CHAPTER V

ON THE ABELIAN GROUP $C_2 \times C_2 \ldots \times C_2$ (N TIMES)

1 Introduction.

Let the abelian group $V_n = C_2 \times C_2 \times \ldots \times C_2$ (n times). It is obvious to see that the order of the abelian group of V_n is 2^n and each non-trivial element of V_n is of order 2. We can let the abelian group V_n be as follows:

$$V_n = \{1, a_1, a_2 \ldots, a_{2^n-1}\}.$$

Then we have

$$a_1^2 = a_2^2 = \cdots a_{2^n-1}^2 = 1.$$

Moreover, the abelian group V_n can be expressed as the irredundant union of subgroups $A_1, A_2, \ldots, A_{2^n-1}$ where

$$A_i = \{1, a_i\}$$

for $\qquad i = 1, 2, \ldots 2^n-1$.

We will consider groups which are homomorphic preimages of the abelian group V_n.

2 Groups Which can be Mapped Homomorphically onto the abelian group V_n.

2.1 Theorem. If a group G can be mapped homomorphically onto the abelian group V_n, then G is an irredundant union of 2^n-1 subgroups, Proof. Let φ be a mapping such that

$$\varphi : \quad G \longrightarrow V_n$$

is an onto homomorphism. Set

$$G_i \quad = \quad \varphi^{-1}(a_i) \cup \varphi^{-1}(1)$$

for $i = 1, 2, \ldots, 2^n-1$. It is easy to see that G_i is a subgroup of G for all i in $1, 2, \ldots, 2^n-1$ and

$$G \quad = \quad \bigcup_{i=1}^{2^n-1} G_i \quad .$$

Finally, if for some $j = 1,2,\ldots n$, $G_j \subset \bigcup_{\substack{k=1 \\ k \neq j}}^{2^n-1} G_k$, then for any y in $G_j \setminus \varphi^{-1}(1)$, y belongs to G_k for some $k = 1,2,\ldots n$ and $k \neq j$, so that

$$1 \neq a_j = \varphi(y) \in \{1, a_k\},$$

which is impossible. Hence the union is irredundant.

Moreover, it is easy to see that the subgroups G_i's which we have constructed, have the following properties:

$$G_i \cap G_j = \varphi^{-1}(1)$$

for all i, j in $1, 2, \ldots 2^n-1$, and $i \neq j$.

In case $n = 2$, we have proved that a group is a 3-group if and only if it can be mapped homomorphically onto the Klein 4-group V which is the abelian group $V_2 = C_2 \times C_2$. However, whether the converse of the Theorem 2.1 holds or not in general has not been solved. But we can prove a partial converse as follows:

2.2 Theorem. Let a group $G = \bigcup_{i=1}^{2^n-1} G_i$ be the irredundant union of its subgroups G_i with the additional properties:

(i) $K = \bigcap_{i=1}^{2^n-1} G_i$ is a normal subgroup of G,

(ii) there exist $g_1, g_2, \dots g_n$ in G such that g_i^2 is in K and $g_i g_j = g_j g_i$ for all i, j in $\{1, 2, \dots n\}$, and

$$\{G_1, G_2, \dots G_{2^n-1}\} = \{A_{i_1 i_2 \dots i_j} / 1 \leqslant i_1 < i_2 \dots < i_j \leqslant n,\ j = 1, 2, \dots n\}$$

where $A_{i_1 i_2 \dots i_j} = [K \cup \{g_{i_1} g_{i_2} \dots g_{i_j}\}]$,

$$1 \leqslant i_1 < i_2 < \dots < i_j \leqslant n,\ j = 1, 2, \dots n.$$

Then G can be mapped homomorphicall onto the abelian group V_n.

Proof. Let φ be a mapping from G onto $G/_K$ as follows:

$$\varphi : G \longrightarrow G/_K,$$

defined by $\qquad g \longmapsto gK.$

Then φ is an onto homomorphism. We only need to show that $G/_K$ is the abelian group V_n. By assumption we have

$$G = \bigcup_{\substack{1 \leqslant i_1 < \dots < i_j \leqslant n \cdot 1 \\ j = 1, 2, \dots n}} [K \cup \{g_{i_1} \dots g_{i_j}\}]$$

is the irredundant union of 2^n-1 subgroups $[K \cup \{g_{i_1} \dots g_{i_j}\}]$. Then the elements of the set

$$B = \left\{ g_{i_1} g_{i_2} \cdots g_{i_j} \;/\; 1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n, \; j = 1,2,\ldots n \right\}$$

are all distinct and each of them does not belong to K. Since g_i^2 is in K for all i in $\{1,2,\ldots n\}$, the square of each element of B must belong to K. From these, we have

$$G_{/K} = \{K\} \cup \{ (g_{i_1} \cdots g_{i_j})K \;/\; 1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n, \; j = 1,2,\ldots n \}$$

and the elements of $G_{/K}$ are all distinct. Since g_i^2 is in K for all i in $\{1,2,\ldots n\}$, $(g_iK)^2 = K$. Then $\{K, g_iK\}$ is a cyclic subgroup of $G_{/K}$ for all i in $\{1,2,\ldots n\}$. From

$$G_{/K} = \{K\} \cup \{ (g_{i_1} \cdots g_{i_j})K \;/\; 1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n, \; j = 1,2,\ldots n \},$$ we have

$$G_{/K} = \{K, g_1K\} \times \{K, g_2K\} \times \cdots \times \{K, g_nK\},$$

since $g_ig_j = g_jg_i$ for all i, j in $\{1, 2, \ldots n\}$, i.e., $G_{/K}$ is an abelian group. So we have $G_{/K}$ is an abelian group of the form $C_2 \times C_2 \times \cdots \times C_2$ (n times).

Hence the theorem is proved completely.

2.3 Remark. We can state Theorem 2.1 and 2.2 in terms of isomorphism as follows:

1. Let G be a group and K a normal subgroup of G. If $G_{/K}$ is isomorphic to the abelian group V_n, then G is an irredundant union of 2^n-1 subgroups and the intersection is K.

2. Let a group $G = \bigcup_{i=1}^{2^n-1} G_i$ be the irredundant union of its

subgroups G_i with the additional properties:

(i) $K = \bigcap_{i=1}^{2^n-1} G_i$ is a normal subgroup of G,

(ii) there exist $g_1, g_2, \dots g_n$ in G such that g_i^2 is in K and $g_i g_j = g_j g_i$ for all i, j in $\{1, 2, \dots n\}$ and

$$\{G_1, G_2 \dots G_{2^n-1}\} = \{A_{i_1 i_2 \dots i_j} / \ 1 \leq i_1 < i_2 < \dots < i_j \leq n, \ j = 1, 2, \dots n\}$$

where $A_{i_1 i_2 \dots i_j} = [K \cup \{g_{i_1} g_{i_2} \dots g_{i_j}\}]$,

$$1 \leq i_1 < i_2 < \dots < i_j \leq n, \ j = 1, 2, \dots n.$$

Then $G/_K$ is isomorphic to the abelian group V_n.