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CHAPTER I

PRELIMINARIES

In this chapter we collect some definitions and theorems used in the later chapters of this thesis. The proofs of the theorems can be found in references [1], [2], [3], [4] and [5].

1.1 Measure and Integral

Suppose X is a locally compact Hausdorff space. The class $\mathcal{B}$ of Borel subsets of X is the smallest family of subsets of X such that

(i) if $B \in \mathcal{B}$, then $X - B \in \mathcal{B}$

(ii) if $B_i \in \mathcal{B}$ $(i = 1,2,3,\ldots)$, then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$

and (iii) if G is an open subsets of X, then $G \in \mathcal{B}$.

A non-negative extended real-valued function μ defined on $\mathcal{B}$ is called a Borel measure if

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$ for any sequence B_i of disjoint sets in $\mathcal{B}$.

If further $\mu(C) < \infty$ for all compact $C \subset X$, then μ is called a Radon measure.

หอสมุดกลาง จุฬาลงกรณ์มหาวิทยาลัย

A function $\psi : X \to \tilde{\mathbb{R}}$ is said to be Borel measurable or simply measurable if for any $\alpha \in \mathbb{R}$, the set $\{x \in X / \psi(x) > \alpha\} \in \mathcal{B}$. Note that a continuous function is measurable. (see [1], page 12)

For measurable function ψ we define its integral with respect to a measure μ as follows :

First suppose ψ is non-negative on X. If s is a measurable simple function on X of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s and F is a Borel subset of X, then we define

$$\int_F s \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap F) .$$

The integral of ψ over F with respect to μ is defined by

$$\int_F \psi \, d\mu = \sup \int_F s \, d\mu ,$$

the supremum being taken over all measurable simple functions s such that $0 \leq s \leq \psi$. The left member is called the Lebesgue integral of ψ over F with respect to μ . The integral may have the value ∞.

If $\psi : X \to \tilde{\mathbb{R}} = [-\infty, \infty]$, we define $\psi_+(x) = \max\{\psi(x), 0\}$ and $\psi_-(x) = -\min\{\psi(x), 0\}$. Then we have ψ_+ and ψ_- are non-negative measurable functions, so the integral of each with respect to μ is defined. If not both $\int \psi_+ \, d\mu$ and $\int \psi_- \, d\mu$ have the value ∞, we define

$$\int \psi d\mu = \int \psi_+ \, d\mu - \int \psi_- \, d\mu \quad .$$

If both are finite, we say that ψ is integrable with respect to μ.

For any real-valued function ψ continuous and having compact support on $\mathbb{R}^n$, the Riemann integral $\int_{\mathbb{R}^n} \psi(x) \, dx$ is well-defined. The unique measure ν such that $\int_{\mathbb{R}^n} \psi(x)dx = \int_{\mathbb{R}^n} \psi(x)d\nu(x)$ is called the <u>Lebesgue measure</u> on $\mathbb{R}^n$ (see [1], page 50). We simply denote the integral $\int f(x)d\,\nu(x)$ by $\int f(x)dx$.

An extended real-valued function μ on the class of all Borel subsets of X is called a <u>signed measure</u> if

(i) $\mu(\emptyset) = 0$

(ii) μ takes at most one of the values $+\infty$, $-\infty$.

(iii) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for any sequence E_i of disjoint Borel sets in X provided that the series in the right hand side absolutely converges.

1.1.1 <u>Theorem</u> Suppose ψ is non-negative measurable function and

$$\nu(F) = \int_F \psi \, d\mu$$

where F is a Borel subset of X.

Then ν is a measure on the class of Borel sets of X, and

$$\int_X h \, d\nu = \int_X h \, \psi \, d\mu$$

for every non-negative measurable h on X.

If   is integrable over X with respect to a Radon measure μ and $\nu(F) = \int_F \psi \, d\mu$, then ν is a signed measure of bounded variation. (see the proof in [1], page 23)

1.1.2 Theorem (Lebesgue Dominated Convergence Theorem). Suppose $\{\psi_n\}$ is a sequence of measurable functions on X such that $\lim_{n\to\infty} \psi_n(x)$ exists for every $x \in X$. If there is an integrable function h such that

$$|\psi_n(x)| \leq |h(x)| \qquad (n = 1,2,\ldots,\ x \in X),$$

then $\lim_{n\to\infty} \int_X \psi_n \, d\mu = \int_X \lim_{n\to\infty} \psi_n \, d\mu$. (see [1], page 26)

1.1.3 Theorem (Tonelli). Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two sigma-finite measure space, and ψ be a non-negative measurable function on X × Y. Then

$$\int_X \int_Y \psi \, d\nu d\mu = \int_Y \int_X \psi \, d\mu \, d\nu . \qquad \text{(see [2], page 270)}$$

We now introduce a measure as defined in Helms [4], page 4. Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\rho > 0$, then $\partial B(y,\rho)$ is the sphere defined by the equation

$$(x_1 - y_1)^2 + \ldots . + (x_n - y_n)^2 = \rho^2$$

Consider a Berel set $M \subset \partial B(y,\rho) \cap \{(x_1,\ldots,x_n)/x_n - y_n \geq 0\}$

Let M_n denote the projection of M onto the subspace $\{(x_1,\ldots,x_n)/x_n = 0\}$;

That is $M_n = \{(x_1,\ldots, x_{n-1},0)/(x_1,\ldots,x_n) \in M\}$.

For each $x \in \partial B(y,\rho)$ let $\gamma = \gamma(x)$ be the angle between the x_n-axis and the outer normal to $\partial B(y,\rho)$ at x.

Then $$\sec\gamma = \frac{1}{\cos\gamma} = \frac{\rho}{x_n - y_n}$$

and $$\sigma(M) = \int_{M_n}\cdots\int \sec\gamma \, dx_1\ldots dx_{n-1}$$

represents the surface area of M.

If $M \subset \partial B(y,\rho) \cap \{(x_1,\ldots,x_n)/x_n - y_n \leq 0\}$, the surface area of M is given by the same integral with $\sec\gamma = -\rho/(x_n - y_n)$. The set function σ defined in this way is a measure, call it the <u>surface area measure</u> on $\partial B(y,\rho)$

1.2 Harmonic Functions

A real-valued function h is said to be <u>harmonic</u> in an open set $D \subset \mathbb{R}^n$ if h has continuous partial derivatives up to second order and $\Delta h = 0$ on D where $\Delta h = \frac{\partial^2 h}{\partial x_1^2} + \cdots + \frac{\partial^2 h}{\partial x_n^2}$ is the Laplacian of h.

If u is a function integrable relative to the surface area measure σ on the boundary $\partial B(y,\rho)$, define

$$L(u;\, y,\rho) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B(y,\rho)} u \, d\sigma \qquad (1.2.1)$$

where σ_n is the surface area of the unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$.

If u is integrable on $B(y,\rho)$ relative to Lebesgue measure ν, define

$$A(u;\, y,\rho) = \frac{1}{\nu_n \rho^n} \int_{B(y,\rho)} u \, dx \qquad (1.2.2)$$

where ν_n is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^n$, i.e. $\nu_n = \nu(B(y,\rho))$.

1.2.3 <u>Theorem</u> If h is harmonic in the open set $D \subset \mathbb{R}^n$, then

$$h(y) = L(h;\, y,\rho) = A(h;\, y,\rho)$$

whenever the compact closure $\bar{B}(y,\rho) \subset D$. (see [4], page 12)

The preceeding theorem can be improved upon by showing that the value of a harmonic function h at an interior point of a ball is a certain weighted average of h over the boundary of the ball.

1.2.4 <u>Theorem</u> (Poisson Integral Formula). If h is harmonic in a neighbourhood of the closure $\bar{B}(y,\rho)$, then for $x \in B(y,\rho)$

$$h(x) = \frac{1}{\sigma_n \rho} \int_{\partial B(y,\rho)} \frac{\rho^2 - \|y-x\|^2}{\|z-x\|^n} h(z)\, d\sigma(z). \qquad ([4], \text{page } 16)$$

1.2.5 <u>Theorem</u> (Picard) If h is a harmonic function in $\mathbb{R}^n$ and either bounded above or bounded below, then h is constant. ([4], page 18)

According to theorem 1.2.3, harmonic functions are mean-valued. A partial converse is available.

1.2.6 Theorem Let u be continuous on the open set D. If for all $x \in D$, $u(x) = L(u; x,\delta)$ for sufficiently small δ, then u is harmonic on D. ([4], page 25)

1.3 Superharmonic Functions

Let u be an extended real-valued function with domain $D \subset \mathbb{R}^n$. The function u is lower semicontinuous (ℓ.s.c) on D if for each $x \in D$, $u(x) = \liminf_{y \to x} u(y)$.

If u is ℓ.s.c and $u > -\infty$ on a compact set, then u is bounded below. ([3], page 111) (1.3.1)

1.3.2 Definition Let D be an open subset of $\mathbb{R}^n$. An extended real-valued function u on D is said to be superharmonic if

(i) u is not identically $+\infty$ on any component of D

(ii) $u > -\infty$ on D

(iii) u is ℓ.s.c. on D

and (iv) for all $x \in D$, $\rho > 0$ with compact closure $\bar{B}(x;\rho) \subset D$,

$$u(x) \geq L(u; x,\rho) .$$

The following properties are drawn from Helms [4] (pages 59, 64, 65, 66, 68).

1.3.3 Theorem If u is superharmonic in a bounded open set D and $\liminf_{z \to x} u(z) \geq 0$ $(x \in \partial D)$, then $u \geq 0$ on D.

1.3.4 Theorem If $\{u_j\}$ is an increasing sequence of seper harmonic functions in an open set D, then on each component of D, $u = \lim_{j\to\infty} u_j$ is either superharmonic or identically $+\infty$.

1.3.5 Theorem Let u be a function defined on an open set $D \subset \mathbb{R}^n$ having continuous second partials. Then u is superharmonic in D if and only if $\Delta u \leq 0$ on D.

1.3.6 Theorem Let D be an open subset of $\mathbb{R}^n$. If u is an extended real-valued function on D satisfying (i),(ii),(iii) in definition 1.3.2 and for each $x \in D$, there is $\delta_x > 0$ such that $B(x; \delta_x) \subset D$ and $u(x) \geq A(u;x,\delta)$ whenever $\delta < \delta_x$, then u is superharmonic in D.

Moreover if u is superharmonic in D and $x \in D$, then $u(x) \geq A(u;x,\delta)$ whenever $B(x;\delta) \subset D$.

1.3.7 Theorem If u is superharmonic in an open set D, then u is finite almost everywhere on D relative to Lebesgue measure and u is Lebesgue integrable on each compact set $K \subset D$, i.e. $-\infty < \int_K u(z)\, dz < \infty$.

1.3.8 Theorem Let u be superharmonic in an open set D and let V be an open set with compact closure $\overline{V} \subset D$.
Then there is an increasing sequence $\{u_j\}$ of superharmonic functions in V having continuous second partials such that

$$u = \lim_{j\to\infty} u_j \quad \text{on } V. \qquad \#$$

The following theorem can be proved by using the theorem 1.3.6.

1.3.9 Theorem Let u be an extended real-valued function on an open set $D \subset \mathbb{R}^n$. If for every $x \in D$, there is an open neighbourhood N_x of x such that u is superharmonic in N_x, then u is superharmonic in D.

1.4 Superharmonic Extensions

An extended real-valued function u is said to be locally bounded below on an open set D if for $x \in D$

$$\lim_{y \to x} \inf u(y) > -\infty . \tag{1.4.1}$$

i.e. for $x \in D$, there are a neighbourhood of x, $k \in \mathbb{R}$ such that $u(y) \geq k \quad (y \in N_x)$.

A set F is said to be a polar set if there are an open set W containing F and a superharmonic function u in W such that $u = +\infty$ on F. (1.4.2)

Example 1 The singleton set $\{x_o\}$ of a point in $\mathbb{R}^n$ is a polar set since the function u_{x_o} defined by

$$u_{x_o}(y) = \begin{cases} +\infty & \text{if } y = x_o \\ -\log \|x_o - y\| & \text{if } y \neq x_o \end{cases} \qquad (n = 2)$$

$$u_{x_o}(y) = \begin{cases} +\infty & \text{if } y = x_o \\ \dfrac{1}{\|x_o - y\|^{n-2}} & \text{if } y \neq x_o \end{cases} \qquad (n \geq 3)$$

is superharmonic in $\mathbb{R}^n$.

Example 2 A line segment in $\mathbb{R}^3$ is a polar set. Consider, for example, the line segment I joining (0,0,0) to (1,0,0) and one-dimensional Lebesgue measure μ on this segment. The function v defined by

$$v(x) = \int_I \frac{1}{\|x-z\|} \, d\mu(z) \qquad (x \in \mathbb{R}^3)$$

is superharmonic in $\mathbb{R}^3$. If $x = (x_1,0,0) \in I$, i.e. $0 \leq x_1 \leq 1$,

$$v(x) = \int_0^1 \frac{1}{|x_1 - z_1|} \, dz_1 = \infty .$$

Hence I is a polar set.

1.4.3 Properties of polar sets

(i) Any subset of a polar set is a polar set.

(ii) If $F \subset \mathbb{R}^n$ is a polar set, then F has n-dimensional Lebesgue measure zero. ([4], page 127)

Now we take up the problem of continuation of a superharmonic function across a polar set .

1.4.4 <u>Theorem</u> Let W be an open set and let F be relatively closed polar subset of W. If u is superharmonic in $W \setminus F$ and locally bounded below on W, then it has a unique superharmonic extension to W. ([4], page 130)