CHAPTER II

SUBHARMONIC AND SUPERHARMONIC FUNCTIONS

We give the definitions and study some properties of subharmonic, superharmonic functions and polar sets in this chapter. Some of the material in this chapter is drawn from [1] and [2].

2.1 Definition. Let G is an open subset of R^n. An extended real-valued function u defined on G is said to be subharmonic on G if it has the following properties :

(i) u is upper semicontinuous on G,

(ii) $u < +\infty$ on G,

(iii) If $x \in G$, then $u(x) \leq L(u:x,\delta)$ whenever $\bar{B}_{x,\delta} \subseteq G$,

(iv) u is not identically $-\infty$ on any component of G.

The function u is superharmonic on G if -u is subharmonic on G.

It is easily seen that u+h is superharmonic if u is superharmonic and h is harmonic on G.

2.2 Definition. A function u defined on an open connected set G obeys the maximum principle if $\sup_{x \in G} u(x)$ is not attained on G unless u is constant. There is a corresponding minimum principle if $\inf_{x \in G} u(x)$ is not attained on G unless u is constant.

2.3 <u>Theorem</u>. If u is continuous on the open connected set G and for each $x \varepsilon G$ there is a $\delta_x > 0$ such that $u(x) = L(u:x,\delta)$ whenever $\bar{B}_{x,\delta} \subset B_{x,\delta_x} \subset G$, then u obeys the maximum and minimum principles.

<u>Proof</u> : Let $M = \{x | x\varepsilon G \text{ and } u(x) = \sup_{y\varepsilon G} u(y)\}$. Since u is continuous, M is a relatively closed subset of G. We shall show that M is open. Let $x\varepsilon M$, then there is a $\delta_x > 0$ such that $u(x) = L(u: x,\delta)$ whenever $\delta < \delta_x$. Let $y \varepsilon B_{x,\delta_x}$ and $\delta_0 = \|x-y\| < \delta_x$ since $u(x) = L(u: x,\delta_0)$, $L(u(x)-u: x,\delta_0) = 0$, but $x\varepsilon M$ so $u(x)-u \geq 0$ on $\partial B_{x,\delta_0}$. Therefore $u(x)-u = 0$ almost everywhere on $\partial B_{x,\delta_0}$. By the continuity of u, $u = u(x)$ on $\partial B_{x,\delta_0}$ and, in particular, $u(y) = u(x)$. This shows that $y\varepsilon M$ and $B_{x,\delta_x} \subset M$ and also M is open. By the connectedness of G either $M = \emptyset$ or $M = G$. If $M = \emptyset$, then $\sup_{y\varepsilon G} u(y)$ is not attained in G and if $M = G$, then u is a constant. This completes the proof that u satisfies the maximum principle. Next, $-u$ satisfies the hypothesis and therefore maximum principle, but the maximum principle for $-u$ is the same as the minimum principle for u.

2.4 <u>Theorem</u>. Let u be continuous on the open set G. If for each $x\varepsilon G$, $u(x) = L(u: x,\delta)$ for all sufficiently small δ, then u is harmonic on G; in particular, $u(x) = L(u: x,\delta) = A(u: x,\delta)$ whenever $\bar{B}_{x,\delta} \subset G$.

<u>Proof</u> : Let $B = B_{x,\rho} \subset \bar{B}_{x,\rho} \subset G$. We define the function v by

$$v = \begin{cases} PI(u,B) & \text{on } B \\ u(y) & \text{on } \partial B. \end{cases}$$

v is continuous on $\bar{B}$ and harmonic on B. For each $y\varepsilon B$, $v(y) = L(v:x,\delta)$

for all sufficiently small δ and by the hypothesis, $u(y) = L(u{:}x,\delta)$. Then $u(y)-v(y) = L(u-v{:}\ x,\delta)$ for all sufficiently small δ. Suppose there is a point $z \in B$ such that $u(z)-v(z) < 0$. Since u-v is continuous on $\bar{B}$, u-v attains its infimum on B. By Theorem 2.3 u-v must be constant on B. Since u-v = 0 on ∂B, u-v = 0 on B, a contradiction. Therefore $u \geq v$ on B. Applying the same argument to v-u, we obtain u = v on B. Since v is harmonic on B, u is harmonic on B and B is arbitrary ball with $\bar{B} \subset G$, then u is harmonic on G.

2.5 Theorem. Let u be a function defined on an open set $G \subset R^n$ having continuous second partials. Then u is superharmonic on G if and only if $\Delta u \leq 0$ on G.

Proof : Assume that $\Delta u \leq 0$ on G. Consider first that $\Delta u < 0$ on $\bar{B} = \bar{B}_{x,\delta} \subset G$. Let

$$h = \begin{cases} PI(u,B) & \text{on } B \\ u & \text{on } \partial B \end{cases}$$

The function h is continuous on $\bar{B}$ and harmonic on B. If we can show that $u \geq h$ on B, then $u(x) \geq L(u{:}\ x,\delta)$ whenever $\bar{B} \subset G$ since $h(x) = PI(u,B) = L(u{:}x,\delta)$ on B. Let w = u-h. Then $\Delta w = \Delta u - \Delta h < 0$ on B, w = 0 on ∂B and w is continuous on $\bar{B}$. Suppose w attains its minimum on $\bar{B}$ at x_0. If $x_0 \in B$, then

$$\left.\frac{\partial^2 w}{\partial x_i^2}\right|_{x_0} \geq 0 , \quad i = 1,2,\ldots,n$$

and $\Delta w(x_0) \geq 0$, a contradiction. Therefore $w = u-h \geq 0$ on $\bar{B}$ and $u \geq h$ on $\bar{B}$. Now suppose that $\Delta u \leq 0$ on B. Letting $q(x) = \|x\|^2$, $\Delta q = 2n$.

For each $\varepsilon > 0$,

$$\Delta(u - \varepsilon q) = \Delta u - \varepsilon \Delta q < 0$$

on G. By the first part of the proof

$$u(x)-\varepsilon q(x) \geq L(u-\varepsilon q\colon x,\delta) = L(u\colon x,\delta)-\varepsilon L(q\colon x,\delta).$$

Letting $\varepsilon \to 0$, we obtain $u(x) \geq L(u\ x,\delta)$ whenever $\bar{B} \subset G$. Therefore u is superharmonic on G.

Conversely, suppose that u is superharmonic and has continuous second partials on G. Then Δu is continuous on G. Let $R_1 = \{x \mid \Delta u(x) > 0\}$. R_1 is an open subset of G. Suppose $R_1 \neq \emptyset$. Then $\Delta(-u) < 0$ on R_1. By sufficientcy part -u is superharmonic or u is subharmonic on R_1. Therefore we have $u(x) \leq L(u\colon x,\delta)$ and $u(x) \geq L(u\colon x,\delta)$ whenever $\bar{B}_{x,\delta} \subset G$ and that $u(x) = L(u\colon x,\delta)$. By Theorem 2.4 u is harmonic and then $\Delta u = 0$, a contradiction. Thus $R_1 = \emptyset$ and $\Delta u \leq 0$ on G.

2.6 Theorem. If u is subharmonic on the open connected set G, then u satisfies the maximum principle on G. If G is bounded open set, u is subharmonic on G, and $\lim_{z \to x} \sup u(z) \leq 0$ for all $x \in \partial G$, then $u \leq 0$ on G.

Proof : To prove the first assertion, suppose that there is a point $x_0 \in G$ such that $u(x_0) = \sup_{x \in G} u(x)$, $-\infty < u(x_0) = \sup_{x \in G} u(x) < +\infty$. Let $M = \{x \mid u(x) = \sup_G u\}$ which is relatively closed subset of G by the upper semicontinuity of u. We shall show that M is open. Let $y \in M$. Since u is subharmonic on G, $u(y) \leq L(u\colon y,\delta)$ whenever $\bar{B}_{y,\delta} \subset G$. Suppose there is a point $z \in B_{y,\delta} \setminus M$. Let $\rho = \|y-z\| < \delta$. Since $y \in M$, $u(y) \geq u(x)$ for all $x \in G$,

$$\frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B_{y,\rho}} u(y)\,d\sigma(x) \geq \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B_{y,\rho}} u(x)\,d\sigma(x)$$

$$u(y) \geq L(u\colon y,\rho).$$

Therefore $u(y) = L(u\colon y,\rho)$ since $u(y) \leq L(u\colon x,\rho)$ whenever $\bar{B}_{y,\rho} \subset \bar{B}_{y,\sigma} \subset G$. So we have $L(u(y)-u\colon y,\rho) = 0$. Since $u(y)-u \geq 0$ on $\partial B_{y,\rho}$, $u(y)-u = 0$ almost everywhere on $\partial B_{y,\rho}$. But since $u(z) < u(y)$, there is an α such that $u(z) < \alpha < u(y)$. By the upper semicontinuity of u there is a neighborhood U of z such that $u < \alpha < u(y)$ on $U \cap \partial B_{y,\rho}$. Therefore $u(y) - u > 0$ on $\partial B_{y,\rho} \cap U$ which has positive surface area, a contradiction. This show that $B_{y,\rho} \subset M$ that is, M is open. Thus $M = \emptyset$ or $M = G$ by the connectedness of G. It follows that u satisfies the maximum principle on G. To prove the second assertion, it suffices to prove that $u \geq 0$ on each component of G, that is, we can assume that G is connected. Suppose there is a point $y \in G$ such that $u(y) < 0$. Then u is not a constant function. Define a function v on $\bar{G}$ by $v(x) = \limsup_{z \to x} u(z)$, $x \in \bar{G}$. Then v is upper semicontinuous on $\bar{G}$, $v \geq 0$ on ∂G, $v(y) < 0$. Since v is upper semicontinuous on $\bar{G}$ which is compact, it must attain a maximum on $\bar{G}$, in fact, on G ; but this contradicts the maximum principle. Therefore $u \geq 0$ on G.

If u and v are superharmonic functions on an open set G, then for showing that u+v is superharmonic, we must show that v cannot be identically $+\infty$ on the set where u is finite.

2.7 Theorem. If G is an open set, u is an extended real-valued function on G satisfying

(i) u is not identically $+\infty$ on any component of G

(ii) $u > -\infty$ on G

(iii) u is lower semicontinuous on G

(iv) there is a $\delta_x > 0$ such that $B_{x,\delta_x} \subset G$ and $u(x) \geq A(u: x,\delta)$

whenever $\delta < \delta_x$, then u is superharmonic on G. Moreover, if u is superharmonic on G and $x \in G$, then $u(x) \geq A(u: x,\delta)$ whenever $B_{x,\delta} \subset G$.

Proof : Let $x \in G$ and $\bar{B}_{x,\delta_x} \subset G$. Since u is lower semicontinuous on $\partial B_{x,\delta_x}$, there is an increasing sequence $\{\phi_j\}$ of continuous function on $\partial B_{x,\delta_x}$ such that $\lim_{j \to \infty} \phi_j = u$ on $\partial B_{x,\delta_x}$. Let

$$h_j = \begin{cases} \phi_j & \text{on } \partial B_{x,\delta_x} \\ PI(\phi_j, B_{x,\delta_x}) & \text{on } B_{x,\delta_x} . \end{cases}$$

Then h_j is continuous on $\bar{B}_{x,\delta_x}$, harmonic on $B_{x,\delta}$ and $u \geq \phi_j = h_j$ on $\partial B_{x,\delta_x}$. Since $h_j - u$ is upper semicontinuous and there is a $\delta_y > 0$ such that $B_{y,\delta_y} \subset B_{x,\delta_x}$ and $(h_j - u)(y) \leq A(h_j - u: y,\delta)$ whenever $\delta < \delta_y$, as in the proof of the first assertion of Theorem 2.6 we can show that $h_j - u$ satisfies the maximum principle on B_{x,δ_x} by substituting $(h_j - u)(y) \leq A(h_j - u: y,\delta)$ for $(h_j - u)(y) \leq L(h_j - u: y,\delta)$. Therefore $h_j - u$ cannot attain its supremum on B_{x,δ_x}. As an upper semicontinuity on the compact set $\bar{B}_{x,\delta_x}$, $h_j - u$ attains its supremum on $\bar{B}_{x,\delta_x}$ and, in fact, on $\partial B_{x,\delta_x}$. Since $h_j - u \leq 0$ on $\partial B_{x,\delta_x}$, $h_j - u \leq 0$ and B_{x,δ_x} and $h_j \leq u$ on B_{x,δ_x}. Therefore

$$u(x) \geq h_j(x) = PI(\phi_j, B_{x,\delta_x})(x) = L(\phi_j : x,\delta)$$

with the latter equality holding since x is the center of the ball B_{x,δ_x}. Since $\phi_j \to u$ on $\partial B_{x,\delta}$, $u(x) \geq L(u: x,\delta)$ by the Lebesgue monotone convergence theorem. This shows that u is superharmonic on G. We show that $u(x) \geq A(u: x,\delta)$ whenever $\bar{B}_{x,\delta} \subset G$. If $u(x) = +\infty$, the inequality is trivially true. Assume $u(x) < +\infty$, for $0 < \rho < \delta$ we have $\bar{B}_{x,\rho} \subset G$ then

$$u(x) \geq L(u: x,\rho)$$

$$\sigma_n \rho^{n-1} u(x) \geq \int_{\partial B_{x,\rho}} u(y)d\sigma(y).$$

Integration over $(0,\delta)$

$$\int_0^\delta \sigma_n \rho^{n-1} u(x) d\rho \geq \int_0^\delta \int_{\partial B_{x,\rho}} u(y)d\sigma(y) = \int_{B_{x,\rho}} u(y)dy$$

$$\frac{\sigma_n \delta^n}{n} u(x) \geq \int_{B_{x,\rho}} u(y)dy .$$

Since $\nu_n = \frac{\sigma_n}{n}$, $u(x) \geq A(u: x,\delta)$ whenever $\bar{B}_{x,\delta} \subset G$.

2.8 <u>Theorem</u>. If u is superharmonic on the open set G, then u is finite almost everywhere on G relative to Lebesgue measure and Lebesgue integrable on each compact set $K \subset G$.

<u>Proof</u> : It suffices to show that u is finite almost everywhere on each component of G. We can assume that G is connected. Since u is not identically $+\infty$ on G, let

$$M = \{x \mid u \text{ is finite almost everywhere on } B_{x,\delta} \subset G \text{ for some } \delta > 0 \text{ and } x \in G\}.$$

M is nonempty since there is at least one point of G such that u is finite and according to Theorem 2.7, u is finite almost everywhere on each ball in G having this point as its center. We can show that M is open and also relatively closed on G. By the connectedness of G and $M \neq \emptyset$ we get $M = G$. For each $x \in M = G$ there corresponds a $B_{x,\delta} \subset G$ on which u is finite almost everywhere. Since such open sets cover G, a countable number of them suffice to cover G. Since u is finite almost everywhere on each elements of a countable covering of G, u is finite almost everywhere on G. To show that u is Lebesgue integrable on each compact set $K \subset G$, there is a finite number of balls with center x_i and radii δ_i such that $\bar{B}_{x_i,\delta_i} \subset G$, $i = 1,\ldots,p$ covers K. Since u is finite almost everywhere on each ball containing x_i, we can assume that $u(x_i) < +\infty$. Then $+\infty > u(x_i) \geq A(u\colon x_i,\delta_i)$ for each x_i. We can assume that $u \geq 0$ on G. Then

$$-\infty < \int_K u(z)dz \leq \sum_{i=1}^{p} \int_{B_{x_i,\delta_i}} u(z)dz \leq \sum_{i=1}^{p} \nu_n \delta_i^n u(x_i) < +\infty$$

and u is integrable on K.

2.9 <u>Theorem</u>. If u is superharmonic on the open set G and B is a ball with $\bar{B} \subset G$, then PI(u,B) is harmonic on B and $u \geq PI(u,B)$ on B.

<u>Proof</u> : We can assume that $u \geq 0$ on ∂B. Since u is lower semicontinuous on ∂B, there is an increasing sequence $\{\emptyset_j\}$ of nonnegative continuous functions on ∂B such that $\lim_{j \to \infty} \emptyset_j = u$ on ∂B. Let

$$v_j = \begin{cases} PI\ (\emptyset_j, B) & \text{on } B \\ \emptyset_j & \text{on } \partial B \end{cases}$$

Then $u \geq \emptyset_j = v_j$ on ∂B, v_j is harmonic on B and v_j is continuous on $\bar{B}$. Since v_j is harmonic on B, $u - v_j$ is superharmonic on B and satisfies minimum principle by Theorem 2.6. Therefore $u - v_j$ cannot attain its infimum on B. Since $u - v_j$ is lower semicontinuous on the compact set $\bar{B}$, $u - v_j$ attains its infimum on $\bar{B}$, infact, on ∂B. Since $u - v \geq 0$ on ∂B, $u - v_j \geq 0$ on $\bar{B}$. The sequence $\{v_j\}$ is an increasing sequence of functionsharmonic on B and $v = \lim_{j \to \infty} v_j$ is either identically $+ \infty$ or harmonic on B by Theorem 1.14. Since u is finite almost everywhere on G and $u \geq v$, v is harmonic on B. It also follows from the Lebesgue monotone convergence theorem that $v = PI(u,B)$ on B.

2.10 <u>Theorem</u>. If u and v are superharmonic on an open set G and $c > 0$, then

(i) cu is superharmonic on G.

(ii) u+v is superharmonic on G.

(iii) min (u,v) is superharmonic on G.

<u>Proof</u> : (i) It is obvious from the definition of superharmonic function. To prove (ii) note that u+v cannot be identically $+ \infty$ on any component U of G since each component has positive Lebesgue measure and both u and v are finite almost everywhere on U. Moreover $u+v > - \infty$ on G since u and v have this property and u+v is lower semicontinuous on G. Since $u(x) \geq L(u: x,\delta)$ and $v(x) \geq L(v: x,\delta)$ whenever $\bar{B}_{x,\delta} \subset G$,

$u(x) + v(x) \geq L(u+v: x,\delta)$ whenever $\bar{B}_{x,\delta} \subset G$. This concludes the proof of (ii). To prove (iii) suppose that $\bar{B}_{x,\delta} \subset G$. Then

$$u(x) \geq L(u: x,\delta) \geq L(\min(u,v): x,\delta)$$

and

$$v(x) \geq L(v: x,\delta) \geq L(\min(u,v): x,\delta)$$

Therefore $\min(u(x), v(x)) \geq L(\min(u,v): x,\delta)$. Moreover, $\min(u,v) > -\infty$ on G and $\min(u,v)$ is lower semicontinuous on G. Thus the proof of (iii) is complete.

2.11 Lemma. Let u be superharmonic on $G \subset R^n$ and suppose that $\bar{B}_{x,\delta} \subset G$. Then $L(u: x,\delta)$ and $A(u: x,\delta)$ are decreasing functions of δ, and

$$\lim_{\delta \to 0} L(u: x,\delta) = \lim_{\delta \to 0} A(u: x,\delta) = u(x).$$

Proof : Let $0 < \delta_1 < \delta_2$. Define a function h on B_{x,δ_2} by $h = PI(u, B_{x,\delta_2})$. Then $u \geq h$ on B_{x,δ_2} by Theorem 2.9. Since $\bar{B}_{x,\delta_1} \subset B_{x,\delta_2}$ and h is harmonic on B_{x,δ_2},

$$L(u:x,\delta_1) \geq L(h: x,\delta_1) = h(x) = L(u: x,\delta_2) .$$

This shows that $L(u: x,\delta)$ is a monotone decreasing function of δ. Since u is lower semicontinuous at $x \in G$, given $\varepsilon > 0$, there is $\delta > 0$ such that $u(y) > u(x)-\varepsilon$ whenever $y \in B_{x,\delta}$ and hence, for $\rho < \delta$

$$L(u: x,\rho) > u(x)-\varepsilon.$$

We have $u(x) \geq L(u: x,\rho)$, so $\lim_{\rho \to 0} L(u: x,\rho) = u(x)$. To show that $A(u: x,\delta)$ is a monotone decreasing function of δ. We consider that

$$A(u: x,\delta) = \frac{1}{\nu_n\delta^n} \int_{B_{x,\delta}} u(x)dx$$

$$= \frac{\sigma_n}{\nu_n\delta^n} \int_0^\delta \rho^{n-1}\left(\frac{1}{\sigma_n\rho^{n-1}} \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} U(\rho,\theta_1,\ldots,\theta_{n-1})\rho^{n-1}\sin^{n-2}\theta_1 \cdots \right.$$

$$\left. \cdots \sin\theta_{n-2}d\theta_1\cdots d\theta_{n-1}\right) d\rho$$

where $u(x) = U(\rho,\theta_1,\ldots,\theta_{n-1})$. By changing variable ρ to $w = \frac{\rho}{\delta}$ we get

$$A(u:x,\delta) = \frac{\sigma_n}{\nu_n} \int_0^1 \frac{(w\delta)^{n-1}}{\delta^n} \left(\frac{1}{\sigma_n(w\delta)^{n-1}} \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} U(w\delta,\theta_1\ldots,\theta_{n-1})(w\delta)^{n-1}\sin^{n-2}\theta_1 \right.$$

$$\left. \cdots \sin\theta_{n-2}d\theta_1\cdots d\theta_{n-1}\right) \delta dw$$

$$= \frac{\sigma_n}{\nu_n} \int_0^1 w^{n-1} L(u: x,w\delta)dw.$$

Since $L(u: x,w\delta)$ is a monotone decreasing function of δ, it follows that $A(u: x,\delta)$ is a monotone decreasing function of δ. For $0 < \rho < \delta$, we have by (1-3) that

$$A(u:x,\delta) = \frac{\sigma_n}{\nu_n\delta^n} \int_0^\delta \rho^{n-1} L(u:x,\rho)d\rho \geq \frac{\sigma_n}{\nu_n\delta^n} \int_0^\delta \rho^{n-1}d\rho L(u:x,\delta) = L(u:x,\delta).$$

Since $u(x) \geq A(u: x,\delta) \geq L(u:x,\delta)$, $\lim_{\delta\to 0} A(u: x,\delta) = u(x)$.

2.12 <u>Theorem</u>. Let u be superharmonic on the open set G and let $\bar{B}_{x,\delta} \subset G$. If $B = B_{x,\delta}$, define

$$v = \begin{cases} PI(u,B) & \text{on } B \\ u & \text{on } G \setminus B \end{cases}$$

then $u \geq v$ on G, v is harmonic on B, and v is superharmonic on G.

<u>Proof</u> : Since $u \geq v$ on B, $u \geq v$ on G and v is harmonic on B. If we can show that v is lower semicontinuous on ∂B, then v is lower semicontinuous on G since v is lower semicontinuous on $G \setminus \partial G$. To do this, we first show that for $x_0 \varepsilon\ \partial B$

$$\liminf_{\substack{y \to x_0 \\ y \,\varepsilon\, B}} v(y) \geq \liminf_{\substack{y \to x_0 \\ y \,\varepsilon\, \partial B}} u(y).$$

Assume that $\liminf_{\substack{y \to x_0 \\ y \,\varepsilon\, \partial B}} u(y) > -\infty$. If k is any number such that $\liminf_{\substack{y \to x_0 \\ y \,\varepsilon\, \partial B}} u(y) \geq k$, then $u(y) \geq k$ for all $y \,\varepsilon\, \partial B$ in a neighborhood of x_0. Suppose that $k < 0$, then $-u(y) \leq -k$ and $-k > 0$. Choose $\varepsilon > 0$ such that $-u(z) \leq -k$ for $z \,\varepsilon\, B_{x_0,\varepsilon} \cap \partial B$. Then for $y \,\varepsilon\, B$

$$PI(-u,B)(y) = \frac{1}{\sigma_n \delta} \int \frac{\delta^2 - \|y-x\|^2}{\|z-y\|^n} (-u(z))d\sigma(z)$$

$$= \frac{1}{\sigma_n \delta} \int_{(\|z-x_0\|<\varepsilon)\cap \partial B} \frac{\delta^2 - \|x-y\|^2}{\|z-y\|^n} (-u(z))d\sigma(z)$$

$$+ \frac{1}{\sigma_n \delta} \int_{(\|z-x_0\|\geq\varepsilon)\cap \partial B} \frac{\delta^2 - \|x-y\|^2}{\|z-y\|^n} (-u(z))\, d\sigma(z).$$

If $A_1(y)$ and $A_2(y)$ denote the first and the second terms, respectively, then

$$A_1(y) \leq \frac{-k}{\sigma_n \delta} \int_{(\|z-x_0\|<\varepsilon)\cap \partial B} \frac{\delta^2 - \|x-y\|^2}{\|z-y\|^n}\, d\sigma(z)$$

$$\leq \frac{-k}{\sigma_n \delta} \int_{\partial B} \frac{\delta^2 - \|x-y\|^2}{\|z-y\|^n}\, d\sigma(z) = -k\, PI(u,B) = -k.$$

Consider $|A_2(y)|$ and $y \in B_{x_0,\varepsilon/2}$. Then $\|z-y\| \geq \varepsilon/_2$ when $\|z-x_0\| \geq \varepsilon$ for if not we would have $\|z-x_0\| \leq \|z-y\|+\|y-x_0\| \leq \varepsilon$. Then for $y \in B_{x_0}, \varepsilon/_2$

$$|A_2(y)| \leq \frac{1}{\sigma_n \delta} \int_{(\|z-x_0\|\geq\varepsilon)\cap\partial B} \frac{\delta^2-\|x-y\|^2}{\|z-y\|^n} |-u(z)|d\sigma(z)$$

$$\leq \frac{\sigma^2-\|x-y\|^2}{\sigma_n \delta(\varepsilon/2)^n} \int_{\partial B} |u(z)|d\sigma(z) \quad .$$

Since $\|y-x\| \to \delta$ as $y \to x_0$, $|A_2(y)| \to 0$ as $y \to x_0$. Therefore

$$\lim_{y \to x_0} \sup PI(-u,B)(y) \leq \lim_{y \to x_0} \sup A_1(y) + \lim_{y \to x_0} \sup A_2(y) \leq -k.$$

Since $\lim\limits_{y \to x_0} \sup PI(-u,B)(y) = -\lim\limits_{y \to x_0} \inf PI(u,B)(y)$,

$$\lim_{y \to x_0} \inf v(y) = \lim_{y \to x_0} \inf PI(u,B)(y) - \lim_{y \to x_0} \sup PI(-u,B)(y) \geq k.$$

But k is any number less than $\lim\limits_{\substack{y \to x_0 \\ y \in \partial B}} \inf u(y)$, $\lim\limits_{\substack{y \to x_0 \\ y \in B}} \inf v(y) \geq \lim\limits_{\substack{y \to x_0 \\ y \in \partial B}} \inf u(y)$.

If $k \geq 0$, then consider $u-k-1$. We have $u-k-1 \geq -1$ in a neighborhood of x_0 and

$$\lim_{y \to x_0} \inf PI(u-k-1,B)(y) = \lim_{y \to x_0} \inf PI(u,B)(y) -k -1.$$

Therefore for any $y \in \partial B$

$$\lim_{\substack{z \to y \\ z \in B}} \inf v(z) \geq \lim_{\substack{z \to y \\ y \in \partial B}} \inf u(z) \geq \lim_{z \to y} \inf u(z) = u(y) = v(y).$$

Since $v = u$ on $G \setminus B$, $\lim\limits_{\substack{z \to y \\ z \in G\setminus B}} \inf v(z) = \lim\limits_{\substack{z \to y \\ z \in G\setminus B}} \inf u(z) \geq \lim\limits_{z \to y} \inf u(z)$

$$= u(y) = v(y).$$

Hence, $\liminf_{z \to y} v(z) \geq v(y)$ and v is lower semicontinuous on $y \in \partial B$.

So we have v is lower semicontinucus on G. For $y \in \partial B$ and $\bar{B}_{y,\rho} \subset G$. We have

$$v(y) = u(y) \geq L(u: y,\rho) \geq L(v: y,\rho)$$

since $u \geq v$ on G. For $y \in G \setminus \partial B$, we get

$$v(y) \geq L(v: y,\rho)$$

whenever $\bar{B}_{y,\rho} \subset G$. Therefore v is superharmonic on G.

Let G be an open set and u be a function on G which is locally integrable. If $\delta > 0$, then G_δ will denote the set $\{x \in G | d(x,\sim G) > \delta\}$. If $\delta,\rho > 0$, then it is easily seen from the inequality of triangle that $G_{\delta+\rho} \subset (G_\delta)_\rho$. Define a function u_δ on G_δ by $u_\delta(x) = A(u:x,\delta)$, $x \in G_\delta$.

2.14 Theorem. If u is superharmonic on G, then u_δ is a continuous superharmonic function on G_δ for each δ and u_δ increases to u as δ tends to zero

Proof : Suppose $x, x_0 \in G_\delta$. Then

$$|u_\delta(x) - u_\delta(x_0)| = \left|\frac{1}{\nu_n \delta^n} \int_{B_{x,\delta}} f(z)dz - \frac{1}{\nu_n \delta^n} \int_{B_{x_0,\delta}} f(z)dz\right|$$

$$\leq \frac{1}{\nu_n \delta^n} \int_{B_{x,\delta} \Delta B_{x_0,\delta}} |f(z)|dz .$$

Since the Lebesgue measure of $B_{x,\delta} \Delta B_{x_0,\delta}$ tends to zero as x tends to x_0, the latter integral tends to zero as x tends to x_0 by the absolute continuity of the indefinite integral with respect to

Lebesgue measure. It follows that $\lim_{x \to x_0} u_\delta(x) = u_\delta(x_0)$ and that u_δ is continuous on G_δ . Suppose $\delta, \rho > 0$ and $x \in G_{\delta+\rho}$. Since $x \in G_{\delta+\rho} \subset (G_\delta)_\rho$, for any $y \in B_{x,\rho}$ we have

$$d(y, \sim G) \geq d(x, \sim G) - d(x,y) > \delta+\rho-\rho = \delta.$$

Therefore $B_{x,\rho} \subset G_\delta$; similarly $B_{x,\delta} \subset G_\rho$. It follows that $A(u_\delta : x,\rho)$ and $A(u_\rho : x,\delta)$ are defined. By Fubini's theorem

$$A(u_\delta : x,\rho) = \frac{1}{\nu_n \rho^n} \int_{B_{x,\rho}} u_\delta(y)dy$$

$$= \frac{1}{\nu_n \rho^n} \int \chi_{B_{x,\rho}}(y)\left(\frac{1}{\nu_n \rho^n} \int \chi_{B_{y,\delta}}(z)u(z)dz\right) dy$$

$$= \frac{1}{\nu_n^2 \rho^n \delta^n} \int u(z)\left(\int \chi_{B_{x,\rho}}(y)\, \chi_{B_{y,\delta}}(z)dy\right)dz .$$

Since $z \in B_{y,\delta}$ if and only if $y \in B_{z,\delta}$,

$$A(u_\delta : x,\rho) = \frac{1}{\nu_n^2 \rho^n \delta^n} \int u(z)(B_{x,\rho} \cap B_{z,\delta})dz$$

$$= \frac{1}{\nu_n^2 \rho^n \delta^n} \int u(z)(B_{x,\delta} \cap B_{z,\rho})dz$$

where the last equality follows from the fact that $B_{x,\rho} \cap B_{z,\delta}$ is a symmetric function of δ and ρ . Therefore

$$A(u_\delta : x,\rho) = \frac{1}{\nu_n^2 \rho^n \delta^n} \int u(z)\left(\int \chi_{B_{x,\delta}}(y) \chi_{B_{y,\rho}}(z)dy\right)dz$$

$$A(u_\delta : x,\rho) = \frac{1}{\nu_n \rho^n} \int \chi_{B_{x,\delta}}(y)\left(\frac{1}{\nu_n \rho^n} \int \chi_{B_{y,\rho}}(z)u(z)dz\right) dy$$

$$= \frac{1}{\nu_n \rho^n} \int_{B_{x,\delta}} u_\rho(y)dy \quad = \quad A(u_\rho : x,\delta)$$

for $x \in G_{\delta+\rho}$. Now consider any $x \in G_\delta$ and choose r such that $x \in G_{\delta+r}$. Then for all $\rho < r$, $d(x,\sim G) > \delta + r > \delta + \rho$ and $x \in G_{\delta+r} \subset G_{\delta+\rho}$. Therefore

$$A(u_\delta : x,\rho) \; = \; A(u_\rho : x,\delta) \; \leq \; A(u: x,\delta) \; = \; u_\delta(x)$$

for all $\rho < r$. It follows that u_δ is superharmonic on G_δ by Theorem 2.7. From Lemma 2.11 u_δ increases to u as δ tends to zero.

2.15 <u>Theorem</u>. If u is continuous on G, then u_δ has continuous first partial derivatives on G_δ. More generally, if u has continuous partial derivatives of order $k \geq 1$ on G, then u_δ has continuous partial derivatives of order k+1 on G_δ.

<u>Proof</u> : We want to show that $\frac{\partial}{\partial x_i} u_\delta(x)$ exists and is continuous, for $x \in G_\delta$ and $i = 1,2,\ldots,n$. Without loss of generality we may suppose $i = 1$. Let $h_1 > 0$ and $h = (h_1,0,\ldots,0)$. Thus

$$\frac{u_\delta(x+h)-u_\delta(x)}{h_1} = \frac{A(u: x+h,\delta) - A(u: x,\delta)}{h_1}$$

$$= \frac{1}{h_1}\left[\frac{1}{\nu_n \delta^n}\left(\int_{B_{x+h,\delta}} u(t)dt - \int_{B_{x,\delta}} u(t)dt\right)\right].$$

So we get

$$\nu_n \delta^n \left\{ \frac{u_\delta(x+h)-u_\delta(x)}{h_1} \right\} = \frac{1}{h_1} \left\{ \int_{B_{x+h,\delta}} u(t)dt - \int_{B_{x,\delta}} u(t)dt \right\} .$$

Let S denote the unit sphere in R^n ; that is, $S = \{\theta | \|\theta\| = 1\}$. If $e_1 = (1,0,\ldots,0)$ is a fixed element in S, let γ_θ denote the angle between θ and e_1. Define $S^{e_1} = \{\theta \in S | \cos_{\gamma_\theta} \geq 0\}$ and $S^{-e_1} = S - S^{e_1}$. Let E_{h_1} be the set of points in $\bigcup_{0 \leq \rho \leq h_1} \{\delta\theta + x + \rho e_1 | \theta \in S^{e_1}\}$ but not in $B_{x,\delta} \bigcup B_{x+h,\delta}$. The former set is just the region swept out by translating a hemisphere in the direction e_1 . Then

$$\nu_n \delta^n \left\{ \frac{u_\delta(x+h)-u_\delta(x)}{h_1} \right\} = \frac{1}{h_1} \int_0^{h_1} \left[\int_S u(x+\rho e_1 + \delta\theta) \cos_{\gamma_\theta} d\sigma(\theta) \right] d\rho .$$

Since u is continuous on G, the inner integral is continuous as a function of ρ in some small interval about $\rho = 0$ and we have

$$\frac{\partial}{\partial x_1} u_\delta(x) = \lim_{h_1 \to 0} \frac{u_\delta(x+h)-u_\delta(x)}{h_1} = \frac{1}{\nu_n \delta^n} \int u(x+\delta\theta) \cos_{\gamma_\theta} d\sigma(\theta).$$

Since the integral on the right is a continuous function of x on G_δ , $\frac{\partial}{\partial x_1} u_\delta$ is continuous on G_δ .

2.16 Theorem. Let u be superharmonic on the open set G and let V be an open set with compact closure $\bar{V} \subset G$. Then there is an increasing sequence $\{v_j\}$ of superharmonic functions on V having continuous second partials such that $u = \lim_{j \to \infty} v_j$ on V.

Proof : Let $3\delta = d(\bar{V}, \sim G)$ and let $\{\delta_j\}$ be a decreasing sequence in $(0,\delta)$ with $\lim_{j \to \infty} \delta_j = 0$. For any $\rho > 0$ and any superharmonic function w on G, define w_ρ on G_ρ as in the preceding theorem. Fix j, u_{δ_j} is superharmonic and $u_{\delta_j} \leq u$ on G_{δ_j}, $(u_{\delta_j})_{\delta_j}$ is superharmonic and $(u_{\delta_j})_{\delta_j} \leq u_{\delta_j} \leq u$ on $(G_{\delta_j})_{\delta_j}$, and $((u_{\delta_j})_{\delta_j})_{\delta_j}$ is superharmonic and $((u_{\delta_j})_{\delta_j})_{\delta_j} \leq (u_{\delta_j})_{\delta_j} \leq u$ on $((G_{\delta_j})_{\delta_j})_{\delta_j}$. $x \in V \subset \bar{V}$ and $d(x, \sim G) \geq 3\delta > 3\delta_j$. Since $((G_{\delta_j})_{\delta_j})_{\delta_j} \supset (G_{\delta_j})_{2\delta_j} \supset G_{3\delta_j}$, $x \in G_{3\delta_j}$. Therefore $V \subset ((G_{\delta_j})_{\delta_j})_{\delta_j}$. Letting $v_j = ((u_{\delta_j})_{\delta_j})_{\delta_j}$ on $((G_{\delta_j})_{\delta_j})_{\delta_j}$, v_j is superharmonic and has continuous second partials on V. Since u_{δ_j} increases to u as δ_j tends to zero, $u_{\delta_j} \leq u_{\delta_{j+1}}$ on G_{δ_j} for any superharmonic function u on G. We have

$$v_j = ((u_{\delta_j})_{\delta_j})_{\delta_j} \leq ((u_{\delta_{j+1}})_{\delta_j})_{\delta_j}$$

$$\leq ((u_{\delta_{j+1}})_{\delta_{j+1}})_{\delta_j}$$

$$\leq ((u_{\delta_{j+1}})_{\delta_{j+1}})_{\delta_{j+1}} = v_{j+1}$$

on $((G_{\delta_j})_{\delta_j})_{\delta_j} \supset V$ and the sequence $\{v_j\}$ is monotone increasing on V. Since $u_{\delta_j} \leq u_{\delta_{j+k}}$ on G_{δ_j} for any superharmonic function u on G and positive integers j, k,

$$((u_{\delta_{j+k}})_{\delta_j})_{\delta_j} \leq ((u_{\delta_{j+k}})_{\delta_{j+k}})_{\delta_{j+k}} = v_{j+k}$$

on $((G_{\delta_{j+k}})_{\delta_j})_{\delta_j}$. Since $u_{\delta_{j+k}} \uparrow u$ on G as $k \to \infty$ by the preceding theorem, $((u_{\delta_{j+k}})_{\delta_j})_{\delta_j} \to (u_{\delta_j})_{\delta_j}$ on $((G_{\delta_j})_{\delta_j}$ as $k \to \infty$. Therefore $(u_{\delta_j})_{\delta_j} \leq \lim_{k \to \infty} v_k$ on $(G_{\delta_j})_{\delta_j}$. Repeating the argument twice, we obtain $u \leq \lim_{k \to \infty} v_k$ on G. Since $v_k = ((u_{\delta_k})_{\delta_k})_{\delta_k}$ on $((R_{\delta_k})_{\delta_k})_{\delta_k}$, $\lim_{k \to \infty} v_k \leq u$ on G ; that is, $u = \lim_{k \to \infty} v_k$ on G. Restricting the function v_j to V, we obtain the desired sequence.

The Kelvin Transformation.

2.17 Definition. Let $B_{y,\rho}$ be a ball and $x \in R^n$. Consider the radial line joining y to x. For $y \neq x$, choose x^* on the radial line so that $\|x^*-y\| \|x-y\| = \rho^2$. x^* is called the inverse of x relative to the sphere $\partial B_{y,\rho}$.

2.18 Definition. The mapping $x \mapsto x^*$, where x^* is the inverse of x relative to $\partial B_{y,\rho}$, defined by

$$x^* = y + \frac{\rho^2}{\|x-y\|^2}(x-y) \qquad (x \neq y)$$

is known as an inversion relative to $\partial B_{y,\rho}$.

2.19 Definition. Let G be an open subset of $R^n \setminus \{y\}$ and let G* be the image of G under the inversion. If u* is a function defined on G*, the equation

$$u(x) \quad = \quad \frac{\rho^{n-2}}{\|x-y\|^{n-2}} \; u^*(x^*)$$

where $\|x-y\|^{n-2} = 1$ if n = 2, defines a function u on G. The mapping $u^* \mapsto u$ defined in this way is called the Kelvin transformation.

2.20 Lemma. If u* is superharmonic and has continuous second partials on G*, where G* is the inversion image of G, then u is superharmonic on G.

Proof : It follows immediately from the fact that

$$\Delta u(x) \quad = \quad \|x^* - y\|^{n+2} \, \Delta^* \, u^*(x^*)$$

where Δ^* denotes the Laplacian with respect to the coordinate x*.

2.21 Theorem. The Kelvin transformation preserves positivity and superharmonicity.

Proof : That positivity is preserved directly from the definition. The superharmonicity of u on G whenever u* is superharmonic on G* (but not necessarily have continuous second partials) follows from Theorem 2.16 and Lemma 2.20.

Polar sets

2.22 <u>Definition</u>. A set $Z \subset R^n$ is said to be a <u>polar set</u> if there is an open set $U \supset Z$ and a function u superharmonic on U such that $u = +\infty$ on Z.

Some properties of polar sets are given as follows :

(i) If $Z \subset R^n$ is a polar set, then Z has Lebesgue measure zero.

(ii) If $Z \subset R^n$ is a polar set, then there is a superharmonic function u on R^n $(n \geq 2)$ such that $u = +\infty$ on Z.

(iii) If $\{Z_j\}$ is a sequence of polar sets, then $\bigcup_j Z_j$ is a polar set.

(iv) The polarity of a set is invariant under translation and rotation.

The proof of (i), (ii) and (iii) can be found in [2] page 127-130.

2.23 <u>Theorem</u>. If the inversion image of a set $Z \subset R^n$ is a polar set, then Z is a polar set.

<u>Proof</u> : Let $y \in R^n$ and a set $Z \subset R^n \setminus \{y\}$. Suppose that Z^* is the image of Z under the inversion relative to $\partial B_{y,\rho}$ and $Z^* \subset R^n \setminus \{y\}$. Since Z^* is a polar set, there is an open set $U^* \supset Z^*$ and a function u^* superharmonic on U^* such that $u^* = +\infty$ on U^*. Let $D^* = U^* \cap (R^n \setminus \{y\})$. $D^* \supset Z^*$ and u^* is superharmonic on D^*. If D denotes the inverse image of D^* under the inversion map and u denotes the Kelvin transformation of u^*, then by Theorem 2.21 u is superharmonic on the set D and $u = +\infty$ on Z. Therefore Z is a polar set.

Examples of polar sets

(i) The singleton set of a point in R^n is a polar set.

Proof : Let x_0 be a fixed point of R^n. The function u_{x_0} which is defined by

$$u_{x_0}(y) = \begin{cases} +\infty & \text{if } x_0 = y \\ -\log \|x_0 - y\| & \text{if } x_0 \neq y \end{cases} \qquad (n = 2)$$

and

$$u_{x_0}(y) = \begin{cases} +\infty & \text{if } x_0 = y \\ \dfrac{1}{\|x_0 - y\|^{n-2}} & \text{if } x_0 \neq y \end{cases} \qquad (n \geq 3)$$

is called the fundamental harmonic function with pole y. u_{x_0} is harmonic on $R^n \setminus \{x_0\}$, $u_{x_0} > 0$ and u_{x_0} is not identically $+\infty$ on R^n.
We see that

$$+\infty = u_{x_0}(x_0) \geq L(u_{x_0} : x_0, \delta)$$

whenever $\bar{B}_{x_0,\delta} \subset R^n$ and $u_{x_0}(y) = L(u_{x_0}; y,\rho)$ whenever $\bar{B}_{y,\rho} \subset R^n \setminus \{x_0\} \subset R^n$.
Therefore u_{x_0} is superharmonic on R^n and $u_{x_0} = +\infty$ on $\{x_0\}$.

(ii) A line segment in $R^n (n \geq 3)$ is a polar set.

<u>Proof</u> : By the property (iv) of a polar set, it suffices to show that the line segment I joining $(a,0,\ldots,0)$ to $(b,0,\ldots 0)$ is a polar set set. Let μ be one dimensional Lebesgue measure on I. We define

$$v(x) = \int_I \frac{1}{\|x-z\|^{n-2}} \, d\mu(z) \qquad x \in R^n$$

$$= \int_a^b \frac{1}{[(x_1-z_1)^2+x_2^2+\ldots+x_n^2]^{\frac{n-2}{2}}} \, dz_1 .$$

Since v is the potential of a finite measure, v is superharmonic on R^n.

Let $x \in I$. Then $x = (x_1,0,\ldots,0)$ where $a < x_1 < b$ and

$$v(x) = \int_a^b \frac{1}{|x_1 - z_1|^{n-2}} \, dz_1$$

$$= \int_a^{z_1} \frac{1}{(x_1 - z_1)^{n-2}} \, dz_1 + \int_{z_1}^b \frac{1}{(z_1 - x_1)^{n-2}} \, dz_1$$

$$= +\infty .$$

Therefore $v = +\infty$ on I and this shows that I is a polar set.

Note that a straight line in R^n ($n \geq 3$) is a polar set since it is a countable union of line segments in $R^n(n \geq 3)$ which are polar sets.

(iii) An intersection of a sphere and a hyperplane in $R^n(n \geq 3)$ is a polar set.

Proof : It suffices to show that the intersection of the sphere centre z with radius δ and the hyperplane x_n = constant is a polar set. Let S be this intersection and $y \in S$. The inversion image of $S \setminus \{y\}$ is a straight line T^* which is a polar set. By Theorem 2.21, $S \setminus \{y\}$ is a polar set. Since $\{y\}$ is a polar set, S is a polar set.

2.24 Theorem. Let G be an open subset of R^n, let u be superharmonic and bounded below on G, and suppose there is a number α such that $\liminf_{y \to z \in \partial G} u(y) \geq \alpha$ except possibly for z in a polar subset Z of ∂G. If $n \geq ?$ and G is bounded, then $u \geq \alpha$ on G.

Proof : Let v be a superharmonic function such that $v = +\infty$ on $Z \subset \partial G$. Since $\bar{G}$ is compact and v is bounded below on $\bar{G}$, we can assume that $v \geq 0$ on $\bar{G}$ by adding a constant to v if necessary. Consider the function $u + \varepsilon v$ which is superharmonic on G for $\varepsilon > 0$. It is easily seen that

$$\liminf_{\substack{y \to z \in \partial G \\ y \in G}} (u + \varepsilon v)(y) \geq \begin{cases} \alpha & \text{if } z \in \partial G \setminus Z \\ +\infty & \text{if } z \in Z. \end{cases}$$

Therefore $\lim\inf_{y\to z\in\partial G} [(u+\epsilon v)(y)-\alpha] \geq 0$ for all $z \in \partial G$. Then $(u+\epsilon v)(y)-\alpha \geq 0$ by Theorem 2.6, that is, $(u+\epsilon v)(y) \geq \alpha$ for all $y \in G$. Suppose $\bar{B}_{x,\delta} \subset G$. Thus $A(u: x,\delta)+ \epsilon A(v: x,\delta) \geq \alpha$. Letting $\epsilon \to 0$ first and then letting $\delta \to 0$ we obtain $u(x) \geq \alpha$ for all $x \in G$.

2.25 Theorem. Let G be an open set and let Z be a relatively closed polar subset of G. If h is harmonic on $G \setminus Z$, continuous and bounded on G, then h is harmonic on G.

Proof : Consider h as a superharmonic on $G\setminus Z$, any $x \in Z$ and $\bar{B}_{x,\delta_x} \subset G$. Let $\{x_j\}$ be a sequence of distinct points in $G\setminus Z$ such that $\lim_{j\to\infty} x_j = x$ and $\lim_{j\to\infty} h(x_j) = h(x)$. Since Z is a polar set, there is a superharmonic function v such that $v = +\infty$ on Z. We can assume that $v(x_j)$ is finite for each j. For $y \in Z$ and $\bar{B}_{y,\rho} \subset G$, $+\infty = (h+\epsilon v)(y) \geq L(h+\epsilon v: y,\rho)$. Since h is superharmonic on $G\setminus Z$, continuous and bounded on G, $h + \epsilon v$ is superharmonic on G. Now $h(x_j)+ \epsilon v(x_j) \geq A(h: x_j,\delta')+ \epsilon A(v: x_j,\delta')$ for every $\epsilon > 0$. Since $v(x_j)$ is finite, we can let $\epsilon \to 0$ to obtain

$$h(x_j) \geq A(h: x_j,\delta') = \frac{1}{\nu_n\delta^n} \int_{B_{x_j,\delta'}} h(y)dy = \frac{1}{\nu_n\delta^n} \int \chi_{B_{x_j,\delta'}}(y)h(y)dy.$$

Since h is bounded below on the compact closure of a neighborhood of x, we can choose δ so that all but finite number of the integrands on the right are bounded below. With this in mind we can apply Fatou's lemma to obtain $h(x) = \lim_{j \to \infty} h(x_j) \geq A(h: x,\delta)$ for all sufficiently small δ. This shows that h is superharmonic on Z. But since h is superharmonic on $G \setminus Z$, h is superharmonic on G. Consider h as a subharmonic on $G \setminus Z$, we then have -h is superharmonic on $G \setminus Z$. Therefore -h is superharmonic on G and h is subharmonic on G. Clearly $h(x) = A(h: x,\delta^*)$ whenever $\bar{B}_{x,\delta^*} \subset G$. Since h is bounded and continuous on G, h is harmonic on G.