CHAPTER II

A PASSAGE FROM $\mathbb{R}^n$ INTO $\mathbb{R}^{n+2}$

In this chapter we construct a passage from any open subset of the unit ball of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$ and study some interesting results on the passage.

2.1 Notation

Let B and ∂B be the open unit ball and the unit sphere in the euclidean n-dimensional space $\mathbb{R}^n$, $n \geqslant 2$. If M and N are in $\mathbb{R}^n$, $\|M-N\|$ denotes the euclidean distance of M from N. We also let $\|x-y\|$ denotes the distance of x from y in $\mathbb{R}^{n+2}$.

Let $E = \{(x_1, \ldots, x_n, 0, 0)/(x_1, \ldots, x_n) \in \partial B\} \subset \mathbb{R}^{n+2}$.

Note that $E = \partial B$ in the sense of 1-1 correspondence,

$$\psi(x_1, \ldots, x_n, 0, 0) = (x_1, \ldots, x_n).$$

For any $x = (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2}$, we assign the number $r > 0$ by letting $r^2 = x_1^2 + \ldots + x_n^2$.

2.2 The passage

2.2.1 Lemma Let x be any point in $\mathbb{R}^{n+2} \setminus E$. Then there exist a function Q from $\mathbb{R}^{n+2} \setminus E$ onto B and a real-valued function λ from $\mathbb{R}^{n+2} \setminus E$ onto $(0,1]$ such that

$$\frac{\|Q(x) - \psi(z)\|}{\|x - z\|} = \lambda^{\frac{1}{2}}(x) \qquad (z \in E).$$

<u>Proof</u> For any $x = (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \setminus E$, define

$$\lambda(x) = \frac{2}{1+\|x\|^2+\sqrt{(1+\|x\|^2)^2 - 4r^2}}$$

where $r^2 = x_1^2 + \ldots + x_n^2$.

Since $1 + \|x\|^2 + \sqrt{(1+\|x\|^2)^2 - 4r^2}$

$$= 1 + \|x\|^2 + \sqrt{1+2(r^2+ x_{n+1}^2 + x_{n+2}^2) + \|x\|^4 - 4r^2}$$

$$\geq 1 + \|x\|^2 + \sqrt{1-2r^2 - 2x_{n+1}^2 - 2x_{n+2}^2 + \|x\|^4}$$

$$= 1 + \|x\|^2 + |1-\|x\|^2| \geq 2.$$

It follows that $\lambda(x) \leq 1$.

Hence λ maps $\mathbb{R}^{n+2} \setminus E$ into $(0,1]$.

Claim that $\lambda r < 1$ and $r^2\lambda^2 - (1+\|x\|^2)\lambda + 1 = 0$ (1)

To show $\lambda r < 1$, we assume $r \neq 0$. (The case $r = 0$ is clear)

Since $(1-r)^2 + x_{n+1}^2 + x_{n+2}^2 + \sqrt{(1+\|x\|^2)^2 - 4r^2} > 0$

$$1+r^2 + x_{n+1}^2 + x_{n+2}^2 + \sqrt{(1+\|x\|^2)^2 - 4r^2} > 2r$$

$$1 + \|x\|^2 + \sqrt{(1+\|x\|^2)^2 - 4r^2} > 2r$$

It follows that $1 > \lambda r$. By replacing $\lambda = 2/(1+\|x\|^2+\sqrt{(1+\|x\|^2)^2 - 4r^2})$

we have $r^2 \lambda^2 - (1+\|x\|^2)\lambda + 1 = 0.$

Now we define the function Q in $\mathbb{R}^{n+2} \setminus E$ by

$$Q(x) = Q(x_1,\dots,x_{n+2}) = (\lambda x_1,\dots,\lambda x_n).$$

The point $Q(x)$ is in B since $\|Q(x)\|^2 = \sum_{i=1}^{n} \lambda^2 x_i^2 = \lambda^2 r^2 < 1.$

Let $z = (\xi_1,\dots,\xi_n,0,0) \in E$. Then $\psi(z) = (\xi_1,\dots,\xi_n)$,

and
$$\frac{\|Q(x)-\psi(z)\|^2}{\|x-z\|^2} = \frac{\sum_{i=1}^{n} (\lambda x_i - \xi_i)^2}{\sum_{i=1}^{n} (x_i-\xi_i)^2 + x_{n+1}^2 + x_{n+2}^2}$$

$$= \frac{\lambda^2 \sum^{n} x_i^2 - 2\lambda \sum^{n} x_i\xi_i + \sum^{n} \xi_i^2}{\sum^{n} x_i^2 - 2\sum^{n} x_i\xi_i + \sum^{n} \xi_i^2 + x_{n+1}^2 + x_{n+2}^2}$$

$$= \frac{\lambda^2 r^2 - 2\lambda \sum^{n} x_i\xi_i + 1}{\|x\|^2 - 2\sum^{n} x_i\xi_i + 1}$$

$$= \frac{(1+\|x\|^2)\lambda - 2\lambda \sum^{n} x_i\xi_i}{\|x\|^2 - 2\sum^{n} x_i\xi_i + 1} \qquad \text{(by (1))}$$

$$= \lambda.$$

The rest of the proof is to show that the function Q is onto B and λ is onto $(0,1]$. First, let $M = (q_1,\dots,q_n) \in B$. The point $x = (q_1,\dots,q_n,0,0) \in \mathbb{R}^{n+2} \setminus E$. Then we have $\lambda(x) = 1$ and $Q(x) = (q_1,\dots,q_n)$. Hence Q is onto B.

For $b \in (0,1]$, let $x = (0,\ldots,0,a) \in \mathbb{R}^{n+2}$ where a satisfies the equation $1 + a^2 = \frac{1}{b}$

Obviously $x \notin E$ and $\lambda(x) == \frac{1}{1+a^2} = b$

This shows that λ is onto $(0,1]$.

The lemma is completely proved ⁎

Moreover λ and Q are continuous on $\mathbb{R}^{n+2} \setminus E$, their partial derivatives exist and continuous for all order, hence we call λ and Q are infinitely differentiable.

We introduce the function f defined by

$$f(x) = \frac{1}{\sigma_n} \int_E \frac{1}{\|x-z\|^n} \, d\sigma(z) \qquad (x \in \mathbb{R}^{n+2} \setminus E)$$

where σ is the surface area measure and σ_n is the surface area of the unit sphere in $\mathbb{R}^n$.

2.2.2 Lemma f is harmonic in $\mathbb{R}^{n+2} \setminus E$.

Proof In stead of f, we will prove that the function g defined by

$$g(x) = \int_E \frac{1}{\|x - z\|^n} \, d\sigma(z)$$

is harmonic in $\mathbb{R}^{n+2} \setminus E$.

We first show that g is finite on $\mathbb{R}^{n+2} \setminus E$.

Let r_o = dist (x,E), distance of x from E $(x \in \mathbb{R}^{n+2} \setminus E)$.

Since E is compact, $r_o > 0$.

Hence for all $z \in E$, $\|x - z\| \geq r_o$ and we have

$$\int_E \frac{1}{\|x-z\|^n} \, d\sigma(z) \leq \int_E \frac{1}{r_o^n} \, d\sigma(z) = \frac{\sigma_n}{r_o^n} < \infty .$$

That is g is finite on E. (1)

Let $x_o \in \mathbb{R}^{n+2} \setminus E$. There is $B(x_o,\delta)$ with $\bar{B}(x_o,\delta) \subset \mathbb{R}^{n+2} \setminus E$.

Let b = distance between E and $\bar{B}(x_o,\delta)$, then $b > 0$.

Choose $x_m \in B(x_o,\delta)$, $m = 1,2,\ldots$ such that $\lim_{m\to\infty} x_m = x_o$.

Then $\|x_m - z\| \geq b$, i.e. $\frac{1}{\|x_m - z\|^n} \leq \frac{1}{b^n}$ $(z \in E)$

Since $\int_E \frac{1}{b^n} \, d\sigma(z) < \infty$, by Lebesgue Dominated Convergence

Theorem (1.1.2) we have

$$\lim_{x\to x_o} \int_E \frac{1}{\|x-z\|^n} \, d\sigma(z) = \lim_{m\to\infty} \int_E \frac{1}{\|x_m-z\|^n} \, d\sigma(z)$$

$$= \int_E \lim_{m\to\infty} \frac{1}{\|x_m-z\|^n} \, d\sigma(z)$$

$$= \int_E \frac{1}{\|x_o-z\|^n} \, d\sigma(z) .$$

That is $\lim_{x \to x_o} g(x) = g(x_o) \qquad (x_o \in \mathbb{R}^{n+2} \setminus E)$.

Hence g is continuous on $\mathbb{R}^{n+2} \setminus E$. (2)

Let $x_o \in \mathbb{R}^{n+2} \setminus E$. For all $\rho < \delta$ where $\bar{B}(x_o;\delta) \quad \mathbb{R}^{n+2} \setminus E$ we have

$$L(g; x_o,\rho) = \frac{1}{\sigma_{n+2}\rho^{n+1}} \int_{\partial B(x_o,\rho)} g(x)d\sigma(x)$$

$$= \frac{1}{\sigma_{n+2}\rho^{n+1}} \int_{\partial B(x_o,\rho)} \int_E \frac{1}{\|x-z\|^n} d\sigma(z)d\sigma(x)$$

$$= \int_E \frac{1}{\sigma_{n+2}\rho^{n+1}} \int_{\partial B(x_o,\rho)} \frac{1}{\|x-z\|^n} d\sigma(x)\, d\sigma(z)$$

where the last equality follows from theorem 1.1.3.

Since the mapping $x \mapsto \frac{1}{\|x-z\|^n}$ is harmonic in $\mathbb{R}^{n+2} \setminus E$

and $\bar{B}(x_o,\rho) \subset \mathbb{R}^{n+2} \setminus E$, by theorem 1.2.3

$$\frac{1}{\sigma_{n+2}\rho^{n+1}} \int_{\partial B(x_o,\rho)} \frac{1}{\|x-z\|^n} d\sigma(z) = \frac{1}{\|x_o-z\|^n}$$

So we obtain

$$L(g;x_o,\rho) = \int_E \frac{1}{\|x_o-z\|^n} d\sigma(z) = g(x_o) \qquad (3)$$

From (1),(2),(3) and theorem 1.2.6, g is harmonic in $\mathbb{R}^{n+2} \setminus E$.

It follows immediately that f is harmonic in $\mathbb{R}^{n+2} \setminus E$ #

We call f the harmonic function associated with E .

2.2.3 <u>Lemma</u> If f is the harmonic function associated with E,

then
$$f(x) = \frac{\lambda^{n/2}(x)}{1-\|Q(x)\|^2} \qquad (x \in \mathbb{R}^{n+2} \setminus E)$$

where λ and Q are the functions given by lemma 2.2.1.

<u>Proof</u> Let $x \in \mathbb{R}^{n+2} \setminus E$. According to lemma 2.2.1,

$$\|Q(x) - \psi(z)\| = \|x - z\|\lambda^{1/2}(x) \qquad (z \in E).$$

Then
$$f(x) = \frac{1}{\sigma_n} \int_E \frac{1}{\|x-z\|^n} \, d\sigma(z)$$

$$= \frac{1}{\sigma_n} \int_E \frac{\lambda^{n/2}(x)}{\|Q(x)-\psi(z)\|^n} \, d\sigma(\psi(z))$$

$$= \frac{\lambda^{n/2}(x)}{1-\|Q(x)\|^2} \frac{1}{\sigma_n} \int_{\partial B} \frac{1-\|Q(x)\|^2}{\|Q(x)-N\|^n} \, d\sigma(N) \quad .$$

By theorem 1.2.4 (Poisson integral formula),

$$\frac{1}{\sigma_n} \int_{\partial B} \frac{1-\|Q(x)\|^2}{\|Q(x)-N\|^n} \, d\sigma(N) = 1$$

Hence the last equation becomes $f(x) = \dfrac{\lambda^{n/2}(x)}{1-\|Q(x)\|^2}$ #

2.2.4 <u>Lemma</u> If λ is the function as in lemma 2.2.1, $\gamma = r^2$

then
$$4\lambda \frac{\partial\lambda}{\partial\gamma} + |\nabla\lambda|^2 = 0$$

where ∇ denotes the (n+2)-dimensional gradient vector operator.

<u>Proof</u> Let $\gamma = x_1^2 + \ldots + x_n^2$, $\tau = x_{n+1}^2 + x_{n+2}^2$

Since $\frac{\partial\lambda}{\partial x_i}(\gamma , \tau) = \frac{\partial\lambda}{\partial\gamma}\frac{\partial\gamma}{\partial x_i} + \frac{\partial\lambda}{\partial\tau}\frac{\partial\tau}{\partial x_i}$

$$= \begin{cases} 2x_i \frac{\partial\lambda}{\partial\gamma} & (1 \leqslant i \leqslant n) \\ 2x_j \frac{\partial\lambda}{\partial\tau} & (j = n+1, n+2) \end{cases}$$

$$\nabla\lambda = (2x_1\frac{\partial\lambda}{\partial\gamma}, \ldots, 2x_n\frac{\partial\lambda}{\partial\gamma}, 2x_{n+1}\frac{\partial\lambda}{\partial\tau}, 2x_{n+2}\frac{\partial\lambda}{\partial\tau})$$

Then $|\nabla\lambda|^2 = 4\gamma \left(\frac{\partial\lambda}{\partial\gamma}\right)^2 + 4\tau \left(\frac{\partial\lambda}{\partial\gamma}\right)^2$ (1)

Return to the proof of lemma 2.2.1, λ satisfies

$$\gamma\lambda^2 < 1 \text{ and } \gamma\lambda^2 - [1 + \gamma + \tau]\lambda + 1 = 0 \qquad (2)$$

Thus $\tau = \gamma\lambda - 1 - \gamma + \frac{1}{\lambda}$.

By differentiating (2) with respect to γ and τ we have

$$\frac{\partial\lambda}{\partial\gamma} = \frac{\lambda^2(1-\lambda)}{\gamma\lambda^2-1} \text{ and } \frac{\partial\lambda}{\partial\tau} = \frac{\lambda^2}{\gamma\lambda^2-1}$$

It is easy to compute $\lambda\frac{\partial\lambda}{\partial\gamma} + \gamma\left(\frac{\partial\lambda}{\partial\gamma}\right)^2 + \tau\left(\frac{\partial\lambda}{\partial\tau}\right)^2 = 0$

Then by (1), $4\lambda\frac{\partial\lambda}{\partial\gamma} + |\nabla\lambda|^2 = 0$ #

2.2.5 <u>Lemma</u> Let $Q = (q_1, \ldots, q_n)$ be the function defined in lemma 2.2.1, that is $q_i(x) = x_i\lambda(x)$ $(i = 1, \ldots, n)$.

Then $|\nabla q_i| = \lambda$, $\nabla q_i \cdot \nabla q_j = 0$ $(1 \leqslant i,j \leqslant n \text{ and } i \neq j)$

where ∇ is the (n+2)-dimensional gradient vector.

Proof Let $x = (x_1,\dots,x_{n+2}) \in R^{n+2} \setminus E$.

Since $$\nabla q_i = \left(\frac{\partial q_i}{\partial x_1},\dots,\frac{\partial q_i}{\partial x_{n+2}}\right) \qquad (i = 1,2,\dots,n).$$

$$= \left(x_i \frac{\partial\lambda}{\partial x_1},\dots, \lambda + x_i \frac{\partial\lambda}{\partial x_i},\dots, x_i \frac{\partial\lambda}{\partial x_{n+2}}\right)$$

Then $$|\nabla q_i|^2 = x_i^2\left[\left(\frac{\partial\lambda}{\partial x_1}\right)^2+\dots+\left(\frac{\partial\lambda}{\partial x_{n+2}}\right)^2\right] + \lambda^2 + 2\lambda\, x_i \frac{\partial\lambda}{\partial x_i}$$

$$= x_i^2\, |\nabla\lambda|^2 + \lambda^2 + 2\lambda x_i\, 2x_i \frac{\partial\lambda}{\partial\gamma}$$

$$= \left(|\nabla\lambda|^2 + 4\lambda \frac{\partial\lambda}{\partial\gamma}\right) x_i^2 + \lambda^2 \quad .$$

By lemma 2.2.4, $|\nabla q_i| = \lambda$.

The remainder of the proof is directly calculated and by lemma 2.2.4 again we have

$$\nabla q_i \cdot \nabla q_j = x_i x_j \sum_{k=1}^{n+2} \left(\frac{\partial\lambda}{\partial x_k}\right)^2 + \lambda x_j \frac{\partial\lambda}{\partial x_i} + \lambda x_i \frac{\partial\lambda}{\partial x_j}$$

$$= x_i x_j\, |\nabla\lambda|^2 + \lambda x_j\, 2x_i \frac{\partial\lambda}{\partial\gamma} + x_i \lambda 2 x_j \frac{\partial\lambda}{\partial\gamma}$$

$$= x_i x_j \left(|\nabla\lambda|^2 + 4\lambda \frac{\partial\lambda}{\partial\gamma}\right) = 0 \qquad \#$$

2.2.6 Lemma If f is the harmonic function associated with E, $Q = (q_1,\dots,q_n)$ is the function as in lemma 2.2.1, then the functions $q_i f$, $i = 1,2,\dots,n$, defined by

$$(q_i f)(x) \quad = \quad q_i(x)\ f(x) \qquad\qquad (x \in \mathbb{R}^{n+2} \setminus E)$$

are harmonic in $\mathbb{R}^{n+2} \setminus E$ and satisfy

$$f \overset{*}{\Delta} q_i + 2(\nabla f \cdot \nabla q_i) \quad = \quad 0$$

where $\overset{*}{\Delta}$ is the (n+2)-dimensional Laplacian operator.

<u>Proof</u> For i = 1,...,n the real-valued function ϕ_i defined by $\phi_i(\zeta_1,\ldots,\zeta_n) = \zeta_i$ is harmonic and $Q(x) \in B$, hence by theorem 1.2.4 (Poisson Integral Formula)

$$\phi_i(Q(x)) \quad = \quad \frac{1}{\sigma_n} \int_{\partial B} \frac{1 - \|Q(x)\|^2}{\|Q(x) - N\|^n}\ \phi_i(N)\ d\sigma(N)\ .$$

Since $\phi_i(Q(x)) \quad = \quad q_i(x)$ and $f(x) \quad = \quad \dfrac{\lambda^{n/2}(x)}{1 - \|Q(x)\|^2}$, the last equation becomes

$$q_i(x)\ f(x) = \frac{1}{\sigma_n} \int_{\partial B} \frac{\lambda^{n/2}(x)}{\|Q(x) - N\|^n}\ \phi_i(N)\ d\sigma(N)\ .$$

By lemma 2.2.1,

$$(q_i f)(x) \quad = \frac{1}{\sigma_n} \int_E \frac{1}{\|x - z\|^n}\ \phi_i(\psi(z))\ d\sigma(z)\ .$$

Let μ be a measure defined by $\mu(F) = \int_F \phi_i(\psi(z))d\sigma(z)$ for all Borel set $F \subset E$. Then μ is a signed measure of bounded variation by Theorem 1.1.1.

Then $(q_i f)(x) = \frac{1}{\sigma_n} \int_E \frac{1}{\|x-z\|^n} \, d\mu(z)$.

Hence the functions $q_i f$ are harmonic in $\mathbb{R}^{n+2} \setminus E$ $(i = 1, \ldots, n)$.

We complete the proof by showing that $f \overset{*}{\Delta} q_i + 2(\nabla f \cdot \nabla q_i) = 0$. Since f and $q_i f$ are harmonic, $\Delta^*(q_i f) = \Delta^* f = 0$. By the fact that

$$\Delta^* (f q_i) = f \, \Delta^* q_i + 2(\nabla f \cdot \nabla q_i) + q_i \Delta^* f$$

Therefore $\quad 0 = f \Delta^* q_i + 2(\nabla f \cdot \nabla q_i)$ #

2.3 Superharmonicity on the passage

In this section we shall study superharmonic properties on the passage from any open subset of the unit ball of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$.

2.3.1 Theorem If u is superharmonic in B and let

$$u^*(x) = f(x)u(Q(x)) \qquad (x \in \mathbb{R}^{n+2} \setminus E)$$

where f is the harmonic function associated with E and Q is the function defined as in lemma 2.2.1, then u^* is superharmonic in $\mathbb{R}^{n+2} \setminus E$.

In particular if u is harmonic in B, then u^* is harmonic in $\mathbb{R}^{n+2} \setminus E$.

The following theorem shows that theorem 2.3.1 has a generalization.

Let ω denote an arbitrary open subset of B, Ω denote the inverse image of ω under Q, i.e. $\Omega = Q^{-1}[\omega]$. Since Q is continuous, Ω is also open. In the case $\omega = B$, we have $\Omega = \mathbb{R}^{n+2} \setminus E$.

2.3.2 <u>Theorem</u> If u is superharmonic in ω and

$$u^*(x) = f(x)u(Q(x)) \qquad (x \in \Omega),$$

then u^* is superharmonic in Ω. In particular if u is harmonic in ω, then u^* is harmonic in Ω.

To prove theorem 2.3.2 and hence theorem 2.3.1, we need the following theorem.

2.3.3 <u>Theorem</u> Let u be a function having continuous second partial derivatives thereon in ω and

$$u^*(x) = f(x)u(Q(x)) \qquad (x \in \Omega)$$

where f is the harmonic function associated with E and Q is the function defined as in lemma 2.2.1.

Then $$\Delta^* u^*(x) = \lambda^2(x)f(x)\Delta u(Q(x))$$

where Δ^* and Δ denote the (n+2)- and n-dimensional Laplacians.

<u>Proof</u> Since f is harmonic and $u^* = fu(Q)$ or $f(u \circ Q)$,

$$\begin{aligned}\Delta^* u^* &= f\Delta^*(u \circ Q) + 2(\nabla f \cdot \nabla(u \circ Q)) + (u \circ Q)\Delta^* f \\ &= f\Delta^*(u \circ Q) + 2(\nabla f \cdot \nabla(u \circ Q)).\end{aligned}$$

Since further $\nabla f \cdot \nabla(u \circ Q) = \sum_{j=1}^{n} (\nabla f \cdot \nabla q_j)\frac{\partial u}{\partial q_j}$ where $Q = (q_1, \ldots, q_n)$,

we get $\Delta^* u^* = f\,\Delta^*(u \circ Q) + 2 \sum_{j=1}^{n} (\nabla f \cdot \nabla q_j)\frac{\partial u}{\partial q_j}$ (1)

Next we claim that $\Delta^*(u \circ Q) = \sum_{j=1}^{n} \frac{\partial u}{\partial q_j} \Delta^* q_j + \lambda^2 \Delta u.$

Infact, since

$$\frac{\partial^2}{\partial x_i^2}(u \circ Q) = \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial x_i} \qquad (i = 1, \ldots, n+2)$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial q_j} \cdot \frac{\partial^2 q_j}{\partial x_i^2} + \sum_{j=1}^{n} \frac{\partial q_j}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial q_j}\right)$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\frac{\partial^2 q_j}{\partial x_i^2} + \sum_{j=1}^{n}\left[\frac{\partial q_j}{\partial x_i} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial q_k \partial q_j}\frac{\partial q_k}{\partial x_i}\right],$$

we get $\Delta^*(u \circ Q) = \sum_{i=1}^{n+2} \frac{\partial^2}{\partial x_i^2}(u \circ Q)$

$$= \sum_{i=1}^{n+2}\sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\frac{\partial^2 q_j}{\partial x_i^2} + \sum_{i=1}^{n+2}\sum_{j=1}^{n}\left[\frac{\partial q_j}{\partial x_i} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial q_k \partial q_j} \cdot \frac{\partial q_k}{\partial x_i}\right]$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\Delta^* q_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial q_k \partial q_j}\left[\sum_{i=1}^{n+2} \frac{\partial q_j}{\partial x_i} \cdot \frac{\partial q_k}{\partial x_i}\right]$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\Delta^* q_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial q_k \partial q_j}(\nabla q_j \cdot \nabla q_k)$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\Delta^* q_j + \frac{\partial^2 u}{\partial q_1^2}\lambda^2 + \ldots + \frac{\partial^2 u}{\partial q_k^2}\lambda^2 \qquad \text{(see 2.2.5)}$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial q_j}\Delta^* q_j + \lambda^2 \Delta u.$$

Therefore the equation (1) becomes

$$\Delta^* u^* = f\left[\sum_{j=1}^{n} \frac{\partial u}{\partial q_j} \Delta^* q_j + \lambda^2 \Delta u\right] + 2\sum_{j=1}^{n} (\nabla f \cdot \nabla q_j)\frac{\partial u}{\partial q_j}$$

$$= \lambda^2 f \, \Delta u + \sum_{j=1}^{n} \left[f\Delta^* q_j + 2(\nabla f \cdot \nabla q_j)\right] \frac{\partial u}{\partial q_j} \quad .$$

By lemma 2.2.6, $\Delta^* u^* = \lambda^2 f \, \Delta u$ #

We end this chapter with the proof of theorem 2.3.2.

In the particular case, if u is harmonic in ω, then $\Delta u = 0$. By theorem 2.3.3, $\Delta^* u^* = 0$. Since u and f have continuous second partials thereon, u^* also has continuous second partials. Hence u^* is harmonic in Ω.

Theorem 2.3.3 also implies theorem 2.3.2 in the case where u is superharmonic and has continuous second partials on ω. By theorem 1.3.5, $\Delta u \leq 0$.
Since $\lambda^2 f > 0$, we have $\Delta^* u^* \leq 0$.
Using theorem 1.3.5 again, u^* is superharmonic in Ω.

To complete the proof of theorem 2.3.2, let $a \in \Omega$. Then $Q(a) \in \omega$. Since ω is open, there is an open ball $B(Q(a), \rho)$ with compact closure $\bar{B}(Q(a), \rho) \subset \omega$. Let $\mathcal{N} = Q^{-1}\left[B(Q(a), \rho)\right]$. $\mathcal{N}$ is an open neighbourhood of a since Q is continuous.
By theorem 1.3.9, it suffices to prove that u is superharmonic in $\mathcal{N}$.

Since u is superharmonic in ω and $\bar{B}(Q(a), \rho) \subset \omega$, by theorem 1.3.8 there is an increasing sequence $\{u_j\}$ of superharmonic functions having continuous second partials such that

$$u = \lim_{j \to \infty} u_j \qquad \text{on } B(Q(a),\rho)$$

For each $j = 1,2,\ldots,,$ let

$$u_j^*(x) = f(x)u_j(Q(x)) \qquad (x \in \mathcal{N}).$$

By theorem 2.3.3, $\Delta^* u_j^*(x) = \lambda^2(x)f(x)\Delta u_j(Q(x))$.

Since u_j is superharmonic, $\Delta u_j \leq 0$. Hence $\Delta^* u_j^* \leq 0$.

By theorem 1.3.5, u_j^* is superharmonic in $\mathcal{N}$.

Since

$$\lim_{j\to\infty} u_j^*(x) = \lim_{j\to\infty} f(x)u_j(Q(x))$$

$$= f(x)u(Q(x)) = u^*(x)$$

and $f > 0$, then u^* is the limit of an increasing sequence $\{u_j^*\}$ of superharmonic functions in $\mathcal{N}$.

By theorem 1.3.4, u^* is either superharmonic or $u^* = \infty$ on each component of $\mathcal{N}$.

Suppose $u^* = \infty$ on a component C of $\mathcal{N}$.

Then $u^*(x) = f(x)u(Q(x))$ and f is finite yield

$u = \infty$ on $Q[C]$ and $\int_{Q[C]} u(M)dM = \infty$.

This is impossible since u is superharmonic in ω and

and $Q[C] \subset \bar{B}(Q(a),\rho) \subset \omega$,

by theorem 1.3.7 $\int_{\bar{B}(Q(a),\rho)} u(M)dM < \infty$.

Hence u^* is superharmonic in $\mathcal{N}$ #