to a case similar to this in the Littlest Higgs model in 5.2.5.4.

where

$$\varepsilon^6 \approx \frac{M_\phi^2 \langle \phi_6 \rangle}{y \Lambda^3} \,. \tag{4.216}$$

Still we will not get the right order of magnitude of the Higgs mass unless we fine tune the mass parameter

$$M_\phi^2 \sim \mathcal{O}(\frac{y^2 \Lambda^2}{g^2}) << \Lambda^2 \,. \tag{4.217}$$

To sum up, we have outlined the strategy to realise the Higgs as a pseudo-Goldstone boson from the global approximate symmetry. Though its mass can be made as low as the electroweak scale, the lack of symmetry protecting it from the radiative correction will generally drives the mass to the $\Lambda^2$ scale again unless parameters are adjusted to extremely fine detail.

In addition to the naturalness problem, electroweak precision tests do not favour this kind of model. The source of the problem is due to the fact that this Georgi-Kaplan model is constructed from the chiral Lagrangian which is bound by the cutoff $\Lambda = 4\pi F$ which is the scale where the Goldstone bosons couple strongly. The requirement that this non-linear sigma model has something to do with electroweak physics is $F \sim 100 - 200\,\text{GeV}$. Then we immediately end up at the cutoff $\Lambda \sim 1 - 2\,\text{TeV}$. This is not surprising at all and can be expected right from the start since there is no symmetry protecting the pseudo Goldstone boson from receiving mass at one-loop. In the next chapter we will see that in the Little Higgs theories, there is a symmetry forbiding the Goldstone from being massive even at one-loop.

# CHAPTER V

# LITTLE HIGGS MODELS

So far we have seen how the little hierarchy problem arises from the framework of the standard model (c.f. section 3.4.2.2) and remains in some theory that tries to go beyond (c.f. B.5). It is considered as a problem once we have convinced ourselves from both theoretical arguments and experimental facts that the Higgs should be light (c.f. 3.2) while the mass scale of the new physics should be large. Also we have studied various interesting tricks or even models that were proposed during the last three decades (c.f. 3.1, 4.1, 4.2, 4.3 ) that provide us ways to avoid parts of the problems. Still, none of them gave satisfactory results. In this chapter, we shall study how the Little Higgs model deals with those problems.

We begin in section 5.1 by re-summarising the little hierarchy problem which is one of the primary tasks for Little Higgs models to solve. Then we will sketch the essential element of the Little Higgs mechanism; namely the collective symmetry breaking. After that we will study the most economical model known as the Littlest Higgs[1] in section 5.2. Finally, in section 5.3, we present some interesting features of the Little Higgs model in the phenomenology side. Conclusions to the Little Higgs will be given in 5.4.

## 5.1 Introduction to the Little Higgs

### 5.1.1 Desired Features of the Little Higgs

We have seen in the sections 3.3 and 3.4 that electroweak precision tests call for *light Higgs* and new physics *beyond* $5-7\,\text{TeV}$. On the other hand, we have seen that the cut-off $\Lambda$ of loop integrals (which also indicates new physics) is required to be lower than $\sim 2\,\text{TeV}$ in order to stabilise electroweak symmetry breaking, making the Higgs naturally light. This little hierarchy problem put stringent constraints on any new physics not only to the Little Higgs, thanks to the advance in the precision tests during the last decade. So the idea is to find a way to push the cutoff of the theory to the safe region, say $\sim 10\,\text{TeV}$ while keeping the Higgs naturally light.

[1]For the outline of the Littlest Higgs, see page 149.

In the section 4.3 on the Georgi-Kaplan model we have seen that the Higgs can be realised as a *pseudo-Goldstone* boson of a nonlinearly realised approximate global symmetry (see also the section 4.1.2). There, this "would-be" Higgs is produced as a Goldstone boson of global symmetry breaking at high-energy scale $F = \Lambda/(4\pi)$. Then explicit symmetry breaking interactions (gauge and Yukawa) made them massive so that none of them survive massless to the physical spectrum. Still, naively breaking the symmetry (explicitly) usually results in the Goldstone bosons masses sensitive to the cut-off $\Lambda$, to one loop, like

$$m^2_{\rm GB} \sim \frac{\Lambda^2}{(4\pi)^2}\,. \tag{5.1}$$

Recall that, the requirement that the masses of the gauge fields should be of order $g^2F^2 \sim g^2v^2$ already forces $F \sim 100 - 200\,\text{GeV}$ and restricts the cut-off of the Georgi-Kaplan model to be $\Lambda = 4\pi F \sim 1 - 2\,\text{TeV}$. As a consequence, everything we have been trying to avoid remains there, both the fine-tuning and the too low lower bound of the new physics (which is not favoured by precision electroweak tests).

The philosophy of the Little Higgs models is to construct a model that has a mechanism to avoid the mass generation of the Goldstone boson at one-loop. A modified version of the Georgi-Kaplan is then very tempting. Then the Goldstone boson will be forced to receive mass (squared) of two-loop order

$$m^2_{GB} \sim \left(\frac{1}{(4\pi)^2}\right)^2 \Lambda^2 = \frac{F^2}{(4\pi)^2} \tag{5.2}$$

which is further suppressed by a factor $1/16\pi^2$. If we can find a way to do so, the mass of the Higgs will come out naturally light for large value of $\Lambda$, say $\Lambda \sim 10\,\text{TeV}$ or more. The other by-product is that we can have the scale $F$ as high as $1\,\text{TeV}$ or so. The immediate benefit of doing so is that in the Little Higgs models the plethora of Goldstone bosons and gauge bosons due to large group needed to accomplish the Little Higgs mechanism (see next section) will be pushed up to the energy scale where no direct detection has reached. This feature is in agreement with what we have argued, in section 3.4.2.3, on the cancellation mechanisms (of quadratic divergent diagrams) between particles of the standard model and the extra heavy particles. Depending on the specific model, the footprints of these particles on electroweak observables may or maynot significantly alter the results of the precision electroweak tests.

The job of focusing on the low-energy degrees of freedom is taken care by a non-linear sigma model. The benefit of using a low-energy effective field theory strategy is that many possibilities are available for us to take, if the low-energy end agrees with the standard model. This is why there are many versions of Georgi-Kaplan models or the Little Higgs models being mass produced. Nevertheless, economical models having one Higgs doublet are preferable. For phenomenological purpose at energies far below the cut-off scale ($\Lambda = 4\pi F$), the origin of the global symmetry breaking should not concern us. We just have to keep working at a scale below this particular cut-off otherwise all higher-order non-perturbative terms become important (and may not be calculable). Nevertheless, we have to keep in mind that we should not work at a scale way too far below the cut-off or take $F \to \infty$ (which is completely unreasonable, in fact) as that would mean we lose the power of the non-linear sigma model. If that is the case, the Higgs from the Little Higgs will be identical to the Higgs from the standard model. So the model will cease to be useful since we cannot tell the Higgs apart and the standard model use less ingredient to explain the same phenomena. We therefore expect some deviations (foe example, by (two-loop) order $\mathcal{O}(\frac{v^2}{F^2})$) from the properties of the Higgs predicted by the standard model for finite $F$.

There are basically two different types of the Little Higgs models if we classify them by the structure of the gauge groups that are broken. One is a class of models where *a number of gauge groups are broken down to the standard model group, using one linear sigma model.* They are referred to as *Product Group Models.* The Littlest Higgs model, by Arkani-Hamed *et al.* [21], that will be focused on in this thesis falls into this category. The other types of the Little Higgs is those where *a single larger gauge group are broken down to the standard model one by a number of sigma models.* These latter kinds are known as the *Simple Group Models.* The Simplest Little Higgs by Schmaltz [35] is one of these models.

### 5.1.2 The Little Higgs Mechanism: Collective Symmetry Breaking

To prevent the one-loop correction to the mass of the Higgs another crucial feature of Little Higgs; namely, the *collective symmetry breaking* is introduced. We will concentrate on the mechanism designed for using with the *product group models* which includes the model that we will study in detail: the Littlest Higgs model. The mechanism can be summarised in words (in the paper by Katz *et al.* [77]) as "*no single term in the Lagrangian breaks **all** the symmetry which is protecting the Higgs mass*". In other words, the mass of the Higgs is protected by introducing another "partially broken" *global symmetries*. One of these global symmetries, alone, must be capable of acting non-linearly[2] on the Higgs doublet in the form

$$H \longrightarrow H + \varepsilon \tag{5.3}$$

and hence preventing the mass term $M^2|H|^2$ from being generated by the radiative corrections. Then explicit breaking interactions are carefully introduced such that each of these coupling, alone, will not break all those global symmetries that protect the Higgs mass. As a result, it is possible to push the quadratic divergence away from one-loop to higher loops, if several explicit symmetry breaking interactions are turned on at the same time.

To make the story a bit less abstract, we consider the gauge sector of the model. The simplest case comes from a model having two independent global symmetry groups $G_1$, $G_2 \in G$; i.e., $[G_1, G_2] = 0$, and gauge each[3] subgroup $G_{W(i)}$ of these $G_i$. When a subgroup is gauged, we have the situation similar to those studied in sections on dynamical symmetry breaking and vacuum alignment (see subsections of 4.1). We see that gauging one subgroup, say $G_{W(1)}$, will leave another global group ($G_2$) survive, and vice versa. The global symmetry is carefully arranged so that each is enough to protect the mass of the Higgs (pseudo Goldstone boson). Denoting $g_i$ as a gauge coupling of $G_{W(i)}$ we find the quadratic divergent mass of the Higgs

$$m_{GB}^2 \sim \frac{g_1^2}{(4\pi)^2}\frac{g_2^2}{(4\pi)^2}\Lambda^2 \tag{5.4}$$

Other Goldstone bosons are not protected by global symmetry and will receive masses sensitive quadratically to the cut-off scale. Therefore, these particles will be "pushed" up to high-energy region of the theory, leaving the Higgs doublet alone in the low-energy spectrum of the non-linear sigma model. Similar situation is happening with the gauge symmetry breaking which is triggered by the global symmetry breaking. Some gauge fields will receive masses of a

[2] Remember that broken symmetry can be realised non-linearly.

[3] This is why the model belongs to the class of "product group". More than one gauge symmetry groups are broken to get the diagonal subgroup corresponding to the electroweak group.

few TeV, while there will be massless gauge fields associated with the survivor (unbroken) subgroup to play the role of electroweak gauge fields. Electroweak spontaneous symmetry breaking is then generated by a Coleman-Weinberg mechanism (see chapter III) and Little Higgs mechanism has a good explanation to keep this scale low and have the Higgs naturally light.

There is another way to view the situation mentioned above from the bottom up. The quadratic divergent diagrams of the Higgs mass due to the standard model particles are always there. Then heavy particles are introduced so as to cancel those diagrams. Unlike the supersymmetry trick where the cancellations occur between particles with different spin statistics (particles and their superpartners) where the "equality of the couplings" is taken care by supersymmetry, in the Little Higgs models the cancellations take place between particles and their "heavier partners" but with the same spin statistics, and the same quantum numbers. Cancellations in this manner require the "persistent" relations between the gauge couplings and are taken care by some careful placement of the gauge generators onto a particular representation[4]. This job is directly related to the collective symmetry breaking mechanism.

There are also severe quadratic divergent diagrams generated by the top quark. It too requires cancellations. A heavy top-like fermion is required, with a definite coupling with the Higgs, to cancel the divergences. The remaining quadratic divergent diagram caused by the self-coupling of the standard model Higgs is also cancelled by the introduction of the heavy partner. We shall see how the cancellations happen in a specific model as we go on.

To sum up, quadratic divergence diagrams from bosons are cancelled by *similar diagrams of other bosons*. Similarly, fermions cancel fermions. Therefore the particle spectrum, though quite rich, may not be as rich as that of the supersymmetry[5]. This may not be a philosophically beautiful outcome but may be easier on the phenomenology side. According to the fine tuning arguments, at TeV scales it is expected to find some heavy pseudo-Goldstone bosons, extra gauge bosons, and "some" extra fermions. In fact, the latter were termed "some" because the fine-tuning argument requires only the top-like partner exists, among the fermioninc partners, due to the fact that severeness

[4]Recall the cancellations of unwanted diagrams from those involved charged-changing current correspond to $W^{\pm}$ and those involved the neutral currents ($Z$) which are not achievable without the $SU(2)$ symmetry relating the coupling constants.

[5]Of course, this is not always the case. But it is likely to happen in this way if we compare "minimal" models from each theory.

quadratic divergences from fermion are mostly due to the standard model top quark.

In this thesis we shall concentrate on the minimal model known as the *Littlest Higgs* and shall see how phenomena described above happen.

## 5.2 The "Littlest Higgs"

In the Littlest model, we shall assume that *for some reason*, a global symmetry $SU(N)$ is broken down to $SO(N)$ which contains the standard model group $SU(2)_L \times U(1)_Y$. The origin of the breaking was not specified in the model but we might think of a possible mechanism initiated by some kinds of fermions transforming under a *real* representation of a strong interaction-like group (e.g., the ultracolour). Since we are requiring that the Higgs be realised as a pseudo-Goldstone boson, we must find a specific group so that some of the Goldstone boson transform under $SU(2) \times U(1)$ like the standard model Higgs doublet. The smallest rank 4 group $SU(4)$ is not qualified because there are no rooms for the Higgs doublet.

So the simplest group to be used is the $SU(5)$ which is the one used in the original Littlest Higgs model proposed by Arkani-Hamed *et al.* [21]. The breaking of $SU(5)/SO(5)$ happens at a scale $F$ by a vacuum expectation value of $1-2\,\mathrm{GeV}$ with the cutoff $\Lambda = 4\pi F$. The other requirement that there must be a room for $SU(2) \times U(1)$ of the standard model in the unbroken subgroup $SO(5)$ is also satisfied by the structure of the group.

First, we will consider the implementation of the salient trick of the Little Higgs; i.e., the collective symmetry breaking. We introduce *two* copies of $SU(2) \times U(1)$ gauge groups. In other words, we gauge two subgroup of $G$, $G_1 = SU(2)_1 \times U(1)_1$ and $G_2 = SU(2)_2 \times U(1)_2$. At the scale $\Lambda_S$, the same condensate that breaks the $SU(5)$ also break the product gauge groups $[SU(2) \times U(1)]^2$ down to the standard model electroweak group. The trick of collective symmetry breaking is accomplished with the requirement that each $G_i$ commutes with a different (global) subgroup $X_i$ of $G$ which, alone, is enough to protect the Goldstone boson from being massive. The group structure is $G \supset G_1 \times X_1 + G_2 \times X_2$. Only when both gauge interactions are turned on will the Goldstone boson be massive. This implies the masses of the Goldstones are proportional to $g_1 g_2 \Lambda^2/(4\pi)^4$ where the $g_i$'s are the gauge couplings. We see right away the extra $\frac{1}{(4\pi)^2} \sim \frac{1}{160}$ factor suppressing this quadratic divergence.

As we have mentioned before, the electroweak symmetry is broken in the second step due to the gauge and Yukawa interactions (which break the global symmetry explicitly). The Coleman-Weinberg mechanism (see 3.1) is used to explain the electroweak symmetry breaking.

This section on the Littlest Higgs is rather long and is organised as follows: First, in the section 5.2.1 we begin by setting up the chiral Lagrangian of the model and work out the gauge sector resulting from the first stage of the global symmetry breaking. After that we will study the application of the collective symmetry breaking from the top-down approach in section 5.2.2, and from the bottom-up (where we can see how the loops cancel) in section 5.2.3. Then, in section 5.2.4, we turn to the quark sector of the model where the extra top quarks are introduced. At that time we will have particles in their gauge eigenstates corresponding to the first stage of gauge symmetry breaking. Then, in 5.2.5, we will consider the second stage; namely, the electroweak symmetry breaking, where we will study the Coleman-Weinberg mechanism. After electroweak symmetry breaks, we need to change the basis of the particles in the theory to that of the "final" eigenstate. This will be done in section 5.2.6.

### 5.2.1 The Sigma Model and Gauge Sector

Once we assumed that the global symmetry breaks, we can start the study with a non-linear sigma model the was introduced in section 4.2.3. We will begin with the Lagrangian (4.177)

$$\mathcal{L} = \frac{F^2}{8}\mathrm{Tr}\left\{\partial_\mu\Sigma^\dagger\partial_\mu\Sigma\right\} , \tag{5.5}$$

where the $\Sigma$ and the "pion" fields are defined in (A.41).

Then we introduce gauge interactions so that the global symmetry is broken explicitly. According to the collective symmetry breaking mechanism, 2 subgroups are gauged, which, in the littlest Higgs model is $[SU(2)\times U(1)]^2$. To leave rooms for global symmetries to protect the pseudo-Goldstone bosons (the Higgs) mass, these gauge generators must be placed at the right location of the $SU(5)$ matrices. Very soon, it will be obvious that the $SU(2)\times U(1)$ subgroups can be gauged as follows:

$$Q_1^a = \left(\begin{array}{c|c} \tau^a & \\ \hline & 0_{3\times 3}\end{array}\right), \qquad Y_1 = \mathrm{diag}(-3,-3,2,2,2)/10\,,$$
$$Q_2^a = \left(\begin{array}{c|c} 0_{3\times 3} & \\ \hline & -\tau^{a*}\end{array}\right), \quad Y_2 = \mathrm{diag}(-2,-2,-2,3,3)/10\,, \tag{5.6}$$

which clearly leave room for a global $SU(3)$ in each subspace. Also notice the appearance of the minus sign in $Q^2$. Gauging the subgroup means we use the gauge covariant derivative

$$D_\mu\Sigma = \partial_\mu\Sigma - \mathrm{i}\sum_{j=1}^{2}\left[g_j W_{j\mu}^a(Q_j^a\Sigma + \Sigma Q_j^{a\mathsf{T}}) + g_j' B_{j\mu}(Y_j\Sigma + \Sigma Y_j^{\mathsf{T}})\right]\,. \tag{5.7}$$

As usual, we will use the following shorthands for the gauge fields[6]

$$W_{i\mu} = W_{i\mu}^a Q_i^a\,, \quad B_{i\mu} = B_{i\mu}Y_i. \tag{5.8}$$

Consequently, the Lagrangian for describing low energy dynamics is

$$\mathcal{L} = \frac{F^2}{8}\mathrm{Tr}\left\{D_\mu\Sigma^\dagger D_\mu\Sigma\right\}\,. \tag{5.9}$$

Having the Lagrangian (5.9) on hand, we can work out the mass term for the gauge fields (after the $\Sigma$ receives a vacuum expectation value). Terms relevant to our consideration are

$$\mathrm{Tr}\sum_{j,k}\left[g_j W_{j\mu}^a(Q_j^a\Sigma_0 + \Sigma_0 Q_j^{a\mathsf{T}})\right]\left[g_j W_{k\mu}^b(Q_k^b\Sigma_0 + \Sigma_0 Q_k^{b\mathsf{T}})\right] \tag{5.10}$$

for $SU(2)$ and

$$\mathrm{Tr}\sum_{j,k}\left[g_j' B_{j\mu}(Y_j\Sigma_0 + \Sigma_0 Y_j^{\mathsf{T}})\right]\left[g_k' B_{k\mu}(Y_k\Sigma_0 + \Sigma_0 Y_k^{\mathsf{T}})\right] \tag{5.11}$$

for $U(1)$ contributions. What remains are brute force. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{F^2}{8}\left[g_1^2 W_{1\mu}^a W_1^{a\mu} - 2g_1g_2W_{1\mu}^aW_2^{a\mu} + g_2^2W_{2\mu}^aW_2^{a\mu}\right]\\ &\quad+\frac{1}{5}\frac{F^2}{8}\left[g_1^2B_{1\mu}B_1^\mu - 2g_1g_2B_{1\mu}B_2^\mu + g_2^2B_{2\mu}B_2^\mu\right]\,.\end{aligned} \tag{5.12}$$

[6]The "hat" notation like $\hat{W}_{i\mu} = W_{i\mu}^a Q_i^a$, introduced in the section B on grand unification theory, will not be used in this section to reduce eyestrain.

Notice the appearances of the numerical factors preceding both terms. They come from expressions like

$$\mathrm{Tr}\left\{\Sigma_0 Q_i Q_j \Sigma_0\right\} \tag{5.13}$$

and

$$\mathrm{Tr}\left\{\Sigma_0 Y_i Y_j \Sigma_0\right\} \tag{5.14}$$

together with some generators being transposed. These terms depends on how we gauge the subgroups with respect to the vacuum. In our choices, the non-zero terms are

$$\begin{aligned}
\mathrm{Tr}\left\{\Sigma_0 Q_i^a Q_j^b \Sigma_0\right\} &= \mathrm{Tr}\left\{\Sigma_0 {Q_i^a}^{\mathsf{T}} {Q_j^b}^{\mathsf{T}} \Sigma_0\right\} = \frac{1}{2}\delta_{ij}\delta^{ab} \\
\mathrm{Tr}\left\{\Sigma_0 Q_i^a \Sigma_0 {Q_j^b}^{\mathsf{T}}\right\} &= \mathrm{Tr}\left\{\Sigma_0 {Q_i^a}^{\mathsf{T}} \Sigma_0 Q_j^b\right\} = -\frac{1}{2}\delta^{ab}\,, \quad i \neq j \\
\mathrm{Tr}\left\{\Sigma_0 Y_i Y_i \Sigma_0\right\} &= \frac{3}{10} \\
\mathrm{Tr}\left\{\Sigma_0 Y_i \Sigma_0 Y_i\right\} &= -\frac{1}{5} \\
\mathrm{Tr}\left\{\Sigma_0 Y_i Y_j \Sigma_0\right\} &= +\frac{1}{5}\,, \quad i \neq j \\
\mathrm{Tr}\left\{\Sigma_0 Y_i \Sigma_0 Y_j\right\} &= -\frac{3}{10}\,, \quad i \neq j
\end{aligned} \tag{5.15}$$

where we have used $\mathrm{Tr}\tau^a\tau^b = 2\delta^{ab}$.

Now we observe that the condensate $\Sigma_0$ breaks the gauge symmetry $[SU(2)\times U(1)]^2$ down to the diagonal subgroup $SU(2)\times U(1)$ corresponding to the generator $Q_1^a + Q_2^a$ and $Y^1 + Y^2$ satisfying the symmetric condition (4.166) of the generators. Since we are expecting a breaking of one of the $SU(2)\times U(1)$'s, we can use the "gauge freedom" in the transformation

$$\Sigma = \mathrm{e}^{\mathrm{i}\alpha\cdot Q_1}\mathrm{e}^{\mathrm{i}\beta Y_1}\mathrm{e}^{\mathrm{i}\gamma\cdot Q_2}\mathrm{e}^{\mathrm{i}\delta Y_2}\mathrm{e}^{\mathrm{i}\Pi\cdot X/F}\Sigma_0\mathrm{e}^{\mathrm{i}\Pi\cdot X^{\mathsf{T}}/F}\mathrm{e}^{\mathrm{i}\alpha\cdot {Q_1}^{\mathsf{T}}}\mathrm{e}^{\mathrm{i}\beta {Y_1}^{\mathsf{T}}}\mathrm{e}^{\mathrm{i}\gamma\cdot {Q_2}^{\mathsf{T}}}\mathrm{e}^{\mathrm{i}\delta {Y_2}^{\mathsf{T}}} \tag{5.16}$$

to remove some of the Goldstone bosons; namely, the triplet $\Omega$ and the singlet $\eta$. So the remaining Goldstones are[7]

$$\Pi \cdot X = \begin{pmatrix} & H^\dagger & \Phi^\dagger \\ H & & H^* \\ \Phi & H^\top & \end{pmatrix} . \tag{5.17}$$

This means we can find a mixing angle (angles, in fact) to rotate the degrees of freedom into the massive and massless gauge fields in a way fairly similar[8] to what we have done in (2.90). Let us call these massive gauge fields $W'$ and $B'$ and the massless ones $W$ and $B$. The original mass matrices defined from (5.12) are

$$M^2_{W_1W_2} = \frac{F^2}{4}\begin{pmatrix} g_1^2 & -g_1g_2 \\ -g_1g_2 & g_2^2 \end{pmatrix} \tag{5.18}$$

$$M^2_{B_1B_2} = \frac{1}{5}\frac{F^2}{4}\begin{pmatrix} g_1'^2 & -g_1'g_2' \\ -g_1'g_2' & g_2'^2 \end{pmatrix} . \tag{5.19}$$

which clearly tell us that we will have massive and massless fields (recall the Weinberg angle in (2.92)) in the mass eigenstates. The new diagonal mass matrices with respect to

$$\mathcal{L} = \frac{1}{2}M_W^2 W^a_\mu W^{a\mu} + \frac{1}{2}M^2_{W'} W'^a_\mu W'^{a\mu} + \frac{1}{2}M_B^2 B^a_\mu B^a_w + \frac{1}{2}M^2_{B'} B'^a_\mu B'^a_w \tag{5.20}$$

are obtained by the massive fields

$$\begin{aligned} W'^\mu &= -\cos\psi W_1^\mu + \sin\psi W_2^\mu \\ B'^\mu &= -\cos\psi' B_1^\mu + \sin\psi' B_2^\mu . \end{aligned} \tag{5.21}$$

where

$$\sin\psi = \frac{g_2}{\sqrt{g_1^2+g_2^2}} \qquad \sin\psi' = \frac{g_2'}{\sqrt{g_1'^2+g_2'^2}} . \tag{5.22}$$

---

[7]Caution: Please note that since we are running out of symbols, in this section we use symbols that are somewhat different from those used in chapter II and B. There we used $H$ for the Higgs doublet and $h$ for the physical Higgs (the real, neutral, one). *Now we use $H$ for the Goldstone bosons that will be eventually be "parts" of the Higgs and $\Phi$ for the triplet.*

[8]One thing to notice about the mixing angles is that unlike the Weinberg angle, they do not correspond to the mixing between the $SU(2)$ and $U(1)$ gauge fields. They characterise the mixing of fields of the same type.

Then the masses of these fields are

$$\begin{aligned} M_{W'} &= \frac{F}{2}\sqrt{g_1^2+g_2^2} = \frac{gF}{\sin 2\psi} \\ M_{B'} &= \frac{F}{2\sqrt{5}}\sqrt{g_1'^2+g_2'^2} = \frac{g'F}{\sqrt{5}\sin 2\psi'} \,. \end{aligned} \tag{5.23}$$

where $g = g_1 g_2/\sqrt{g_1^2+g_2^2}$ and $g' = g_1' g_2'/\sqrt{g_1'^2+g_2'^2}$ are the $SU(2)_L$ and $U(1)_Y$ couplings in the standard model, respectively. Consequently, the remaining massless gauge fields at this stage of symmetry breaking are fields orthogonal to (5.21):

$$\begin{aligned} W^\mu &= \sin\psi W_1^\mu + \cos\psi W_2^\mu \\ B^\mu &= \sin\psi' B_1^\mu + \cos\psi' B_2^\mu \,, \end{aligned} \tag{5.24}$$

with

$$\begin{aligned} M_W &= 0 \\ M_B &= 0\,, \end{aligned} \tag{5.25}$$

which will be identified with the gauge fields of the electroweak interaction. The gauge fields in (5.23), (5.23), (5.24) and (5.24) are living in their mass eigenstates just in the region between the global symmetry breaking scale $\Lambda$ and the electroweak symmetry scale, say $v$. The electroweak symmetry breaking will generate vacuum expectation values for some of the Goldstone bosons and hence further introduce mixings between them.

### 5.2.2 Collective Symmetry Breaking in the Littlest Higgs Model

Having the pictures of the Goldstone bosons matrix, and the structure of the gauge coupling in mind, we can turn to the important feature of the model - the collective symmetry breaking in the gauge sector, mentioned in the introduction. The idea that the gauge symmetry is broken down to its diagonal subgroup $SU(2)\times U(1)$ suggests that we can concentrate on the relevant degrees of freedom of the Goldstone boson matrix

$$\Pi^z X^z = \begin{pmatrix} & & \pi^-/\sqrt{2} & \phi^{--} & \frac{\phi^-}{\sqrt{2}} \\ & & \pi^{0*}/2 & \frac{\phi^-}{\sqrt{2}} & \phi^{0*} \\ \pi^+/\sqrt{2} & \pi^{0*}/2 & & \pi^-/\sqrt{2} & \pi^0/2 \\ \phi^{++} & \frac{\phi^+}{\sqrt{2}} & \pi^+/\sqrt{2} & & \\ \frac{\phi^+}{\sqrt{2}} & \phi^0 & \pi^0/2 & & , \end{pmatrix} \tag{5.26}$$

where the missing elements (the triplet, and the singlet) can be feed to the gauge fields. With this matrix, we can see how the global symmetries "protect" the mass of the Higgs doublet by acting non-linearly on it so the mass terms like $M_2 H^\dagger H$ are forbidden due to a "shift" symmetry (see the section 4.2). By observing the form of the generators in (5.14) we see that the both $Q_1^a$ and $Y_1$ commute with the $SU(3)$ generators embedded in the lower-right block of the $SU(5)$ matrices. Similarly, there is also $SU(3) \in SU(5)$ generators in the upper-left block commuting with the $Q_2^a$ and $Y_2$. These $SU(3)$ global symmetry, left survived by some subset of the broken generators, manifest when one species of the gauge interactions are turned off. First consider the Higgs (a Goldstone boson at the moment) alone

$$\Pi \cdot X \supset \begin{pmatrix} & & h^- & & \\ & & h^{0*} & & \\ h^+ & h^0 & & h^- & h^{0*} \\ & & h^+ & & \\ & & h^0 & & \end{pmatrix}, \tag{5.27}$$

where we have changed the normalisations for convenience. Observe that in each block the Higgs fields can be written in terms of the $SU(3)$ generators as, taking $h^- = h^{+*}$,

$$\begin{pmatrix} 0 & 0 & h^{+*} \\ 0 & 0 & h^{0*} \\ h^+ & h^0 & 0 \end{pmatrix} = \mathrm{Re}(h^+)\lambda_4 + \mathrm{Im}(h^+)\lambda_5 + \mathrm{Re}(h^0)\lambda_6 + \mathrm{Im}(h^0)\lambda_7 \tag{5.28}$$

and, similarly,

$$\begin{pmatrix} 0 & h^{+*} & h^{0*} \\ h^+ & 0 & 0 \\ h^- & 0 & 0 \end{pmatrix} = \mathrm{Re}(h^+)\lambda_1 + \mathrm{Im}(h^+)\lambda_2 + \mathrm{Re}(h^0)\lambda_4 + \mathrm{Im}(h^0)\lambda_5 \tag{5.29}$$

In this way, we can turn off the gauge couplings one by one and see how the residue global $SU(3)$ generators act on the Higgs doublet. First, let us switch off the $G_{W(1)}$ gauge fields. Next, recall that the Gellman-matrices $\lambda_a$ of the $SU(3)$ are embedded in the $SU(5)$ generators as

$$\tilde{\lambda}^a = \begin{pmatrix} \lambda^a & 0 \\ 0 & 0_{2\times 2} \end{pmatrix}, \tag{5.30}$$

which we have named it the $SU(3)_2$ in the introduction section. Expanding the exponentials of $\Sigma$, we find the series with respect to powers of $F$

$$\Sigma = \Sigma_0 + \frac{2\mathrm{i}}{F}(\Pi \cdot X)\Sigma_0 - \frac{2}{F^2}(\Pi \cdot X)^2\Sigma_0 + \cdots \tag{5.31}$$

we then find the action of the global $SU(3)_1 \subset SU(5)$ on the $\Sigma$

$$\begin{aligned} \Sigma' &= \mathrm{e}^{2\mathrm{i}\Pi'\cdot X/F}\Sigma_0 = e^{\mathrm{i}\theta\cdot\tilde{\lambda}}\mathrm{e}^{2\mathrm{i}\Pi\cdot X/F}\Sigma_0 e^{\mathrm{i}\theta\cdot\tilde{\lambda}^{\mathsf{T}}} \\ &\approx \left(\mathbb{1} + \mathrm{i}\theta^a \begin{pmatrix} \lambda^a & 0 \\ 0 & 0_{2\times 2} \end{pmatrix}\right) \\ &\qquad \left\{\Sigma_0 + \frac{2\mathrm{i}}{F}(\Pi \cdot X)\Sigma_0\right\}\left(\mathbb{1} + \mathrm{i}\theta^a \begin{pmatrix} \lambda^{a\mathsf{T}} & 0 \\ 0 & 0_{2\times 2} \end{pmatrix}\right) \\ &= \Sigma_0 + \mathrm{i}\Sigma_0\theta^a \begin{pmatrix} \lambda^{a\mathsf{T}} & 0 \\ 0 & 0_{2\times 2} \end{pmatrix} + \mathrm{i}\theta^a \begin{pmatrix} \lambda^a & 0 \\ 0 & 0_{2\times 2} \end{pmatrix}\Sigma_0 + \frac{2\mathrm{i}}{F}\Pi \cdot X\Sigma_0 \end{aligned} \tag{5.32}$$

Since terms proportional to $\lambda^1$, $\lambda^2$, $\lambda^3$ and $\lambda^8$ do not harm the Higgs doublet, let us, for simplicity, take only the $\theta^4, \ldots, \theta^7$ to be non-zero,. Using the Higgs doublet defined in (5.27), we find

$$\begin{aligned} \Sigma_0 + \frac{2\mathrm{i}}{F}\Pi' \cdot X\Sigma_0 \;\approx\; & \Sigma_0 + \begin{pmatrix} 0 & 0 & \theta_{45}^* & 0 & 0 \\ 0 & 0 & \theta_{67}^* & 0 & 0 \\ -\theta_{45}^* & -\theta_{67}^* & 0 & \theta_{45} & \theta_{67} \\ 0 & 0 & -\theta_{45} & 0 & 0 \\ 0 & 0 & -\theta_{67} & 0 & 0 \end{pmatrix} \\ & + \frac{2\mathrm{i}}{F}\Pi \cdot X\Sigma_0 \end{aligned} \tag{5.33}$$

where

$$\theta_{45} = \theta^4 + \mathrm{i}\theta^5, \qquad \theta_{67} = \theta^6 + \mathrm{i}\theta^7 \,. \tag{5.34}$$

We did not work out the last term, $F\Pi \cdot X\Sigma_0$, in (5.33) because these terms will not contribute to the Higgs' mass. Alone, the transformations induced non-linearly on the Higgs fields

$$\begin{aligned} h^+ &\rightarrow h^+ - F\theta^*_{45} + \ldots \\ h^0 &\rightarrow h^0 - F\theta^*_{67} + \ldots \end{aligned} \tag{5.35}$$

are enough to guarantee their masslessness. In a similar manner, the lower-right global $SU(3)$ also protects the Higgs by acting non-linearly on the doublet.

Let us return to (5.32) and keep all the $\theta$'s in the transformation and as well as recover the $\Phi$ triplet so that we can find a more general form of the transformation. The global $SU(3)_1$ (lower-right) symmetry with parameters $\epsilon \sim F\theta$ acts on $H$ and $\Phi$ as

$$G_{W(1)}: \begin{cases} H_i \longrightarrow H_i + \epsilon_i + \ldots \\ \Phi_{ij} \longrightarrow \Phi_{ij} - \mathrm{i}(\epsilon_i H_j + \epsilon_j H_i) + \ldots \end{cases} \tag{5.36}$$

and similarly for the $SU(3)_2$ (upper-left) with parameter $\eta$ we have

$$G_{W(1)}: \begin{cases} H_i \longrightarrow H_i + \eta_i + \ldots \\ \Phi_{ij} \longrightarrow \Phi_{ij} + \mathrm{i}(\eta_i H_j + \eta_j H_i) + \ldots \end{cases} \tag{5.37}$$

where $i, j$ run over component indices of each field. The appearances of the $H$ comes from the small $\mathbb{1}_{2\times 2}$ from $\Sigma_0$.

### 5.2.3 Bottom-up Approach of the Collective Symmetry Breaking

We have seen how the cancellation goes in the context of collective symmetry breaking which can be considered as the top-down approach. Now we will use the bottom-up strategy instead and workout the terms in the Lagrangian that can lead to quadratic divergences of the Higgs mass. This can be done by expanding terms contributing to $H^\dagger H$. In the realm of the standard model we have seen the quadratic divergences due to electroweak gauge fields. So we will make sure that in the case of Little Higgs the contributions from the gauge field corresponding to each $G_{W(i)}$ alone is harmless and their collective effect will not generate vertices leading to quadratic divergences when 1-loop effects are taken into account.

The strategy in this section is simple: we collect terms from the

Lagrangian (5.9) that are proportional to $AAS^\dagger S$ where $A$ stands for $W_i^a$ or $B_i$ and $S$ stands for scalar fields. The remaining tasks are straightforward The Lagrangian to begin with is

$$\mathcal{L} = \frac{F^2}{8}\mathrm{Tr}\Bigg\{ \Bigg[\partial_\mu\Sigma - \mathrm{i}\sum_{j=1}^{2}\Big[g_j W^a_{j\mu}(Q^a_j\Sigma + \Sigma Q^{a\,\mathsf{T}}_j) + g'_j B_{j\mu}(Y_j\Sigma + \Sigma Y_j^{\mathsf{T}})\Big]\Bigg]^\dagger \times \Bigg[\partial_\mu\Sigma - \mathrm{i}\sum_{j=1}^{2}\Big[g_j W^a_{j\mu}(Q^a_j\Sigma + \Sigma Q^{a\,\mathsf{T}}_j) + g'_j B_{j\mu}(Y_j\Sigma + \Sigma Y_j^{\mathsf{T}})\Big]\Bigg]\Bigg\}. \tag{5.38}$$

First let us consider the contributions from $WWH^\dagger H$ diagrams. They come from

$$\begin{aligned}
\mathcal{L}_{\Sigma-W_{1,2}} &= \frac{F^2}{8}\mathrm{Tr}\Big\{ \Big[-\mathrm{i}g_1(W_{1\mu}\Sigma + \Sigma W_{1\mu}^{\mathsf{T}}) - \mathrm{i}g_2(W_{2\mu}\Sigma + \Sigma W_{2\mu}^{\mathsf{T}})\Big]^\dagger \\
&\qquad \Big[-\mathrm{i}g_1(W_1^\mu\Sigma + \Sigma W^{\mu\,\mathsf{T}}_1) - \mathrm{i}g_2(W_2^\mu\Sigma + \Sigma W^{\mu\,\mathsf{T}}_2)\Big]\Big\} \\
&= \frac{F^2}{8}\mathrm{Tr}\Big\{ \Big[\mathrm{i}g_1(\Sigma^\dagger W_{1\mu} + W^*_{1\mu}\Sigma^\dagger) + \mathrm{i}g_2(\Sigma^\dagger W_{2\mu} + W^*_{2\mu}\Sigma^\dagger)\Big] \\
&\qquad \Big[-\mathrm{i}g_1(W_1^\mu\Sigma + \Sigma W^{\mu\,\mathsf{T}}_1) - \mathrm{i}g_2(W_2^\mu\Sigma + \Sigma W^{\mu\,\mathsf{T}}_2)\Big]\Big\} \\
&= \frac{F^2}{8}\mathrm{Tr}\Big\{ g_1^2(\Sigma^\dagger W_{1\mu} + W^*_{1\mu}\Sigma^\dagger)(W_1^\mu\Sigma + \Sigma W_1^{\mu\,\mathsf{T}}) \\
&\qquad + g_2^2(\Sigma^\dagger W_{2\mu} + W^*_{2\mu}\Sigma^\dagger)(W_2^\mu\Sigma + \Sigma W_2^{\mu\,\mathsf{T}})\Big\} + \mathcal{L}_{g_1g_2}
\end{aligned} \tag{5.39}$$

Notice that we have separated out the terms involving $g_1g_2$ into $\mathcal{L}_{g_1g_2}$. These mixing terms obviously vanish when one of the couplings is turned off. Then

$$\begin{aligned}
\mathcal{L}_{\Sigma-W_{1,2}} &= \frac{F^2}{8}g_1^2\mathrm{Tr}\Big\{\Sigma^\dagger W_{1\mu}\Sigma W_1^{\mu\,\mathsf{T}} + \Sigma^\dagger W_{1\mu}\Sigma W_1^{\mu *}\Big\} \\
&\quad + \frac{F^2}{8}g_2^2\mathrm{Tr}\Big\{\Sigma^\dagger W_{2\mu}\Sigma W_2^{\mu\,\mathsf{T}} + \Sigma^\dagger W_{2\mu}\Sigma W_2^{\mu *}\Big\} \\
&\quad + \mathcal{L}_{g_1g_2} + \mathrm{Tr}\{WW\}\,\text{terms} \\
&= \frac{F^2}{8}g_1^2\mathrm{Tr}\Big\{\Sigma^\dagger Q_1^a\Sigma Q_1^{b\,\mathsf{T}} + \Sigma^\dagger Q_1^a\Sigma Q_1^{b\,\mathsf{T}}\Big\}W^a_{1\mu}W_1^{b\mu} \\
&\quad + \frac{F^2}{8}g_2^2\mathrm{Tr}\Big\{\Sigma^\dagger Q_2^a\Sigma Q_2^{b\,\mathsf{T}} + \Sigma^\dagger Q_2^a\Sigma Q_2^{b\,\mathsf{T}}\Big\}W^a_{2\mu}W_2^{b\mu} \\
&\quad + \mathcal{L}_{g_1g_2} + \mathrm{Tr}\{WW\}\,\text{terms} \\
&= \frac{F^2}{4}g_1^2\mathrm{Tr}\Big\{\Sigma^\dagger Q_1^a\Sigma Q_1^{b\,\mathsf{T}}\Big\}W^a_{1\mu}W_1^{b\mu} + \frac{F^2}{4}g_2^2\mathrm{Tr}\Big\{\Sigma^\dagger Q_2^a\Sigma Q_2^{b\,\mathsf{T}}\Big\}W^a_{2\mu}W_2^{b\mu} \\
&\quad + \mathcal{L}_{g_1g_2} + \mathrm{Tr}\{WW\}\,\text{terms}
\end{aligned} \tag{5.40}$$

Terms like Tr$WW$ are truncated here because they are not relevant to the $WW\Sigma\Sigma$ loops. Now we are left with two similar expressions in (5.40). At this step we can explicitly work out contributions from each set of gauge fields (corresponding to $Q_1$ and $Q_2$) separately. Then we can check whether turning on one of them introduces the quadratic divergence diagrams to the Higgs mass. Consider the $g_1$ terms. Expanding the $\Sigma$ (we will put the $W^a_{1\mu}W^{b\mu}_1$ and other factors back later) and collect terms containing 2 $\Sigma$'s

$$\begin{aligned}
\mathcal{L}_{\Sigma\Sigma W_1 W_1} \quad &\propto \quad \mathrm{Tr}\Big\{\Sigma_0\left[1+\frac{2\mathrm{i}}{F}(\Pi\cdot X)-\frac{2}{F^2}(\Pi\cdot X)^2\right]Q^a_1 \\
&\qquad\qquad \left[1-\frac{2\mathrm{i}}{F}(\Pi\cdot X)-\frac{2}{F^2}(\Pi\cdot X)^2\right]\Sigma_0 Q^{b\,\mathsf{T}}_1\Big\} \\
&\rightarrow \quad \mathrm{Tr}\Big\{Q^a_1\left(-\frac{2}{F^2}(\Pi\cdot X)^2\right)\Sigma_0 Q^{b\,\mathsf{T}}_1\Sigma_0+\left(-\frac{2}{F^2}(\Pi\cdot X)^2\right)Q^a_1\Sigma_0 Q^{b\,\mathsf{T}}_1\Sigma_0 \\
&\qquad +\frac{2\mathrm{i}}{F}(\Pi\cdot X)Q^a_1\frac{(-2\mathrm{i})}{F}(\Pi\cdot X)\Sigma_0 Q^{b\,\mathsf{T}}_1\Sigma_0\Big\}+\text{others} \qquad (5.41)
\end{aligned}$$

where "others" means interactions containing more or less than two $\Sigma$'s. Then

$$\begin{aligned}
\mathcal{L}_{\Sigma\Sigma W_1 W_1} &= -\frac{2}{F^2}\mathrm{Tr}\Big\{(\Pi\cdot X)^2\Sigma_0 Q^{b\,\mathsf{T}}_1\Sigma_0 Q^a_1\Big\}-\frac{2}{F^2}\mathrm{Tr}\Big\{(\Pi\cdot X)^2 Q^a_1\Sigma_0 Q^{b\,\mathsf{T}}_1\Sigma_0\Big\} \\
&\qquad +\frac{4}{F^2}\mathrm{Tr}\Big\{(\Pi\cdot X)Q^a_1(\Pi\cdot X)\Sigma_0 Q^{b\,\mathsf{T}}_1\Sigma_0\Big\}+\text{others} \qquad (5.42) \\
&= \frac{4}{F^2}\mathrm{Tr}\Big\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H\frac{\tau^a}{2}\Phi^\dagger\frac{\tau^{b\,\mathsf{T}}}{2} \\ 0 & 0 & \Phi\frac{\tau^a}{2}\Phi^\dagger\frac{\tau^{b\,\mathsf{T}}}{2}\end{pmatrix}\Big\}+\text{others} \\
&= \frac{1}{F^2}\mathrm{Tr}\left\{\Phi\tau^a\Phi^\dagger\tau^{b\,\mathsf{T}}\right\}+\text{others} \qquad (5.43)
\end{aligned}$$

The first two terms in (5.42) are zero due to the vanishing product $Q^a\Sigma_0 Q^{b\,\mathsf{T}}\Sigma_0$. Gathering similar terms corresponding to the $Q_2$ we find

$$\mathcal{L}_{\Sigma\Sigma W_1 W_2}=\frac{g_1^2}{4}\mathrm{Tr}\left\{\Phi\tau^a\Phi^\dagger\tau^{b\,\mathsf{T}}\right\}W^a_{1\mu}W^{b\mu}_1+\frac{g_2^2}{4}\mathrm{Tr}\left\{\Phi\tau^a\Phi^\dagger\tau^{b\,\mathsf{T}}\right\}W^a_{2\mu}W^{b\mu}_2\,, \quad (5.44)$$

which shows that the terms that are potentially contributing to quadratic divergent diagrams like $W_iW_i\mathrm{Tr}H^\dagger H$ are absent. The Higgs mass term does not receive 1-loop contributions from either $W^1$ or $W^2$ interaction (via $Q^1, Q^2$). It is even better to see large one-loop correction to the $\Phi$'s mass appearing since this implies that they prefer to be as massive as the largest mass scale of the theory and we do not have to worry about them so soon. This $\Phi$'s mass also reflects the fact that the mass of $\Phi$ is not protected by the extra global symmetry $SU(3)$.

Now we have to come back to the terms in $\mathcal{L}_{g_1 g_2}$ in (5.40) that was omitted from the above consideration. They will tell us whether the "collective" effect of turning both gauge interactions on adds severe quadratic divergence diagrams to the Higgs mass. The cancellations between the light $W$ and heavy fields $W'$; i.e., between the gauge bosons in their mass eigenstates[9], should manifest due to the presence of the mixing. To show such cancellations, we have made the following substitutions (to the $W, W'$ eigenstates)

$$\begin{aligned} W_1^{a\mu} &= -cW'^{a\mu} + sW^{a\mu} \\ W_2^{a\mu} &= sW'^{a\mu} + cW^{a\mu}, \end{aligned} \tag{5.45}$$

as well as

$$g = \frac{g_2 g_1}{\sqrt{g_1^2 + g_2^2}} = g_2 \cos\psi = g_2 c = g_1 \sin\psi = g_1 s \tag{5.46}$$

and put them into $\mathcal{L}_{g_1 g_2}$ in (5.40). First we write the $\mathcal{L}_{g_1 g_2}$, neglecting the $\Sigma^\dagger \Sigma$ terms in the trace:

$$\begin{aligned} \mathcal{L}_{g_1 g_2} &= \frac{F^2}{8} \mathrm{Tr}\Big\{ \Big[ \mathrm{i}g_1(\Sigma^\dagger W_{1\mu} + W_{1\mu}^* \Sigma^\dagger) + \mathrm{i}g_2(\Sigma^\dagger W_{2\mu} + W_{2\mu}^* \Sigma^\dagger) \Big] \\ &\qquad \Big[ -\mathrm{i}g_1(W_1^\mu \Sigma + \Sigma W^{\mu\,\mathsf{T}}_{\ 1}) - \mathrm{i}g_2(W_2^\mu \Sigma + \Sigma W^{\mu\,\mathsf{T}}_{\ 2}) \Big] \Big\} \\ &= \frac{g_1 g_2 F^2}{8} W_1^a W_2^b \mathrm{Tr}\Big\{ \Sigma^\dagger Q_1^a \Sigma Q_2^{b\,\mathsf{T}} + Q_1^{a\,\mathsf{T}} \Sigma^\dagger Q_2^b \Sigma \Big\} \\ &\quad + \frac{g_1 g_2 F^2}{8} W_1^a W_2^b \mathrm{Tr}\Big\{ \Sigma^\dagger Q_2^a \Sigma Q_1^{b\,\mathsf{T}} + Q_2^{a\,\mathsf{T}} \Sigma^\dagger Q_1^b \Sigma \Big\} \end{aligned} \tag{5.47}$$

[9] In the Littlest Higgs, and other models of Little Higgs in general, there are more than one stage of gauge symmetry breaking. So there will be more than one mass eigenstates of the gauge fields (two in our case).

$$
\begin{aligned}
\mathcal{L}_{g_1g_2} &= \frac{g_1g_2F^2}{8}W_1^aW_2^b\mathrm{Tr}\Big\{2\Sigma^\dagger Q_1^a\Sigma Q_2^{b\,\mathsf{T}}\Big\} + \frac{g_1g_2F^2}{8}W_1^aW_2^b\mathrm{Tr}\Big\{2Q_1^{a\,\mathsf{T}}\Sigma^\dagger Q_2^b\Sigma\Big\} \\
&= \frac{g_1g_2F^2}{8}(W_1^aW_2^b + W_2^aW_1^b)\mathrm{Tr}\Big\{2\Sigma^\dagger Q_1^a\Sigma Q_2^{b\,\mathsf{T}}\Big\} \\
&\approx \frac{g_1g_2F^2}{8}(W_1^aW_2^b + W_2^aW_1^b)\mathrm{Tr}\Big\{2\Sigma_0\left[1+\frac{2\mathrm{i}}{F}(\Pi\cdot X) - \frac{2}{F^2}(\Pi\cdot X)^2\right]Q_1^a \\
&\qquad \left[1-\frac{2\mathrm{i}}{F}(\Pi\cdot X) - \frac{2}{F^2}(\Pi\cdot X)^2\right]\Sigma_0 Q_2^{b\,\mathsf{T}}\Big\} \\
&= -\frac{g_1g_2}{2}(W_1^aW_2^b + W_2^aW_1^b)\mathrm{Tr}\Big\{(\Pi\cdot X)^2\Sigma_0 Q_2^{b\,\mathsf{T}}\Sigma_0 Q_1^a\Big\} \\
&\quad -\frac{g_1g_2}{2}(W_1^aW_2^b + W_2^aW_1^b)\mathrm{Tr}\Big\{(\Pi\cdot X)^2 Q_1^a\Sigma_0 Q_2^{b\,\mathsf{T}}\Sigma_0\Big\} \\
&\quad +g_1g_2(W_1^aW_2^b + W_2^aW_1^b)\mathrm{Tr}\Big\{(\Pi\cdot X)Q_1^a(\Pi\cdot X)\Sigma_0 Q_2^{b\,\mathsf{T}}\Sigma_0\Big\} + \text{others}
\end{aligned}
\tag{5.48}
$$

Note that

$$
(\Pi\cdot X)^2 = \begin{pmatrix} H^\dagger H + \Phi^\dagger\Phi & \Phi^\dagger H^\mathsf{T} & H^\dagger H^* \\ H^*\Phi & HH^\dagger + H^*H^\mathsf{T} & H\Phi^\dagger \\ H^\mathsf{T}H & \Phi H^\dagger & \Phi\Phi^\dagger + H^\mathsf{T}H^* \end{pmatrix}. \tag{5.49}
$$

So

$$
\begin{aligned}
(\Pi\cdot X)^2\Sigma_0 Q_2^{b\,\mathsf{T}}\Sigma_0 Q_1^a &= (\Pi\cdot X)^2\begin{pmatrix} \tau_2^{b\,\mathsf{T}}\tau_1^a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} (H^\dagger H + \Phi^\dagger\Phi)\tau_2^{b\,\mathsf{T}}\tau_1^a & 0 & 0 \\ H^*\Phi\tau_2^{b\,\mathsf{T}}\tau_1^a & 0 & 0 \\ H^\mathsf{T}H\tau_2^{b\,\mathsf{T}}\tau_1^a & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{5.50}
$$

leads to

$$
\mathrm{Tr}\Big\{(\Pi\cdot X)^2\Sigma_0 Q_2^{b\,\mathsf{T}}\Sigma_0 Q_1^a\Big\} = -\mathrm{Tr}\left\{(H^\dagger H + \Phi^\dagger\Phi)\delta^{ab}\right\}. \tag{5.51}
$$

Therefore,

$$
\begin{aligned}
-\frac{g_1 g_2}{2}&(W_1^a W_2^b + W_2^a W_1^b)\mathrm{Tr}\Big\{(\Pi\cdot X)^2 \Sigma_0 {Q_2^b}^{\mathsf{T}} \Sigma_0 Q_1^a\Big\} \\
&= \frac{g_1 g_2}{2}(W_1^a W_2^b + W_2^a W_1^b)\mathrm{Tr}\left\{(H^\dagger H + \Phi^\dagger \Phi)\delta^{ab}\right\} \\
&= \frac{g^2}{2sc}\left[-2csW'^a W'^b + 2(s^2-c^2)W^a W'^b + 2scW'^a W'^b\right] \\
&\quad \mathrm{Tr}\left\{(H^\dagger H + \Phi^\dagger \Phi)\delta^{ab}\right\} . \qquad (5.52)
\end{aligned}
$$

Calculations of the remaining terms in (5.48) can proceed in very similar manners so we are not going to show them here. Collecting the results, together with the $g'gW'W$ version of (5.44):

$$
\begin{aligned}
(5.44) \quad = \quad & \frac{g^2}{4}\mathrm{Tr}\left\{\Phi\tau^a\Phi^\dagger {\tau^b}^{\mathsf{T}}\right\}\left[\frac{(c^4+s^4)}{s^2c^2}\right] W_\mu^a W^{b\mu} \\
& + \frac{g^2}{4}\mathrm{Tr}\left\{\Phi\tau^a\Phi^\dagger {\tau^b}^{\mathsf{T}}\right\}\left[2\frac{(s^2-c^2)}{sc}\right] W_\mu^a W'^{b\mu} \qquad (5.53)
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
\mathcal{L}_{\Sigma\Sigma W'W} \quad = \quad & \frac{g^2}{4}\left[W_\mu^a W^{b\mu} - \frac{(c^2-s^2)}{sc}W_\mu^a W'^{b\mu}\right] \\
& \qquad \times \mathrm{Tr}\left\{H^\dagger H \delta^{ab} + 2\Phi^\dagger\Phi\delta^{ab} + 2\sigma^a\Phi^\dagger\sigma^{bT}\Phi\right\} \\
& - \frac{g^2}{4}\Bigg[W_\mu'^a W'^{a\mu}\mathrm{Tr}\left\{H^\dagger H + 2\Phi^\dagger\Phi\right\} \\
& \quad - \frac{(c^4+s^4)}{2s^2c^2}W_\mu'^a W'^{b\mu}\mathrm{Tr}\left\{2\sigma^a\Phi^\dagger\sigma^{bT}\Phi\right\}\Bigg] , \qquad (5.54)
\end{aligned}
$$

which shows the collective symmetry breaking at work. While the terms corresponding to $\mathrm{Tr}H^\dagger H$ in (5.54) do not cancel in the Lagrangian, the one-loop diagrams constructed from them cancel precisely since they both have the couplings $g^2$, but with opposite signs. Also notice the factor $\cot 2\psi = (cot\psi - tan\psi)/2 = (c^2-s^2)/2sc$ in the $WW'HH$ coupling which is a unique feature of the Little Higgs model.

Once we have done the $SU(2)$ case, what we have to do in order to work out the $U(1)$ contributions is only making some modifications to the results found above. The relevant parts of the $U(1)$ interaction deduced from (5.38) are similar to those found in (5.40). So we paste them here

$$\mathcal{L}_{\Sigma-B} = \frac{F^2}{4}g_1'^2\mathrm{Tr}\Big\{\Sigma^\dagger Y_1\Sigma Y_1\Big\}B_{1\mu}B_1^\mu + \frac{F^2}{4}g_2'^2\mathrm{Tr}\Big\{\Sigma^\dagger Y_2\Sigma Y_2\Big\}B_{2\mu}B_2^\mu + \mathcal{L}_{g_1'g_2'} + \mathrm{Tr}\{BB\}\,\mathrm{terms}\,. \tag{5.55}$$

Again we have separated the mixing terms within $\mathcal{L}_{g_1'g_2'}$ and neglected terms without Goldstone bosons interactions. What remain are the explicit calculations. The pure $g_1'$ terms contribute

$$\mathcal{L}_{\Sigma\Sigma B_1B_1} \propto -\frac{2}{F^2}\mathrm{Tr}\Big\{(\Pi\cdot X)^2\Sigma_0Y_1\Sigma_0Y_1\Big\} - \frac{2}{F^2}\mathrm{Tr}\Big\{(\Pi\cdot X)^2Y_1\Sigma_0Y_1\Sigma_0\Big\} + \frac{4}{F^2}\mathrm{Tr}\Big\{(\Pi\cdot X)Y_1(\Pi\cdot X)\Sigma_0Y_1\Sigma_0\Big\} + \mathrm{others}\,. \tag{5.56}$$

Notice that now there is no reasons that the first two terms should vanish (due to the diagonal structure of the $Y_1$). We find

$$\begin{aligned}
&\mathrm{Tr}\Big\{(\Pi\cdot X)^2\Sigma_0Y_1\Sigma_0Y_1\Big\}\\
&= \frac{1}{100}\mathrm{Tr}\left\{\begin{pmatrix} H^\dagger H+\Phi^\dagger\Phi & \Phi^\dagger H^\top & H^\dagger H^* \\ H^*\Phi & HH^\dagger + H^*H^\top & H\Phi^\dagger \\ H^\top H & \Phi H^\dagger & \Phi\Phi^\dagger + H^\top H^* \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{1}\end{pmatrix}\right\}\\
&= \frac{1}{100}\mathrm{Tr}\Big\{-6(H^\dagger H+\Phi^\dagger\Phi) + 4(HH^\dagger + H^*H^\top) - 6(\Phi\Phi^\dagger + H^\top H^*)\Big\}\\
&= \mathrm{Tr}\Big\{(\Pi\cdot X)^2Y_1\Sigma_0Y_1\Sigma_0\Big\}
\end{aligned} \tag{5.57}$$

where $\mathbb{1}$ indicates a $2\times 2$ unit matrix. So our job is to make sure the $HH$ terms from the second line in (5.56) cancel these terms. We find

$$\begin{aligned}
&\mathrm{Tr}\Big\{(\Pi\cdot X)Y_1(\Pi\cdot X)\Sigma_0Y_1\Sigma_0\Big\}\\
&= \frac{1}{100}\mathrm{Tr}\Big\{8(H^\dagger H+\Phi^\dagger\Phi) - 12HH^\dagger + 8H^*H^\top + 18\Phi\Phi^\dagger - 12H^\top H^*\Big\}\,.
\end{aligned} \tag{5.58}$$

Then (5.56) turns to

$$\begin{aligned}
\mathcal{L}_{\Sigma\Sigma B_1B_1} &\propto -\mathrm{Tr}\Big\{-12(H^\dagger H+\Phi^\dagger\Phi) + 8(HH^\dagger + H^*H^\top) - 12(\Phi\Phi^\dagger + H^\top H^*)\Big\}\\
&\quad +\mathrm{Tr}\Big\{8(H^\dagger H+\Phi^\dagger\Phi) - 12HH^\dagger + 8H^*H^\top + 18\Phi\Phi^\dagger - 12H^\top H^*\Big\}\\
&= \mathrm{Tr}\left\{50\Phi^\dagger\Phi\right\}\,.
\end{aligned} \tag{5.59}$$

It is clear that the contributions from the $B_2$ field should be the same way. We have to stress again that these cancellations for each type are not fluke (i.e., valid only in some particular gauge choice) but are the results of some definite of way of gauging the subgroup of the theory.

To see cancellations between the $B$ and $B'$ fields, we proceed in a way similar to that used in the case of $W$ fields. After some straightforward, though tedious, calculations we eventually arrive at

$$\begin{aligned}\mathcal{L}_{\Sigma\Sigma BB'} &= g'^2\left[B_\mu B^\mu - \frac{(c'^2 - s'^2)}{s'c'}B_\mu B'^\mu\right]\mathrm{Tr}\left[\frac{1}{4}H^\dagger H + \Phi^\dagger\Phi\right] \\ &\quad -g'^2\left[B'_\mu B'^\mu \mathrm{Tr}\left[\frac{1}{4}H^\dagger H\right] - \frac{(c'^2 - s'^2)^2}{4s'^2c'^2}B'_\mu B'^\mu \mathrm{Tr}\left[\Phi^\dagger\Phi\right]\right], \end{aligned} \tag{5.60}$$

which also shows the cancellations when one-loop diagrams are taken into account. Notice that we have used

$$g' = \frac{g_1'^2 g_2'^2}{\sqrt{g_1'^2 + g_2'^2}}. \tag{5.61}$$

This equation, together with, (5.46) can be rewritten as

$$\begin{aligned}\frac{1}{g_1^2} + \frac{1}{g_2^2} &= \frac{1}{g^2} \\ \frac{1}{g_1'^2} + \frac{1}{g_2'^2} &= \frac{1}{g'^2}. \end{aligned} \tag{5.62}$$

These relations (or their original form) will be helpful as their right-hand side are measurable within the standard model.


**Figure 5.1:** Cancellations between the $U(1)$ gauge fields

### 5.2.4 Fermions Cancel Fermions

So far we have seen how the trick of collective symmetry works in the scalar and gauge sector of the theory. In this section, we shall turn to the fermion sector of the theory where the most severe quadratic divergence contribution from standard model top quark lies in. We will write the top-bottom weak doublet and the right-handed weak singlet[10] as

$$q_3 = \begin{pmatrix} t_3 \\ b_3 \end{pmatrix} \quad \text{and } u_3'^c \tag{5.63}$$

It is expected that the structure of the fermion-Goldstone boson (Yukawa) interaction will be fairly complicated (unfamiliar indeed) thanks to the non-linear realisation of the Higgs. So we should list some of the features that we anticipate before we move on. The very first one of such is that there should be two additional heavy quark fields. Top-like they must be, or their contributions will not mean anything to the diagrams from the standard model top. We shall call them $T$ and $T'^c$. Let us associate them the quantum numbers $(\mathbf{3}, \mathbf{1})_{Y_i}$ and $(\bar{\mathbf{3}}, \mathbf{1})_{-Y_i}$. In this way they transform as vectors in $SU(2)$ (with correct $U(1)$ charge) and their mass term

$$yFT^cT'^c \tag{5.64}$$

will not spoil the standard model's electroweak symmetry. (We will reserve the symbol $T^c$ for later use.) Therefore, they are legitimately massive with mass $\mathcal{O}(\text{TeV})$. In addition, since they were "born to be" the standard model top "cancellers" we do not need similar particles for the up and charm family due to the fine-tuning arguments.

Recall that Yukawa interaction breaks a global symmetry explicitly. We basically have more than one options to make sure that the Higgs remains massless, with mechanism invoked in similar manners with collective symmetry in the gauge sector. We can add a Yukawa coupling with the fermion such that the extra $SU(3)$ global symmetry is recovered when the Yukawa interaction is turned off (zero Yukawa coupling). In other words, turning off one of the two couplings should result in different extra global symmetry in the same sense as what we have done in the gauge section. In addition, we can also demand that

[10] For the reason that we shall see, this $u_3'^c$ is not necessarily the top. Also note that now we use the small $q_i$ for the quark doublet instead of $Q_i$ to ad void ambiguities.

the standard model fermions be charged under only one of the gauge groups, say the $SU(2)_1$. Having zero $SU(2)_2$ charges means that turning the $SU(1)_1$ off will completely remove interactions between Goldstone bosons and fermions. To keep the heavy fermions $T$ and $T'^c$ being weak singlets; i.e., singlets under $Q_1^a + Q_2^a$, we have to let them stay at the dead centre with respect to the $SU(5)$ matrices. Then let us collect the heavy fermions together with the standard model quarks in an $SU(3)$ "royal" triplet

$$\chi = \begin{pmatrix} b_3 \\ t_3 \\ T \end{pmatrix} = \begin{pmatrix} \tau^2 q_3 \\ T \end{pmatrix} \tag{5.65}$$

and let this triplet Yukawa interacts with the Goldstone bosons and the "right-handed" $T^c$ in an $SU(3)$ symmetric way. It is not obvious (at least for the author) but is still easy to verify that at tree level, the 3-Yukawa coupling (rather than 4-Yukawa one) like $thu_3'^c$ can be generated by the Lagrangian ([21]):

$$\frac{1}{2} y_1 F \varepsilon_{ijk} \varepsilon_{xy} \chi_i \Sigma_{jx} \Sigma_{ky} u_3'^c \tag{5.66}$$

where the antisymmetric tensors $\varepsilon_{ijk}$ and $\varepsilon_{xy}$, together with $i, j, k = 1, 2, 3$ and $x, y = 4, 5$, are introduced to ensure the desired pairings betweens the $SU(3)$ triplet, the singlet fermions, and Goldstone bosons; i.e.,

$$y_1 (t_3 \ b_3) H t_3^c \,. \tag{5.67}$$

To digest the expression (5.66) a bit, let us first notice that the $\Sigma_{jx}$ denotes the $3 \times 2$ upper right block of $\Sigma$. To lowest (non-trivial) order, the $2 \times 2$ upper-right block of $\Sigma$ is just an identity matrix $\mathbb{1}_{2\times 2}$. Then the antisymmetric coupling between the two $\Sigma$ will single out only one (rather than 2 from 2 $\Sigma$'s) Goldstone boson field out of the product $\varepsilon^{ijk} \Sigma_{j4} \Sigma_{k5}$.

With (5.66), the interactions between fermions and Goldstone bosons can be extracted to desired order. Let us write out the $\Sigma_{jx}$ (or $\Sigma_{ky}$), extracted from the relevant components of $2\mathrm{i}(\Pi \cdot X)\Sigma_0/F$ (the upper-right corner), to the first order:

$$\Sigma_{jx} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\mathrm{i}\sqrt{2}}{F} h^+ & \frac{\mathrm{i}\sqrt{2}}{F} h^0 \end{pmatrix} . \tag{5.68}$$

The contributions from the next order do not look very clean

$$\Sigma_{jx,2^{nd}\text{order}} \propto -\frac{4}{F^2}\begin{pmatrix} h^+h^- + \frac{\phi^+\phi^-}{2} + \phi^{--}\phi^{++} & h^0h^- + \frac{\phi^0\phi^-}{\sqrt{2}} + \frac{\phi^+\phi^{--}}{\sqrt{2}} \\ h^+h^{0*} + \frac{\phi^+\phi^{0*}}{\sqrt{2}} + \frac{\phi^-\phi^{++}}{\sqrt{2}} & h^0h^{0*} + \phi^0\phi^{0*} + \frac{\phi^+\phi^-}{2} \\ \frac{h^{0*}\phi^+}{\sqrt{2}} + h^-\phi^{++} & \frac{h^-\phi^+}{\sqrt{2}} + h^{0*}\phi^0 \end{pmatrix}. \quad (5.69)$$

All together, the Lagrangian for Yukawa interactions becomes

$$\mathcal{L}_t = \frac{1}{2}y_1 F\varepsilon_{ijk}\varepsilon_{xy}\chi_i\Sigma_{jx}\Sigma_{ky}u_3^{'c} + y_2FT^cT^{'c} \quad (5.70)$$

which is, after some expansions,

$$\begin{aligned} \mathcal{L}_t \;=\; & y_2 fTT^{'c} \\ & +iy_1\left\{-b_3\left[\sqrt{2}h^+ + \frac{\mathrm{i}}{F}(\sqrt{2}h^-\phi^{++} + h^{0*}\phi^+)\right]u_3^{'c}\right. \\ & -t_3\left[\sqrt{2}h^0 + \frac{\mathrm{i}}{F}(h^-\phi^+ + \sqrt{2}h^{0*}\phi^0)\right]u_3^{'c} \\ & \left.+T\left[-\mathrm{i}F + \frac{\mathrm{i}}{F}(h^+h^- + h^0h^{0*} + 2\phi^{++}\phi^{--} + 2\phi^+\phi^- + 2\phi^0\phi^{0*})\right]u_3^{'c}\right\} \\ & +\text{h.c.} \end{aligned} \quad (5.71)$$

Let us inspect this expression more closely. First observe that $y_1$ respects the global $SU(3)_2$ (in the upper-left block)[11] and breaks the other.. Next, we see the couplings between $u_3^{'c}$ and both $t_3$ and $T$. This means that the $u_3^{'c}$ cannot just be the usual standard model top quark (right-handed), otherwise we would have to remove $T$ and face with quadratic divergences from the top because the $T$ would not couple with the top.

Interaction vertices can be extracted from the Lagrangian (5.71). In addition to the usual 3-Yukawa coupling $ffh$ ($f$ stands for fermions and $h$ stands for scalars), there are the $ffhh$ interactions from the second order expansions with a suppression factor $1/F$. The $ff$ interaction is also present. By inspecting the $2^{nd}$ line in (5.71), we get the first order $ffH$ coupling

$$-\mathrm{i}y_1\sqrt{2}t^3u_3^{'c} \quad (5.72)$$

and for the last line we get another first order coupling

$$y_1FTu_3^{'c} \quad (5.73)$$

[11] Recall that we partition the group as $G \supset SU(2)_1 \times SU(3)_1 + SU(2)_2 \times SU(3)_2$.

and the second order one:

$$-\frac{y_1}{F} T h^0 h^{0*} u_3'^c \,. \tag{5.74}$$

Notice the crucial extra minus sign in the second equation. These four-point interactions can be curled up into a loop like that shown in fig. 5.2.

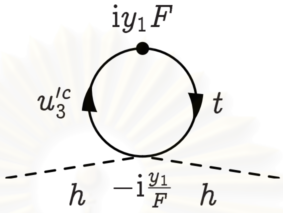



**Figure 5.2:** Fermion loop from the 2nd order interaction.

There are two loops from the interaction like (5.74). Though we cannot identify the fermions we have on hand with physical particles, we can construct fermion loops and watch the cancellations occur. These loops are shown in

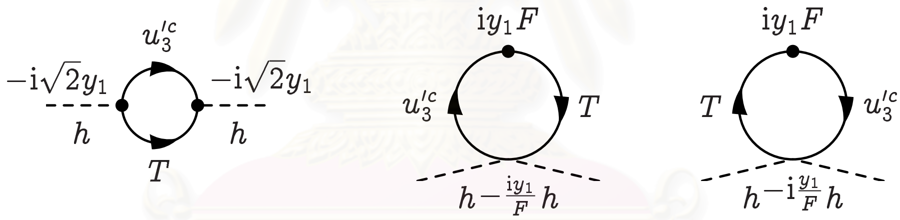



**Figure 5.3:** Cancellations contributed from the extra "top"

In addition, the mass of $T$ comes from interacting with both the $u_3'^c$ and the $T'^c$. These mixings suggest that we introduce

$$\begin{aligned} T^c &\equiv \frac{1}{\sqrt{y_1^2 + y_2^2}} (y_1 u_3'^c + y_2 T'^c) \\ u_3^c &\equiv \frac{1}{\sqrt{y_1^2 + y_2^2}} (-y_1 T'^c + y_2 u_3'^c) \,. \end{aligned} \tag{5.75}$$

Their inversions are

$$\begin{aligned} T'^c &\equiv \frac{1}{\sqrt{y_1^2 + y_2^2}} (-y_1 u_3^c + y_2 T^c) \\ u_3'^c &\equiv \frac{1}{\sqrt{y_1^2 + y_2^2}} (y_1 T^c + y_2 u_3^c) \,, \end{aligned} \tag{5.76}$$

which, after a few algebras changes (5.71) to

$$
\begin{aligned}
\mathcal{L}_t &= F\sqrt{y_1^2+y_2^2}TT^c \\
&+\frac{\sqrt{2}iy_1^2}{\sqrt{y_1^2+y_2^2}}\left[b_3h^+T^c+t_3h^0T^c\right]-\frac{\sqrt{2}iy_1y_2}{\sqrt{y_1^2+y_2^2}}\left[b_3h^+u_3^c+t_3h^0u_3^c\right] \\
&+\frac{y_1^2}{\sqrt{y_1^2+y_2^2}}\frac{1}{F}\Big[b_3(\sqrt{2}h^-\phi^{++}+h^{0*}\phi^+)T^c+t_3(h^-\phi^++\sqrt{2}h^{0*}\phi^0)T^c \\
&\quad -T(h^+h^-+h^0h^{0*}+2\phi^{++}\phi^{--}+2\phi^+\phi^-+2\phi^0\phi^{0*})T^c\Big] \\
&+\frac{y_1y_2}{\sqrt{y_1^2+y_2^2}}\frac{1}{F}\Big[b_3(\sqrt{2}h^-\phi^{++}+h^{0*}\phi^+)u_3^c+t_3(h^-\phi^++\sqrt{2}h^{0*}\phi^0)u_3^c \\
&\quad -T(h^+h^-+h^0h^{0*}+2\phi^{++}\phi^{--}+2\phi^+\phi^-+2\phi^0\phi^{0*})u_3^c\Big]+\text{h.c.}
\end{aligned}
\tag{5.77}
$$

and we can now interpret the $F\sqrt{y_1^2+y_2^2}TT^c$ as a mass term for the heavy fermion; i.e.,

$$
F\sqrt{y_1^2+y_2^2}=m_T\,, \tag{5.78}
$$

which clearly prefers an $F\sim\mathcal{O}(\text{TeV})$ scale. It is also helpful to define new Yukawa couplings

$$
y_t=\frac{\sqrt{2}y_1y_2}{\sqrt{y_1^2+y_2^2}} \tag{5.79}
$$

and

$$
y_T=\frac{\sqrt{2}y_1^2}{\sqrt{y_1^2+y_2^2}}\,, \tag{5.80}
$$

so that we can rewrite (5.77) in the new basis

$$
\begin{aligned}
\mathcal{L}_t &= m_TTT^c \\
&-y_T\tilde{q}_3HT^c-y_t\tilde{q}_3Hu_3^c \\
&+y_T\frac{-\mathrm{i}}{F}\left[\tilde{q}_3\Phi H^*T^c\right]+y_t\frac{-\mathrm{i}}{F}\left[\tilde{q}_3\Phi H^*u_3^c\right]-\frac{1}{F\sqrt{2}}H^\dagger HT\left[y_TT^c+y_tu_3^c\right] \\
&-\frac{\sqrt{2}}{F}(y_TTT^c+y_tTu_3^c)(\phi^{++}\phi^{--}+\phi^+\phi^-+\phi^0\phi^{0*})+\text{h.c.}\,,
\end{aligned}
\tag{5.81}
$$

where the $t_3$, $b_3$ and $u_3^c$ are manifestly massless (at tree level). Therefore we can identify them with the standard model quarks; i.e.,

$$t_3 \leftrightarrow t_L\,, \qquad b_3 \leftrightarrow b_L\,, \qquad u_3^c \leftrightarrow t_R\,. \tag{5.82}$$

In addition, in these new basis, we can rewrite the mass of the heavy quarks as

$$m_T = \frac{y_t^2 + y_T^2}{y_T} F\,, \tag{5.83}$$

which should serve as a useful test of the model once the heavy top is found since $F$ can be measured from other processes involving gauge interactions, etc.

The fermion-Higgs vertices are readily read off from (5.81). They are shown in fig. 5.4. Then the relevant loops to the quadratic divergences due to

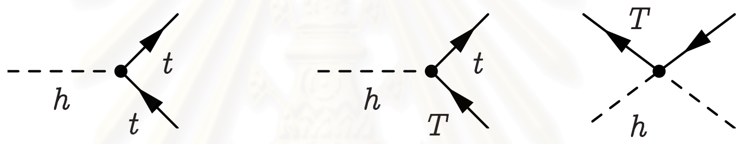



**Figure 5.4:** Heavy fermion vertices

fermions are shown in fig. 5.5 (with arrows removed).

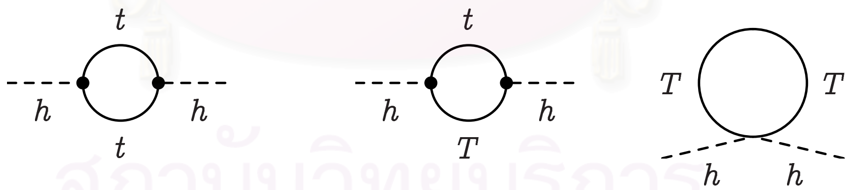



**Figure 5.5:** Cancellations contributed from the extra "top" in mass eigenbasis.

To see how the cancellation goes in this case, we will evaluate the contributions from the loops in fig. 5.5. It is clear that they are proportional to

$$\begin{aligned}
\mathrm{t-t\,loop} &\propto -y_t^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2} \\
\mathrm{t-T\,loop} &\propto -y_T^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m_T^2} \\
\mathrm{T-T\,loop} &\propto +\frac{y_T}{F} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{m_T}{k^2-m_T^2}\,.
\end{aligned} \tag{5.84}$$

Altogether, they give, to leading diverging terms

$$-y_t^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}-y_T^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m_T^2}+\frac{y_T}{F}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{m_T}{k^2-m_T^2}$$
$$=-\frac{y_Tm_T}{F}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}+\frac{y_T}{F}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{m_T}{k^2-m_T^2}\,, \tag{5.85}$$

where we have used the relation (5.83). Clearly the dangerous parts cancelled. The remaining terms ($m_T$ dependence) that we neglected above leads to the logarithmic correction to the Higgs mass

$$\delta m_h^2=-3\frac{y_1^2y_2^2F^2}{8\pi^2}\ln\frac{\Lambda^2}{m_T^2} \tag{5.86}$$

or, in terms of the physical fields,

$$\delta m_h^2=-3\frac{y_t^2m_T^2}{8\pi^2}\ln\frac{\Lambda^2}{m_T^2}\,. \tag{5.87}$$

If the heavy top is too heavy, the fine-tuning problem will resurface (similar to the “stop” in minimal supersymmetric standard model).

We also need interaction terms for other fermions as they interact with the standard model Higgs. Fine-tuning argument does not require the existences of the heavy partners (other than those for the top quark) and the Lagrangian is in the form

$$\mathcal{L}=\frac{1}{2}\lambda_dF\varepsilon_{ijk}\varepsilon_{xy}\chi_i\Sigma_{jx}^*\Sigma_{ky}^*d^c+\text{h.c.}\,. \tag{5.88}$$

where now $\chi_3=0$. For example, recall the partially upside down triplet (with one member missing)

$$\chi_{1i}=\begin{pmatrix}d_1\\u_1\\0\end{pmatrix},\qquad \chi_{2i}=\begin{pmatrix}s_1\\c_1\\0\end{pmatrix}. \tag{5.89}$$

For the expression (5.66) to be neutral, we must provide suitable $U(1)_1\times U(1)_2$ charges definitions of the fermions. The constraint between the two $U(1)$'s is

$$Y_1+Y_2=Y_{\mathrm{SM}} \tag{5.90}$$

which is not very restrictive and is clear to leave one parameter unfixed. The free parameters can be fixed if we demand that the theory is anomaly free. Still, doing so is not a very good idea as we are considering the model as an effective field theory - more fermions to fix or to introduce anomaly problem may exist already at the high-energy end of the theory. So it is best to leave things the way they were. The $U(1)$ (hypercharges) assignments of the fermions are listed in the table

| | $Q$ | $u'^c$ | $d^c$ | $L$ | $e^c$ | $\tilde{t}$ | $\tilde{t}'^c$ |
|---|---|---|---|---|---|---|---|
| $Y_1$ | $-\frac{3}{10}-Y_u$ | $Y_u$ | $\frac{3}{5}+Y_u$ | $\frac{3}{10}-Y_e$ | $Y_e$ | $\frac{1}{5}-Y_u$ | $-\frac{1}{5}+Y_u$ |
| $Y_2$ | $\frac{7}{15}+Y_u$ | $-\frac{2}{3}-Y_u$ | $-\frac{4}{15}-Y_u$ | $-\frac{4}{5}+Y_e$ | $1-Y_e$ | $\frac{7}{15}+Y_u$ | $-\frac{7}{15}-Y_u$ |

**Table 5.1:** Hypercharge assignments of fermions. Two free parameters $Y_u$, $Y_e$ can be removed using the anomaly free condition. The table is taken from Han *et al.* [3].

### 5.2.5 Electroweak Symmetry Breaking

The first task of the Littlest Higgs model is accomplished. What we have on hands are bunches of particles; those with masses of order $F$ and those massless particles which are the ingredients of the usual standard model. Still, the model will not be useful unless we can reproduce electroweak symmetry breaking. So we will evaluate the potential of the Goldstone bosons generated radiatively from loop corrections and workout the new mass eigenstates of the Goldstone bosons and the gauge fields.

#### 5.2.5.1 Coleman-Weinberg Potential

Recall that our Higgs was born as a pseudo-Goldstone boson. So its potential is basically zero at tree-level. Explicit symmetry breaking terms (the gauge coupling, for example) bring some effective interactions like the mass term and the additional coupling (between 4 Higgs and 2 gauge fields) via quantum effects. Electroweak symmetry breaking is then triggered by the radiatively generated Mexican-hat type potential (negative and positive coefficients of $H^\dagger H$ and $(H^\dagger H)^2$ respectively). The Coleman-Weinberg mechanism is a good candidate for explaining this. There are basically two types of the radiative corrections from the Coleman-Weinberg mechanism to be described here. One is logarithmic and the other is quadratic. The logarithmic part is easy since we have sketched major parts of the calculations in various sections, especially in section 3.1.4. Moreover, as is well-known, the logarithmic part is not very severe (comparing to the quadratic divergent part). They will become important (i.e., comparable) when the two-loop quadratic divergent calculations are considered though. However, the 2-loop calculations are out of the scope of this thesis.

#### 5.2.5.2 Coleman-Weinberg Potential: Logarithmic

Now we will begin with the radiative correction from the interactions between the Higgs and gauge bosons. Recall (3.55), the logarithmically modified potential from the gauge bosons

$$V_{\rm g,CW}^{\rm log} = \frac{3}{64\pi^2}\,{\rm Tr} M_g^4(\Sigma)\,\ln\frac{M_g^2(\Sigma)}{\Lambda^2} \tag{5.91}$$

The $M_g^2$ is the mass matrix of the fields in the presence of the background $\Sigma$ (think of $\phi_c$ in section 3.1.1) which can be found by expanding the covariant derivative in the Lagrangian (5.5). The result is (recall that the mass term is proportional to the second derivative of a potential)

$$\delta m_{h,g}^2 = \frac{3}{64\pi^2}\left(3g^2 M_{W'}^2 \ln\frac{\Lambda^2}{M_{W'}^2} + g'^2 M_{B'}^2 \ln\frac{\Lambda^2}{M_{B'}^2}\right) . \tag{5.92}$$

Still the logarithmic corrections from the gauge fields are not very severe comparing to the effects from the top quark. This is mainly due to large Yukawa coupling of the top. Taking another formula from the section 3.1.4, the top loop contribution is

$$V_{\rm t,CW}^{\rm log} = -\frac{3}{16\pi^2}\,{\rm Tr}\left(M_t(\Sigma)M_t^\dagger(\Sigma)\right)^2\,\ln\frac{M_t(\Sigma)M_t^\dagger(\Sigma)}{\Lambda^2} . \tag{5.93}$$

which (recall the loop calculation in the previous section) evidently results in

$$\delta m_{h,t}^2 = -\frac{3y_t^2 m_T^2}{8\pi^2}\ln\frac{\Lambda^2}{m_T^2} \tag{5.94}$$

where negative contribution to the mass can be traced back to the fermion loop. It is this contribution from the top that triggers electroweak symmetry breaking. Notice that the scalar (mainly the triplet) also contributes logarithmic correction but is, again, overwhelm by the top effects. Such the effect can be written as

$$\delta m_{h,s}^2 = \frac{\lambda}{16\pi^2} M_\Phi^2 \ln\frac{\Lambda^2}{M_\Phi^2} . \tag{5.95}$$

#### 5.2.5.3 Coleman-Weinberg Potential: Quadratic

Now we will move to the potential due to the quadratically divergent loops with gauge fields running in. The one-loop contribution from gauge fields can be written as (recall (3.32) and the section 4.3)

$$V_{g,CW}^{\text{quad}} = \frac{\Lambda^2}{(4\pi)^2}\text{Tr}M_g^2(\Sigma)\,, \tag{5.96}$$

where $(4\pi)^2$ is a generic one-loop factor. Let us concentrate on the $SU(2)$ interaction. The $\text{Tr}M_g^2(\Sigma)$ can be evaluated from the kinetic term

$$\text{Tr}M_g^2(\Sigma) = F^2\sum_i g_i^2\text{Tr}\left[(Q_i^a\Sigma)^*Q_i^a\Sigma\right] \tag{5.97}$$

We can also think of the (5.96) as the results of the vertex

$$\begin{aligned}
&\frac{g_i^2F^2}{8}\text{Tr}\left[(Q_i^a\Sigma+\Sigma Q_i^{a\mathsf{T}})^{dg}(Q_i^a\Sigma+\Sigma Q_i^{a\mathsf{T}})\right]W_i^{a\mu}W_{i\mu}^a\\
&=\frac{g_i^2F^2}{8}\text{Tr}\left[\Sigma^\dagger Q_i^a\Sigma Q_i^{a\mathsf{T}}+\Sigma Q_i^{a*}\Sigma^\dagger Q_i^a\right]W_i^{a\mu}W_{i\mu}^a\\
&=\frac{g_i^2F^2}{4}\text{Tr}\left[(Q_i^a\Sigma)^*Q_i^a\Sigma\right]W_i^{a\mu}W_{i\mu}^a\,,
\end{aligned} \tag{5.98}$$

where we have used $\Sigma^{\mathsf{T}} = \Sigma^*$. Then the quadratic divergent contribution from $W$ running in the loop yields the factor $\Lambda^2/(4\pi)^2$; i.e.,

$$\frac{g_i^2F^2}{4}\text{Tr}\left[(Q_i^a\Sigma)^*Q_i^a\Sigma\right]\frac{\Lambda^2}{(4\pi)^2}\,. \tag{5.99}$$

Consequently the effective potential

$$V_{g,CW}^{\text{quad}} \supset a\frac{g_i^2F^4}{8}\text{Tr}\left[(Q_i^a\Sigma)^*Q_i^a\Sigma\right]\,, \tag{5.100}$$

where we have used $\Lambda = 4\pi F$ and a $UV$-dependent coefficient $a/2 \sim \mathcal{O}(1)$ (which contains the $(4\pi)^2$ ) has been introduced. The $U(1)$ contribution shows up in a very similar manner so that we finally get

$$\begin{aligned}
V_{g,CW}^{\text{quad}} &= a\frac{F^2}{8}\left\{g_i^2\text{Tr}\left[(Q_i^a\Sigma)^*Q_i^a\Sigma\right]+g_i^{\prime 2}\text{Tr}\left[(Y_i\Sigma)^*Y_i\Sigma\right]\right\}\\
&= a\frac{F^2}{8}\left\{g_1^2\text{Tr}\left[(Q_1^a\Sigma)^*Q_1^a\Sigma\right]+g_1^{\prime 2}\text{Tr}\left[(Y_1\Sigma)^*Y_1\Sigma\right]\right\}\\
&\quad +a\frac{F^2}{8}\left\{g_2^2\text{Tr}\left[(Q_2^a\Sigma)^*Q_2^a\Sigma\right]+g_2^{\prime 2}\text{Tr}\left[(Y_2\Sigma)^*Y_2\Sigma\right]\right\}\,.
\end{aligned} \tag{5.101}$$

In order to extract some useful information from (5.101) we expand the $\Sigma$ and keep terms involving more than one Goldstone bosons in $V_{g,CW}^{\text{quad}}$. Each gauge interaction (i.e., those labelled 1 and 2) will allow different form of the $H$ and $\Phi$ appearing in, depending on the transformation properties under the global symmetry $SU(3)_i$ it manifests (see (5.36) and (5.37)). In other words, the operators involved must be $SU(3)_i$ preserving interactions. The result is ([21], [3])

$$\begin{aligned}
V_{g,CW}^{\text{quad}} &= \frac{a}{2}(g_1^2 + g_1^{'2})F^2 \left|\Phi_{ij} - \frac{i}{2F}(H_iH_j + H_jH_i)\right|^2 \\
&\quad + \frac{a}{2}(g_2^2 + g_2^{'2})F^2 \left|\Phi_{ij} + \frac{i}{2F}(H_iH_j + H_jH_i)\right|^2 + \dots \\
&= \frac{a}{2}(g_1^2 + g_1^{'2})F^2 \Big[\Phi_{ij}^*\Phi_{ij} - \frac{i}{2F}\Phi_{ij}^*H_iH_j - \frac{i}{2F}\Phi_{ij}^*H_jH_i \\
&\qquad + \frac{i}{2F}H_i^*H_j^*\Phi_{ij} + \frac{i}{2F}H_j^*H_i^*\Phi_{ij} \\
&\qquad + \frac{1}{4F^2}(H_i^*H_j^* + H_j^*H_i^*)(H_iH_j + H_jH_i)\Big] \\
&\quad + \frac{a}{2}(g_2^2 + g_2^{'2})F^2 \Big[\Phi_{ij}^*\Phi_{ij} + \frac{i}{2F}\Phi_{ij}^*H_iH_j + \frac{i}{2F}\Phi_{ij}^*H_jH_i \\
&\qquad - \frac{i}{2F}H_i^*H_j^*\Phi_{ij} - \frac{i}{2F}H_j^*H_i^*\Phi_{ij} \\
&\qquad + \frac{1}{4F^2}(H_i^*H_j^* + H_j^*H_i^*)(H_iH_j + H_jH_i)\Big] + \dots
\end{aligned} \tag{5.102}$$

where $H_i$, $\Phi_{ij}$ denote the field components. Then we see right away that gauge loops contribute to the triplet masses

$$M_\Phi \simeq a\sqrt{g_1^2 + g_1^{'2}}F \sim \mathcal{O}(\text{TeV})\,. \tag{5.103}$$

Notice that we have not taken into account the contributions from the top quark.

The case where we have the heavy fermion running in the loop is quite similar. Recall that the Yukawa interaction between fermions and Goldstone bosons in (5.70) preserves the upper-left global symmetry; i.e., the $SU(3)_2$. So the Coleman-Weinberg potential is, with the fermion loop factor $-1$,

$$V_{t,CW}^{\text{quad}} = -\frac{a'}{2}y_1^2F^2\left|\Phi_{ij} + \frac{i}{2F}(H_iH_j + H_jH_i)\right|^2 + \dots\,. \tag{5.104}$$

This potential is constructed from the Lagrangian (see Arkani-Hamed *et al.* [21])

$$V_{t,CW} = -a'\frac{1}{4}\lambda_1^2 F^4 \Sigma_{iw}\varepsilon^{wx}\Sigma_{jx}\varepsilon^{ijk}\varepsilon_{kmn}\Sigma^{*my}\varepsilon_{yz}\Sigma^{*nz}\,. \tag{5.105}$$

Notice how the triplet $\Phi$ receives contributions from the Yukawa interaction.

To get the couplings, we will cast the Lagrangians on hand in the general form

$$\begin{aligned} V_{CW} &= \lambda_2 F^2 \mathrm{Tr}\left(\Phi^\dagger\Phi\right) + \mathrm{i}\lambda_3 F\left(H\Phi^* H^\top - H^*\Phi H^\dagger\right) \\ &\quad -\mu^2 HH^\dagger + \lambda_4 (HH^\dagger)^2 \\ &= V_{g,CW}^{\mathrm{quad}} + V_{t,CW}^{\mathrm{quad}} - \mu^2 HH^\dagger\,. \end{aligned} \tag{5.106}$$

Notice that we cannot do much with the $\mu^2 HH^\dagger$ term in the Coleman-Weinberg potential. Collective symmetry breaking trick pushes the mass of $H$ to two-loop which is of order

$$\mu_{2-\mathrm{loop}}^{2(\mathrm{quad})} \sim \frac{\Lambda^2}{(16\pi^2)^2} \approx \frac{(4\pi F)^2}{(16\pi^2)^2} = \frac{F^2}{16\pi^2}\,. \tag{5.107}$$

This term competes with the logarithmic contribution from one-loop; i.e.,

$$\mu_{1-\mathrm{loop}}^{2(\log)} = \frac{F^2\ln(\Lambda^2/F^2)}{16\pi^2} \tag{5.108}$$

The situation is beyond the power of the Coleman-Weinberg technique at one-loop. Therefore we have to take $\mu^2$ as a free parameter and the condition that electroweak symmetry breaks is $\mu^2 > 0$. We then have, from (5.102),

$$
\begin{aligned}
V_{CW} &= -\mu^2 HH^\dagger \\
&\quad +\frac{a}{2}(g_1^2+g_1^{'2})F^2\left[\mathrm{Tr}\Phi^\dagger\Phi+\frac{\mathrm{i}}{2F}\left(h\Phi^\dagger H^\top - H^*\Phi H^\dagger\right)+\frac{1}{4F^2}(HH^\dagger)^2\right] \\
&\quad +\frac{a}{2}(g_2^2+g_2^{'2})F^2\left[\mathrm{Tr}\Phi^\dagger\Phi-\frac{\mathrm{i}}{2F}\left(h\Phi^\dagger H^\top - H^*\Phi H^\dagger\right)+\frac{1}{4F^2}(HH^\dagger)^2\right] \\
&\quad +8a'y_1^2F^2\left[\mathrm{Tr}\Phi^\dagger\Phi-\frac{\mathrm{i}}{2F}\left(h\Phi^\dagger H^\top - H^*\Phi H^\dagger\right)+\frac{1}{4F^2}(HH^\dagger)^2\right]+\ldots \\
&= -\mu^2 HH^\dagger+\left[\frac{a}{2}\left\{\frac{g^2}{s^2c^2}+\frac{g'^2}{s'^2c'^2}\right\}+8a'y_1^2\right]F^2\mathrm{Tr}\Phi^\dagger\Phi \\
&\quad +\mathrm{i}\left[-\frac{a}{4}\left\{g^2\frac{(c^2-s^2)}{s^2c^2}+g'^2\frac{(c'^2-s'^2)}{s'^2c'^2}\right\}+4a'y_1^2\right] \\
&\qquad\qquad \times F\left(H\Phi^\dagger H^\top - H^*\Phi H^\dagger\right) \\
&\quad +\frac{1}{4}\left[\frac{a}{2}\left\{\frac{g^2}{s^2c^2}+\frac{g'^2}{s'^2c'^2}\right\}+8a'y_1^2\right](HH^\dagger)^2\,,
\end{aligned}
\tag{5.109}
$$

which enables us to extract the couplings

$$
\begin{aligned}
\lambda_2 &= \frac{a}{2}\left[\frac{g^2}{s^2c^2}+\frac{g'^2}{s'^2c'^2}\right]+8a'\lambda_1^2, \\
\lambda_3 &= -\frac{a}{4}\left[g^2\frac{(c^2-s^2)}{s^2c^2}+g'^2\frac{(c'^2-s'^2)}{s'^2c'^2}\right]+4a'y_1^2, \\
\lambda_4 &= \frac{a}{8}\left[\frac{g^2}{s^2c^2}+\frac{g'^2}{s'^2c'^2}\right]+2a'y_1^2\,.
\end{aligned}
\tag{5.110}
$$

So, the quadratically divergent Coleman-Weinberg potential is expressed in terms of these couplings as

$$
\begin{aligned}
V_{CW} &= -\mu^2 HH^\dagger+\lambda_4(HH^\dagger)^2 \\
&\quad +\lambda_2F^2\mathrm{Tr}\Phi^\dagger\Phi+\mathrm{i}\lambda_3F\left(H\Phi^\dagger H^\top - H^*\Phi H^\dagger\right)\,.
\end{aligned}
\tag{5.111}
$$



To workout the true minimum, let us introduce the real elements of the fields as follows:

$$
\begin{aligned}
H^\top &= \begin{pmatrix} h_1+\mathrm{i}h_2 \\ h_3+\mathrm{i}h_4 \end{pmatrix} \\
\Phi &= \begin{pmatrix} \phi_1+\mathrm{i}\phi_2 & \frac{1}{\sqrt{2}}(\phi_3+\mathrm{i}\phi_4) \\ \frac{1}{\sqrt{2}}(\phi_3+\mathrm{i}\phi_4) & \phi_5+\mathrm{i}\phi_6 \end{pmatrix}
\end{aligned}
\tag{5.112}
$$

so that we have

$$\mathrm{Tr}\Phi^{\dagger}\Phi = \phi_i\phi_i \tag{5.113}$$

and

$$\begin{aligned} H\Phi^{\dagger}H^{\mathsf{T}} &= (h_1^2+2\mathrm{i}h_1h_2-h_2^2)(\phi_1-\mathrm{i}\phi_2)+\sqrt{2}(h_1+\mathrm{i}h_2)(h_3+\mathrm{i}h_4)(\phi_3-\mathrm{i}\phi_4) \\ &\quad +(h_3^2+2\mathrm{i}h_3h_4-h_4^2)(\phi_5-\mathrm{i}\phi_6)\,. \end{aligned} \tag{5.114}$$

as well as

$$H^*\Phi H^* = (H\Phi^{\dagger}H^{\mathsf{T}})^*\,. \tag{5.115}$$

Consequently, there are only real numbers left in the potential

$$\begin{aligned} V_{CW} &= -\mu^2\left(h_1^2+h_2^2+h_3^2+h_4^2\right)+\lambda_4\left(h_1^4+h_2^4+h_3^4+h_4^4\right) \\ &\quad \lambda_2F^2\left(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2+\phi_5^2+\phi_6^2\right) \\ &\quad +2\lambda_4\left(h_1^2h_2^2+h_1^2h_3^2+h_1^2h_4^2+h_2^2h_3^2+h_2^2h_4^2+h_3^2h_4^2\right) \\ &\quad +2\lambda_3F\Big(-2\phi_1h_1h_2+\phi_2h_1^2-\sqrt{2}\phi_3h_3h_2-\phi_2h_2^2-\phi_3h_4h_1 \\ &\qquad +\sqrt{2}\phi_4h_3h_1-\sqrt{2}\phi_4h_4h_2-2\phi_5h_3h_4+\phi_6h_3^2-\phi_6h_4^2\Big) \end{aligned} \tag{5.116}$$

If we are expecting that electroweak symmetry breaks when $\mu^2 > 0$, the potential (5.116) must be extremum when the vacuum points in the "neutral" direction; i.e., the $SU(2)\times U(1)\to U(1)_{em}$ direction,

$$\langle H^{\mathsf{T}}\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{5.117}$$

and[12]

$$\mathrm{i}\langle\Phi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & v' \end{pmatrix}\,. \tag{5.118}$$

which are equivalent to

$$\langle h_3\rangle = \frac{v}{\sqrt{2}}\,, \qquad \langle\phi_6\rangle = -v' \tag{5.119}$$

[12] The factor i is introduced for later convenience.

and

$$\begin{aligned}\langle h_i \rangle &= 0\,; \qquad \text{i} \neq 3 &(5.120)\\ \langle \phi_j \rangle &= 0\,; \qquad j \neq 6\,. &(5.121)\end{aligned}$$

The validity of these conditions can be directly verified by taking the considering the first derivative of $V_{CW}$ in (5.116) and imposing the vacuum conditions; i.e.,

$$\left.\frac{\partial V_{CW}}{\partial h_i}\right|_0 = 0\,, \qquad \left.\frac{\partial V_{CW}}{\partial \phi_i}\right|_0 = 0\,. \tag{5.122}$$

As a quick check, we will have a look at terms relating to $\phi_6$ and $h_3$. The "$\phi_6$" term is easy

$$\frac{\partial V_{CW}}{\partial \phi_6} = 2\lambda_2 F^2 \phi_6 + 2\lambda_3 F(h_3^2 - h_4^2) \tag{5.123}$$

while the "$h_3$" terms are a bit more involved

$$\begin{aligned}\frac{\partial V_{CW}}{\partial h_1} &= 2\sqrt{2}\lambda_3 F\phi_4 h_3 + 4\lambda_4 h_1 h_3^2 + \text{others} &(5.124)\\ \frac{\partial V_{CW}}{\partial h_2} &= -2\sqrt{2}\lambda_3 F\phi_3 h_3 + 4\lambda_4 h_2 h_3^2 + \text{others} &(5.125)\end{aligned}$$

$$\begin{aligned}\frac{\partial V_{CW}}{\partial h_3} &= -2\mu^2 h_3 + 4\lambda_3 F\phi_6 h_3 + 4\lambda_4 h_3(h_1^2 + h_2^2 + h_3^2 + h_4^2)\\ &\quad +\text{others} &(5.126)\\ \frac{\partial V_{CW}}{\partial h_4} &= -4\lambda_3 F(\phi_5 h_3 + \phi_6 h_4) + 4\lambda_4 h_3^3 h_4 + \text{others} &(5.127)\end{aligned}$$

where "others" denotes terms that do not even contain the $h_3$ or $\phi_6$. In fact the first three terms in (5.117) vanish altogether by the conditions (5.120) and (5.121). The survivors of these conditions are

$$\begin{aligned}\left.\frac{\partial V_{CW}}{\partial \phi_6}\right|_0 &= 2\lambda_2 F^2\langle\phi_6\rangle + 2\lambda_3 F\langle h_3\rangle^2\\ \left.\frac{\partial V_{CW}}{\partial h_4}\right|_0 &= 4\lambda_3 F\langle\phi_6\rangle\langle h_3\rangle - 2\mu^2\langle h_3\rangle + 4\lambda_4\langle h_3\rangle^3\,. &(5.128)\end{aligned}$$

So the conditions (5.119) are satisfied when

$$v^2 = \frac{\mu^2}{\lambda_4 - \lambda_3^2/\lambda_2}\,, \tag{5.129}$$

and

$$v' = \frac{\lambda_3}{\lambda_2}\frac{v^2}{2F}. \tag{5.130}$$

Notice that the relation (5.129) tells us that the free parameter $\mu^2$ should be of order $\sim (100\,\text{GeV})^2$ as we are expecting that $M_h^2 \approx 2\mu^2 = 2v^2(\lambda_4 - \lambda_3^2/\lambda_2)$. The considerations posted around the equation (5.109), stating that $\mu^2 \sim \frac{F^2}{16\pi^2}$, therefore suggest the hierarchy

$$v^2 \sim \frac{F^2}{16\pi^2} \sim \frac{\Lambda^2}{(16\pi^2)^2}\,. \tag{5.131}$$

Notice that this results are obtainable from dimensional analysis. Since the factor $\frac{F^2}{16\pi^2}$ and $v^2 \sim \mu^2$ have their origins from the collective symmetry arguments, the separation between the two scales characterised by $v^2$ and $\Lambda^2$ can be said to be "natural". In other words, the scale $v \sim 100\,\text{GeV} - 200\,\text{GeV}$ is naturally produced, without the requirement of fine-tunings, for $\Lambda$ in the range

$$\Lambda \sim 10\,\text{TeV} - 30\,\text{TeV}\,. \tag{5.132}$$

Now we can trade the 3 couplings ($\lambda$'s) with the parameters $\mu^2$, $v$, and $v'$

$$\lambda_2 = \frac{4\mu^2 v^2}{v^4 - 16F^2 v'^2} \tag{5.133}$$

$$\lambda_3 = \frac{8Fv'\mu^2}{v^4 - 16F^2 v'^2} \tag{5.134}$$

$$\lambda_4 = \frac{\lambda_2}{4} = \frac{\mu^2 v^2}{v^4 - 16F^2 v'^2} \tag{5.135}$$

### 5.2.5.4 Gauge and Mass Eigenstates of the Goldstone Bosons

Once there is the second stage of symmetry breaking (electroweak), we must deal with the new mass eigenstates of the gauge fields. Before doing that, we also have to find the relations between the gauge and mass eigenstates of the Goldstone bosons in order to collect the physical particles (remember that 3 Goldstone bosons will be eaten by $W^\pm$ and $Z$).

There are 10 real fields representing the Goldstone bosons; namely the $h_i$'s and $\phi_j$'s. So we are expecting a $10 \times 10$ mass matrices with $\phi_i$ and $h_i$ mixed inside:

$$M_{ab}^2 = \frac{1}{2}\frac{\partial^2 V_{CW}}{\partial^a\xi\partial^b\xi}\bigg|_0 \tag{5.136}$$

where

$$\xi^a = \begin{cases} h_a\,; & a = 1,\ldots,4 \\ \phi_a\,; & a = 5,\ldots,10 \end{cases}. \tag{5.137}$$

Obviously this $M^2$ is a symmetric. We will partition the matrix $M^2$ as follows:

$$M^2 = \begin{pmatrix} M^2_{UL(4\times4)} & M^2_{UR(4\times6)} \\ M^2_{LL(6\times4)} & M^2_{LR(6\times6)} \end{pmatrix} \tag{5.138}$$

which is still not-diagonal. We have found that the a $4\times4$ upper-left (UL) block[13] is

$$M^2_{UL} = \begin{pmatrix} -\mu^2+v^2\lambda_4 & 0 & 0 & 0 \\ 0 & -\mu^2+v^2\lambda_4 & 0 & 0 \\ 0 & 0 & -\mu^2+3v^2\lambda_4-2F\lambda_3 v' & 0 \\ 0 & 0 & 0 & -\mu^2+v^2\lambda_4+2F\lambda_3 v' \end{pmatrix} \tag{5.139}$$

while the $6\times6$ lower-right (LR) block is

$$M^2_{LR} = \begin{pmatrix} F^2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & F^2\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & F^2\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & F^2\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & F^2\lambda_2 \end{pmatrix}. \tag{5.140}$$

[13]The full $10\times10$ matrix is given in (C.69) page 257.

Still, there is the off-diagonal block. We present the $4 \times 6$ upper-right (UR) block (the lower-left is just a mirror image as $M^2$ is symmetric)

$$M^2_{UR} = \begin{pmatrix} 0 & 0 & 0 & Fv\lambda_3 & 0 & 0 \\ 0 & 0 & -Fv\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}Fv\lambda_3 \\ 0 & 0 & 0 & 0 & -\sqrt{2}Fv\lambda_3 & 0 \end{pmatrix}. \tag{5.141}$$

It is not straightforward to diagonalise the $M^2$ in this form. However, since there are very few off-diagonal elements, we can "re-shuffle" them to get a more desirable matrix. For example, consider the element $\xi^3 = h_3$ and $\xi^4 = h_4$. We see that we can construct a $2 \times 2$ block matrix by switching between $\xi^4$ and $\xi^{10}$; i.e., setting $\tilde{\xi}^4 = \xi^{10} = \phi^6$, $\tilde{\xi}^{10} = \xi^4 = h_4$ and $\tilde{\xi}^3 = \xi^3 = h_3$. We obtain the $h_3 - \phi_6$ mixing matrix

$$\begin{aligned} \tilde{M}^2_{(h_3\phi_6)} &= \begin{pmatrix} -\mu^2 + 3v^2\lambda_4 - 2F\lambda_3 v' & \sqrt{2}Fv\lambda_3 \\ \sqrt{2}Fv\lambda_3 & F^2\lambda_2 \end{pmatrix} \\ &= \frac{\mu^2}{v^4 - 16F^2v'^2} \begin{pmatrix} -v^4 + 16F^2v'^2 + 3v^4 - 16F^2v'^2 & 8\sqrt{2}F^2vv' \\ 8\sqrt{2}F^2vv' & 4F^2v^2 \end{pmatrix} \\ &= \frac{2\mu^2}{v^4 - 16F^2v'^2} \begin{pmatrix} v^4 & 4\sqrt{2}F^2vv' \\ 4\sqrt{2}F^2vv' & 2F^2v^2 \end{pmatrix}. \end{aligned} \tag{5.142}$$

where we have used (5.133), (5.134) and (5.135). By recalling (5.130) we further approximate $v' = v^2/2F$. Then we get

$$\begin{aligned} \tilde{M}_1 &\approx 2\mu^2 \\ \tilde{M}_2 &\approx \frac{4\mu^2F^2\left(v^2 + \frac{v^4}{2F^2}\right)}{v^4 - 16F^2v'^2}. \end{aligned} \tag{5.143}$$

Since the mixing that we are dealing with at the moment is that between $h_3$ and $\phi_6$ and we know that the position of $h_3$ in the doublet (i.e., its quantum number) is just that of the Higgs in the standard model, we can then safely say that

$$M^2_{\tilde{h}} \approx 2\mu^2 \tag{5.144}$$

$$M^2_{\Phi} \approx \frac{2M^2_{\tilde{h}}F^2\left(v^2 + \frac{v^4}{2F^2}\right)}{v^4 - 16F^2v'^2}. \tag{5.145}$$

Notice that we have introduced the notation $\tilde{h}$ for the physical Higgs; i.e., in its mass eigenstate after electroweak symmetry is broken.

Next, we will have a look at other blocks. A quick glance that there are 8 off-diagonal elements of the matrix $M^2$ (c.f. (C.69)) tells us that there will be one $2 \times 2$ diagonal matrix left after the reshuffle of the basis. That matrix, corresponding to the field $\phi_1$ and $\phi_2$ ($\xi^5$ and $\xi^6$), is

$$M^2_{\phi_1\phi_2} = M^2_{\tilde{\Phi}^{++}} = \begin{pmatrix} F^2\lambda_2 & 0 \\ 0 & F^2\lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{2M^2_{\tilde{h}}F^2}{v^4-16F^2v'^2} & 0 \\ 0 & \frac{2M^2_{\tilde{h}}F^2}{v^4-16F^2v'^2} \end{pmatrix}. \tag{5.146}$$

In the new basis, we will use

$$\tilde{\Phi}^{++} = \Phi/\mathrm{i}\,. \tag{5.147}$$

The remaining eigenvalues of the $M^2$ are fairly non-trivial (still, brute force is always possible). Hence, we will use some heuristic arguments to "guess" what should they be, rather than diagonalise them explicitly.

At this stage we have paired up 4 rows and columns corresponding to $\xi^4 - \xi^{10}$ and $\xi^5 - \xi^6$, we then have 6 pairs to go. However, notice that as there are 3 diagonal elements $F^2\lambda_2 = 2M^2_{\tilde{h}}F^2$ left unpaired, the remaining $2 \times 2$ blocks will be of the form

$$\begin{pmatrix} F^2\lambda_2 & \alpha F v\lambda_3 \\ \alpha F v\lambda_3 & X \end{pmatrix} \tag{5.148}$$

where $\alpha$ is a numerical factor (say, $\sqrt{2}$) and $X$ is one of the following elements

$$-\mu^2 + v^2 \tag{5.149}$$

$$-\mu^2 + v^2\lambda_4 + 2F\lambda_3 v' \tag{5.150}$$

$$F^2\lambda_2 \tag{5.151}$$

Also notice that we cannot let the $h_1$ and $h_2$ stay at the same positions in the matrix $M^2$ because of their off-diagonal elements. Therefore, in the new basis, there is no $2 \times 2$ sub-matrix having $-\mu^2 + v^2$ in both of the diagonal elements. Similar argument also applies to the case of having $F^2\lambda_2$ in both diagonal elements. Consequently all the remaining mixings will be of the $h - \phi$

type. One of them is

$$\tilde{M}_1^2 = \begin{pmatrix} F^2\lambda_2 & \alpha_1 F v \lambda_3 \\ \alpha_1 F v \lambda_3 & -\mu^2 + v^2\lambda_4 + 2F\lambda_3 v' \end{pmatrix} \tag{5.152}$$

and the remaining two are

$$\tilde{M}_2^2 = \begin{pmatrix} F^2\lambda_2 & \alpha_2 F v \lambda_3 \\ \alpha_2 F v \lambda_3 & -\mu^2 + v^2 \end{pmatrix} . \tag{5.153}$$

By using (5.133), (5.134), and (5.135), it is easy[14] to see that one of the eigenvalues of either $\tilde{M}_1^2$ or $\tilde{M}_2^2$ will be zero while the other are proportional to

$$\frac{4\mu^2 F^2 \left(v^2 + \alpha' v'^2\right)}{v^4 - 16F^2 v'^2} , \tag{5.154}$$

where $\alpha'$ is a numerical factor (depending on the members of the matrices). Finally, we can set up another convention that further simplifies things: we arrange the matrix so that the mixings above occur between fields in the real and imaginary part in the same manner that $h_3$ and $\phi_6$ does (recall $h_3 + \mathrm{i}h_4$ and $\phi_5 + \mathrm{i}\phi_6$). This trick will help us arrange the mixing angles more easily.

So far we have considered the mixings of the $h_i$ and $\phi_i$ in the "final" mass eigenstates defined by the electroweak symmetry breaking. We see that masses of the members of the Higgs doublets in these mass eigenstates are zero except for the one that corresponds to the physical Higgs of the standard model. Such the three massless members are the Goldstone bosons that survive massless throughout both stages of symmetry breaking. They are exact Goldstone bosons. Of course, they will not make it to the physical spectrum as they will be "eaten" by the electroweak gauge fields. In addition, the triplet Goldstone bosons $\Phi$ are now massive as expected. At tree level, the mass eigenstates of $\Phi$ are degenerate where (c.f. (5.112)),

$$M_\Phi^2 \approx \lambda_2 F^2 \approx \frac{2M_{\tilde{h}}^2 F^2 v^2}{v^4 - 16F^2 v'^2} . \tag{5.155}$$

This immediately points out the conditions

$$v^4 > 16F^2 v'^2 , \tag{5.156}$$

[14]Remember that the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are $\frac{1}{2}(a+c) \pm \frac{1}{2}\sqrt{(a-c)^2 + 4b^2}$.

or

$$\frac{v'^2}{v^2} < \frac{v^2}{16F^2}\,, \tag{5.157}$$

which tell us how much the contributions from the triplet $\Phi$ get suppressed. It also tells us that the vacuum expectation value $v'$ of the triplet is fairly smaller than that of the Higgs doublet; for example, $v' \sim 5 - 20\,\text{GeV}$.

### 5.2.6 After Electroweak Symmetry Breaking

Once the electroweak symmetry is broken, both the Goldstone bosons and the gauge bosons will have their new mass eigenstates. They can be obtained by diagonalising their corresponding mass matrices and evaluating the mixing angles in the similar fashion to those shown in the previous section.

#### 5.2.6.1 Final Mass Eigenstates of the Goldstone Bosons

In this section we will follow up what we have left in the previous section on the mass eigenstates of the Goldstone bosons. Let us denote the mixing between $h_3$ and $\phi_6$ as[15]

$$\begin{pmatrix} h_3 \\ \phi_6 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{pmatrix} \begin{pmatrix} \tilde{h} \\ \tilde{\Phi}^0 \end{pmatrix} \tag{5.158}$$

where $c_0$ and $s_0$ correspond to the mixing angles, and $\tilde{h}$ is the Higgs of the standard model. Still we need to find how the other two components mix; namely, the $h_4$ and $\phi_5$ in terms of the "to be eaten" Goldstone bosons $\tilde{G}^0$ and the neutral pseudoscalar $\tilde{\Phi}^P$. We write

$$\begin{pmatrix} h_4 \\ \phi_5 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c_P & -s_P \\ s_P & c_P \end{pmatrix} \begin{pmatrix} \tilde{G}^0 \\ \tilde{\Phi}^P \end{pmatrix} \tag{5.159}$$

which results in new real and imaginary parts, in terms of the mass eigenstates,

$$h^0 = \frac{1}{\sqrt{2}}\left(c_0\tilde{h} - s_0\tilde{\Phi}^0\right) + \mathrm{i}\frac{1}{\sqrt{2}}\left(c_p\tilde{G}^0 - s_P\tilde{\Phi}^P + v\right) \tag{5.160}$$

$$\phi^0 = \frac{1}{\sqrt{2}}\left(s_p\tilde{G}^0 + c_P\tilde{\Phi}^P\right) + \mathrm{i}\frac{1}{\sqrt{2}}\left(s_0\tilde{h} + c_0\tilde{\Phi}^0 + \sqrt{2}v'\right)\,. \tag{5.161}$$

[15] We follow notations of sines and cosines used in Han *et al.* [3].

Then the next step is to write out the mixing angles[16]. Using the formulas in the footnote below and the matrix elements; for example, from (5.142), we can get the rough approximations

$$\begin{aligned} c_0^2 &\approx 1 - 8\frac{v'^2}{v^2} \\ s_0^2 &\approx 8\frac{v'^2}{v^2} \end{aligned} \tag{5.165}$$

or $c_0 = 1 - 4\frac{v'^2}{v^2}$ and $s_0 = 2\sqrt{2}\frac{v'}{v}$ . Then, the other mixing angles can be found in a very similar manners. They are

$$\begin{aligned} c_P^2 &\approx 1 - 8\frac{v'^2}{v^2} \\ s_P^2 &\approx 8\frac{v'^2}{v^2}\,. \end{aligned} \tag{5.166}$$

Next, we define the final mass eigenstates of the charged fields

$$h^+ = c_+\tilde{G}^+ - s_+\tilde{\Phi}^+ \tag{5.167}$$

$$\mathrm{i}\phi^+ = \left(s_+\tilde{G}^+ + c_+\tilde{\Phi}^+\right)\,. \tag{5.168}$$

as well as

$$\mathrm{i}\phi^{++} = \tilde{\Phi}^{++}\,. \tag{5.169}$$

[16] Recall that if $m_1$ and $m_2$ are the eigenvalues of a matrix $M$, then there is a matrix $U$ that diagonalises the matrix $M$ according to

$$\begin{aligned} UMU^{\mathsf{T}} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} a+c-\sqrt{(a-c)^2+4b^2} & 0 \\ 0 & a+c+\sqrt{(a-c)^2+4b^2} \end{pmatrix}\,. \end{aligned} \tag{5.162}$$

Then we can solve for the mixing angles $(\cos\theta,\,\sin\theta)$ as functions of $a, b, c$ as

$$\cos^2\theta = \left[(a-c) + \sqrt{(a-c)^2+4b^2}\right]/2\sqrt{(a-c)^2+4b^2} \approx 1 - b^2/(a-c)^2\,, \tag{5.163}$$

and $\sin^2\theta \approx b^2/(a-c)^2$. Besides, we have

$$\tan 2\theta = \frac{2b}{a-c}\,. \tag{5.164}$$

Then we get the mixing angles

$$\begin{aligned} c_+^2 &\approx 1 - 4\frac{v'^2}{v^2} \\ s_+^2 &\approx 4\frac{v'^2}{v^2}\,. \end{aligned} \tag{5.170}$$

### 5.2.6.2 Final Mass Eigenstates of the Gauge Fields

Once we have specified the vacuum expectation value of the Goldstone bosons fields, we can evaluate the mass of the gauge fields from kinetic terms (in the covariant derivative). Since the calculations in this section are rather tedious (though straightforward), we will not reproduces the results found in the literature by Han *et al.* [3], for example. Instead, we will try, more or less, to use heuristic arguments that can guide us to the results and also try to analyse them.

Now the Goldstone boson matrix is

$$\Pi \cdot X\Big|_{EW} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{v}{2} & 0 & v' \\ 0 & \frac{v}{2} & 0 & 0 & \frac{v}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{v'}{2} & \frac{v}{2} & 0 & 0 \end{pmatrix} \tag{5.171}$$

Then

$$\Sigma_{EW} \equiv e^{i2\Pi\cdot X\big|_{EW}/F}\Sigma_0 = \Sigma_0 + \frac{1}{F}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2iv' & iv & 0 & 0 \\ 0 & iv & 0 & 0 & iv \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iv & 0 & 2iv' \end{pmatrix} - \frac{1}{F^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & v^2 & 2vv' & 0 & 4\left(\frac{v^2}{4}+v'^2\right) \\ 0 & 2vv' & 2v^2 & 0 & 2vv' \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4\left(\frac{v^2}{4}+v'^2\right) & 2vv' & 0 & v^2 \end{pmatrix}. \tag{5.172}$$

So

$$\Sigma_{EW} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{v^2}{F^2}+\frac{2iv'}{F} & \frac{iv}{F}-\frac{2vv'}{F^2} & 0 & 1-\frac{\left(v^2+4v'^2\right)}{F^2} \\ 0 & \frac{iv}{F}-\frac{2vv'}{F^2} & 1-\frac{2v^2}{F^2} & 0 & \frac{iv}{F}-\frac{2vv'}{F^2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{\left(v^2+4v'^2\right)}{F^2} & \frac{iv}{F}-\frac{2vv'}{F^2} & 0 & -\frac{v^2}{F^2}+\frac{2iv'}{F} \end{pmatrix} + \mathcal{O}(\frac{1}{F^3}) \quad (5.173)$$

This matrix will be put into the usual kinetic term

$$\mathcal{L}_{\Sigma_{EW}} = \frac{F^2}{8}\mathrm{Tr}\left\{D_\mu\Sigma^\dagger D^\mu\Sigma\right\}\Big|_{\Sigma_{EW}} \quad (5.174)$$

To the lowest order (include only $\Sigma_0$), we obviously recover the mass matrices that were obtained in section 5.2.1. Thus, it will be helpful if the calculation is performed based on the mass eigenstates so we can easily see the corrections from electroweak symmetry breaking.

Since the calculations will be painstaking (as we shall see, the author will take results from other sources), we will try some heuristic arguments based on our considerations done in the previous chapter. First, recall that to first non-zero order (the $\mathcal{O}(g^2v^2)$), there will be interactions between the Higgs and the Goldstone bosons while the other pseudo-Goldstone bosons are massive and do not interact with the gauge bosons [17]. Thus, it can be expected that the form of all the corrections (to the gauge bosons mass matrices) to the first order $\mathcal{O}(v^2)$ will be proportional to

$$\frac{\beta}{4}g^2v^2 \quad \text{or} \quad \frac{\beta}{4}g'^2v^2\,, \text{etc...} \quad (5.175)$$

where $\beta$, standing for various forms of all possible coefficients, can be a function of the mixing angles $(c, c', s, s')$. The factor $1/4$ can always be obtained with proper definitions of the vacuum expectation value $v$ and $v'$. In addition, the factors $g, g'$ should go along in a consistent manners with the corresponding gauge fields ($gg'v^2WB'$, for example).

[17] Still, we have seen that electroweak symmetry breaking introduced mixings between the two types of the pseudo-Goldstone bosons. The effects of mixings and hence the dependencies on $v'$ will show up in higher orders.

Then our heuristic arguments tell us that the mass terms "should" look like

$$\begin{aligned}
\mathcal{L}_{\Sigma_{EW}}\Big|_{\mathcal{O}(g^2v^2)} \overset{?}{\approx} & \frac{1}{2}\left[M_{W'}^2-\frac{1}{4}g^2v^2\right]W'^a_\mu W'^{a\mu}+\frac{1}{2}\left[M_{B'}^2-\frac{1}{4}g'^2v^2\right]B'_\mu B'^\mu \\
& +\frac{1}{2}\left(\frac{1}{4}g^2v^2\right)W^a_\mu W^{a\mu}+\frac{1}{2}\left(\frac{1}{4}g'^2v^2\right)B_\mu B^\mu \\
& +\frac{1}{2}\left(\frac{\beta_1}{4}g^2v^2\right)W^a_\mu W'^{a\mu}+\frac{1}{2}\left(\frac{\beta_2}{4}g'^2v^2\right)B_\mu B'^{mu} \\
& +\frac{1}{2}\left(\frac{1}{4}gg'v^2\right)\left[\beta_3 W^3_\mu B^\mu+\beta_4 W'^3_\mu B'^\mu+\beta_5 W'^3_\mu B^\mu+\beta_6 W^3_\mu B'^\mu\right],
\end{aligned} \tag{5.176}$$

where $\beta_i$ stands for a function of the mixing angles (from the first stage of symmetry breaking). Also note the differrent definitions of masses of the charged and the neutral gauge fields.

The higher order effects will introduce the splitting between the masses of neutral and charged gauge fields. This is not beyond our expectation since there are mixings between the Goldstone bosons (the $H$ and the $\Phi$) which were eventually eaten by the gauge fields (see the section 4.1 on dynamical symmetry breaking and vacuum alignment). Here, we will show how to multiply matrices and collect various terms here, and instead will refer to the result found by Han *et al.* in [3]. The mass terms in the unmixed basis are

$$\begin{aligned}
\mathcal{L}_{\Sigma_{EW}} = & \frac{1}{2}W'^a_\mu W'^{a\mu}\left[M_{W'}^2-\frac{1}{4}g^2v^2\right]+\frac{1}{2}B'_\mu B'^\mu\left[M_{B'}^2-\frac{1}{4}g'^2v^2\right] \\
& +W^+_\mu W^{-\mu}\left[\frac{1}{4}g^2v^2\left(1-\frac{v^2}{6F^2}+4\frac{v'^2}{v^2}\right)\right] \\
& +\frac{1}{2}W^3_\mu W^{3\mu}\left[\frac{1}{4}g^2v^2\left(1-\frac{v^2}{6F^2}+8\frac{v'^2}{v^2}\right)\right] \\
& +\frac{1}{2}B_\mu B^\mu\left[\frac{1}{4}g'^2v^2\left(1-\frac{v^2}{6F^2}+8\frac{v'^2}{v^2}\right)\right] \\
& +W^3_\mu B^\mu\left[\frac{1}{4}gg'v^2\left(1-\frac{v^2}{6F^2}+8\frac{v'^2}{v^2}\right)\right] \\
& -W^a_\mu W'^{a\mu}\left[\frac{1}{4}g^2v^2\frac{(c^2-s^2)}{2sc}\right]-B_\mu B'^\mu\left[\frac{1}{4}g'^2v^2\frac{(c'^2-s'^2)}{2s'c'}\right] \\
& -W'^3_\mu B^\mu\left[\frac{1}{4}gg'v^2\frac{(c^2-s^2)}{2sc}\right]-W^3_\mu B'^\mu\left[\frac{1}{4}gg'v^2\frac{(c'^2-s'^2)}{2s'c'}\right] \\
& +W'^3_\mu B'^\mu\left[-\frac{1}{8}gg'v^2\left(\frac{cs'}{sc'}+\frac{sc'}{cs'}\right)\right].
\end{aligned} \tag{5.177}$$

We should keep in mind the condition (5.157) that though $\frac{v'^2}{v^2} < \frac{v^2}{16F^2}$, the $\mathcal{O}(\frac{v'^2}{v^2})$ effects are not negligible, and so do $\mathcal{O}(\frac{v^4}{F^2})$ terms. These effects will show up in the masses of the light gauge bosons (in their new mass eigenstates).

Now let us consider the charged $W, W'$ bosons. Denoting, $m_W^2 = \frac{1}{4}g^2v^2$, we have the $4 \times 4$ mass matrix

$$\begin{pmatrix} -m_W^2 + M_{W'}^2 & -\frac{m_W^2(c^2-s^2)}{2cs} \\ -\frac{m_W^2(c^2-s^2)}{2cs} & m_W^2\left(1 - \frac{v^2}{6F^2} + \frac{4v'^2}{v^2}\right) \end{pmatrix} \otimes \mathbb{1}_{2\times 2} \tag{5.178}$$

where we have used the basis ($W'^+$, $W^+$, $W'^-$, $W^-$). Then observe that the off-diagonal terms are considerably smaller than the $M_{W'}^2$. So we can easily evaluate the eigenvalues. Let us denote the eigenvalues of this matrix as $M_{W_H^\pm}$ and $M_{W_L^\pm}$, for heavy and light gauge bosons respectively. We then find,

$$\begin{aligned} M_{W_H^\pm}^2 &= M_{W'}^2 - m_W^2 + \frac{m_W^4(c^2-s^2)^2/4c^2s^2}{m_W^2\left(-\frac{1}{2} + \frac{v^2}{6F^2} - \frac{4v'^2}{v^2}\right) + M_{W'}^2} \\ &= M_{W'}^2 - m_W^2 + \mathcal{O}(\frac{m_W^4}{M_{W'}^2}) \end{aligned} \tag{5.179}$$

and

$$M_{W_L^\pm}^2 = m_W^2\left[1 - \frac{v^2}{F^2}\left(\frac{1}{6} + \frac{1}{4}(c^2-s^2)^2\right) + 4\frac{v'^2}{v^2}\right]. \tag{5.180}$$

Note that both mass terms come with different orders of magnitude. We have neglected the $\mathcal{O}(\frac{m_W^4}{M_{W'}^2}) \sim \mathcal{O}(\frac{v^4}{F^2})$ as it is small comparing with the mass of $M_{W'}^2$ in the first term, while we keep terms up to $\mathcal{O}(\frac{v^4}{F^2})$ in the eigenvalue (mass) of the light gauge field. At this point, it should become clear that had we neglected the $\mathcal{O}(\frac{v^4}{F^2})$ terms in (5.177), we would have faced with the "too simple" mass term of the light gauge bosons

$$M_{W_L} = m_W \qquad \text{(wrong)} \tag{5.181}$$

which is inconsistent with the mass of the heavy gauge boson; one shows mixings, the other does not. Consequently, it is easy to verify (direct substitutions) that the final charged gauge fields mass eigenstates $W_H, W_L$ are given by

$$\begin{aligned} W_H^\pm &= W'^\pm - \frac{v^2}{2F^2}sc(c^2-s^2)W^\pm \\ W_L^\pm &= \frac{v^2}{2F^2}sc(c^2-s^2)W'^\pm + W^\pm \end{aligned} \tag{5.182}$$

Now, let us turn to the mixings of the neutral gauge bosons. The situation is rather involved as, according to (5.177), the mass matrix is not reducible to the $2\times 2$ block matrix (direct product with $\mathbb{1}_{2\times 2}$). Let us denote $m_Z = gv/2c_W = m_W/c_W = g'v/2s_W$ where $c_W$ is (the cosine of) the Weinberg angle

$$c_W = \frac{g}{\sqrt{g^2+g'^2}} \tag{5.183}$$

$$s_W = \frac{g'}{\sqrt{g^2+g'^2}}\,. \tag{5.184}$$

Again, let us note the frequently used relations

$$\frac{g'^2}{g^2}m_W^2 = s_W^2 m_Z^2\,. \tag{5.185}$$

The $4\times 4$ mass matrix found is listed in (C.4). With a quick glance at (C.4) we see right away that one of the eigenvalues should be zero (located in the lower-right $2\times 2$ block).

We require that electroweak symmetry breaks down to $U(1)_{em}$. Thus, one zero eigenvalue of the mass matrix (C.4) is expected. Notice that the appearance of the zero mass gauge field (the photon) is fairly non-trivial especially if we try to diagonalise the $4\times 4$ mass matrix (C.4) directly. Still, it might be helpful if we write the $2\times 2$ lower-right block of the mass matrix (C.4) as

$$M_{LR}^2 = \begin{pmatrix} m_W^2 K & -s_W m_W m_Z K \\ -s_W m_W m_Z K & s_W^2 m_Z^2 K \end{pmatrix} \tag{5.186}$$

where

$$K \equiv \left(1 - \frac{v^2}{6F^2} + 8\frac{v'^2}{v^2}\right)\,. \tag{5.187}$$

Notice that we cannot just push this $2\times 2$ block $M_{LR}^2$ out since there are off-diagonal terms in the $4\times 4$ matrix. However, it can be seen that the off-diagonal $2\times 2$ matrices of (C.4) have one zero eigenvalue and is by itself diagonalisable. So our heuristic argument suggests that we can somehow rearrange (transform) the basis such that we can separate one of the element, say the one corresponding to lower-right most ($s_W^2 m_Z^2 K$), from the first two components, so that this component mixes with only the members of the lower-right $2\times 2$ matrix. Therefore, the zero eigenvalue in (5.186), if there is any, is an exact zero. The other one (i.e., the physical $Z$), which must be non-zero by construction, must have some terms that are mixed with the other elements from the heavy gauge fields masses. Clearly this matrix $M^2_{LR}$ has eigenvalues,

$$0 \quad \text{and}\, (m^2_W + s^2_W m^2_Z)K\,, \tag{5.188}$$

and therefore tells us that there is a final mass eigenstate that represents the photon.

With the above reasonings, we can begin with the mass eigenstate of the photon

$$A^\mu_L = s_W W^{3\mu} + c_W B^\mu\,. \tag{5.189}$$

The other mass eigenstate of the light gauge field (the $Z$) must be orthogonal to (5.189) as well as the other two eigenstates of the heavy fields. Thus, it is best to separate out terms that mix with the $A_L$ from others. So we follow Han *et al.* [3] and introduce the notation (that is consistent with the eigenvalues)

$$Z^\mu_L \;=\; c_W W^{3\mu} - s_W B^\mu + x^{W'}_Z \frac{v^2}{F^2} W'^{3\mu} + x^{B'}_Z \frac{v^2}{F^2} B'^\mu \tag{5.190}$$

where the $x^{W'}_Z \frac{v^2}{F^2}$ and $x^{B'}_Z \frac{v^2}{F^2}$ will play the role of the mixing "angles", which also explicitly tell us that the mixing between light and heavy gauge field are suppressed by a factor $\frac{v^2}{F^2}$. With these mass eigenstates, we can further look for the form of those of the heavy fields. Denoting, the heavy fields as $Z_H$ and $A_H$, and demanding that they are orthogonal to each other and to the light gauge fields, we write

$$Z^\mu_H \;=\; W'^3 - x_H \frac{v^2}{F^2} B'^\mu - x^{W'}_Z \frac{v^2}{F^2}(c_W W^{3\mu} - s_W B^\mu) \tag{5.191}$$

$$A^\mu_H \;=\; B'^\mu + x_H \frac{v^2}{F^2} W'^{3\mu} - x^{B'}_Z \frac{v^2}{F^2}(c_W W^{3\mu} - s_W B^\mu)\,, \tag{5.192}$$

where $x_H$ characterises how the heavy fields ($W^3$ and $B$) mix. It is found that

$$\begin{aligned} x^{W'}_Z &= -\frac{1}{2c_W} sc(c^2 - s^2) \\ x^{B'}_Z &= -\frac{5}{2s_W} s'c'(c'^2 - s'^2)\,. \end{aligned} \tag{5.193}$$

Then, for the light fields, the eigenvalues are

$$M_{Z_L}^2 = m_Z^2\left[1-\frac{v^2}{F^2}\left(\frac{1}{6}+\frac{1}{4}(c^2-s^2)^2+\frac{5}{4}(c'^2-s'^2)^2\right)+8\frac{v'^2}{v^2}\right] \quad (5.194)$$

$$M_{A_L} = 0\,. \quad (5.195)$$

Next, eigenvalue for the heavy neutral $Z$ field is found to be

$$M_{Z_H}^2 = M_{W'}^2 - m_W^2 - m_Z^2 s_W^2 \frac{5}{2}\frac{gg'}{s'c'}\frac{sc(c^2s'^2+s^2c'^2)}{(5g^2s'^2c'^2-g'^2s^2c^2)}\,. \quad (5.196)$$

So we define

$$x_H = \frac{5}{2}gg'\frac{scs'c'(c^2s'^2+s^2c'^2)}{(5g^2s'^2c'^2-g'^2s^2c^2)}\,. \quad (5.197)$$

Then (5.196) becomes

$$\begin{aligned} M_{Z_H}^2 &= M_{W'}^2 - m_W^2 - m_Z^2 s_W^2 \frac{x_H}{s'^2c'^2} \\ &= M_{W'}^2 - m_W^2\left(1+\frac{s_W^2}{c_W^2}\frac{x_H}{s'^2c'^2}\right)\,. \end{aligned} \quad (5.198)$$

The other eigenvalue of the heavy field is

$$\begin{aligned} M_{A_H}^2 &= \frac{1}{5}M_{B'}^2 - m_Z^2 s_W^2 + m_W^2\frac{x_H}{s^2c^2} \\ &= \frac{1}{5}M_{B'}^2 - m_Z^2 s_W^2\left(1-\frac{c_W^2}{s_W^2}\frac{x_H}{s^2c^2}\right)\,. \end{aligned} \quad (5.199)$$

#### 5.2.6.3 Final Mass Eigenstates of the Top Quarks

Recall that the top quarks (the top and its partners) interact via the interaction term in (5.70). After electroweak symmetry breaks, the pseudo Goldstones are in their new mass eigenstates, the top quarks mass eigenstates will further mix. So let us write the mass term in the Lagrangian as

$$\mathcal{L}_t = -m_t t_L t_R^c - M_T T_L T_R^c\,. \quad (5.200)$$

However, we know that the mass matrix of the fermions in the gauge eigenstates basis is, in general, neither symmetric nor hermitian (but the $M_f M_f^\dagger$ is hermitian and positive). But once we assume that the original mass matrix is rewritable as a product of a unitary matrix and a hermitian matrix, it is possible to perform a biunitary transformation

$$S^{\dagger} M_f T = M_{\text{diagonal}} \tag{5.201}$$

where both $S$ and $T$ are unitary[18]. This means left-handed and right-handed fermions have different relations between the mass eigenstates and the gauge eigenstates. With (5.201), the mass term in the Lagrangian (for fermions $\psi$, in general) is diagonalised as follows:

$$\bar{\psi}'_L S S^{\dagger} M_f T T^{\dagger} \psi_R = \bar{\psi}_L M_{\text{diagonal}} \psi_R \tag{5.202}$$

where $\psi'_L = S\psi_L$ and $\psi'_R = T\psi_R$.

Still, this is not the whole story. We know that other fermions in the theory also have similar mixings between gauge and mass eigenstates (recall the CKM matrix). Then by requiring them to interact with the pseudo Goldstone bosons in the same way as the tops, we need to make some further adjustments with parameters (like the couplings) or else the hierarchy of the fermions will come out wrong. This will introduce more complexities (and ambiguities) into the model. Nevertheless, recall that fermions in the first two generations do not bring up severe quadratic divergences like the top quark does. Then the Yukawa couplings of fermions in those generations do not need to be protected by the global symmetries that are encoded in the interaction (5.70). Thus, one of the ways to simplifies the consideration is to assume that the fermions interact with the Goldstone bosons via other types of operator[19]; for example,

$$\bar{Q}_L H (X)^r (Y)^s u^c \tag{5.203}$$

where $X$ and $Y$ are some components of the $\Sigma$ that receives vacuum expectation value of order $F$ and $r, s$ are integers. If we do so, the by-product we get is the extra freedom for choosing the $U(1)$ charges of fermions in the first two generations because there is no global symmetries to fix them as those in the top quark case.

With various possibilities for the quark couplings described above, it is best to simplify things by concentrating on the top quark alone. We denote the top quark and its partner in the gauge eigenstates by $T_L, T_R$ and $t_L, t_R$ and write

[18] For more information on the biunitary transformation, see, for example, Cheng and Li [50].

[19] See Csáki *et al.* [78] for the modifications of the interactions of the Yukawa couplings and the $U(1)$ charges of the first two generations.

the mixings as

$$t_L = c_L t_3 - s_L \tilde{t} \tag{5.204}$$

$$T_L = s_L t_3 + c_L \tilde{t} \tag{5.205}$$

and

$$t_R = c_R u_3'^c - s_R \tilde{t}'^c \tag{5.206}$$

$$T_R = s_R u_3'^c + c_R \tilde{t}'^c \tag{5.207}$$

where the left-handed and right-handed fields are rotated with different parameters $s_L$ and $s_R$. Notice that we use $u_3'^c$ instead of $u_3'$ since we have to start over and expand $\Sigma$ (with the electroweak expectation values) from the Yukawa interaction Lagrangian. Also observe that now we will use $T_R$ (not $T_R^c$). Then the top quark mass matrix is

$$M_t = \begin{pmatrix} y_1 F(\sin\frac{v}{F})/\sqrt{2} & y_1 F(1+\cos\frac{v}{F})/2 \\ 0 & y_2 F \end{pmatrix}, \tag{5.208}$$

giving

$$M_t M_t^\dagger = \begin{pmatrix} y_1^2 F^2(\bar{s}^2/2 + (1+\bar{c})^2/4) & y_1^2 F^2 \bar{s}(1+\bar{c})/(2\sqrt{2}) \\ y_1^2 F^2 \bar{s}(1+\bar{c})/(2\sqrt{2}) & y_2^2 F^2 \end{pmatrix} \tag{5.209}$$

which is clearly Hermitian. Notice that we have used $\bar{s} = \sin(v/F)$ and $\bar{c} = \cos(v/F)$. Then the mixing angles are

$$s_L \simeq \left(\frac{y_1^2}{y_1^2 + y_2^2}\right)\frac{v}{F} \tag{5.210}$$

for the left-handed quarks, and

$$s_R \simeq \bar{s}\left[1 - \bar{c}^2\left(\frac{1}{2} - \left(\frac{y_1^2}{y_1^2 + y_2^2}\right)\right)\frac{v^2}{F^2}\right]. \tag{5.211}$$

for the right-handed ones. The mass of of the tops in their new eigenstates are[20]

$$m_{t_L} \simeq \frac{y_t v}{\sqrt{2}}\left[1+\beta\frac{v^2}{F^2}\frac{y_1^2}{y_1^2+y_2^2}\right] \quad (5.212)$$

$$M_{T_L} \simeq \sqrt{y_1^2+y_2^2}F\left[1-\beta\frac{v^2}{F^2}\frac{y_1^2}{y_1^2+y_2^2}\right] \quad (5.213)$$

where $\beta$ is a numerical value of $\mathcal{O}(1)$. Notice that the left-handed quarks will not mix unless the electroweak symmetry breaks, unlike the right-handed ones which always do. Actually, we are already aware of this fact from the mixing formula (5.75) which tells us the the right-handed (top-like) singlet appearing in the Lagrangian (5.70) does mix with the right-handed heavy quark.

## 5.3 A Survey on the Phenomenology and Issues of the Little Higgs

So far we have studied various aspects of the model building of the Littlest Higgs. Currently, in 2007, we are at the time where the next generation accelerators like the LHC (CERN) is scheduled to begin its operation within the near future. We have seen that there are some free parameters and some new particles needed to be introduced in the model and hence opening possibilities for the experiments as there are more processes to choose. Thus, the Little Higgs should be interesting in the sense of experiments. In addition, since the model have a fairly large number of parameters comparing with the standard model, we generally expect to see some footprints or some theoretical constraints that allow us to distinguish the Little Higgs model from others (or the model is not useful at all as it needs more ingredients to predict similar things). In this section we will describe some of the interesting phenomenological results we can get from the model. The relevant articles that provide insightful information on the Little Higgs phenomenology include those by Han *et al.* [3, 33] and Perelstein *et al.* [79]. It is highly recommended that the interested readers consult these nice articles.

### 5.3.1 Unitarity and the Cut-Off

First we begin with our claim on the cut-off of the theory that the cut-off scale, which indicates strongly interacting systems or new physics (with spontaneous symmetry breaking, for example), is pushed up to the two-loop

[20] We use the subscript $L$ for the quarks in their final eigenstates because we run out of some proper indices.

order at $\mathcal{O}(10\,\text{TeV})$. Recall that this is due to the factor $\Lambda \simeq 4\pi F$ which is obtained from naive dimensional analysis. However, we also have seen that in the electroweak theory, unitary calls for new physics at $2\,\text{TeV}$ rather than $4\pi v \simeq 3\text{TeV}$ as expected. The situation is quite similar in the Littlest Higgs and there are now more pseudo Goldstone bosons to produce various forms of unitarity violating diagrams (also recall that the gauge bosons scattering can also be described by using Goldston bosons at high energies). The detail analysis on a general Little Higgs models performed by Chang and He [80] shows that the unitarity bound $\Lambda_U$ comes as soon as $3-4\text{TeV}$ if we set $F \sim 1\,\text{TeV}$. Actually, this low unitarity bound is more satisfactory as it means that the high-energy limit of the theory and the new physics (recall the technicolour model) can be reached soon. Unfortunately, we shall see in the next section that the Littlest Higgs is very tightly constrained by the precision electroweak tests and it is not likely that $F$ be that low.

### 5.3.2 Bound on the $F$ from Precision Electroweak Tests

Let us begin with some unsatisfactory fact. Consider the heavy gauge bosons that are partners of those in the standard model. Recalling, (5.180) and (5.194) for the masses of the light gauge fields, we see that their ratio is

$$\frac{M_{W_L^2}}{c_W^2 M_{Z_L}^2} \simeq \left[1 + \frac{v^2}{F^2}\frac{5}{4}(c'^2 - s'^2)^2 - 4\frac{v'^2}{v^2}\right], \qquad (5.214)$$

which is more than obvious that we have lost the custodial symmetry (see section 2.3.5). Unfortunately, this fact alone is enough to put the theory into a very tiny corner of the parameter space and will eventually rules out the Littlest Higgs (not the whole class of models). Nevertheless, this should not be beyond our expectation at all, not only from the appearances of the mass terms, but also from the fact that there is no room left for the global $SU(2)$ symmetry to act as the custodial symmetry. The collective symmetry breaking results in gauging all the possible two $SU(2)$ subspace when the matrix under consideration is $SU(5)$. Therefore there are no global symmetry left to prevent the $\rho$ from being altered. In addition, since the source of the custodial symmetry breaking lies in the order $\mathcal{O}(\frac{v^2}{F^2})$ (or $\mathcal{O}(\frac{v'^2}{v^2})$), we look for the culprit at the same order. By recalling the (2.160) we see that the possible source of the problem is the effective "non-doublet" from of the Higgs. Thus, we will focus on the triplet $\Phi$. By expanding the $\Sigma$ to the second order letting the fields receive electroweak vacuum values, that the neutral component of the triplet contain the $HH^\dagger$ (and hence the electroweak vacuum expectation value), acting like the Higgs triplet. This custodial symmetry $SU(2)$) can be used as an essential tool to test if the Little Higgs can evade the precision electroweak tests.

The modification (5.214) also tells us that either the second and the third terms miraculously cancel, we need $v' << v$ and $c' \sim s'$. Notice that we cannot take the mixing angles to be too small because that would mean strong couplings of the gauge interactions (for example, recall $c' = g'/g'_2$). The lowest values are usually taken to be $c' \gtrsim 0.1$. Csáki *et al.* have done the fittings related to precision electroweak tests of the Littlest Higgs in [6], and we will present some of their results. They evaluate the bound of the "decay constant" $F$ as a function of $c'$ (or $c$) within various values of[21] $a$. The example of the fit is shown in figure 5.6. They used $M_H = 115\,\text{GeV}$ which was found to yield the lowest bound and is consistent with the current excluded Higgs mass. In short, what they basically found is

$$F \gtrsim 4\,\text{TeV} \quad (95\%\,\text{C.L.}). \tag{5.215}$$

On the one hand, the larger-than-expected lower bound of the $F$ pushes the cutoff of the theory (i.e., the scale of the new physics) up to $3 \times 4 \sim 12\,\text{TeV}$ or to $4\pi \times 4 \sim 50\text{TeV}$ which is plausible from the precision electroweak tests point of view. On the other hand, the high value of $F$ means that the mass of the top partner is raised up, to $M_T \gtrsim 5 - 6\,\text{TeV}$, and hence the return of the naturalness problem. If it is the case, there would be the need of fine-tuning to $1 - 2\%$ level in order to get the Higgs light. In addition, let us recall that even though the quadratic divergent diagram cancels, the top quark, as well as other particles, contribute to the Higgs mass squared the logarithmic terms. For the top-quark we have (recall (5.94))

$$\delta m^2_{h,t} = -\frac{3y_t^2 m_T^2}{8\pi^2} \ln \frac{\Lambda^2}{m_T^2} \tag{5.216}$$

which is fairly harmful if the mass $m_T$ of the heavy top is too high. The situation is now worse than the previous case. The primary goal to solve the naturalness problem is gone.

[21]The convention of the $a$ in the Coleman-Weinberg potential may be different form paper to paper. In this thesis, we use 1/2 of the $a$ in Csáki *et al.* [6].

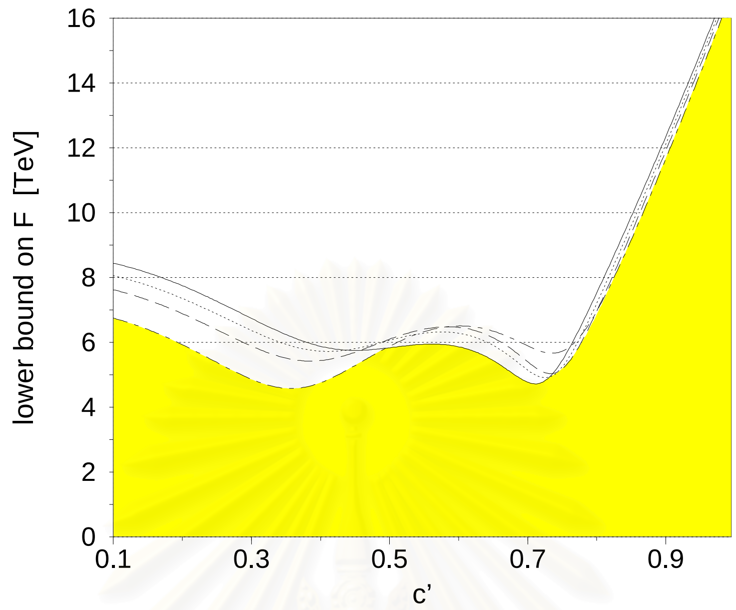



**Figure 5.6:** The region of parameters excluded (below the curves) at 95% C.L. where $c$ varies from 0.1 (shown in solid line) to 0.99 (dot-dashes line). The shaded region is totally excluded. (Taken from [6].)

### 5.3.3 Particles in the Littlest Higgs Model

The particle spectrum of the Littlest Higgs model, as an economical model, is interesting as the new particles, which were introduced to cancel the severe quadratic divergences, can be probed within the reach of the next generation accelerators (for example, the LHC or the ILC). We will point out some features of the Littlest Higgs model that can be used to distinguish it from others; namely the heavy top and the extra pseudo Goldstone bosons (the triplet $\Phi$).

#### 5.3.3.1 Heavy Tops

The heavy top plays very important role in cancelling the quadratic divregent contributions from the top of the standard model. Besides, due to the fine-tuning arguments, the heavy top must show itself up before $\sim 2$ TeV. Recalling (5.213), we find

$$M_{T_L} \sim \sqrt{y_1^2 + y_2^2} F \lesssim \sqrt{2} F \qquad (5.217)$$

However, taking the value $F$ suggested by the previous section, the bound rises beyond the limit of acceptable fine-tuning. Anyway, this is rather interesting as it will still be within the probing range of the LHC if we take $F \sim 1\,\text{TeV}$. Beyond that, recall that there is the equation (5.83)

$$m_T = \frac{y_t^2 + y_T^2}{y_T} F\,, \tag{5.218}$$

which serves as the unique feature of the Little Higgs obeyed by every Little Higgs model (See Perelstein *et al.* [79]). The quadratic divergences cancellation mechanism will not work without this relation. All the four parameters are either known or can be measured (probably indirectly) if the heavy top is found. This heavy top can be produced either alone (via $W$ gauge field and the other quark in the family) or in pair (via pair-production) at the LHC.

The fact that the heavy top lives in the same multiplet with $t$ and $b$ of the standard model helps a little. By little we mean we have to mess with hadronic process and the results generally come with wide uncertainties (comparing to processes in the lepton colliders). Since the heavy top mass is of TeV order, the uncertainties may be as high as hundreds GeV's ([79]). The decay width of the $T_H$ is dominated by the decay products $th$, $tZ$ and $bW^+$ where others come with $v^2/F^2$ suppression factor. In the same paper, Perelstein *et al.* found

$$\Gamma(T \to th) \simeq \frac{m_T y_T^2}{64\pi} \simeq \Gamma(T \to tZ) \simeq \frac{\Gamma(T \to bW^+)}{2}\,, \tag{5.219}$$

where the $T \to bW^+$ covers half of the fraction. This implies the (approximate) total decay width of the $T$

$$\Gamma_T \simeq \frac{y_T^2}{16\pi} m_T\,. \tag{5.220}$$

Suppose we take $m_T \simeq 2\,\text{TeV}$ and $y_t \sim y_T$ (which is legitimate) the width becomes $\Gamma_T \sim 50\,\text{GeV}$ which is rather small but still distinctive.

#### 5.3.3.2 The Light Higgs and the Heavy Scalars

Since the goal of the Little Higgs model is to provide some explanations for having a light scalar (i.e., the Higgs), then once the Higgs is discovered, it may or may not be easy to distinguish the Littest Higgs from other models. For example, naively we have to take the range of the Higgs mass predicted by other models like supersymmetry into account. Then we have to check whether the (discovered) Higgs mass falls into the common range of the mass shared between Little Higgs and others or not. If it does, then it is difficult and we need to perform further investigations to distinguish between them or we have to look elsewhere. In addition, notice that the correction to the mass of the Higgs comes with the factor $\mathcal{O}(v^2/F^2)$ which is of a few percent order. This also adds difficulties for distinguishing between the Higgs from the Little Higgs and the standard model.

The existence of extra scalars is a unique feature of the Little Higgs models. Furthermore, we can even use the scalars to distinguish one Little Higgs from another. The larger the global symmetry group is the more the scalars to be found at some TeV scales. Nevertheless, the naturalness argument tells us that the extra scalar can be rather heavy since the quadratic divergent diagram from the standard model Higgs quartic coupling is not very severe. By recalling that 10% fine-tunings calls for $M_\Phi \lesssim 5 - 10\,\text{TeV}$, we then see that a direct production of the heavy Higgs may not be easy to carry out. Still there is nothing saying explicitly that the $\Phi$ cannot be as light as $1 - 2\,\text{TeV}$ so the possibility of direct production cannot be left out.

In the littlest Higgs model we have a triplet consisting of one neutral and two charged (the $+$ and the $++$). The interesting process concerning the triplet is the $W_L^+ W_L^+$ scattering where the the triplet can contribute $\Phi^{++} \leftrightarrow W^+ W^+$.

#### 5.3.3.3 Heavy Gauge Bosons

Heavy gauge bosons are common features of physics beyond the standard model due to large gauge group structure and symmetry breaking. Still the TeV size of these gauge fields make them available for the next generation accelerators and there are some features that can let us distinguish from others. They will allow the measurements of some important parameters of the model; for example, the "decay constant" $F$ and the additional parameters from the gauge sector like $\tan\psi = \frac{s}{c} = \frac{g_2}{g_1}$ or $\tan\psi' = \frac{s'}{c'} = \frac{g_2'}{g_1'}$. In addition, if we take only the $SU(2)$-like gauge fields into account, we are then left with only 2 important free parameters; namely, $F$ and $\tan\psi$. This means we need few experiments to deal with these parameters. For example, from (5.23) and (5.23) we have, to leading order,

$$\begin{aligned} M_{W_H} &= M_{Z_H} \simeq \frac{0.65F}{\sin 2\psi} \\ M_{A_H} &= \simeq \frac{0.16F}{\sin 2\psi'} \,. \end{aligned} \tag{5.221}$$

Then we find a problem, a very crude approximation on (5.23) reveals that

$$M_{A_H} \lesssim \frac{F}{\sqrt{10}} \tag{5.222}$$

which can be somewhere below a TeV, depending on the value $F$. The mass of this "heavy" gauge boson ranges from $\sim 400\,\text{GeV}$ to $\sim 1\,\text{TeV}$. This is not friendly with the precision electroweak tests at all. Light gauge fields of hundreds GeV scale, if exist, should show up as contributions to electroweak observables.

## 5.4 Conclusions on the Little Higgs model

So far we have studied the Littlest Higgs which is the most economical model in its class that is proposed to resolve the naturalness problem while remains perturbative up to about 10 TeV and hence being "friendly" with electroweak precision measurements. We have seen how the mass of the Higgs is protected at tree-level by non-linear realisation of the global symmetry and at one-loop from quadratic divergences by the collective symmetry breaking. In addition, masses of other particles are generated via explicit symmetry breaking of the global symmetry in a way that is consistent with the collective symmetry breaking. The cancellations between quadratic divergent diagrams were evaluated in detail from the point of view of both the non-linear realisation of the symmetry (on the transformations of the Higgs) and the loop-corrections from various particles. Various mass eigenstates of the physical particles, both before and after electroweak symmetry breaks, were evaluated to some detail. In the last section we presented some important phenomenological features of the Littlest Higgs model.

Unfortunately, the minimal model showed some signs of inconsistencies with both the fine-tuning and precision electroweak measurements. The amount of fine-tunings are worse than advertised and the contributions from the partners to the standard model particles are not very well controlled. These problems, however, did not rule out the whole classes of the Little Higgs models. Though the Littlest Higgs by itself has some problems, it can be used as a prototype and can be easily modified to more sophisticated model. There were many Little Higgs models developed along the path provided by the Littlest Higgs with the aim of resolving the specific problem in mind. The extensions are usually done by extending the global symmetry group, or using more groups so that there are more rooms for additional symmetries to be used. The Littles Higgs with the custodial $SU(2)$ symmetry by Chang and Wacker [81, 82] fall in the this kind of examples. Another example is the Littlest Higgs with T-Parity by Cheng and Low [24, 23, 83] where there is an additional mechanism preventing the electroweak observables from receiving tree-level contributions from the heavy particles which are usually the source of the problems with precision electroweak measurements.

The Littlest Higgs, along with many of the Little Higgs models, come without any explanations on its ultraviolet (UV) completion; roughly speaking, without mentioning how the global symmetry breaks. One of the interesting ideas on the UV completion was inspired by the theory concerning ultracolour interactions presented in section 4.1 and 4.3 where ultrafermions transform under the real representation of a $SO(7)$ and the Little Higgs becomes a composite particle. That model is proposed with the name *composite Littlest Higg*s by Katz *et al.* [77].

Many, if not most, particles predicted by the Little Higgs models are expected to be within the reach of the next generation accelerators especially the LHC and the ILC. With some state-of-the-art design on the experiments, various properties of the physics to be as an extention of the standard models can be studied. Physicists are very confident that the standard model is not the end of the story and something interestng must be discovered soon in the LHC. The Higgs is very likely to be found. With the upcoming experiments, we will know whether it is Little Higgs, supersymmetry, or something else that has its role in particle physics.

# CHAPTER VI

# CONCLUSIONS

In this review-type thesis, we have introduced the Little Higgs models (especially the Littlest Higgs) in a natural way by gathering essential ingredients step by step; filling the gap left out by most of the review papers of the Little Higgs.

We began by bringing up the relevant ingredients of electroweak sector of the standard models and studied the effects of loop-corrections to the mass of the Higgs with additional helps from the Coleman-Weinberg mechanism. This lead to the theoretical bound on the mass of the Higgs. In addition, to convince the readers that the Higgs should be light, we have presented supportive findings from the precision electroweak measurements. In this way, we have transparently introduced the problems of naturalness and fine-tuning concerning the Higgs (and other elementary scalars) and finally the Little Hierarchy problem of the standard model. In the appendix we also briefly presented the ideas of unification of the gauge couplings which convinces us that the standard model should be thought of as an effective field theory of some fundamental theory lying below the Planck scale. Some aspects of the $SU(5)$ grand unification theory are introduced so that we have picked up how to deal with representations of particles, running of the couplings, and the Big Hierarchy problem.

Once we have formulated the Little Hierarchy problem, we presented some interesting extensions or alternatives of various mechanisms of the standard model. Dynamical symmetry breaking mechanism was investigated in some detail where we have learned how to implement the BEH mechanism without elementary scalars. Besides, the section on dynamical symmetry breaking has provided us a nice way to understand the problem of vacuum alignment and how the Goldstone boson becomes massive due to the introduction of explicit global symmetry breaking interactions. Then we studied the method for concentrating on the low-energy degrees of freedom (the Goldstone bosons) of a theory via the non-linear realisations of a symmetry together with the non-linear sigma model. We finished the introductory parts by briefly outlining the pre-Little Higgs model; i.e., the Georgi-Kaplan models.

In the chapter on the Little Higgs, we showed how the general Little Higgs models solved the little hierarchy problem. We have focused our detail study on the Littlest Higgs; including the collective symmetry breaking mechanism and the cancellations of the quadratic divergent diagrams, and gauge and mass eigenstates of various particles. Some possible phenomenological properties, or hopes, for distinguishing the Little Higgs models from others were discussed. The section dedicated to serve as a conclusion on the Little Higgs models is presented at the end of chapter V.

Though this article does not serve as a self-contained introduction to the Little Higgs as there are several topics that we have left from our discussions (as mentioned in the section 1.2.2) for the reason that their comprehensive analyses take space and time, it is still hoped that various aspects presented in this thesis are sufficient, or at least satisfactory, to give the readers some "feelings" of the non-supersymmetric physics that can be thought of as an extension of standard model.

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# APPENDICES

# APPENDIX A

# SUPPLEMENTARY MATERIALS

In this appendix we present some important and interesting topics that do not fit anywhere in the thesis. In section A.1 we very briefly discuss some basic ideas of symmetry that are relevant to our studies in this thesis. Basic concepts of representations of a group is summarised in section A.2. Then in A.3, we turn to discuss why the space of $G/H$, which is referred to in spontaneous symmetry breaking phenomena, is symmetric. Finally in section A.4, we present an alternative, easier, method to evaluate the effective potential.

## A.1 A Few Words on Symmetry

The form of transformations of the fields $\phi^i(x)\,,(i=1,\dots,N)$ in the Lagrangian that is usually relevant to most physical phenomena is the (linear) unitary one; i.e.,

$$\phi \longrightarrow \phi + \mathrm{i}\epsilon^a T^a \phi\,. \tag{A.1}$$

$T^a\,(a = 1,\dots,m)$ is called the generator of the transformation and can be written in the form of $N \times N$ matrices (acting on the index $i$). What defines the generators is the commutation relation

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c \tag{A.2}$$

where $f^{abc}$ is the Lie-algebra structure constant. The $U(1)$ generators are defined by a set of generators that commute with everything else. The remaining members having non-zero commutators among themselves are called simple subalgebra. The latter can also be adjusted to satisfy the relation



$$f^{acd} f^{bcd} = k\delta^{ab} \tag{A.3}$$

which remains valid in any representation. A particular value of $k$ will lead to the desired form of the kinetic term. The first priority is given to the kinetic term because it is one having the largest possible symmetry. Given a specific type of the field (scalar, fermionic, etc...) we can deduce the symmetry of the Lagrangian by requiring that the kinetic term be invariant under a linear transformation like (A.1)[1]. Then, the mass term can break the symmetry into its subgroup if the mass (squared) matrix does not commute with the generator of the group. The interaction terms will further restrict the symmetry down to a smaller subgroup.

The logic introduced above also applies to the case of $N$ Dirac fields $\psi_i$ grouped into an $N$ component vector $\Psi$. Nevertheless, the situation is slightly different here. At first it looks as if the largest symmetry of the kinetic term

$$\mathrm{i}\overline{\Psi}\partial\!\!\!/\Psi \tag{A.4}$$

is $SU(2)\times U(1)$. However, by recalling that the left- and right-handed fermionic field can be constructed from the Dirac field $\Psi$, we see that the kinetic term (A.4) becomes

$$\mathrm{i}\overline{\Psi}_L\partial\!\!\!/\Psi_L + \mathrm{i}\overline{\Psi}_R\partial\!\!\!/\Psi_R \tag{A.5}$$

which means that the chiral fields $\Psi_L$ and $\Psi_R$ are allowed to transform differently under the $SU(2)\times U(1)$ transformations. Still, this is not all we can do. By making use of the charge conjugated field

$$\Psi^c = C\Psi^* \tag{A.6}$$

where $C^2 = 1$, $C^\dagger = C$, $C\gamma^{\mu *}C = -\gamma^\mu$, we can rewrite the charged conjugated right-handed field as a left-handed charged conjugated field with the opposite $U(1)$ "charge" (not necessary the electromagnetic one); i.e.,

$$(\Psi_R)^c = (\Psi^c)_L\,. \tag{A.7}$$

Since all the fields are now left-handed, they can be grouped together in

$$\Upsilon = \begin{pmatrix}\Psi_L \\ (\Psi^c)_L\end{pmatrix} \tag{A.8}$$

and hence the kinetic term (A.5) becomes

$$\mathrm{i}\overline{\Upsilon}\partial\!\!\!/\Upsilon\,, \tag{A.9}$$

which clearly possess $SU(2N)\times U(1)$ symmetry. Since fermions and anti-fermions are allowed to mix in (A.9), the $SU(2N)\times U(1)$ defined above is not

[1]A set of $N$ massless real scalar field may provide a good example. The reality of the field and the invariant of the kinetic term under (A.1) automatically implies that the symmetry is $SO(N)$.

legitimate in most cases in the standard model (where we need to distinguish between particles and anti-particles, especially when they have charges).

## A.2 Representations of a Group

This section summarises some basic facts on the representation of a group.

Before we get to the formal results, let us recall a loose meaning of the representation. The idea of representation becomes useful when we have some ideas of the group for the problem in mind. The choice of the group is a phenomenological question. Suppose that we are dealing with force described by a group of $N \times N$ unitary matrices having determinant 1; i.e., the $SU(N)$. Then we say that a particle experiencing this force transforms under some representations of $SU(N)$. In other words, its state is described by a vector in some vector spaces where the elements of $SU(N)$ acts as unitary operators. Obviously, the simplest vector space can be constructed from a $N$ column vector, resulting in the *fundamental representation*. This also brings up the "conjugate" representation where the $N \times N$ unitary matrices act on a row vector from the right. The conjugate representation may or may not be "equivalent" to the fundamental representation. In addition, we can also "sandwich" a traceless hermitian matrix between two $SU(N)$ elements in a specific way, which results in the adjoint representation.

In gauge theories, we have to deal with at least two kinds of symmetries; namely, the gauge symmetry and the flavour symmetry. The first being a local type while the second is of a global one (and not necessarily continuous, in fact), referring to a symmetry of the theory that are not (yet) gauged[2]. Still they are more or less related. The action of a flavour symmetry generator must resulting in mapping the gauge fields into themselves. So the gauge fields before and after mapping must belong to the same irreducible representation of the gauge group. This means that generators of a flavour symmetry must commute with all the gauge generators; i.e., they act on different spaces and hence we can associate different indices for both. A quantum number for a specific flavour generator for some of the representations of the gauge group is now possible because the flavour generators behave like a constant with respect to the gauge generators. Consequently, suppressing these indices and putting all fermions into the left-handed version, the kinetic term is a sum of all the kinetic terms from different

[2]There is no reason that the global flavour symmetry be the same as the global symmetry correspond to the gauge symmetry.

(irreducible) representations:

$$\sum_{\mathcal{R}} \bar{\Psi}_{\mathcal{R}L} \not{D} \Psi_{\mathcal{R}L} \,. \tag{A.10}$$

Each $\Psi_{\mathcal{R}L}$ behaves as a vector in a flavour space. Then the rotation group corresponding to the dimension $d_{\mathcal{R}}$ of the flavour space is the $SU(d_{\mathcal{R}})$.

Now let us move on to the formal properties. A representation (denoted by $\mathcal{R}$ here) of a group with the structure $f^{abc}$ is specified by a set of traceless Hermitian matrices $T^a_{\mathcal{R}}$ having dimension $d_{\mathcal{R}} \times d_{\mathcal{R}}$. $T^a_{\mathcal{R}}$ will be qualified for being a representation only if they satisfy

$$[T^a_{\mathcal{R}}, T^b_{\mathcal{R}}] = \mathrm{i} f^{abc} T^c_{\mathcal{R}} \tag{A.11}$$

which are defined by the original generators $T^a$ of the group defined by the fundamental representation. Two representations are said to be *equivalent* (physical properties described by them are indistinguishable) if the exists a transformation governed by a fixed unitary matrix

$$X^{-1} T^a_{\mathcal{R}} X = T^a_{\mathcal{R}'} \tag{A.12}$$

for all $a$. If this transformation reduces $T^a_{\mathcal{R}}$ to the block diagonal form

$$X^{-1} T^a_{\mathcal{R}} X = \begin{pmatrix} t^a_{r1} & & \\ & t^a_{r2} & \\ & & \\ & & t^a_{rn} \end{pmatrix} , \tag{A.13}$$

the representation is said to be *reducible* and we write

$$\mathcal{R} = r_1 \oplus r_2 \oplus \cdots \oplus r_n \,. \tag{A.14}$$

Otherwise, the representation is *irreducible.* In the latter case, we can define the *quartic Casimir invariant* $C(\mathcal{R})$ by

$$(T^a_{\mathcal{R}} T^a_{\mathcal{R}})_{ij} = C(\mathcal{R}) \delta_{ij} \,. \tag{A.15}$$

This follows from $[T^a T^a, T^b] = 0$. Moreover, for any representation, the generators can always be adjusted so that

$$\mathrm{Tr}(T^a_{\mathcal{R}} T^b_{\mathcal{R}}) = T(\mathcal{R}) \delta^{ab} \tag{A.16}$$

where $T(\mathcal{R})$ is called the *index* of the representation. Taking the trace of the representation indices $i$, $j$ we find

$$\mathrm{Tr}\,(T^a_{\mathcal{R}}T^a_{\mathcal{R}})_{ij} = C(\mathcal{R})d_{\mathcal{R}}\,. \tag{A.17}$$

Summing over the generators ($a$, $b$) in (A.16) eventually results in

$$C(\mathcal{R})d_{\mathcal{R}} = T(\mathcal{R})d_G\,. \tag{A.18}$$

Observe that (A.11) also implies

$$[T^{a*}_{\mathcal{R}}, T^{b*}_{\mathcal{R}}] = -\mathrm{i}f^{abc}T^{c*}_{\mathcal{R}} \tag{A.19}$$

or

$$[-T^{a*}_{\mathcal{R}}, -T^{b*}_{\mathcal{R}}] = \mathrm{i}f^{abc}(-T^{c*}_{\mathcal{R}}) \tag{A.20}$$

which means $-T^{a*}_{\mathcal{R}}$ also obeys the commutation relation(A.11). If $T^a_{\mathcal{R}}$ and $-T^{a*}_{\mathcal{R}}$ are equivalent the representation $\mathcal{R}$ is said to be *real*. If there exists a matrix $V$ such that

$$-T^{a*}_{\mathcal{R}} = V^{-1}T^a_{\mathcal{R}}V \tag{A.21}$$

with $V \neq I$ for all $a$ then it is *pseudo-real*. For example, $V = \sigma_2$ for the fundamental representation of $SU(2)$. When both conditions fail, the representation is said to be *complex*. The complex conjugation representation $\overline{\mathcal{R}}$ is defined by $T^a_{\overline{\mathcal{R}}} = -T^{a*}_{\mathcal{R}}$.

Now, notice that (A.11) and (A.16) tell us that

$$f^{abc} = \frac{-\mathrm{i}}{T(\mathcal{R})}\mathrm{Tr}\left\{[T^a_{\mathcal{R}}, T^b_{\mathcal{R}}]T^c_{\mathcal{R}}\right\} \tag{A.22}$$

which means $f^{abc}$ is completely anti-symmetric (the trace on the right-hand side is cyclic). This also shows that the structure constant $f^{abc}$ is an *invariant symbol* of the group. Now, we can define the *adjoint representation* by

$$(T^a_A)^{bc} = -\mathrm{i}f^{abc} \tag{A.23}$$

which automatically implies that $(T^a_A)^{bc}$ is Hermitian. The $T^a_A$ satisfies the commutation relation (A.11). By construction, the dimension of the adjoint representation is equal to the representation of the group. This leads to

$$T(A) = C(A)\,. \tag{A.24}$$

When the anti-commutator is used instead of the commutator in (A.22), the anomaly coefficient of the representation $A(\mathcal{R})$ can be defined:

$$A(\mathcal{R})d^{abc} = \frac{1}{2}\mathrm{Tr}\left\{[T^a_{\mathcal{R}}, T^b_{\mathcal{R}}]T^c_{\mathcal{R}}\right\} . \tag{A.25}$$

Notice that now $d^{abc}$ is totally symmetric. Then $(T^a_{\overline{\mathcal{R}}})_i{}^j = -(T^a_{\mathcal{R}})^j{}_i$ yields the important relation

$$A(\overline{\mathcal{R}}) = -A(\mathcal{R})\,. \tag{A.26}$$

This plays an important role when we want to find the representations of a group that are anomaly-free. $A(\mathcal{R})$ automatically vanishes in the real or pseudo-real representations.

When the real representation and its complex conjugate one are not equivalent, it is helpful to use up and down indices - up for the fundamental representation and down for its conjugate:

$$\phi^{i^\dagger} = (\phi^\dagger)_i \equiv \overline{\phi}_i\,. \tag{A.27}$$

Then the elements of generator $T^a_{\mathcal{R}}$ are written as $(T^a_{\mathcal{R}})^i{}_j$. This means for the conjugate representation: $(T^a_{\overline{\mathcal{R}}})_i{}^j = -(T^a_{\mathcal{R}})^j{}_i$. Then it is easy to see that $\overline{\phi}_i\phi^i$ is an invariant symbol. The other important invariant symbol is the $\delta^j_i$. We find

$$\delta^i_j \longrightarrow (1 + \mathrm{i}\theta^a T^a_{\mathcal{R}})^i{}_k(1 + \mathrm{i}\theta^a T^a_{\overline{\mathcal{R}}})_j{}^l\delta^k_l = \delta^i_j + \mathcal{O}(\theta^2)\,, \tag{A.28}$$

which means that a *singlet* (or *trivial*) representation is always there in $\overline{\mathcal{R}}\otimes\mathcal{R}$; i.e.,

$$\overline{\mathcal{R}}\otimes\mathcal{R} = 1\oplus\cdots\,. \tag{A.29}$$

Since the generator matrix $T^a_{\mathcal{R}}$, carrying an additional index from the adjoint representation $(a)$, is also an invariant symbol, we can write

$$\overline{\mathcal{R}}\otimes\mathcal{R}\otimes A = 1\oplus\cdots\ . \tag{A.30}$$

With the helps of (A.29) and the fact that $\overline{A} = A$, this leads to the fact that the product $\overline{\mathcal{R}}\otimes\mathcal{R}$ always contain the adjoint representation:

$$\overline{\mathcal{R}}\otimes\mathcal{R} = 1\oplus A\oplus\cdots\,. \tag{A.31}$$

When $\mathcal{R}$ is the fundamental representation $N$ or the $SU(N)$, (A.31) reduces to

$$\overline{N} \otimes N = 1 \oplus A\,. \tag{A.32}$$

For example, $\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} + \mathbf{8}\,.$

Next we will look at some of the important applications of the representations. First we know that weak interaction teats left-handed and right-handed fermions differently. But trying to introducing a right-handed as a charge conjugate of the left-handed one leads to a behaviour under a gauge transformation like

$$\psi_R^c \longrightarrow \psi_R^c + \mathrm{i}\varepsilon^\alpha(-T^{\alpha *})\psi_R^c \tag{A.33}$$

if

$$\psi_L \longrightarrow \psi_L + \mathrm{i}\varepsilon^\alpha T^\alpha \psi_L\,. \tag{A.34}$$

So we may or may not get the equivalent transformation rule for the right-handed fields depending on what kind of representations of the gauge symmetry group these fermions transform under. If the representation is real, the generators $T^\alpha$ are all imaginary and antisymmetric. Hence left and right-handed fermions in a real representation transform the same way.

### A.2.1 The Meson Octet

Now we will consider the construction of the meson octet. Let us begin with the quark triplet which transforms under a fundamental representation of $SU(3)$

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \tag{A.35}$$

as well as its conjugate

$$\psi = \begin{pmatrix} \bar{u}, & \bar{d}, & \bar{s} \end{pmatrix}\,. \tag{A.36}$$

Since the matrix $\psi\bar{\psi}$ contain both singlet and octet parts, we will consider

$$\psi\bar{\psi} - \frac{1}{3}\mathbb{1}_{3\times3}\mathrm{Tr}\psi\bar{\psi} = \begin{pmatrix} (2u\bar{u} - d\bar{d} - s\bar{s})/3 & u\bar{d} & u\bar{s} \\ d\bar{u} & (-u\bar{u} + 2d\bar{d} - s\bar{s})/3 & d\bar{s} \\ s\bar{u} & s\bar{d} & (-u\bar{u} - d\bar{d} + 2s\bar{s})/3 \end{pmatrix} \tag{A.37}$$

In this quark basis, we can identify the states with the physical particles. They can be written as

$$\begin{aligned} \psi\bar{\psi} - \frac{1}{3}\mathbb{1}_{3\times3}\mathrm{Tr}\psi\bar{\psi} & \propto \Pi\cdot X \\ &= \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -2\eta/\sqrt{6} \end{pmatrix} \end{aligned} \tag{A.38}$$

where $\pi^0 \propto (u\bar{u} - d\bar{d})/\sqrt{2}$ and $\eta \propto (u\bar{u} - d\bar{d} - 2s\bar{s})/\sqrt{6}$. Their transformation properties will be made clear if we collapse them into pieces

$$\Pi\cdot X \propto \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ & \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} & \\ & & \end{pmatrix} + \begin{pmatrix} & & K^+ \\ & & K^0 \\ K^- & \bar{K}^0 & \end{pmatrix} + \begin{pmatrix} \frac{\eta}{\sqrt{6}} & & \\ & \frac{\eta}{\sqrt{6}} & \\ & & -2\frac{\eta}{\sqrt{6}} \end{pmatrix}, \tag{A.39}$$

or write them in terms of the Gell-Mann matrices, given in C.3.1,

$$\Pi\cdot X \propto \sum_{a=1}^{3} \pi^a\lambda^a + \sum_{a=4}^{7} K^a\lambda^a + \eta\lambda^8\,. \tag{A.40}$$

Notice that the kaons transform as a complex doublet under the $SU(2)$ subgroup (the doublet structure shows up right away after we “hide” the $s$ quark dependence). They will plays the role of the standard model Higgs doublet in many sections of this thesis.

For reference, we also write down the "pion" matrix for the $SU(5)/SO(5)$ case

$$\Pi \cdot X = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{\omega^0}{\sqrt{2}} - \frac{\eta}{\sqrt{10}} & -\omega^+ & H^+ & -\mathrm{i}\sqrt{2}\phi^{++} & -i\sqrt{2}\phi^+ \\ -\omega^- & \frac{\omega^0}{\sqrt{2}} - \frac{\eta}{\sqrt{10}} & H^0 & -\mathrm{i}\phi^+ & -\mathrm{i}\phi^0 + \phi_P^0 \\ H^- & H^{0*} & \sqrt{8/5}\eta & H^+ & H^0 \\ \mathrm{i}\sqrt{2}\phi^{--} & \mathrm{i}\phi^- & H^- & -\frac{\omega^0}{\sqrt{2}} - \frac{\eta}{\sqrt{10}} & -\omega^- \\ \mathrm{i}\phi^- & \mathrm{i}\phi^0 + \phi_P^0 & H^{0*} & -\omega^+ & \frac{\omega^0}{\sqrt{2}} - \frac{\eta}{\sqrt{10}} \end{pmatrix} . \tag{A.41}$$

## A.3 The Space $G/H$ is Symmetric

In this section, we will study the behaviour of unbroken ($Y^i$) and broken ($X^z$)generators, for a symmetry breaking $G \to H$ in a more general way, and show that the space $G/H$ spanned by $X^z$ is symmetric. This happens in most of the cases when we deal with spontaneous symmetry breaking and worth some discussions.

Starting with a particular vacuum, we can always partition the $G$-generators $T^a$ into the broken and unbroken ones, depending on how they act on the specific vacuum. A proper normalisation scheme can be given to these generators. By assumption, the unbroken generators $Y^i$ have the following Lie algebra

$$[Y^i, Y^j] = \mathrm{i}f^{ijk}Y^k \,. \tag{A.42}$$

Since we know that $Y^i$'s and $X^z$'s are orthogonal; $\mathrm{Tr}(Y^i X^z) = 0$, the condition

$$\mathrm{Tr}\left\{Y^i[Y^j, X^z]\right\} = \mathrm{Tr}\left\{[Y^i, Y^j]X^z\right\} \sim \mathrm{Tr}(YX) = 0 \tag{A.43}$$

implies that the commutator between the two kinds of generators are

$$[Y^i, X^z] = \mathrm{i}f^{izx}X^x \,. \tag{A.44}$$

The problem is that without further condition, we do not have any constraint for the commutators of the broken generators (they do not necessary form a group). So we must write

$$[X^x, X^z] = \mathrm{i}f^{xzi}Y^i + f^{xzw}X^w \,. \tag{A.45}$$

To go on, we "assume" that the broken and unbroken generators behave under a parity operation $\mathcal{P}$ as follows:

$$\mathcal{P}(X) = -X \qquad \mathcal{P}(T)\,, \tag{A.46}$$

which seem to be phenomenologically acceptable. So the "algebra" of the broken generators becomes

$$[X^w, X^z] = \mathrm{i} f^{wzi} Y^i \tag{A.47}$$

so that in this sense, the (coset) space defined by $X^z$ is symmetric.

## A.4 The Return of the Tadpoles

In this appendix we present another interesting technique to evaluate the effective potential. Instead of expanding of the effective action in the way we have done in (3.21), we will expand it about another arbitrary point (now non-zero, but not necessary the minimum of the potential), namely $\phi_c = \omega$, i.e.,

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \Gamma^{(n)}_{\phi-\omega}(x_1, \ldots, x_n)[\phi_c(x_1) - \omega] \ldots [\phi_c(x_n) - \omega]\,, \tag{A.48}$$

so that the $\Gamma^{(n)}_{\phi-\omega}$'s are now the proper vertex functions for a new theory which the fields $\phi_c$'s are replaced by $\phi' \equiv \phi_c - \omega$. Using the same reasoning as those used in arriving at (3.24), we see that the effective potential (for the new "shifted" theory) becomes

$$V_{\text{eff}}(\phi_c) = -\sum_{n=1}^{\infty} \frac{1}{n!} \tilde{\Gamma}^{(n)}_{\phi-\omega}(0, \ldots, 0)[\phi_c(x) - \omega]^n\,. \tag{A.49}$$

Notice that the theory is shifted in the sense that the Lagrangian now contains the new field $\phi'$ which gives rise to new vertices depending on $\omega$. According to (A.49) we see that

$$\left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\omega}\right|_{\phi_c=\omega} = \Gamma^{(1)}_{\phi-\omega} \tag{A.50}$$

where $\Gamma^{(1)}_{\phi-\omega}$ is i times the tadpole diagram of the shifted theory. Therefore, the effective potential can be recovered by evaluating the tadpole diagram in the shifted theory, integrate with respect to $\omega$, and finally replace $\omega$ with $\phi_c$. This starts to look good as we now need a diagram (the tadpole for the $\phi_c$) instead of an infinite number of diagrams as we did in previous sections.

To see how this works, let us consider a system of massless scalar field with potential $V(\phi_c) = \frac{\lambda}{4!}\phi_c^4$, which becomes, for the shifted theory,

$$V(\phi_c) = \frac{\lambda}{4!}\phi_c^4 - \frac{\lambda}{3!}w\phi_c^3 + \frac{\lambda}{4}\omega^2\phi_c^2 - \frac{\lambda}{3!}w^3\phi_c + \frac{\lambda}{4!}w^4 \,. \tag{A.51}$$

We can extract the $\phi\phi\phi$ vertex as $-\mathrm{i}\lambda\omega$, and the effective $\sigma-$dependent mass (squared) for $\phi_c$ as $\lambda\omega^2/3$. So the tadpole diagram in Fig.(A.1) contributes

$$-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\lambda\omega}{k^2+\lambda\omega^2/2} \tag{A.52}$$

where the factor 1/2 is the symmetry factor. After integrating with respect to

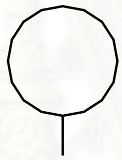

**Figure A.1:** A scalar tadpole diagram.

$w$, multiplying by i, and replace $\omega$ with $\phi_c$ as prescribed above, we find

$$\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left[k^2+\lambda\phi_c^2/2\right] \tag{A.53}$$

which yields the same result, apart from some irrelevant constants and the dropped $\hbar$, as those obtain in (3.32) which used the diagrammatic method.

Let us see the application to the loop diagram having gauge fields running inside. We consider the gauge-scalar interaction term

$$\frac{1}{2}e^2\phi^2A^\mu A_\mu \tag{A.54}$$

so that the "shifted" theory contains

$$\frac{1}{2}e^2\phi^2A^\mu A_\mu + \frac{1}{2}e^2\omega^2A^\mu A_\mu - e^2\omega\phi A^\mu A_\mu \,. \tag{A.55}$$

Thus the gauge-gauge-scalar and the mass terms for the shifted theory are $-\mathrm{i}e^2\omega g_{\mu\nu}$ and $e^2\omega^2$, respectively. The tadpole contribution from the gauge field is shown in Fig.(A.2). Now we have the tadpole diagram multiplied by i

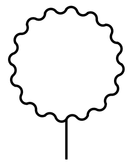

**Figure A.2:** A gauge boson tadpole diagram.

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} e^2\omega g_{\mu\nu} \frac{g^{\mu\nu} - k^\mu k^\nu/k^2}{k^2 + e^2\omega^2} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{3e^2\omega}{k^2 + e^2\omega^2} \tag{A.56}$$

which yields, after integrating with respect to $\omega$ and setting $\omega = \phi_c$,

$$V_g = \frac{3}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \ln(k^2 + e^2\phi_c^2)\,, \tag{A.57}$$

hence

$$V_{\text{eff}}(\phi_c) = \frac{3e^4(M)}{64\pi^2} \phi_c^4 \ln \frac{\phi_c^2}{M^2} - \frac{25}{6}\,. \tag{A.58}$$

Notice that the factor 25/6 can be removed (absorbed to somewhere else, to be more precise), for future convenience, by choosing a new subtraction point $M$; i.e.,

$$V_{\text{eff}}(\phi_c) = \frac{3e^4(M)}{64\pi^2} \phi_c^4 \ln \frac{\phi_c^2}{M^2}\,. \tag{A.59}$$

# APPENDIX B

# THE $SU(5)$ GRAND UNIFICATION THEORY

Most of the topics that are dealt with in this appendix are related to the unified gauge theory. However, our main goal here is only to gather the basic structures of the theory including particle multiplets, gauge interactions, and symmetry breaking. Then we can study the *Big Hierarchy Problem* at the end.

This appendix is organised as follows. We begin in section B.1 by investigating the structure of the group $SU(5)$ so that we can pick up the appropriate representations for describing particles in the standard model. Then in section B.2 we introduce the gauge structure to the theory and construct a gauge invariant Lagrangian. After that we can set up the scene of spontaneous symmetry breaking in section B.3, where the appropriate scale of the $SU(5)$ breakings will be investigated using simple renormalisation group equations in section B.4. Finally we can talk about the hierarchy problem in section B.5.

We start with a hope in mind that all the known (gauge) interactions of the standard model; namely, the electroweak and strong interactions, described collectively by the product group $SU(3)_C \times SU(2)_L \times U(1)_Y$, can be somehow unified into a gauge theory that relies on a larger symmetry group, called $G$. This $G$ must contain the standard model group as a subgroup. In addition $G$ is preferably a simple one. Then it is hoped that the larger gauge group may help to deal with some problems that the standard model cannot provide answers; for example, the non-integral electric charge of quarks (and their relations to leptons charges which must be "assumed" in the context of the standard model) and the existence of many "copies" of quark and lepton families.

These difficulties may be resolved by introducing a larger gauge group of the unified theory which allows the possibilities of having quarks and leptons in the same representation. The minimum requirement for doing so is that the group of interest must be simple or at least has a simple group containing the standard model group as a subgroup. In addition, to accommodate the charge quantisation problem, we look for a gauge theory depicted by a simple Lie group which have a particle multiplet that allows the correct relations of charges of particles in that multiplet. This implies that the group we seek for must allow complex representations, or it will not have a room for the representation of $SU(3) \times SU(2) \times U(1)$ (which is complex). Moreover, we know that the standard model group is of rank four since it contains four commuting generators. Thus the group $G$ must, at least, have that rank. These arguments rule out many possibilities, leaving the smallest group having rank four which is the $SU(5)$.

## B.1 The Group Structures and the Particle Contents

Now let us consider the structure of $SU(5)$. It is clear that here we have two five dimensional representations. This tells us that it is not possible to put all the standard model particles into the fundamental representations, which is one of the shortcomings of the $SU(5)$. The group has $5^2 - 1 = 24$ generators where four of them are diagonal and traceless. The forms of the generators are not unique and they can be recombined (using linear combinations) to give the desired forms depending on the groups that we want to embed in the $SU(5)$. Nevertheless, the definition of generators must get along with experimental facts. We know that weak interaction is colourblind (and the $SU(3)$ strong interaction does not "know" the existence of the electroweak). This requires that the $SU(3)$ generators have zero eigenvalues for the leptons components and the $SU(2) \times U(1)$ generators behave as unit matrices (or zeroes) with respect to the $SU(3)$ generators. Thus, we assign the first three indices of the $SU(5)$ to the $SU(3)$ and the last to indices to the $SU(2)$. Then we call the 24 generators $\lambda_i$, and put the first eight for the $SU(3)$ and the last three for the $SU(2)$. The generators for the $SU(3) \subset SU(5)$; i.e., the $\lambda_1 \cdots \lambda_8$ can take the usual forms (the Gell-Mann's matrices); for example, the diagonal ones are

$$\lambda_3 \equiv \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & -1 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \quad \lambda_8 \equiv \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & & \\ 0 & \frac{1}{\sqrt{3}} & 0 & & \\ 0 & 0 & \frac{-2}{\sqrt{3}} & & \\ & & & & \\ & & & & \end{pmatrix}. \tag{B.1}$$

All the $SU(2) \subset SU(5)$ generators contain the Pauli matrices; i.e.,

$$\lambda_{20+i} \equiv \left(\begin{array}{ccc:c} 0 & & & \\ & 0 & & \\ & & 0 & \\ \hdashline & & & \tau_i \end{array}\right). \tag{B.2}$$

We see that this particular partitioning of the fundamental representation of the $SU(5)$; namely[1]

$$\mathbf{5} \to (\mathbf{3}, \mathbf{1}, C_1) \oplus (\mathbf{1}, \mathbf{2}, C_2)\,, \tag{B.3}$$

induces the specific elements of the $U(1)$ generator which must be both a unit matrix with respect to the $SU(3)$ and $SU(2)$, and traceless. This means that

$$(\mathbf{3}, \mathbf{1}, -1/3) \oplus (\mathbf{1}, \mathbf{2}, 1/2)\,,$$

is allowed (up to an overall $U(1)$ factor) while

$$(\mathbf{3}, \mathbf{1}, 1/3) \oplus (\mathbf{1}, \mathbf{2}, 1/2)\,,$$

is not. So we choose

$$\lambda_{24} = \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{B.4}$$

Notice that all the generators of the $SU(5)$ used here are constructed so as to satisfy

$$\mathrm{Tr}\lambda_a \lambda_b = 2\delta_{ab} \tag{B.5}$$

which is necessary to reproduce the correct (conventional) factor in the kinetic term. For future reference, let us use the tensor notation, the general representation of $SU(5)$, $\psi^{i_1 \cdots i_p}_{j1 \cdots j_q}$, consisting of the fundamental $\psi^i$ and the conjugate fundamental representations $\bar{\phi}_j = (\psi^j)^*$ transforms as

$$\psi^{i_1 \cdots i_p}_{j1 \cdots j_q} = \left(U^{i_1}_{k_1} \cdots U^{i_p}_{k_p}\right) \left(U^{l_1}_{j_1} \cdots U^{l_q}_{j_q}\right) \psi^{k_1 \cdots k_p}_{l_1 \cdots l_q} \tag{B.6}$$

where matrix $U$ contains the generators $\lambda^a$ defined above; i.e.,

$$U^i_j = \exp\left\{-\mathrm{i}\alpha^a \frac{[\lambda^a]^i_j}{2}\right\}. \tag{B.7}$$

Now let us see if the arrangements of the representation that we have just done fit with the particle content of the standard model. From the standard

[1] We use the notation $(\mathbf{A}, \mathbf{B}, C)$, where $\mathbf{A}$, $\mathbf{B}$, and $C$ stand for the representations of $SU(3)$, $SU(2)$, and $U(1)$ respectively.

model, the 15 fermions for each family live in

$$(\mathbf{3},\mathbf{2},1/3)_L\,,(\mathbf{3},\mathbf{1},4/3)_R\,,(\mathbf{3},\mathbf{1},-2/3)_R\,,(\mathbf{1},\mathbf{2},-1)_L\,,(\mathbf{1},\mathbf{1},-2)_R\,, \tag{B.8}$$

or if we regard everybody as left-handed:

$$(\mathbf{3},\mathbf{2},1/3)_L\,,(\bar{\mathbf{3}},\mathbf{1},-4/3)_L\,,(\bar{\mathbf{3}},\mathbf{1},2/3)_L\,,(\mathbf{1},\mathbf{2},-1)_L\,,(\mathbf{1},\mathbf{1},2)_L\,, \tag{B.9}$$

where the right-handed fields are replaced by the left-handed ones via charge conjugation

$$(\psi_R)^c = (\psi^c)_L \equiv \psi^c_L. \tag{B.10}$$

It is important to emphasise that the concept of chirality and electric charges will make sense only after the electroweak symmetry is broken. We only use them as bookkeeping devices for the moment. Note that the charge-conjugated fields must be defined in the same ways as that of the Higgs (see (2.134)); i.e.,

$$(L^e_L)^c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \nu^c_e \\ e^c \end{pmatrix}_R = \begin{pmatrix} e^c \\ -\nu^c_e \end{pmatrix}_R , \tag{B.11}$$

which is required to get the appropriate $T_3$ values for the $SU(2)$ doublet, otherwise it would be the $e^c$ that possesses $T^3 = -\frac{1}{2}$. So we see that a slight modification of the generator $\lambda_{24}$ to

$$Y \equiv \sqrt{\frac{5}{3}}\lambda_{24} = \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} , \tag{B.12}$$

leads to the electric charge

$$Q = \lambda_{23} + \frac{Y}{2} = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix} , \tag{B.13}$$

where $\lambda_{23}$ is the usual $SU(2)$ isospin (in $SU(5)$). Evidently, the charge $Q$ here commutes with our $SU(3)$ generators. It is then clear that the $\mathbf{5}$ (the fundamental representation)

$$\mathbf{5} = (\mathbf{3}, \mathbf{1}, -2/3) \oplus (\mathbf{1}, \mathbf{2}, 1) \tag{B.14}$$

contains the right-handed down-type quarks and the right-handed antiparticles of leptons. Then the conjugated representation

$$\bar{\mathbf{5}} = (\bar{\mathbf{3}}, \mathbf{1}, 2/3) \oplus (\mathbf{1}, \mathbf{2}, -1) \tag{B.15}$$

contains the standard model particles (i.e., $d_R$, $e^-$, and $\nu_e$). As a convention we use separate indices in tensor notations, $\alpha, \beta, \ldots$ for $SU(3)$ and $r, s, \ldots$ for $SU(2)$ as follows:

$$\mathbf{5} : \psi^i = \{\psi^\alpha, \psi^r\} \, , \tag{B.16}$$

where, as is obvious, $i = 1, \ldots, 5$, $\alpha = 1, 2, 3$, and $r = 4, 5$. In other words, we have

$$\mathbf{5} : (\psi_R)^c = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^c \\ -\nu_e^c \end{pmatrix}_R \, , \quad \text{or} \quad \bar{\mathbf{5}} : \psi_L = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{pmatrix}_L \, . \tag{B.17}$$

where the subscript of the quark field $d$ denotes colour. It is important to keep in mind that by putting quarks and leptons altogether in a multiplet, we have to accept that they are indistinguishable at very high energy scales where the $SU(5)$ symmetry is valid. However, we know that the masses of quarks and leptons are far from being similar. Then it is obvious that we must find a way to properly describe the breaking of the $SU(5)$ into the standard model group, $SU(3) \times SU(2) \times U(1)$. In the tensor notation we find that the operator $Q$ defined in (B.13) has eigenvalues

$$Q^{[5]}(\bar{\phi}_i) = -Q_j^{[5]} \delta_{ij} \tag{B.18}$$

where $i$ here refers to the components of the $SU(5)$ spinor and the eigenvalues of $Q$. Consequently, (B.17) and the vanishing trace of (B.13) partially answer the question of the charge quantisation in the standard model.

To find room for the remaining 10 particles, it is obvious that the next representation we should try is the direct product of the fundamental ones. Since the dimensions of the symmetric and antisymmetric parts of the $\mathbf{5} \times \mathbf{5}$ (which can be written as $5 \times 5$ matrix) are $\mathbf{15}$ and $\mathbf{10}$ respectively, the $(\mathbf{5} \times \mathbf{5})_A$ is then our target. By recalling that

$$\mathbf{3}\times\mathbf{3}=\mathbf{6}_S+\bar{\mathbf{3}}_A\,,\qquad \text{and}\qquad \mathbf{2}\times\mathbf{2}=\mathbf{3}_S+\mathbf{1}_A\,, \tag{B.19}$$

we find

$$\mathbf{10}_A \;=\; (\bar{\mathbf{3}},\mathbf{1},-4/3)+(\mathbf{3},\mathbf{2},1/3)_A+(\mathbf{1},\mathbf{1},2)\,, \tag{B.20}$$

$$\mathbf{15}_S \;=\; (\bar{\mathbf{6}},\mathbf{1},-4/3)+(\mathbf{3},\mathbf{2},1/3)_S+(\mathbf{1},\mathbf{3},2)\,, \tag{B.21}$$

The $\mathbf{10}$ fits well with the remaining particles. The next task is then to locate the right place for each particle in the $\mathbf{10}$. First, the antisymmetric tensor for the $\mathbf{10}$ can be constructed from the $\mathbf{5}$; i.e., the $\psi^i$, as follow

$$\psi^{ij}=\psi^i\psi^j-\psi^j\psi^i\,. \tag{B.22}$$

So

$$\mathbf{10}:\psi^{ij}=\left\{\psi^{\alpha\beta},\psi^{\alpha r}\psi^{45}\right\}\,. \tag{B.23}$$

It is follows that the charge operators from each representation add up like usual $U(1)$ (diagonal) generators; namely,

$$Q^{[\mathbf{10}]}(\psi^{kl})=Q^k+Q^l\equiv Q^{kl}\,, \tag{B.24}$$

and hence leading to the symmetric operator

$$Q^{[\mathbf{10}]}=\begin{pmatrix} & -\frac{2}{3} & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & & 1 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & \end{pmatrix}\,. \tag{B.25}$$

Similarly,

$$Y^{[\mathbf{10}]}=\begin{pmatrix} & -\frac{4}{3} & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{4}{3} & & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{4}{3} & -\frac{4}{3} & & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 2 & \end{pmatrix}\,, \tag{B.26}$$

which clearly show how to throw the remaining 10 particles into them. Those particles are $(\mathbf{3}, \mathbf{2}, 1/3)$, $(\bar{\mathbf{3}}, \mathbf{1}, -4/3)$, and $(\mathbf{1}, \mathbf{1}, 2)$. Still, the sign of the fields inside the $\psi^{[\mathbf{10}]}$ are not fixed at this stage, but can be done so when we consider the mass eigenstates. Following Langacker [84], we use

$$\psi^{\alpha\beta} = \frac{1}{\sqrt{2}} \varepsilon^{\alpha\beta\gamma} u_L^{c,\gamma} \, , \tag{B.27}$$

and write

$$\mathbf{10} : \ \Psi^{[\mathbf{10}]} = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc|cc} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ \hline u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{array} \right)_L . \tag{B.28}$$

We can also write the 10

$$\overline{\mathbf{10}} : \ \Psi^{[\overline{\mathbf{10}}]} = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc|cc} 0 & u_3 & -u_2 & -u_1^c & -d_1^c \\ -u_3 & 0 & u_1 & -u_2^c & -d_2^c \\ u_2 & -u_1 & 0 & -u_3^c & -d_3^c \\ \hline u_1^c & u_2^c & u_3^c & 0 & -e^- \\ d_1^c & d_2^c & d_3^c & e^- & 0 \end{array} \right)_R , \tag{B.29}$$

which, playing the similar role as the $\mathbf{5}$, describes anti-particles. We finally have the multiplets for all the particles in the first family of the standard model. The similar constructions can be done for the other two families. Notice that there is no room left for the right-handed neutrino in the $\bar{\mathbf{5}} + \mathbf{10}$ representations. The simplest possibility to have massive neutrinos is to introduce the $\nu_R$ as a singlet of $SU(5)$. However, we will not consider the case of massive neutrinos here.

According to the appearance of the multiplets we have on hand, it is clear that there is (or, at least, should be) one quark family corresponding to each lepton family. Moreover, the appearance of quarks and the corresponding antiquarks in the same multiplet means that it is possible that a particle like a proton decays. This is one of the prominent prediction of the theory (which, however, eventually ruled out the model).

## B.2 The $SU(5)$ Gauge Sector

The group $SU(5)$ has 24 gauge fields which live in the adjoint representation of the group that is the non-singlet part (the $\mathbf{24}$) of $\bar{\mathbf{5}} \times \mathbf{5} = \mathbf{24} + \mathbf{1}$. Let us define the matrix field as

$$\hat{A}_\mu \equiv \sum_{a=1}^{a=24} \frac{\lambda^a}{2} A_\mu^a \tag{B.30}$$

where $A_\mu^a$ are the $SU(5)$ gauge fields and denote $(A_\mu)^i_j \equiv (\hat{A}_\mu)_{ij}$. For example, in the $SU(3)$ case we have $A_3^1 = (A^4 - A^5)/\sqrt{2}$. Then due to the way the $SU(5)$ representations are partitioned, we know that the $SU(5)$ gauge fields must contain the standard model ones; namely the $(\mathbf{8},\mathbf{1})$ for $SU(3)$, and the $(\mathbf{1},\mathbf{3})$ for $SU(2)\times U(1)$. This can be easily verified by using $\overline{\mathbf{3}}\times\mathbf{3} = \mathbf{8}+\mathbf{1}$ and (B.19),

$$\begin{aligned} \overline{\mathbf{5}}\times\mathbf{5} &= \left[(\overline{\mathbf{3}},\mathbf{1},2/3)+(\mathbf{1},\mathbf{2},-1)\right]+\left[(\mathbf{3},\mathbf{1},-2/3)+(\mathbf{1},\mathbf{2},1)\right] \\ &= (\mathbf{8},\mathbf{1},0)+(\overline{\mathbf{3}},\mathbf{2},5/6)+(\mathbf{3},\mathbf{2},-5/6)+(\mathbf{1},\mathbf{3},0)+2(\mathbf{1},\mathbf{1},0) \end{aligned} \tag{B.31}$$

which leads to the decomposition of the gauge bosons

$$\mathbf{24} = (\mathbf{8},\mathbf{1},0)+(\mathbf{1},\mathbf{3},0)+(\mathbf{1},\mathbf{1},0)+(\overline{\mathbf{3}},\mathbf{2},5/6)+(\mathbf{3},\mathbf{2},-5/6)\,, \tag{B.32}$$

or

$$\hat{A}_\mu \supset \left\{(G_\mu^a)^\alpha_\beta\,,(W_\mu^a)^r_s\,,B_\mu\,,(A_\mu^a)^\alpha_r\,,(A_\mu^a)^r_\alpha\right\}\,, \tag{B.33}$$

with the fields ordered according to (B.32). Here we are still using the representation indices $\alpha$ and $r$ in the same way as in the previous section. Observe that the last 12 gauge fields carry both flavour and colour and form coloured isospin doublets as expected. Literatures usually call the fields $(A^\mu)^4_\alpha = X^\mu_\alpha$ and $(A^\mu)^5_\alpha = Y^\mu_\alpha$ which are collected in

$$(\mathbf{3},\mathbf{2}) \to \begin{pmatrix} X_1^\mu & X_2^\mu & X_3^\mu \\ Y_1^\mu & Y_2^\mu & Y_3^\mu \end{pmatrix} \tag{B.34}$$

and their "antiparticles"

$$(\overline{\mathbf{3}},\mathbf{2}) \to \begin{pmatrix} X_1^{c\mu} & Y_1^{c\mu} \\ X_2^{c\mu} & Y_2^{c\mu} \\ X_3^{c\mu} & Y_3^{c\mu} \end{pmatrix} \tag{B.35}$$

where the subscripts label colours. The $U(1)$ gauge field $(A_\mu^{24})$ couples with $\lambda_{24}$ in (B.4) not the $Y$ in (B.12) while the $SU(3)$ and $SU(2)$ gauge fields are constructed from the generators of their corresponding subgroups in the usual way. Consequently, we have

$$\hat{\mathrm{A}}_\mu = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|cc} & \frac{1}{\sqrt{2}}\sum_{a=1}^{8} G_\mu^a\lambda^a & & X_{1\mu}^c & Y_{1\mu}^c \\ & & & X_{2\mu}^c & Y_{2\mu}^c \\ & & & X_{3\mu}^c & Y_{3\mu}^c \\ \hline X_{1\mu} & X_{2\mu} & X_{3\mu} & \frac{W_\mu^3}{\sqrt{2}} & W_\mu^+ \\ Y_{1\mu} & Y_{2\mu} & Y_{3\mu} & W_\mu^- & -\frac{W_\mu^3}{\sqrt{2}} \end{array}\right) + \frac{A_\mu^{24}}{2}\sqrt{\frac{3}{5}}\begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{B.36}$$

where the $Z$ and the photon are constructed from the equations resemble (2.90) and (2.91)

$$Z_\mu = -A_\mu^{24}\sin\theta_W + A_\mu^{23}\cos\theta_W \tag{B.37}$$

$$A_\mu = A_\mu^{24}\cos\theta_W + A_\mu^{23}\sin\theta_W\,. \tag{B.38}$$

where $W^3 = A^{23}$, etc.

Now, let us promote the transformation (B.7) into a local version taken care by

$$U(x) = e^{-\mathrm{i}g_5\theta^a(x)\lambda^a/2}\,. \tag{B.39}$$

Then we can write the covariant derivative for the $SU(5)$, in a general form, as

$$\mathrm{D}_\mu = \partial_\mu - \mathrm{i}g_5 A_\mu^a\frac{\lambda^a}{2}\,, \tag{B.40}$$

and for its $SU(3)\times SU(2)\times U(1)$ subgroup

$$\mathrm{D}_\mu = \partial_\mu - \mathrm{i}g_s\frac{\lambda^\alpha}{2}G_\mu^\alpha - \mathrm{i}g\frac{\tau^r}{2}W_\mu^r - \mathrm{i}g'\frac{Y}{2}B_\mu\,, \tag{B.41}$$

where $g_s$ denotes the QCD coupling. When acting on the fundamental representation of $SU(5)$, the covariant derivative (B.40) takes the usual form; i.e.,

$$(\mathrm{D}_\mu^{[5]}\psi)^i = \left[\delta_j^i\partial_\mu - \mathrm{i}\frac{g_5}{\sqrt{2}}(A_\mu)_j^i\right]\psi^j\,, \tag{B.42}$$

or simply

$$\mathrm{D}_\mu^{[5]}\Psi^{[5]} = \partial_\mu\Psi^{[5]} - \mathrm{i}g_5\hat{\mathrm{A}}_\mu\Psi^{[5]}\,. \tag{B.43}$$

Similarly, we have

$$(\mathrm{D}_\mu^{[5]}\overline{\phi})_i = \left[\delta_i^j\partial_\mu + \mathrm{i}\frac{g_5}{\sqrt{2}}(A_\mu)_i^j\right]\overline{\phi}_i \tag{B.44}$$

for the conjugate of the fundamental representation. The structure of the covariant derivative depends on the representation it is acting on. So this is why we introduced the superscript [5] for $\mathrm{D}_\mu$. However, as it is usually clear from the context what the "target" of the covariant derivative is, we will omit that superscript from now on. These definitions of the covariant derivative control how the gauge fields transform; i.e.,

$$\hat{\mathrm{A}}_\mu \longrightarrow U\hat{\mathrm{A}}_\mu U^{-1} - \frac{\mathrm{i}}{g_5}U\partial_\mu U^{-1}\,, \tag{B.45}$$

so as to ensure that $\mathrm{D}_\mu\psi$ transforms in the same way as the fundamental representation. Consequently, the "kinetic term" for the $\psi^{[5]}$ becomes

$$\mathcal{L}^{[5]} = \mathrm{i}\bar{\Psi}^{[5]}\not{D}\Psi^{[5]}\,. \tag{B.46}$$

The part for the **10** is a just bit more involved. Referring to (B.22), we see that the $\Psi^{[10]}$ transforms as

$$\psi^{ij} \quad\longrightarrow\quad U_k^iU_l^j\left(\psi^k\psi^l - \psi^l\psi^k\right) = U_k^i\psi^{kl}(U^\mathsf{T})_j^l \tag{B.47}$$

or $\Psi^{[10]} \to U\Psi^{[10]}U^\mathsf{T}$. This implies, using (B.7), that

$$\begin{aligned}\partial_\mu\Psi^{[10]} \quad&\to\quad U\left(\partial_\mu\Psi^{[10]}\right)U^\mathsf{T} + \left(\partial_\mu U\right)\Psi^{[10]}U^\mathsf{T} + U\Psi^{[10]}\left(\partial_\mu U^\mathsf{T}\right)\\ &=\quad \Psi^{[10]}\left(\partial_\mu\Psi^{[10]}\right)U^\mathsf{T}\\ &\qquad -\mathrm{i}g_5U\left[\partial_\mu\theta^a\frac{\lambda^a}{2}\Psi^{[10]} + \Psi^{[10]}\partial_\mu\theta^a\frac{\lambda^{\mathsf{T}a}}{2}\right]U^\mathsf{T}\,,\end{aligned} \tag{B.48}$$

and suggesting the covariant derivative for the **10**; namely,

$$\begin{aligned}\mathrm{D}_\mu\Psi^{[10]} \quad&=\quad \partial_\mu\Psi^{[10]} - \mathrm{i}g_5\left[A_\mu^a\frac{\lambda^a}{2}\Psi^{[10]} + \Psi^{[10]}A_\mu^a\frac{\lambda^{\mathsf{T}a}}{2}\right]\\ &=\quad \partial_\mu\Psi^{[10]} - \mathrm{i}g_5\left[\hat{\mathrm{A}}_\mu\Psi^{[10]} + \Psi^{[10]}\hat{\mathrm{A}}_\mu\right]\,,\end{aligned} \tag{B.49}$$

or

$$(\mathrm{D}_\mu\psi)^{ij} = \partial_\mu\psi^{ij} - \mathrm{i}\frac{g_5}{2}\left[(A_\mu)_k^i\psi^{kj} + (A_\mu)_k^j\psi^{ik}\right]\,, \tag{B.50}$$

which ensures that the transformation

$$\mathrm{D}_\mu\Psi^{[10]} \longrightarrow U(x)\mathrm{D}_\mu\Psi^{[10]}U^\mathsf{T}(x)\,, \tag{B.51}$$

is always satisfied. So the Lagrangian for the $\mathbf{10}$ is

$$\mathcal{L}^{[10]} = \mathrm{iTr}\left[\overline{\Psi}^{[10]} \not{D} \Psi^{[10]}\right] . \tag{B.52}$$

Consequently, we arrive at the Lagrangian for the kinetic part of the fermions (and their interactions with gauge fields)

$$\begin{aligned}
\mathcal{L}_{\text{kin}} &= \mathrm{i}\overline{\Psi}^{[5]}\not{D}\Psi^{[5]} + \mathrm{iTr}\left[\overline{\Psi}^{[10]}\not{D}\Psi^{[10]}\right] = \mathrm{i}(\overline{\psi^c_R})_i(\not{D}\psi^c_R)^i + \mathrm{i}(\overline{\psi_L})_{ij}(\not{D}\psi_L)^{ij} \\
&= (\overline{\psi^c_R})_i\left[\mathrm{i}\delta^i_j\not{\partial} + \frac{g_5}{2}(\not{A})^i_j\right](\psi^c_R)^j \\
&\quad +(\overline{\psi^c_L})_{ij}\left[\mathrm{i}\delta^i_k\not{\partial}(\psi^c_L)^{kj} + \frac{g_5}{2}\left\{(\not{A})^i_k(\psi^c_L)^{kj} + (\not{A})^j_k(\psi^c_L)^{ik}\right\}\right] \\
&= (\overline{\psi^c_R})_i\left[\mathrm{i}\delta^i_j\not{\partial} + \frac{g_5}{2}(\not{A})^i_j\right](\psi^c_R)^j + (\overline{\psi^c_L})_{ij}\left[\mathrm{i}\delta^i_k\not{\partial} + 2\frac{g_5}{2}(\not{A})^i_k\right](\psi^c_L)^{kj},
\end{aligned} \tag{B.53}$$

which, upon extracting some terms out; for example,

$$\frac{g_5}{2}(\overline{\psi^c_R})_\alpha(\not{A})^\alpha_s(\psi^c_R)^s = \frac{g_5}{2}\sum_{i=1}^{3}\bar{d}_{Ri}\not{X}^c_i e^+_R - \frac{g_5}{2}\sum_{i=1}^{3}\bar{d}_{Ri}\not{Y}^c_i \nu^c_R \tag{B.54}$$

convinces us that in this $SU(5)$ theory, a proton decay (to leptons) is possible with the $X^\mu$ and $Y^\mu$ as mediators.

## B.3 The Breaking of $SU(5)$ Part I

The $SU(5)$ is clearly not the exact symmetry of nature. In this section we will see how the $SU(5)$ symmetry is spontaneously broken into the "$3-2-1$" symmetry of the standard model where some gauge fields become massive via the BEH (Higgs) mechanism. Since nobody has seen a quark turns itself into a leptons or vice versa, then it has to be assumed that the $SU(5)$ gauge mediators, if exist at all, must be very heavy so that they cannot only escaped the current detectors but also gives reasonable predictions (decay rates, for instance) that agree with experiments. Another constraint to the symmetry breaking is that the $SU(3)_C$ and $U(1)_{em}$ must survive as we believe that they are exact.

We see that to get from $SU(5)$ to the $SU(3)\times U(1)$ theory, the symmetry must be broken via two stages and hence two "Higgs" particles are required. One of them is the standard model Higgs that taking care of the lower energy breaking stage $SU(3)\times SU(2)\times U(1) \to SU(3)_C \times U(1)_{em}$. The other corresponds to the breaking of the $SU(5)$. So this heavy Higgs must have very large vacuum expectation value so as to guarantee that 12 of the $SU(5)$ gauge bosons are heavy.

This is required since we have not yet seen them, as well as their effects. Let the vacuum expectation value of this heavy scalar field be of order $M_{GUT}$, then the masses of the heavy gauge bosons $M_X$ will be of this order. Hereafter, we will invert the argument by saying that the energy scale of the symmetry breaking of $SU(5)$ will be of order $M_X$. Let us call this latter scalar field $\Sigma$. In addition, it is required that when this $\Sigma$ develops a non-zero vacuum expectation value, the potential must still be invariant under the sub group $H = SU(3)\times SU(2)\times U(1)$. Therefore, the $\Sigma$ must contain a singlet when its representation is decomposed with respect to $H$.

One of the candidates for the GUT breaking Higgs, which is also found to be the simplest one, is obviously the adjoint representation (which is the same as that of the $SU(5)$ gauge bosons)

$$\mathbf{24}: \qquad \hat{\Sigma} = \sum \frac{\lambda^a}{2}\Sigma^a . \tag{B.55}$$

(note that it is a traceless hermitian matrix). The renormalisable scalar potential that does not depend on its overall sign (a convention; i.e., $\Sigma \to -\Sigma$ symmetry) can be written as

$$V(\hat{\Sigma}) = -\frac{\mu^2}{2}\mathrm{Tr}\hat{\Sigma}^2 + \frac{a}{4}\left[\mathrm{Tr}\hat{\Sigma}^2\right]^2 + \frac{b}{2}\mathrm{Tr}\hat{\Sigma}^4 , \tag{B.56}$$

where

$$Tr\hat{\Sigma}^2 = \Sigma^j_i\Sigma^i_j , \qquad \mathrm{Tr}\hat{\Sigma}^4 = \Sigma^j_i\Sigma^k_j\Sigma^l_k\Sigma^i_l . \tag{B.57}$$

For the electroweak symmetry breaking, the lighter Higgs can be a quintet $\Phi$ (the fundamental representation)

$$\mathbf{5} = (\mathbf{3},\mathbf{1},-\frac{1}{3}) + (\mathbf{1},\mathbf{2},\frac{1}{2}): \qquad \Phi^a = (\Phi^\alpha, H), \tag{B.58}$$

where the $H$ is the usual standard model (GWS) Higgs doublet (remember that $\alpha = 1,2,3$). The potential for the $\Phi$ is assumed to be the symmetric one ($\Phi \to -\Phi$)

$$V(\Phi) = -\frac{\mu_\Phi^2}{2}(\Phi^\dagger\Phi) + \frac{\lambda_\Phi}{4}(\Phi^\dagger\Phi)^2 \tag{B.59}$$

to prevent $\Phi^3$ interactions. Notice that the interaction between $\hat{\Sigma}$ and $H$; namely,

$$V_{\Sigma\Phi} = \alpha(\mathrm{Tr}\hat{\Sigma}^2)\Phi^\dagger\Phi + \beta(\Phi^\dagger\hat{\Sigma}^2\Phi) \tag{B.60}$$

are also possible. The most general potential is therefore

$$V(\hat{\Sigma},\Phi) = -\frac{\mu^2}{2}\mathrm{Tr}\hat{\Sigma}^2 + \frac{a}{4}\left[\mathrm{Tr}\hat{\Sigma}^2\right]^2 + \frac{b}{2}\mathrm{Tr}\hat{\Sigma}^4 - \frac{\mu_\Phi^2}{2}(\Phi^\dagger\Phi) + \frac{\lambda_\Phi}{4}(\Phi^\dagger\Phi)^2 + \alpha(\mathrm{Tr}\hat{\Sigma}^2)\Phi^\dagger\Phi + \beta(\Phi^\dagger\hat{\Sigma}^2\Phi)\,, \qquad (\text{B.61})$$

As there is a desert between the two symmetry breaking scales, the effects of the $\hat{\Phi}$, which we assume to be light, on the $\hat{\Sigma}$ should be negligible. In addition, as we have said earlier, the $\hat{\Phi}$ must live in some subgroup of the survival subgroups from the first step of symmetry breaking. This sets a constraint on the form of $\Phi$.

At this stage we will treat the two stages of symmetry breaking separately. Let us concentrate on the $SU(5)$ breaking part, with $\mu_\Phi^2 > 0$. Though the $SU(3)\times SU(2)\times U(1)$ is embedded in $SU(5)$, it is not the only possible choice when SSB occurs. The parameter that "decides" the which subgroup to break to is the $b$ in (B.56). The rough idea of the influence from the parameter $b$ on the group $SU(N)$ is as follows. The $\hat{\Sigma}$, being a traceless Hermitian matrix, can be diagonalised by an $SU(N)$ transformation to a matrix having elements that are real numbers. So the potential (B.56) can be written as

$$V(\Sigma) \to -\frac{\mu_\Sigma^2}{2}\sum\phi_i^2 + \frac{a}{4}\left(\sum\phi_i^2\right)^2 + \frac{b}{2}\sum\phi_i^4 - \xi\sum\phi_i \qquad (\text{B.62})$$

where $\xi$ is the Lagrange multiplier introduced to ensure that the matrix is traceless. Upon minimising of the potential, we get a set of cubic equations and hence there are three different roots (with constraints). Put differently,

$$SU(N) \longrightarrow SU(N_1)\times SU(N_2)\times SU(N-N_1-N_2)\,. \qquad (\text{B.63})$$

It is found that (see Langacker [84]) for $b>0$,

$$SU(N) \longrightarrow SU(N_1)\times SU(N-N_1)\times U(1) \qquad (\text{B.64})$$

together with $N_1 = N/2$ or $N_1 = (N+1)/2$ for $N$ even or odd respectively. When $b<0$ the symmetry breaking pattern is

$$SU(N) \longrightarrow SU(N-1)\times U(1)\,. \qquad (\text{B.65})$$

Now let us come back to the $SU(5)$ and concentrate on the case $b>0$. The vacuum expectation value that is invariant under the $SU(3)\times SU(2)\times U(1)$

is[2]

$$\langle 0|\hat{\Sigma}|0\rangle = \mathrm{diag}\left(v, v, v, -\frac{3}{2}v, -\frac{3}{2}v\right) . \tag{B.66}$$

In addition, this vacuum expectation value also constraints the value of $a$ which is $a > -7b/15$ (otherwise the potential would not be bounded from below). We can write the potential in terms of this $\langle 0|\hat{\Sigma}|0\rangle$ and minimise it with respect to the parameter $v$. This leads to the condition $v^2 = \frac{2\mu_\Sigma^2}{15a+7b}$ which we will rewrite as

$$\mu_\Sigma^2 = \frac{15a}{2}v^2 + \frac{7b}{2}v^2 \tag{B.67}$$

for future references. Moreover, group theory also tells us that the gauge fields $X$'s and $Y$'s will be massive as the generators associated with them do not commute with (B.66). These mass terms can be obtained from the gauge invariant kinetic terms.

To construct a kinetic term of the Higgs $\hat{\Sigma}$, recall that fields in an adjoint representation will have components transforming as vectors. Then the usual form of the gauge the covariant derivative of a scalar field $\hat{\Sigma}$ in a vector representation $\Sigma^a$, defined by $\hat{\Sigma} = \Sigma^a \frac{\lambda^a}{2}$

$$D_\mu \Sigma^a = \partial_\mu \Sigma^a - \mathrm{i} g_5 A_\mu^{[\mathrm{adj}]} \Sigma^a , \tag{B.68}$$

where $A_\mu^{[\mathrm{adj}]} = T^a A_\mu^a$, can be traced back to that for a field in the adjoint representation; namely,

$$D_\mu \hat{\Sigma} = \partial_\mu \hat{\Sigma} - \mathrm{i} g_5 [\hat{A}_\mu, \hat{\Sigma}] . \tag{B.69}$$

The appearance of the minus sign, instead of the plus sign (in other words, a commutator instead of an anti-commutator) as used for the $\mathbf{10}$ is due to the fact that the adjoint representation is constructed from $\overline{\mathbf{5}} \times \mathbf{5} \supset \mathbf{24}$ while the $\mathbf{10}$ is from $\mathbf{5} \times \mathbf{5} \supset \mathbf{10}$.

Now we can evaluate the mass terms of the gauge fields which are given by the term in a Lagrangian that is quite similar to (2.72); i.e.,

$$\frac{1}{2}\mathrm{Tr}\left\{ D_\mu \langle\hat{\Sigma}\rangle^\dagger D^\mu \langle\hat{\Sigma}\rangle \right\} = \frac{g_5^2}{2}\mathrm{Tr}\left\{ ([\hat{A}_\mu, \langle\hat{\Sigma}\rangle])^2 \right\} \tag{B.70}$$

With the choice of the vacuum given by (B.66), it is then clear from the commutator appearing in (B.70) that the standard model gauge bosons remain

[2] As usual, this vacuum expectation value can always be reached using an $SU(5)$ transformation.

massless in this stage of SSB. Since the field $\Sigma$ appears in the potential (B.56) via $\mathrm{Tr}\hat{\Sigma}^n$, other possible choices, reachable via the unitary transformation $\langle\hat{\Sigma}\rangle \to U\langle\hat{\Sigma}\rangle U^{-1}$ are equally possible and maybe convenient in some cases.

To work out the masses of the gauge field, we first note that

$$\langle\hat{\Sigma}\rangle = -\frac{3v}{2}Y\,, \tag{B.71}$$

which results in

$$\frac{g_5^2}{2}\mathrm{Tr}\left\{([\hat{A}_\mu, \langle\hat{\Sigma}\rangle])^2\right\} = \frac{1}{2}\frac{9v^2g_5^2}{4}A_\mu^a A^{a\mu}\mathrm{Tr}\left\{([\lambda^a, Y])^2\right\}\,. \tag{B.72}$$

Those $A_{ik}^\mu$ surviving the commutators are ones corresponding to $A_{\alpha r}^\mu$ (the $X$ and the $Y$ bosons) together with factors $\pm\frac{5}{3}$. Therefore,

$$\frac{9v^2g_5^2}{8}(2)\frac{25}{9}\left(X^{i\mu\dagger}X_\mu^i + Y^{i\mu\dagger}Y_\mu^i\right) = \frac{25v^2g_5^2}{4}\left(X^{i\mu\dagger}X_\mu^i + Y^{i\mu\dagger}Y_\mu^i\right)\,, \tag{B.73}$$

where $i$ denotes colours (the factor 2 comes from the trace), which leads to the mass terms

$$M_X^2 = M_Y^2 = \frac{25v^2g_5^2}{4}\,. \tag{B.74}$$

Now let us turn to the next stage of the symmetry breaking. Observe that the potential (B.59) has a non-zero vacuum expectation value (squared)

$$v_0^2 = \Phi_0^\dagger\Phi_0 = \frac{2\mu_\Phi^2}{\lambda_\Phi}\,, \tag{B.75}$$

which we will rewrite in terms of the mass of $\Phi$ as

$$\mu_\Phi^2 = \frac{\lambda_\Phi}{2}v_0^2 \tag{B.76}$$

Recall that in the fundamental representation, we can arrange

$$\Phi = \begin{pmatrix} \Phi^{(1)} \\ \Phi^{(2)} \\ \Phi^{(3)} \\ h^+ \\ h^0 \end{pmatrix}\,. \tag{B.77}$$

Still this does not completely fix the form of the $\Phi_0$ as the vacuum can point in any direction of the **5**. Nevertheless, the assumption that the $SU(3)_C$ being exact forces the $\Phi_0$ to be in the 4 or 5 direction. Moreover, since we know that the $SU(2) \times U(1)$ will eventually break down to the $U(1)_{em}$, we then plant the non-vanishing expectation value to the neutral component

$$\Phi_0 = \Phi_0^{(5)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_0/\sqrt{2} \end{pmatrix} . \tag{B.78}$$

Then the spontaneous symmetry breaking should proceed in the way fairly similar to the case in electroweak symmetry breaking. However, it might be obvious that the process is not that simple. We are dealing with 2 fundamental scalars in the theory and trying to claim that their vacuum expectation values are so different so as to suppress the effect of the heavy particles at low energy scale ($M_W$). As we have seen in section 3.4.2, there is no mechanism to keep a scalar particle naturally light. Then, it is more "natural" for the masses of these scalars to have the same order of magnitude, namely $M_\Sigma \sim M_\Phi \sim M_X$. The thought of grand unification, however, forces us to take the standard model as a low energy effective theory and we are left with no choices but to force the masses of the two scalars to be so different. Failure to do so may result too large contributions to the vacuum expectation value of $\hat{\Sigma}$ from the light scalar $\Phi$ via the cross couplings in (B.60). The need to force the two scalars, namely the **24** and **5**, inevitably leads to the *gauge hierarchy problem* or the *Big hierarchy problem*.

The hierarchy problem mentioned above is not the only problem we have. As the multiplet of the light Higgs is extended to the **5**, it can initiate the transition between quarks and leptons via Yukawa couplings. Hence this provides another way for a proton to decay. Consequently, it is not only the $\hat{\Sigma}$, but also the triplet $\Phi^{(i)}$ for $i = 1, 2, 3$, that receives mass of the $M_X$ scale. In fact, it is the cross couplings (B.60) that take care of this job. A particular structure of the matrix $\hat{\Sigma}$ will provide $M_X$ scale masses to the triplet while leaving the standard model Higgs light.

We will pause the treatment of the spontaneous symmetry of the $SU(5)$ for a moment and discuss briefly on the grand unification scale so that we can have a feeling of how severe the (big) hierarchy problem is.

## B.4 Where Does This Happen?

Our "guess" that there exists a large unified gauge group like $SU(5)$ simply means that the couplings of the standard model must meet each other at a particular energy scale. Such a scale should be fairly large due to the slow running of the couplings. In this section we try to be more qualitative and see what this energy level is and what consequences it leads to. There are basically two ways to look at the situation: one is from dimensional grounds with some rough approximations to see what we can expect from the experimental side, the other is to use the renormalisation group analysis to estimate the unification scale.

First, let us consider the somewhat naive dimensional analysis. The process we will consider is the proton decay (which is predicted from the theory). Still, the result will be fairly reliable if we *first* make an assumption that the unification scale be much larger from the weak scale, which should be so that we have not seen any footprints of the GUT physics or the $X$ and $Y$ bosons. This helps further simplify the rough estimation in general, because the initial and final particles are hadrons and the intermediate interactions involve a number of particles; for example, the fundamental fermions and the heavy gauge bosons. The argument is similar to what happened in the transition from Fermi's theory to Glashow-Weinberg-Salam's theory; but taken the opposite way around. In other words, we take the standard model as a low energy effective theory of a grand unification theory, which we do not understand yet. This allows us to approximate the transformation from quarks to leptons via the $X$ boson as Fermi's point interaction; i.e., the local version of the interaction at low energies should be capable of "replacing" the non-local interactions, using only the building blocks of the standard model. Thus we just introduce the dimension-6 operator like

$$\left(\frac{g_5}{M_X}\right)^2 \bar{u}\gamma^\mu u\bar{e}\gamma_\mu d \tag{B.79}$$

with many indices suppressed. Still, this allows us to use the typical decay width analogous to that of the Fermi's model (e.g., the muon decay):

$$\Gamma \propto \left(\frac{g_5}{M_X}\right)^4 m_p^5 \tag{B.80}$$

where $m_p$ is the mass of a proton. Then the lifetime can be calculated from $\tau = \Gamma^{-1}$. Since, it is well-known that a proton does not decay or at least lives for a very long time, its lifetime $\tau_p$ should be greater than $10^{31}$ years. Consequently, we get the typical unification scale

$$M_X \approx 10^{14} - 10^{15}\mathrm{GeV}\,, \tag{B.81}$$

which is still moderately lower than the Planck scale (so we do not have to worry much about gravity). However, it brings us a "desert" of about 12 orders of magnitude between the weak and the grand unification scale.

Before we discuss about the running of the couplings, let us first identify the $SU(5)$ gauge coupling with the coupling of the electroweak subgroup $SU(2)\times(1)$. Consider the relevant interaction terms deduced from the covariant derivative $\mathrm{D}_\mu$ in (B.40) we find that the $U(1)_Y$ coupling $g'$ for the electroweak theory is related to the $g_5$ by

$$g' = \sqrt{\frac{3}{5}}g_5\,. \tag{B.82}$$

This identification is crucial in order to look for the unification scale otherwise the running of the $U(1)_{em}$ coupling will not meet others. To see this let us recall that in a non-Abelian group like $SU(5)$, we can fix the normalisation of the generators of the group by an equation similar to (B.5). However, we do not have such the Lie algebraic relation to fix the normalisation of the $U(1)$ generator (and we have put by hand further assumptions such as the unit charges of protons or electrons). At this point we immediately find a by-product. The Weinberg angle is *predicted at* the unification scale to be

$$\sin^2\theta_W = \frac{g'^2}{g_2^2 + g'^2} = \frac{3}{8}\,, \tag{B.83}$$

where the "scale-down" relation requires the renormalisation group analysis which also depends on the value of the unification energy scale.

Let us call the gauge couplings as

$$\alpha_i = \frac{g_i^2}{4\pi} \tag{B.84}$$

where $i = 1, 2, 3$ denoting the gauge group $U(1)$, $SU(2)$, and $SU(3)$ respectively. The way these couplings run is affected by the particles content and their representations which is mathematically determined by the renormalisation group equations

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}\ln\mu^2} = -\beta_i\alpha_i^2 + \mathcal{O}(\alpha_i^3)\,. \tag{B.85}$$

The $\beta_i$ is the usual coefficient in the beta function:

$$\beta_i = -\frac{1}{4\pi}\left[\frac{11}{3}C_2(G_i) - \frac{2}{3}\sum_f T(R_f) - \frac{1}{3}\sum_s T(R_s)\right] \tag{B.86}$$

where $C(G_i)$ is the eigenvalue of the quartic Casimir operator and $T(R)$ is the index for each representation; $n_f$ denoting the number of fermions in each representation. Note that for the $SU(N)$ group we have $T(N) = \frac{1}{2}$ and $T(A) = N$ for the fundamental and the adjoint representations, respectively. For further information, we also note that $T_{SO(N)}(N) = 2$ and $T_{SO(N)}(A) = 2N - 4$ for the fundamental and the adjoint representations of $SO(N)$ respectively.

From now on, we will always assume that there are three families ($n_g = 3$). The estimation has to be done with care as it means will neglect every possibility of finding new particles before the $M_X$ scale which affects the renormalisation group equations. This includes the heavy gauge bosons and the heavy scalar altogether. In other words, we have to assume that at these heavy particles beyond the $M_X$ scale can be "integrated out" which implies that our estimation remains valid only if we consider $\mu^2 << M_X^2$. Consequently, we find that for strong interaction group $SU(3)_C$, where quarks live in the fundamental representation

$$\beta_3 = -\frac{1}{4\pi}\left[\frac{11}{3}\times 3 - \frac{2}{3}2\times n_g \times \frac{1}{2}\right] = -\frac{9}{4\pi} \tag{B.87}$$

where we have used $n_g = 3$ in the last step. Similarly, the $\beta$ for the $SU(2)_L$ is

$$\beta_2 = -\frac{1}{4\pi}\left[\frac{11}{3}\times 2 - \frac{2}{3}\times\frac{1}{2}\times 4\times n_g \times\frac{1}{2} - \frac{1}{3}\times\frac{1}{2}\right] = -\frac{4}{3\pi} \tag{B.88}$$

where we have neglected the contribution from the Higgs in the last step. Notice the extra factor $\frac{1}{2}$, which is there to assure that we count only the left-handed fermions. Finally, for the $U(1)_Y$ we recall, from (A.16), that $T(R_f) = 2\mathrm{Tr}(Y^2)$. So $T(R_f) = \frac{20}{3}$ per family

$$\beta_1 = -\frac{1}{4\pi}\left[-\frac{2}{3}\times\frac{1}{2}\times\frac{20}{3}n_g - \frac{1}{6}\right] = \frac{5}{3\pi} \tag{B.89}$$

Consequently, the solution to (B.85), to one-loop, is

$$\frac{1}{\alpha_i(\mu^2)} = \frac{1}{\alpha_i(M_X^2)} + \frac{\beta_i}{4\pi}\ln\left(\frac{\mu^2}{M_X^2}\right)\,. \tag{B.90}$$

By requiring that all the couplings are equal at the unification scale ($\alpha_3 = \alpha_2 =$

$\frac{5}{3}\alpha_1 = \alpha_5$) we can arrange the $\alpha_5(M_X)$ and $M_X$ in terms of the known values at $M_Z$ (electroweak) scale as follows:

$$\begin{aligned}
\frac{1}{\alpha_3(M_Z^2)} &= \frac{1}{\alpha_5(M_X^2)} + \frac{\beta_3}{4\pi}\ln\left(\frac{M_Z^2}{M_X^2}\right) \\
\frac{\sin^2\theta_W(M_Z^2)}{\alpha(M_Z^2)} &= \frac{1}{\alpha_5(M_X^2)} + \frac{\beta_2}{4\pi}\ln\left(\frac{M_Z^2}{M_X^2}\right) \\
\frac{\cos^2\theta_W(M_Z^2)}{\frac{5}{3}\alpha(M_Z^2)} &= \frac{1}{\alpha_5(M_X^2)} + \frac{\beta_1}{4\pi}\ln\left(\frac{M_Z^2}{M_X^2}\right)
\end{aligned} \tag{B.91}$$

where we have used $g_1 = e/\cos\theta_W$ and $g_2 = e/\sin\theta_W$. These equations are readily solvable, we find

$$\frac{1}{\alpha_5(M_X^2)} = \frac{1}{\beta_1 + \beta_2 - \frac{8}{3}\beta_3}\left(-\frac{\beta_3}{\alpha(M_Z^2)} + \frac{\beta_1+\beta_2}{\alpha_3(M_Z^2)}\right) \tag{B.92}$$

as well as

$$\ln\left(\frac{M_Z^2}{M_X^2}\right) = \frac{4\pi}{\beta_1 + \beta_2 - \frac{8}{3}\beta_3}\left(\frac{1}{\alpha(M_Z^2)} - \frac{8/3}{\alpha_3(M_Z^2)}\right). \tag{B.93}$$

Taking the approximate values from [63]; namely, $\alpha(M_Z^2) = 1/128$, $\alpha_3(M_Z^2) = 1/8.48$, and $\sin^2\theta_W(M_Z^2) = 0.231$, together with (B.87-B.89), we find that

$$\alpha_5(M_X^2) = 1/41.5 \qquad \text{and} \qquad M_X \approx 10^{15}\text{GeV}\,, \tag{B.94}$$

which agrees, to some degrees, with our "guess" value of $M_X$ for the proton decay. Though this agreement convinces us that our assumption that nothing shows up between the electroweak and the unification may be sensible, it brings us an obvious problem: why the two scales are so different? To see the difficulties the hierarchy problem brings to us, we go back to the Higgs sector of the $SU(5)$.

## B.5 The Breaking of $SU(5)$ Part II: The Big Hierarchy Problem

Let us return to the cross coupling between $\hat{\Sigma}$ and $\Phi$. Since we know that there is a desert between the two symmetry breaking scales, the extra $SU(2)$-breaking term added to the vacuum expectation value of the heavy scalar field $\hat{\Sigma}$ should be small. As usual, we can parametrise the $SU(2)$-breaking part by a diagonal matrix proportional to $\tau^3 = \lambda_{23}$ (recall Cartan subalgebra). So (see Buras *et al.*

[85])

$$\begin{aligned}\langle 0|\hat{\Sigma}|0\rangle &= \text{diag}\left(v, v, v, -\frac{3}{2}v, -\frac{3}{2}v\right) - \varepsilon\frac{\lambda_{23}}{2}v \\ &= \text{diag}\left(v, v, v, (-\frac{3}{2} - \frac{\varepsilon}{2})v, (-\frac{3}{2} + \frac{\varepsilon}{2})v\right) . \end{aligned} \tag{B.95}$$

Since the $SU(3)\times SU(2)\times U(1)$ symmetry is restored by taking $\lambda' = 0$ we should have, to lowest non-zero order,

$$\varepsilon \propto \lambda' \left(\frac{v_0}{v}\right)^2 \sim \mathcal{O}(10^{-24}), \tag{B.96}$$

where $K$ is some constant that should somehow proportional to $a^{-1}$, $b^{-1}$ an so on, in order to keep the $SU(2)$ breaking effect of $\hat{\Sigma}$ small comparing to that of $\Phi$ (i.e., $\varepsilon v << v_0$). The potential $V(\hat{\Sigma}_0, \Phi_0)$ becomes a complicated function of the vacuum configuration related variables $v^2$, $v_0^2$, $\varepsilon$ and $\mu_\Sigma^2$, $\mu_\Phi^2$, $\alpha$, $\beta$, $a$, $b$. Now we can minimise it with respect to the parameters $v$, $v_0$, $\varepsilon$. The explicit form of the potential as well as the calculations will be tedious but straightforward. So we will not working them out here. The results are (again, see Buras *et al.* [85]),

$$\mu_\Sigma^2 = \frac{15a}{2}v^2 + \frac{7b}{2}v^2 + \alpha v_0^2 + \frac{9}{30}\beta v_0^2 \tag{B.97}$$

$$\mu_\Phi^2 = \frac{\lambda_\Phi}{2}v_0^2 + 15\alpha v^2 + \frac{7}{2}\beta v^2 - 3\varepsilon\beta v^2 \tag{B.98}$$

which are the slight modifications of (B.67) and (B.76) respectively. In addition

$$\varepsilon \approx \frac{9\beta}{20b}\left(\frac{v_0}{v}\right)^2 , \tag{B.99}$$

guarantees that the effects of $SU(2)\times U(1)$ breaking at the $M_X$ scale is negligible. However, the unnaturally smallness of $v_0/v$ clearly lead to the problem in (B.98). We know that the cancellations, which keep $v_0^2$ small, between the terms on the right-hand side of

$$\frac{\lambda_\Phi}{2}v_0^2 = \mu_\Phi^2 - v^2\left[15\alpha + \left(\frac{7}{2} - 3\varepsilon\right)\beta\right] \tag{B.100}$$

will never happen in a natural way as they require fine-tunings of the parameters to one part in $10^{24}$.

Before we leave this section, Let us have a look at a sketch of the unification of couplings as shown in Fig. B.1 taken from Dienes [7]. As the plot was made in the time where people have enough data from experiments, we can then check whether the three couplings really meet at a point. Note that the thickness of the lines show uncertainties from experiments, and that the indices of the couplings $\alpha_i$ denote the corresponding group, as well as that $\alpha_1$ already included the factor 5/3.

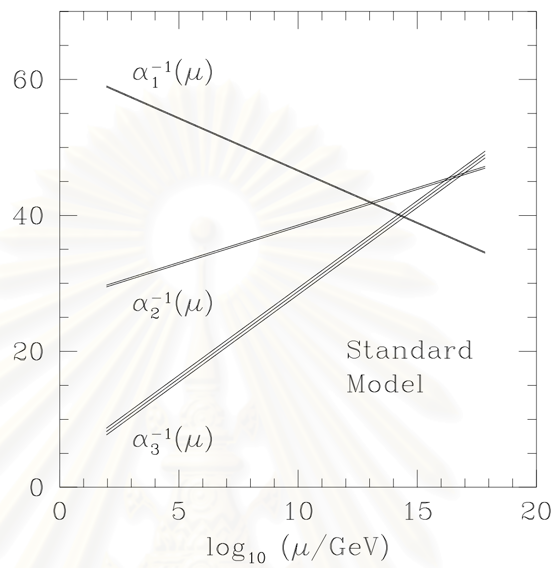


**Figure B.1:** The sketch shows how couplings of the standard model *almost unify* ([7]).

As we shall see from the figure, the couplings do not not meet (in contrast to what people used to think in the 70's). By the way, this is not a bad news. If we insist on having unified interaction, a theory (e.g., $SO(10)$ unification, supersymmetry, etc.) to be a candidate to explain or support it must provides more particles to "bend" the running of the couplings to the desired unification point. In fact, it opens possibilities for many types of particle physics beyond the standard model as "the desert is not that boring".

# APPENDIX C

# MATHEMATICAL FORMULAE

In this appendix we present some important mathematical formulae that are frequently referred to (maybe implicitly). In addition, in section C.3, we present several generators the groups that we used in the thesis.

## C.1 Dirac $\gamma$ Matrices

Formulae in this appendix are taken from the book by Quigg [39].
Useful identities

$$[\gamma^\mu\gamma^\nu,\gamma^\rho] \equiv \gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu = 2(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}) \tag{C.1}$$

$$\gamma^\mu\gamma_\nu\gamma_\mu = -2\gamma_\nu \tag{C.2}$$

$$\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\mu = 4g_{\nu\rho} \tag{C.3}$$

$$\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu = -2\gamma_\sigma\gamma_\rho\gamma_\nu \tag{C.4}$$

$$\gamma^\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\tau\gamma_\mu = 2(\gamma_\tau\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho\gamma_\nu\gamma_\tau) \tag{C.5}$$

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{\mathrm{i}}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{C.6}$$

$$\gamma^5\gamma^\sigma = \frac{\mathrm{i}}{3!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho \tag{C.7}$$

Frequently used identities:

$$(1-\gamma^5)^2 = 2(1-\gamma^5) \tag{C.8}$$

$$\gamma^\mu(1-\gamma^5) = 2\frac{1+\gamma^5}{2}\gamma^\mu\frac{1-\gamma^5}{2} \tag{C.9}$$

Trace technology:

$$\begin{aligned}
\mathrm{Tr}[\gamma_\mu] &= 0 && \text{(C.10)}\\
\mathrm{Tr}[\text{odd no. of } \gamma' s] &= 0 && \text{(C.11)}\\
\mathrm{Tr}[\gamma_\mu \gamma_\nu] &= 4 g_{\mu\nu} && \text{(C.12)}\\
\mathrm{Tr}[\not{a}\not{b}] &= 4 a \cdot b && \text{(C.13)}\\
\mathrm{Tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] &= 4[g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}] && \text{(C.14)}\\
\mathrm{Tr}[\gamma^5] &= 0 && \text{(C.15)}\\
\mathrm{Tr}[\gamma^5 \gamma_\mu] &= 0 && \text{(C.16)}\\
\mathrm{Tr}[\gamma^5 \gamma_\mu \gamma_\nu] &= 0 && \text{(C.17)}\\
\mathrm{Tr}[\gamma^5 \gamma_\mu \gamma_\nu \gamma_\rho] &= 0 && \text{(C.18)}\\
\mathrm{Tr}[\gamma^5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] &= 4\mathrm{i} \varepsilon_{\mu\nu\rho\sigma} && \text{(C.19)}
\end{aligned}$$

In addition we present the formulae for the $\varepsilon$ tensor:

$$\begin{aligned}
-\varepsilon^{\alpha\lambda\mu\nu} \varepsilon_{\alpha\rho\sigma\tau} &= \delta^\lambda_\rho (\delta^\mu_\sigma \delta^\nu_\tau - \delta^\mu_\tau \delta^\nu_\sigma) - \delta^\lambda_\sigma (\delta^\mu_\rho \delta^\nu_\tau - \delta^\mu_\tau \delta^\nu_\rho)\\
&\quad + \delta^\lambda_\tau (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) && \text{(C.20)}\\
-\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\sigma\tau} &= 2(\delta^\mu_\sigma \delta^\nu_\tau - \delta^\mu_\tau \delta^\nu_\sigma) && \text{(C.21)}\\
-\varepsilon^{\alpha\beta\gamma\nu} \varepsilon_{\alpha\beta\gamma\tau} &= 6 \delta^\nu_\tau && \text{(C.22)}
\end{aligned}$$

## C.2 Feynman Parametrisation

The formula

$$\int \frac{\mathrm{d}x}{[ax + b(1-x)^2]^2} = \frac{x}{b[(a-b)x + b]} \tag{C.23}$$

yields

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}x}{[ax + b(1-x)^2]^2}\,. \tag{C.24}$$

Then the general formula is obtained by successive differentiations:

$$\begin{aligned}
\frac{1}{a_1 a_2 \ldots a_n} &= \Gamma(n) \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \ldots \int_0^{1-x_1-\ldots x_{n-1}} \mathrm{d}x_{n-1}\\
&\qquad \frac{1}{[a x_1 + a_2 x_2 + \ldots + a_n(1 - x_1 - \ldots - x_{n-1})]^n}\,. && \text{(C.25)}
\end{aligned}$$

Most of the integrals concerning us can be deduced from the *Wicked rotated* version:

$$I_{r,m} = \int \frac{\mathrm{d}^4 k}{(2\pi)^d} \frac{k^{2r}}{[k^2 - C + \mathrm{i}\varepsilon]^m} \mathrm{i}(-\mathrm{i})^{r-m} \int \frac{\mathrm{d}^4 k_E}{(2\pi)^d} \frac{k_E^{2r}}{[k_E^2 + C]^m} \tag{C.26}$$

where $C$ is positive. Note that

$$\int \mathrm{d}^d k_E = \int \mathrm{d}|k||k|^{d-1} \mathrm{d}\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int \mathrm{d}|k||k|^{d-1}, \tag{C.27}$$

where $|k| = \sqrt{k_E^0 k_E^0 + \mathbf{k}\cdot\mathbf{k}}$. The integral $I_{r,m}$ can be evaluated

$$I_{r,m} = \mathrm{i}\frac{(-1)^{r-m}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\varepsilon/2} C^{2+r-m} \frac{\Gamma(2 + r - \frac{\varepsilon}{2}}{\Gamma(2 - \frac{\varepsilon}{2})} \frac{\Gamma(m - r - 2 + \frac{\varepsilon}{2}}{\Gamma(m)}. \tag{C.28}$$

The following integral is usually encountered

$$\begin{aligned} I_{0,2} &= \frac{\mathrm{i}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\varepsilon/2} \frac{2\Gamma(1 + \frac{\varepsilon}{2}}{\varepsilon} \\ &= \frac{\mathrm{i}}{16\pi^2}[\Delta_\varepsilon - \ln C + \mathcal{O}(\varepsilon)] \end{aligned} \tag{C.29}$$

where

$$\Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma + \ln 4\pi \tag{C.30}$$

and $\gamma$ is the Euler-Mascheroni constant. The other one is the "tadpole":

$$I_{0,1} = \frac{\mathrm{i}}{16\pi^2} C(1 + \Delta_\varepsilon - \ln C). \tag{C.31}$$

Consider the integral involving spacetime indices

$$I_n^{\mu_1\cdots\mu_p} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{\mu_1} \ldots k^{\mu_p}}{[k^2 + 2k\cdot P - M^2 + \mathrm{i}\varepsilon]^n}. \tag{C.32}$$

We have

$$I_1^\mu = 0 \tag{C.33}$$

$$I_1^{\mu\nu} = \frac{\mathrm{i}}{16\pi^2}\frac{1}{8} C_1^2 g^{\mu\nu}(3 + 2\Delta_\varepsilon - 2\ln C_1) \tag{C.34}$$

$$I_2^\mu = \frac{\mathrm{i}}{16\pi^2}(-\Delta_\varepsilon + \ln C_2) P_{(2)}^\mu \tag{C.35}$$

$$I_2^{\mu\nu} = \frac{\mathrm{i}}{16\pi^2}\frac{1}{2}\left[C g^{\mu\nu}(1 + \Delta_\varepsilon - \ln C_2) + 2(\Delta_\varepsilon - \Delta_\varepsilon) P_{(2)}^\mu P_{(2)}^\nu\right], \tag{C.36}$$

where

$$P^{\mu}_{(1)} = 0 \tag{C.37}$$
$$C_1 = m^2 \tag{C.38}$$
$$P^{\mu}_{(2)} = x r_1^{\mu} \tag{C.39}$$
$$C_2 = x^2 r_1^2 + (1-x) m_0^2 + x m_1^2 - x r_1^2 \,, \tag{C.40}$$

with $r_i$ and $m_i$ defined as the momenta running in the loop (related to external momenta) and its corresponding "mass" in the sense of

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^d} \frac{k^{\mu_1} \cdots k^{\mu_p}}{D_0 \cdots D_{n-1}} \tag{C.41}$$

where

$$D_i = (k + r_i)^2 - m_i^2 + \mathrm{i}\varepsilon \tag{C.42}$$

and

$$r_0 = \sum_1^n p_i = 0 \tag{C.43}$$
$$r_j = \sum_1^j p_i \,, \qquad j = 1, \ldots, n-1 \,. \tag{C.44}$$

## C.3 Various Symmetry Generators

#### C.3.0.1 Pauli Matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.45}$$

#### C.3.0.2 Triplet Representation of Isospin

$$T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ o & 1 & 0 \end{pmatrix} \tag{C.46}$$

$$T^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tag{C.47}$$

#### C.3.0.3 $SU(2)$ Real Representation

$$T^1 = \begin{pmatrix} & & & \mathrm{i} \\ & & -\mathrm{i} & \\ & \mathrm{i} & & \\ -\mathrm{i} & & & \end{pmatrix}, \quad T^2 = \left(\begin{array}{cc|cc} & & -\mathrm{i} & \\ & & & -\mathrm{i} \\ \hline \mathrm{i} & & & \\ & \mathrm{i} & & \end{array}\right)$$

$$T^3 = \left(\begin{array}{cc|cc} & \mathrm{i} & & \\ -\mathrm{i} & & & \\ \hline & & & -\mathrm{i} \\ & & \mathrm{i} & \end{array}\right) \tag{C.48}$$

#### C.3.0.4 $SU(2) \times SU(2) \sim SO(4)$ Real Representation

$$
T_V^1 = \left(\begin{array}{cc|cc} & & & \\ & & -\mathrm{i} & \\ \hline & \mathrm{i} & & \\ & & & \end{array}\right) \qquad T_V^2 = \left(\begin{array}{cc|cc} & & \mathrm{i} & \\ & & & \\ \hline -\mathrm{i} & & & \\ & & & \end{array}\right)
$$

$$
T_V^3 = \left(\begin{array}{cc|cc} & -\mathrm{i} & & \\ \mathrm{i} & & & \\ \hline & & & \\ & & & \end{array}\right) \tag{C.49}
$$

$$
T_A^a = \left(\begin{array}{cc|cc} & & & \mathrm{i} \\ & & & \\ \hline & & & \\ -\mathrm{i} & & & \end{array}\right), \quad T_A^2 = \left(\begin{array}{cc|cc} & & & \\ & & & \mathrm{i} \\ \hline & & & \\ & -\mathrm{i} & & \end{array}\right)
$$

$$
T_A^3 = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & \mathrm{i} \\ & & -\mathrm{i} & \end{array}\right) \tag{C.50}
$$

### C.3.1 Gell-Mann Matrices

Observe that the patterns of the matrices are extremely easy to remember. We have the Pauli matrices embeded in the upper-left block:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.51}$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{C.52}$$

as well as some elements "similar" to the Pauli matrices spread elsewhere:

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \tag{C.53}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \tag{C.54}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{C.55}$$

### C.3.2 $SU(5)$ Generators

We can use exactly the same philosophy to "memorise" the $SU(5)$ generators[1] We will use $T^a = \frac{\lambda^a}{2}$ where $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta^{ab}$. We shall use the same symbols with the Gell-Mann matrices since the different should be clear from the context. We only try to distinguish between them when it is necessary. The first 8 generators

[1]Observe that there are 14 symmetric "real" generators and 10 antisymmetric "complex" generators.

are just the Gell-Mann matrices embedded in $5 \times 5$ matrices.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.56}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.57}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.58}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.59}$$

The $2 \times 2$ lower-right block contains the Pauli matrices

$$\lambda_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \lambda_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix} \tag{C.60}$$

$$\lambda_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.61}$$

Then we will the remaining blocks with 1, 1 or i, −i:

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.62}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.63}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.64}$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.65}$$

$$\lambda_{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix} \tag{C.66}$$

$$\lambda_{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \lambda_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix} \tag{C.67}$$

The last diagonal generator is

$$\lambda_{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} . \tag{C.68}$$

## C.4 The Mass Eigenstate Matrices

The non-diagonalised non-transformed matrix $M^2$ from (5.136) of the Goldstone bosons, and the other $M^2$ of the neutral gauge fields (5.177) are

$$M^2 =$$

$$\begin{bmatrix}
-\mu^2+v^2\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & Fv\lambda_3 & 0 & 0 \\
0 & -\mu^2+v^2\lambda_4 & 0 & 0 & 0 & 0 & -Fv\lambda_3 & 0 & 0 & 0 \\
0 & 0 & -\mu^2+3v^2\lambda_4-2F\lambda_3 v' & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}Fv\lambda_3 \\
0 & 0 & 0 & -\mu^2+v^2\lambda_4+2F\lambda_3 v' & 0 & 0 & 0 & 0 & -\sqrt{2}Fv\lambda_3 & 0 \\
0 & 0 & 0 & 0 & F^2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & F^2\lambda_2 & 0 & 0 & 0 & 0 \\
0 & -Fv\lambda_3 & 0 & 0 & 0 & 0 & F^2\lambda_2 & 0 & 0 & 0 \\
Fv\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & F^2\lambda_2 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{2}Fv\lambda_3 & 0 & 0 & 0 & 0 & F^2\lambda_2 & 0 \\
0 & 0 & \sqrt{2}Fv\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & F^2\lambda_2
\end{bmatrix} \tag{C.69}$$

$$M^2_{WW'BB'} = \begin{pmatrix}
-m_W^2+M_{W'}^2 & -\frac{1}{8}gg'v^2\left(\frac{sc'}{cs'}+\frac{cs'}{sc'}\right) & -\frac{1}{8}g^2v^2cs\left(c^2-s^2\right) & -\frac{1}{8}gg'v^2cs\left(c^2-s^2\right) \\
-\frac{1}{8}gg'v^2\left(\frac{sc'}{cs'}+\frac{cs'}{sc'}\right) & -m_Z^2 s_W^2+M_{B'}^2 & -\frac{1}{8}gg'v^2cs\left(c^2-s^2\right) & -\frac{1}{8}g'^2v^2cs\left(c^2-s^2\right) \\
-\frac{1}{8}g^2v^2cs\left(c^2-s^2\right) & -\frac{1}{8}gg'v^2cs\left(c^2-s^2\right) & m_W^2\left(1-\frac{v^2}{6F^2}+\frac{8v'^2}{v^2}\right) & -\frac{1}{4}gg'v^2\left(1-\frac{v^2}{6F^2}+\frac{8v'^2}{v^2}\right) \\
-\frac{1}{8}g'v^2cs\left(c^2-s^2\right) & -\frac{1}{8}g'^2v^2cs\left(c^2-s^2\right) & -\frac{1}{4}gg'v^2\left(1-\frac{v^2}{6F^2}+\frac{8v'^2}{v^2}\right) & m_Z^2 s_W^2\left(1-\frac{v^2}{6F^2}+\frac{8v'^2}{v^2}\right)
\end{pmatrix} \tag{C.70}$$

# VITAE

Mr. Pawin Ittisamai was born in Bangkok, Thailand, in June 30$^{th}$, 1982. After graduated from Satit Chulalongkorn University Demonstration School in 2000, he received his Bachelor's degree (with honour) in physics from Chulalongkorn University in 2004. In his bachelor degree project, he studied applications of the Dirac constrained dynamics on the Nambu-Goto and Polyakov actions in classical string theory. Since 2003, his studies have been supported by the Institute for the Promotion of Teaching Science and Technology via a scholarship under the Development and Promotion of Science and Technology Talents Project (DPST). His research interests include theoretical elementary particle physics especially those concerning physics beyond the standard model and dynamical symmetry breaking.