Chapter V

AVERAGED MODEL GREEN FUNCTION

In this chapter, we will use the same model as Bezák to find the averaged Green function of disordered systems, but we will not use his method; instead, we will first use the path integrals formalism and then synthesize this with the cumulant theory developed by Kubo[16].

Kubo pointed out that the series may be written as a series in the exponent, i.e.

$$\left\langle \exp(\xi x) \right\rangle = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \mu_n = \exp\left\{ \sum_{n=1}^{\infty} \frac{\xi^n}{n!} K_n \right\}, \quad \text{---(5.1)}$$

where μ_n is the n^{th} moment and K_n is the n^{th} cumulant.

Before applying Eq. (5.1) to our problem, let us give a rough indication of how it can be derived.

Since $$\left\langle \exp(\xi x) \right\rangle = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \mu_n = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \langle x^n \rangle$$

$$= 1 + \frac{\xi \langle x \rangle}{1!} + \frac{\xi^2 \langle x^2 \rangle}{2!} + \cdots + \frac{\xi^n \langle x^n \rangle}{n!},$$

therefore $$\ln\left\langle \exp(\xi x) \right\rangle = \ln\left[1 + \left\{ \frac{\xi \langle x \rangle}{1!} + \frac{\xi^2 \langle x^2 \rangle}{2!} + \cdots + \frac{\xi^n \langle x^n \rangle}{n!} \right\} \right]$$

$$= \left\{ \frac{\xi \langle x \rangle}{1!} + \frac{\xi^2 \langle x^2 \rangle}{2!} + \cdots + \frac{\xi^n \langle x^n \rangle}{n!} \right\}$$

$$+ \frac{\left\{ \frac{\xi \langle x \rangle}{1!} + \frac{\xi^2 \langle x^2 \rangle}{2!} + \cdots + \frac{\xi^n \langle x^n \rangle}{n!} \right\}^3}{3!}$$

$$+ \cdots \cdots \quad \text{---(5.2)}$$

If we expand every term in Eq.(5.2), and then rearrange the terms into a new form consisting of the sum of all terms having the same power of ξ, then Eq.(5.2) can be written as

$$\ln\langle \exp(\xi x)\rangle = \sum_{n=1}^{\infty} \frac{\xi^n}{n!} K_n \;, \qquad \text{-------- (5.3)}$$

where K_n consists of the sum of different powers in $\langle x\rangle$. K_n is defined as a kind of average, the cumulant average. By taking the exponential on both sides of Eq.(5.3), we obtain

$$\langle \exp(\xi x)\rangle = \exp\left\{\sum_{n=1}^{\infty} \frac{\xi^n}{n!} K_n\right\} .$$

The relationship between the cumulants and the moments can be derived from Eq.(5.1).

For brievity, the derivation procedure will not be given, only the result will be encode.

$$\frac{K_n}{n!} = \sum_{\substack{\{n_i\} \\ (\sum_i i n_i = n)}} (-1)^{\sum_i n_i - 1} \left(\sum_i n_i - 1\right)! \prod_i \left[\frac{1}{n_i!}\left\{\frac{\mu_i}{i!}\right\}^{n_i}\right] \text{.....(5.4)}$$

The meaning of the restriction in Eq.(5.4) is that the sum over all sets numbers $\{n_i\}$ must satisfy $\sum_i i n_i = n$. Thus the resulting values of the moments provide information for calculating the cumulants. In order to clarify the restriction in Eq.(5.4) and the method of finding the relation between cumulants and moments, it will be useful to follow one example that of determining the second cumulant K_2 .

Since n = 2, therefore $\sum_i in_1 = 2$,
i.e. $1n_1 + 2n_2 = 2$
which implies that $n_1 = 0$ and $n_2 = 1$, or $n_1 = 2$ and $n_2 = 0$.

Thus for K_2, Eq.(5.3) has to be summed over two sets of numbers, $\{0,1\}$ and $\{2,0\}$.

For the set $\{0,1\}$, $\sum_i n_i - 1 = (0+1)-1 = 0$.

For the set $\{2,0\}$, $\sum_i n_i - 1 = (2+0)-1 = 1$.

$$\text{Thus} \quad \frac{K_2}{2!} = (-1)^0 (0)! \frac{1}{0!}\left(\frac{\langle x\rangle}{1!}\right)^0 \frac{1}{1!}\left(\frac{\langle x^2\rangle}{2!}\right)^1$$

$$+ (-1)^1 (1)! \frac{1}{2!}\left(\frac{\langle x\rangle}{1!}\right)^2 \frac{1}{0!}\left(\frac{\langle x^2\rangle}{2!}\right)^0$$

$$= \frac{\langle x^2\rangle}{2!} - \frac{\langle x\rangle^2}{2!}.$$

The other cumulants can be similarly obtained.

For convenience, we will simplify some terms as follows :-

$$\langle x_j\rangle_c = \langle x_j\rangle$$

$$\langle x_j^2\rangle_c = \langle x_j^2\rangle - \langle x_j\rangle^2$$

$$\langle x_j x_\ell\rangle_c = \langle x_j x_\ell\rangle - \langle x_j\rangle\langle x_\ell\rangle$$

$$\langle x_j x_k x_\ell\rangle_c = \langle x_j x_k x_\ell\rangle - \Big\{\langle x_j\rangle\langle x_k x_\ell\rangle + \langle x_k\rangle\langle x_\ell x_j\rangle + \langle x_\ell\rangle\langle x_j x_k\rangle\Big\} + 2\langle x_j\rangle\langle x_k\rangle\langle x_\ell\rangle$$

Having thus briefly introduced the meaning of cumulant average and how it can be expressed in terms of moments, let us now go back to the problem of determining the averaged time-dependent Green function of disordered systems.

Since we will use the same model as Bezák, we can therefore similarly obtain

$$\left\langle G(\underline{r},\underline{r}';\beta)\right\rangle = \exp\left(\frac{\eta^2\beta^2}{2}\right)\int \mathcal{N}\,\mathcal{D}(\text{path})\,\exp\left[-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\,\dot{\underline{r}}^2(\tau) - \frac{\eta^2}{2\hbar^2L^2}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]\right]$$

as in Eq.(4.7).

This equation can also be written as

$$\left\langle G(\underline{r},\underline{r}';\beta)\right\rangle = \exp\left(\frac{\eta^2\beta^2}{2}\right)\int \mathcal{N}\,\mathcal{D}(\text{path})\,\exp\left[-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\,\dot{\underline{r}}^2(\tau) - \frac{m\omega_G^2}{4\hbar^2\beta}\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right] . \qquad \text{--------- (5.5)}$$

Eq.(5.5) can be looked upon as the Green function of an electron interacting with a non-local potential $[r(\tau)-r(\tau')]^2$ We can rewrite Eq.(5.5) as

$$\left\langle G(\underline{r},\underline{r}';\beta)\right\rangle = \exp\left(\frac{\eta^2\beta^2}{2}\right) G_0(\underline{r},\underline{r}';\beta)\, E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right.\right] , \qquad \text{----- (5.6)}$$

where $G_0(\underline{r}, \underline{r}'; \beta) = \int \mathcal{N} \mathcal{D}(\text{path}) \exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}^2(\tau)\right\}$,

and E is an expectation defined by

$$E\left[\exp\left\{F[\underline{r}(\tau)]\right\}\right] = \frac{\int \mathcal{N}\mathcal{D}(\text{path}) \exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}^2(\tau)\right\} \exp\left\{F[\underline{r}(\tau)]\right\}}{\int \mathcal{N}\mathcal{D}(\text{path}) \exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\, \dot{\underline{r}}^2(\tau)\right\}}$$

Let us consider the evaluation of $E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right) \iint d\tau\, d\tau' \left[\underline{r}(\tau) - \underline{r}(\tau')\right]^2\right\}\right]$

Since
$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{r}(\tau) - \underline{r}(\tau')\right]^2\right\}\right]$$

$$= E\left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^n \int d\tau_1 \int d\tau_1' \cdots \int d\tau_n \int d\tau_n' \left[\underline{r}(\tau_1) - \underline{r}(\tau_1')\right]^2 \cdots \left[\underline{r}(\tau_n) - \underline{r}(\tau_n')\right]^2\right]$$

therefore by using the property of cumulant in Eq.(5.1), we obtain

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta}d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\}\right] = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^n\int d\tau_1\int d\tau_1' \cdots\cdots \int d\tau_n\int d\tau_n'\, E_c\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2 \cdots\cdots \left[\underline{r}(\tau_n)-\underline{r}(\tau_n')\right]^2\right\}\right., \quad \text{---}(5.7)$$

where E_c denotes the cumulant average. By keeping only the first cumulant, Eq.(5.7) becomes

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta}d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1'\, E_c\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2\right\}\right\}.$$

Since $E_c\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2\right\} = E\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2\right\}$,

therefore

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta}d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1'\,\frac{\int \mathcal{N}\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta}d\tau\,\dot{\underline{r}}^2(\tau)\right\}\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2}{\int \mathcal{N}\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta}d\tau\,\dot{\underline{r}}^2(\tau)\right\}}\right\}.$$

Since

$$\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2 = \left[-\frac{\partial^2}{\partial\underline{k}^2}\exp\left\{i\,\underline{k}\cdot\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]\right\}\right]_{\underline{k}=0},$$

therefore

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta}d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1'\,\frac{\int \mathcal{N}\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta}d\tau\,\dot{\underline{r}}^2(\tau)\right\}\left[-\frac{\partial^2}{\partial\underline{k}^2}\exp\left\{i\underline{k}\cdot\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]\right\}\right]_{\underline{k}}}{\int \mathcal{N}\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int d\tau\,\dot{\underline{r}}^2(\tau)\right\}}\right)$$

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left[\mathbf{r}(\tau)-\mathbf{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1' \frac{\left[\left(-\frac{\partial^2}{\partial \mathbf{k}^2}\right)\int \mathcal{N}\,\mathcal{D}(\text{path})\exp\frac{1}{\hbar}\left\{-\frac{m}{2}\int_0^{\hbar\beta} d\tau\,\dot{\mathbf{r}}^2(\tau) + i\hbar\,\mathbf{k}.\left[\mathbf{r}(\tau_1)-\mathbf{r}(\tau_1')\right]\right\}\right]_{\mathbf{k}=0}}{\int \mathcal{N}\,\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int d\tau\,\dot{\mathbf{r}}^2(\tau)\right\}}\right\}$$

Since $\quad i\hbar\,\mathbf{k}.\left[\mathbf{r}(\tau_1)-\mathbf{r}(\tau_1')\right] = \int i\hbar\left(\delta(\tau-\tau_1)-\delta(\tau-\tau_1')\right)\mathbf{k}.\mathbf{r}(\tau)\, d\tau,$

therefore

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left[\mathbf{r}(\tau)-\mathbf{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1' \frac{\left[\left(-\frac{\partial^2}{\partial \mathbf{k}^2}\right)\int \mathcal{N}\,\mathcal{D}(\text{path})\exp\frac{1}{\hbar}\left\{-\frac{m}{2}\int_0^{\hbar\beta} d\tau\,\dot{\mathbf{r}}^2(\tau) + \int_0^{\hbar\beta} f_1(\tau)\,\mathbf{r}(\tau)\, d\tau\right\}\right]_{\mathbf{k}=0}}{\int \mathcal{N}\,\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int d\tau\,\dot{\mathbf{r}}^2(\tau)\right\}}\right\} \qquad \text{---------- (5.8)}$$

where $\quad f_1(\tau) = i\hbar\,\mathbf{k}\left(\delta(\tau-\tau_1)-\delta(\tau-\tau_1')\right).$

Let us use the method described at the end of chapter III, i.e. writing $\underline{r}(\tau)=\underline{r}_{cl}+\underline{y}(\tau)$, then the path integrals in Eq.(5.8) can be reduced to a product of two functions. Consequently Eq. (5.8) becomes

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left[\mathbf{r}(\tau)-\mathbf{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1' \frac{\left[-\frac{\partial^2}{\partial \mathbf{k}^2}\exp\left(\frac{1}{\hbar}S_{cl}'\right)\int \mathcal{N}\,\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\,\dot{\mathbf{y}}^2(\tau)\right\}\right]_{\mathbf{k}=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)\int \mathcal{N}\,\mathcal{D}(\text{path})\exp\left\{-\frac{m}{2\hbar}\int_0^{\hbar\beta} d\tau\,\dot{\mathbf{y}}^2(\tau)\right\}}\right\},$$

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau' \left[\underline{x}(\tau)-\underline{x}(\tau')\right]^2\right\}\right]$$

$$= \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1 \int d\tau_1' \frac{\left[-\frac{\partial^2}{\partial \underline{k}^2}\exp\left(\frac{1}{\hbar}S'_{cl}\right)\right]_{\underline{k}=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)}\right\}, \quad \text{-----}(5.9)$$

where $S'_{cl} = \int_0^{\hbar\beta}\left\{-\frac{m}{2}\dot{\underline{x}}_c^2(\tau) + f_1(\tau)\underline{x}_c(\tau)\right\}d\tau$,

$$S_{cl} = \int_0^{\hbar\beta}\left\{-\frac{m}{2}\dot{\underline{x}}_c^2(\tau)\right\}d\tau .$$

Since $\delta S'_{cl} = \int_0^{\hbar\beta} d\tau\, m\dot{\underline{x}}_c\delta\dot{\underline{x}}_c + \int_0^{\hbar\beta} d\tau\, f_1(\tau)\delta\underline{x}_c(\tau)$

$$= -m\dot{\underline{x}}_c\delta\underline{x}_c\Big|_0^{\hbar\beta} + \int_0^{\hbar\beta} d\tau\, m\ddot{\underline{x}}_c\delta\underline{x}_c + \int_0^{\hbar\beta} d\tau\, f_1(\tau)\delta\underline{x}_c = 0,$$

therefore $m\ddot{\underline{x}}_c + f_1(\tau) = 0$,

$$\ddot{\underline{x}}_c = -\frac{f_1(\tau)}{m} = -\frac{i\hbar\underline{k}}{m}\left\{\delta(\tau-\tau_1)-\delta(\tau-\tau_1')\right\} . \quad \text{----}(5.$$

Integrating Eq.(5.10) with respect to τ, we obtain

$$\underline{\dot{r}}_c(\tau) = \underline{\dot{r}}_c(0) - \frac{i\hbar \underline{k}}{m}\left[H(\tau-\tau_1) - H(\tau-\tau_1')\right]. \quad \text{-------}(5.11)$$

Integrating Eq. (5.11) with respect to τ , we obtain

$$\underline{r}_c(\tau) = \underline{r}_c(0) + \underline{\dot{r}}_c(0)\tau - \frac{i\hbar\underline{k}}{m}\Big((\tau-\tau_1)H(\tau-\tau_1) - (\tau-\tau_1')H(\tau-\tau_1')\Big). \quad \text{----}(5.$$

By applying the boundary conditions $\underline{r}_c(0) = \underline{r}'$ and $\underline{r}_c(\hbar\beta) = \underline{r}$, $\underline{r}_c(0)$ and $\underline{\dot{r}}_c(0)$ can be obtained,

$$\underline{r}_c(0) = \underline{r}',$$

$$\underline{\dot{r}}_c(0) = \frac{\underline{r}-\underline{r}'}{\hbar\beta} - \frac{i\underline{k}}{m\beta}(\tau_1-\tau_1').$$

Substituting for $\underline{r}_c(0)$ and $\underline{\dot{r}}_c(0)$ in Eq.(5.12), we obtain

$$\underline{r}_c(\tau) = \underline{r}' + \frac{(\underline{r}-\underline{r}')\tau}{\hbar\beta} - \frac{i\underline{k}(\tau_1-\tau_1')\tau}{m\beta} - \frac{i\hbar\underline{k}}{m}\Big((\tau-\tau_1)H(\tau-\tau_1) - (\tau-\tau_1') H(\tau-\tau_1')$$

Since $$S'_{cl} = -\frac{m}{2}\int_0^{\hbar\beta} d\tau\, \underline{\dot{r}}^2(\tau) + \int_0^{\hbar\beta} d\tau\, \underline{f}_1(\tau)\underline{r}(\tau)$$

$$= -\frac{m}{2}\underline{\dot{r}}\,\underline{r}\Big|_0^{\hbar\beta} + \frac{m}{2}\int_0^{\hbar\beta} d\tau\, \underline{r}\,\underline{\ddot{r}} + \int_0^{\hbar\beta} d\tau\, \underline{f}_1(\tau)\underline{r}(\tau),$$

ans since $$\underline{\ddot{r}}_c(\tau) = -\frac{\underline{f}_1(\tau)}{m},$$

therefore $$S'_{cl} = -\frac{m}{2}\left[\underline{\dot{r}}(\hbar\beta)\underline{r}(\hbar\beta) - \underline{\dot{r}}(0)\underline{r}(0)\right] + \frac{1}{2}\int_0^{\hbar\beta} d\tau\, \underline{f}_1(\tau)\underline{r}(\tau).$$

$$\text{-------}(5.13)$$

Substituting for $\tau = \hbar\beta$ and $\tau = 0$ in Eq.(5.11) and in Eq.(5.12), we obtain

$$\underline{\dot{x}}(\hbar\beta) = \frac{\underline{x}-\underline{x}'}{\hbar\beta} - \frac{i\underline{k}}{m\beta}(\tau_1-\tau_1') = \underline{\dot{x}}(0),$$

$$\underline{x}(\hbar\beta) = \underline{x}, \quad \underline{x}(0) = \underline{x}'.$$

Thus $$\underline{\dot{x}}(\hbar\beta)\underline{x}(\hbar\beta) - \underline{\dot{x}}(0)\underline{x}(0) = \frac{(\underline{x}-\underline{x}')^2}{\hbar\beta} - \frac{i\underline{k}}{m\beta}(\underline{x}-\underline{x}')(\tau_1-\tau_1')$$

$$\text{-----}(5.14)$$

Multiplying Eq.(5.12) by $f_1(\tau)$ and integrating over τ from 0 to $\hbar\beta$, we obtain

$$\int_0^{\hbar\beta} f_1(\tau)\underline{x}(\tau)\,d\tau = \frac{i\underline{k}}{\beta}(\underline{x}-\underline{x}')(\tau_1-\tau_1') - \frac{\hbar^2 k^2}{m\beta}(\tau_1-\tau_1')^2 - \frac{\hbar^2 k^2}{m}(\tau_1-\tau_1')H(\tau_1-\tau_1')$$
$$- \frac{\hbar^2 k^2}{m}(\tau_1'-\tau_1)H(\tau_1'-\tau_1). \qquad \text{-----}(5.15)$$

Substituting for Eq. (5.14) and Eq. (5.15) in Eq.(5.13), we obtain

$$S'_{cl} = -\frac{m}{2}\frac{(\underline{x}-\underline{x}')^2}{\hbar\beta} + \frac{i\underline{k}}{\beta}(\underline{x}-\underline{x}')(\tau_1-\tau_1') + \frac{\hbar k^2}{2m\beta}(\tau_1-\tau_1')^2$$
$$-\frac{\hbar^2 k^2}{2m}(\tau_1-\tau_1')H(\tau_1-\tau_1') - \frac{\hbar^2 k^2}{2m}(\tau_1'-\tau_1)H(\tau_1'-\tau_1).$$

Similarly, we can obtain

$$S_{cl} = \frac{m}{2}\frac{(\underline{x}-\underline{x}')^2}{\hbar\beta}.$$

Differentiating $\exp(\frac{1}{\hbar}S_{cl})$ with respect to $\underline{k}$ two times and then evaluating at $\underline{k}=0$, we obtain

$$\left[-\frac{\partial^2}{\partial \underline{k}^2}\exp\left(\frac{1}{\hbar}S'_{cl}\right)\right]_{\underline{k}=0} = \exp\left\{\frac{1}{\hbar}\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar\beta}\right)\right\}\left[\frac{1}{\hbar^2\beta^2}(\underline{r}-\underline{r}')^2(\tau_1-\tau_1')^2-\frac{(\tau_1-\tau_1')^2}{m\beta}+\frac{\hbar}{m}(\tau_1-\tau_1')H(\tau_1-\tau_1')+\frac{\hbar}{m}(\tau_1'-\tau_1)H(\tau_1'-\tau_1)\right].$$

Thus

$$\frac{\left[-\frac{\partial^2}{\partial \underline{k}^2}\exp\left(\frac{1}{\hbar}S'_{cl}\right)\right]_{\underline{k}=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)} = \left[\frac{1}{\hbar^2\beta^2}(\underline{r}-\underline{r}')^2(\tau_1-\tau_1')^2-\frac{(\tau_1-\tau_1')^2}{m\beta}+\frac{\hbar}{m}(\tau_1-\tau_1')H(\tau_1-\tau_1')+\frac{\hbar}{m}(\tau_1'-\tau_1)H(\tau_1'-\tau_1)\right]. \quad \text{------}(5.16)$$

Integrating Eq.(5.16) over τ_1 and τ_1', we obtain

$$\int_0^{\hbar\beta} d\tau_1 \int_0^{\hbar\beta} d\tau_1' \frac{\left[-\frac{\partial^2}{\partial \underline{k}^2}\exp\left(\frac{1}{\hbar}S'_{cl}\right)\right]_{\underline{k}=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)} = \frac{1}{6}\left\{(\underline{r}-\underline{r}')^2\hbar^2\beta^2+\frac{\hbar^4\beta^3}{m}\right\}. \quad \text{-------}(5.17)$$

Substituting for Eq. (5.17) in Eq.(5.9) , we obtain

$$E\left[\exp\left\{\left(-\frac{m\omega_c^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta}d\tau\,d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_c^2}{4\hbar^2\beta}\right)\frac{1}{6}\left[(\underline{r}-\underline{r}')^2\hbar^2\beta^2+\frac{\hbar^4\beta^3}{m}\right]\right\}. \quad \text{-------}(5.18)$$

In chapter III, we have

$$G_0(\underline{r},\underline{r}';\beta) = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\exp\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\right).$$

Substituting for $G_0(\underline{r},\underline{r}';\beta)$ and for Eq.(5.18) in Eq.(5.6), we obtain

$$\left\langle G(\underline{r},\underline{r}';\beta)\right\rangle = \exp\left(\frac{\eta^2\beta^2}{2}\right)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\exp\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\right)\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\frac{1}{6}\left[(\underline{r}-\underline{r}')^2\hbar^2\beta^2+\frac{\hbar^4\beta^3}{m}\right]\right\}$$

$$= \exp\left(\frac{\eta^2\beta^2}{2}\right)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\exp\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\right)\exp\left\{\left(\frac{\omega_G^2\hbar^2\beta^2}{12}\right)\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}-\frac{1}{2}\right)\right\}$$

$$= \exp\left(\frac{\eta^2\beta^2}{2}\right)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\exp\left\{-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\left(1+\frac{\omega_G^2\hbar^2\beta^2}{12}\right)\right\}\exp\left(-\frac{\omega_G^2\hbar^3\beta^2}{24}\right) \quad \text{.} \quad \text{-------} \quad (5.19)$$

Let us consider the evaluation of Eq.(5.7) by keeping the cumulant up to the second order. Eq.(5.7) thus becomes

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta}d\tau\, d\tau'\left[\underline{r}(\tau)-\underline{r}(\tau')\right]^2\right\}\right] = \exp\left[\sum_{n=1}^{2}\frac{1}{n!}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^n\int d\tau_1\int d\tau_1'\cdots\int d\tau_n\int d\tau_n'\, E_c\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2\cdots\left[\underline{r}(\tau_n)-\underline{r}(\tau_n')\right]^2\right\}\right]$$

$$= \exp\left[\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int d\tau_1\int d\tau_1'\, E_c\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2\right\} + \frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\, E_c\left\{\left[\underline{r}(\tau_1)-\underline{r}(\tau_1')\right]^2\left[\underline{r}(\tau_2)-\right.\right.$$

$$\text{-------} \quad (5.20)$$

The first term in Eq.(5.20) has already been calculated and the result is in the Eq.(5.18) .

Since $$E_c\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\} = E\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\} - E\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\right\}E\left\{\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\},$$

therefore the second term in Eq.(5.20) becomes

$$\exp\left[\frac{1}{2}\left(-\frac{m\omega_0^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\, E_c\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\}\right] = \exp\left[\frac{1}{2}\left(-\frac{m\omega_0^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\, E\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\}\right.$$

$$\left.-\frac{1}{2}\left(-\frac{m\omega_0^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\, E\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\right\}E\left\{\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\}\right]$$

$$= \exp\left[\frac{1}{2}\left(-\frac{m\omega_0^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\left\{\frac{\left[\frac{\partial^2}{\partial\underline{k}^2}\frac{\partial^2}{\partial\underline{k}'^2}\exp\left(\frac{1}{\hbar}S''_{cl}\right)\right]_{\underline{k}=\underline{k}'=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)} - \frac{\left[-\frac{\partial^2}{\partial\underline{k}^2}\exp\left(\frac{1}{\hbar}S'^{(\tau_1)}_{cl}\right)\right]_{\underline{k}=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)}\,\frac{\left[-\frac{\partial^2}{\partial\underline{k}'^2}\exp\left(\frac{1}{\hbar}S'^{(\tau_2)}_{cl}\right)\right]_{\underline{k}'=0}}{\exp\left(\frac{1}{\hbar}S_{cl}\right)}\right\}\right]$$

where $$S''_{cl} = \int_0^{\hbar\beta}\left\{-\frac{m}{2}\dot{\underline{x}}_c^2(\tau) + \left(f_1(\tau)+f_2(\tau)\right)\underline{x}_c(\tau)\right\}d\tau,$$

$$S'_{cl} = \int_0^{\hbar\beta}\left\{-\frac{m}{2}\dot{\underline{x}}_c^2(\tau) + f_1(\tau)\right\}\underline{x}_c(\tau)\,d\tau,$$

$$S_{cl} = \int_0^{\hbar\beta}\left\{-\frac{m}{2}\dot{\underline{x}}_c^2(\tau)\right\}d\tau, \quad f_1(\tau) = i\hbar\underline{k}\left(\delta(\tau-\tau_1)-\delta(\tau-\tau_1')\right), \quad f_2(\tau) = i\hbar\underline{k}'\left(\delta(\tau-\tau_2)-\delta(\tau-\tau_2')\right).$$

$$\exp\left[\frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\; E_c\left\{\left[\underline{x}(\tau_1)-\underline{x}(\tau_1')\right]^2\left[\underline{x}(\tau_2)-\underline{x}(\tau_2')\right]^2\right\}\right]$$

$$= \exp\left[\frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2\int d\tau_1\int d\tau_1'\int d\tau_2\int d\tau_2'\left\{\frac{1}{\hbar^4\beta^4}(\underline{x}-\underline{x}')^4(\tau_2-\tau_2')^2(\tau_1-\tau_1')^2 - \left(\frac{1}{\hbar^2\beta^2}(\underline{x}-\underline{x}')^2(\tau_1-\tau_1')^2 - \frac{(\tau_1-\tau_1')^2}{m\beta} + \frac{\hbar}{m}(\tau_1-\tau_1')H(\tau_1-\tau_1') + \frac{\hbar}{m}(\tau_1'-\tau_1)H(\tau_1'-\tau_1)\right.\right.\right.$$

$$\left.\left.\left. - \left(\frac{1}{\hbar^2\beta^2}(\underline{x}-\underline{x}')^2(\tau_2-\tau_2')^2 - \frac{(\tau_2-\tau_2')^2}{m\beta} + \frac{\hbar}{m}(\tau_2-\tau_2')H(\tau_2-\tau_2') + \frac{\hbar}{m}(\tau_2'-\tau_2)H(\tau_2'-\tau_2)\right)\right\}\right]$$

$$= \exp\left[\frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2\left\{\frac{1}{36}(\underline{x}-\underline{x}')^4\hbar^4\beta^4 - \frac{1}{36}\left[(\underline{x}-\underline{x}')^4\hbar^4\beta^4 + \frac{2\hbar^6\beta^5}{m}(\underline{x}-\underline{x}')^2 - \frac{\hbar^8\beta^6}{m^2}\right]\right\}\right]$$

$$= \exp\left[\frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2\frac{1}{36}\left\{-\frac{2\hbar^6\beta^5}{m}(\underline{x}-\underline{x}')^2 + \frac{\hbar^8\beta^6}{m^2}\right\}\right]. \qquad \text{-------- } (5.21)$$

Substituting for Eq.(5.18) and (5.21) in Eq.(5.20), we obtain

$$E\left[\exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\int_0^{\hbar\beta}\int_0^{\hbar\beta} d\tau\, d\tau'\left[\underline{x}(\tau)-\underline{x}(\tau')\right]^2\right\}\right] = \exp\left\{\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)\frac{1}{6}\left[(\underline{x}-\underline{x}')^2\hbar^2\beta^2 + \frac{\hbar^4\beta^3}{m}\right]\right\}\exp\left\{\frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2\frac{1}{36}\left(-\frac{2\hbar^6\beta^5}{m}(\underline{x}-\underline{x}')^2 + \frac{\hbar^8\beta^6}{m^2}\right)\right\}$$

$$\text{---------- } (5.22)$$

Substituting for G ($\underline{r}, \underline{r}'; \beta$) and Eq.(5.22) in Eq.(5.6), we obtain

$$\langle G(\underline{r}, \underline{r}'; \beta)\rangle = \exp\left(\frac{\eta^2\beta^2}{2}\right)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left\{\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\right)\left(1+\frac{\omega_G^2\hbar^2\beta^2}{12}\right)\right\} \exp\left(-\frac{\omega_G^2\hbar^2\beta^2}{24}\right) \exp\left\{\frac{1}{2}\left(-\frac{m\omega_G^2}{4\hbar^2\beta}\right)^2 \frac{1}{36}\left(-\frac{2\hbar^6\beta^5}{m}(\underline{r}-\underline{r}')^2 + \frac{\hbar^8\beta^6}{m^2}\right)\right\}$$

$$= \exp\left(\frac{\eta^2\beta^2}{2}\right)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left\{\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\right)\left(1+\frac{\omega_G^2\hbar^2\beta^2}{12}\right)\right\} \exp\left(-\frac{\omega_G^2\hbar^2\beta^2}{24}\right) \exp\left\{\left(-\frac{\omega_G^4\hbar^4\beta^4}{288}\right)\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}+\frac{1}{4}\right)\right\}$$

$$= \exp\left(\frac{\eta^2\beta^2}{2}\right)\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left\{\left(-\frac{m(\underline{r}-\underline{r}')^2}{2\hbar^2\beta}\right)\left(1+\frac{\omega_G^2\hbar^2\beta^2}{12}-\frac{\omega_G^4\hbar^4\beta^4}{288}\right)\right\} \exp\left(-\frac{\omega_G^2\hbar^2\beta^2}{24}-\frac{\omega_G^4\hbar^4\beta^4}{1152}\right).$$

Thus, we have evaluated the averaged Green function of disordered systems by keeping the cumulant up to the second order. The correspondence between this result and that of Bezák will be discussed in the next chapter.