## CHAPTER V

## TRANSFORMATIONS OF WIENER INTEGRALS UNDER TRANSLATIONS

The purpose of this chapter is to show how the Wiener integral transforms under a translation, and we consider special cases of translations which seem to lead to rather interesting results. Moreover, we also find a necessary and sufficient condition under which the integral is invariant.

Definition 5.1. Let n be a positive integer, denote by $t_j$ the point $t_j = \frac{j}{n}$, and for any $y \in C$ define a polygonalized form of y by the relations

$$L_n y(t) = y(t_j) + \frac{y(t_{j+1}) - y(t_j)}{t_{j+1} - t_j} \cdot (t - t_j) \; ; \text{ for } t_j \leq t \leq t_{j+1},$$

$$j = 0, 1, \ldots, n-1.$$

i.e. for $y \in C$, $L_n(y)$ is defined to be equal to y at each $t_j, j = 0, \ldots, n$ and linear on each of the intervals $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, n-1$. It follows that $L_n(y) \in C$.

Theorem 5.2. Let F be a functional defined on C, let F be continuous in the sense that $\lim_{k \to \infty} F(y_k) = F(y_0)$ whenever $\{y_k\}$ is any sequence in C which converges uniformly in $[0,1]$ to $y_0 \in C$. Then the functional $y \longmapsto F(L_n(y))$ defined on C is Wiener measurable and there exists a continuous extended real-valued function $H(\xi_1, \ldots, \xi_n)$ defined on $R^n$ such that

$$F(L_n(y)) \;=\; H(y(t_1),\ldots,y(t_n)).$$

Proof. We divide the proof into 3 steps :

Step 1. Since for any $(\xi_1,\ldots,\xi_n) \in R^n$ there exists $y \in C$ such that $y(t_1) = \xi_1,\ldots, y(t_n) = \xi_n$ and $L_n(y)$ depends only upon the values of y at the n points $t_1,\ldots,t_n$, it follows that by defining H as follow :

$$H(y(t_1),\ldots, y(t_n)) \;=\; F(L_n(y)) .$$

Thus H is an extended real-valued function of n real variables $\xi_1,\ldots,\xi_n$. Then this is well defined, since if there exists $z \in C$ such that $z(t_1) = \xi_1,\ldots,z(t_n) = \xi_n$ we will have that $L_n(z) = L_n(y)$ and hence $F(L_n(z)) = F(L_n(y))$.

Step 2. We show the continuity of H.

Let $\lim_{k\to\infty} u_k = u_0$ where $u_k = (u_{k1},\ldots, u_{kn}) \in R^n$ for $k = 0,1,\ldots$. Then for each k, there exists $y_k \in C$ such that $y_k(t_i) = u_{ki}$ and is linear on each $[t_{i-1}, t_i]$ where $t_i = \frac{i}{n}$, $i = 1,\ldots,n$. Since $\lim_{k\to\infty} u_k = u_0$, we have that $\lim_{k\to\infty} (y_k(t_1),\ldots,y_k(t_n)) = (y_0(t_1),\ldots,y_0(t_n))$ and hence $\lim_{k\to\infty} y_k(t_i) = y_0(t_i)$ for all $i = 1,\ldots,n$. ................(1)

But for any point $t \in [0,1]$, $t_{i-1} \le t \le t_i$ for some i, it follows from the linearity of each $y_k$ on $[t_{i-1}, t_i]$ that

$$y_k(t) \;=\; y_k(t_{i-1}) + \frac{y_k(t_i)-y_k(t_{i-1})}{t_i - t_{i-1}} \cdot (t-t_i) \;,\quad k = 0,1,\ldots ,$$

Thus, for any point $t \in [0,1]$

$$y_k(t)-y_0(t) = \frac{t-t_i}{t_i-t_{i-1}}\left[(y_k(t_i) - y_0(t_i)) + (y_0(t_{i-1}) - y_k(t_{i-1}))\right]$$

$$+ (y_k(t_{i-1})- y_0(t_{i-1})).$$

But then

$$|y_k(t)-y_0(t)| \leq |y_k(t_{i-1})-y_0(t_{i-1})| + |y_k(t_i)-y_0(t_i)| + |y_k(t_{i-1})-y_0(t_{i-1})| .$$

It follows from (1) that $\{y_k\}$ converges uniformly to $y_0$ in $[0,1]$ and hence by hypothesis,

$$\lim_{k\to\infty} F(y_k) = F(y_0) . \qquad \text{..................(2)}$$

Also, by Definition 5.1 we have $L_n(y_k) = y_k$, $k = 0,1,\ldots$ and hence $F(y_k) = F(L_n(y_k)) = H(y_k(t_1),\ldots, y_k(t_n)) = H(u_{k1},\ldots,u_{kn})$, $k = 0,1,\ldots$. It follows from (2) that $\lim_{k\to\infty} H(u_k) = H(u_0)$ and hence H is continuous.

<u>Step 3</u>. It remains to show the Wiener measurability of $y \longmapsto F(L_n(y))$. Since for any real number $\alpha_0$,

$$\begin{aligned} \theta_{\alpha_0} &= \{ y \in C : F(L_n(y)) > \alpha_0 \} \\ &= \{ y \in C : H(y(t_1),\ldots, y(t_n)) > \alpha_0 \} \\ &= \{ y \in C : (y(t_1),\ldots, y(t_n)) \in E \} \end{aligned}$$

where $E = \{ u \in R^n : H(u) > \alpha_0 \}$ which is a Borel set since H is continuous. Thus $\theta_{\alpha_0} \in \mathcal{J}$ and hence is Wiener measurable. Therefore the functional $y \longmapsto F(L_n(y))$ is Wiener measurable, since $\alpha_0$ is arbitrary.

<u>Q.E.D.</u>

Definition 5.3. Let $A \subseteq C$ and let $x_0 \in C$. We define

$$A - x_0 = \{x \in C : x = y - x_0, y \in A\}.$$

Theorem 5.4. If $A \in \mathcal{B}(C)$ then $A - x_0 \in \mathcal{B}(C)$.

Proof. We divide the proof into 3 steps :

Step 1. We show that $(A-x_0)' = A' - x_0$, where $(A-x_0)' = C \sim (A-x_0)$ and $A' = C \sim A$. If $x \in (A-x_0)'$, then $x \notin A-x_0$, so that $x + x_0 \notin A$. Therefore $x + x_0 \in A'$ and hence $x \in A' - x_0$. Conversely if $x \in A' - x_0$, then $x + x_0 \in A'$, so that $x + x_0 \notin A$. Therefore $x \notin A - x_0$ and hence $x \in (A-x_0)'$.

Step 2. We show that $\bigcup_{i=1}^{\infty} (A_i - x_0) = \bigcup_{i=1}^{\infty} A_i - x_0$. If $x \in \bigcup_{i=1}^{\infty}(A_i - x_0)$, then $x \in A_i - x_0$ for some i. But then $x + x_0 \in \bigcup_{i=1}^{\infty} A_i$ and hence $x \in \bigcup_{i=1}^{\infty} A_i - x_0$. Conversely, if $x \in \bigcup_{i=1}^{\infty} A_i - x_0$, then $x + x_0 \in \bigcup_{i=1}^{\infty} A_i$. But then $x + x_0 \in A_i$ for some i and hence $x \in \bigcup_{i=1}^{\infty}(A_i - x_0)$.

Step 3. Let $\mathcal{A} = \{A \in \mathcal{B}(C) : A-x_0 \in \mathcal{B}(C)\}$. Then

(i) Since $y \mapsto y-x_0$ is a homeomorphism on C, it follows that if $A \subseteq C$ is open then $A-x_0$ is open and hence $A-x_0 \in \mathcal{B}(C)$. Therefore $\mathcal{A}$ contains all open sets in C.

(ii) Let $A \in \mathcal{A}$, then A and $A-x_0 \in \mathcal{B}(C)$. Since $\mathcal{B}(C)$ is a $\sigma$-algebra, $A'$ and $(A-x_0)' \in \mathcal{B}(C)$. It follows from step 1 that $A' \in \mathcal{A}$.

(iii) Let $A_i \in \mathcal{A}$ for $i = 1,2,\ldots$, then $A_i$ and $A_i - x_0 \in \mathcal{B}(C)$ for all i. Since $\mathcal{B}(C)$ is a $\sigma$-algebra, $\bigcup_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty}(A_i - x_0) \in \mathcal{B}(C)$. It follows from step 2 that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

From (i), (ii) and (iii) we have that $\mathcal{A}$ is a $\sigma$-algebra containing all open sets in C and must therefore contain the cellection $\mathcal{B}(C)$ of all Borel sets, since $\mathcal{B}(C)$ is the smallest $\sigma$-algebra containing open sets. Hence $\mathcal{A} = \mathcal{B}(C)$.

Q.E.D.

Theorem 5.5. Let F be a functional defined and Wiener integrable over C. Let F be bounded in any uniformly bounded subset of C and let F be continuous in the sense as in Lemma 5.2.

Let $x_0 \in C$ be a given function with a first derivative $x_0'$ of bounded variation on $[0,1]$. Then under the translation

$$y \longmapsto y - x_0 \quad , \quad y \in C \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

the Wiener integral undergoes the transformation

$$\int_C F(y)dW_c(y) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\}\int_C F(x+x_0)\exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\}dW_c(x)$$

Remark 5.6. Let $x_0 \in C$ satisfy the condition in Theorem 5.5. Then the functional S defined on C by

$$S(x) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt - \frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\}$$

is Wiener measurable.

Proof. Let $\{x_n\}$ be any sequence in C which converges uniformly to x in [0,1]. Then according to (1.51) and (1.55) and the fact that $x_n(0) = 0$ for all n, we have

$$\lim_{n\to\infty} \int_0^1 x_0'(t)dx_n(t) = \lim_{n\to\infty} x_0'(1)x_n(1) - \lim_{n\to\infty} \int_0^1 x_n(t)dx_0'(t)$$

$$= x_0'(1)x(1) - \int_0^1 x(t)dx_0'(t)$$

$$= \int_0^1 x_0'(t)dx(t).$$

It follows that $\lim_{n\to\infty} S(x_n) = S(x)$. Thus the functional S is continuous and hence by Theorem 4.3 is Wiener measurable.

Q.E.D.

In order to prove the theorem, we need the following Lemmas :

Lemma 5.7. Let $x, x_0 \in C$ and let $x_0$ satisfy the condition in Theorem 5.5. Let $0 = t_0 < t_1 < \ldots < t_n = 1$, where $t_j = \frac{j}{n}$ for $j = 0,\ldots,n$. Then

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left\{ \frac{(x_0(t_j)-x_0(t_{j-1}))^2}{ct_j - ct_{j-1}} + \frac{2(x_0(t_j)-x_0(t_{j-1}))(x(t_j)-x(t_{j-1}))}{ct_j - ct_{j-1}} \right\}$$

$$= \frac{1}{c}\int_0^1 (x_0'(t))^2 dt + \frac{2}{c}\int_0^1 x_0'(t)dx(t),$$

and the convergence being bounded in x for all x in any uniformly bounded set.

(Remark : Since n now varies, it must be mentioned that the points $t_j = \frac{j}{n}$ vary with n. For the sake of simplicity in writing, we do not add another index n to the $t_j$.)

Proof. Let $\psi(x;t) = x_0(t) + 2x(t)$ and

$$P_n(x) = \sum_{j=1}^{n} \frac{x_0(t_j)-x_0(t_{j-1})}{ct_j - ct_{j-1}} \left\{\psi(x;t_j) - \psi(x;t_{j-1})\right\}.$$

Then by the mean-value theorem,

$$P_n(x) = \frac{1}{c}\sum_{j=1}^{n} x_0'(t_j^*) \left\{\psi(x;t_j) - \psi(x;t_{j-1})\right\},$$

for some $t_j^*$ such that $t_{j-1} \leq t_j^* \leq t_j$.

Since $x_0'$ is of bounded variation and $\psi$ is continuous in t, it follows that $\int_0^1 \psi(x;t)dx_0'(t)$ exists and hence by (1.51), $\int_0^1 x_0'(t)d\psi(x;t)$ exists. According to the definition of Riemann-Stieltjes integral, $P_n(x)$ approaches the integral $\frac{1}{c}\int_0^1 x_0'(t)d\psi(x;t)$ as $n \to \infty$.

But $\sum_{j=1}^{n} \frac{x_0(t_j)-x_0(t_{j-1})}{ct_j - ct_{j-1}} \left\{\psi(x\ ;\ t_j) - \psi(x\ ;\ t_{j-1})\right\}$

$$= \sum_{j=1}^{n}\left\{\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{ct_j - ct_{j-1}} + 2\,\frac{(x_0(t_j)-x_0(t_{j-1}))(x(t_j)-x(t_{j-1}))}{ct_j - ct_{j-1}}\right\},$$

so we have that as $n \to \infty$, the right hand side of the above equality becomes $\frac{1}{c}\int_0^1 x_0'(t)d\psi(x;t) = \frac{1}{c}\int_0^1 (x_0'(t))^2 dt + \frac{2}{c}\int_0^1 x_0'(t)dx(t)$.

It remains to show that the convergence is bounded in x for all x in any uniformly bounded set. Let B be any uniformly bounded set of C; i.e., there is a constant $K_1 = K_1(B)$ such that

$$|x(t)| \leq K_1 \qquad \text{for all } x \in B \text{ and } t \in [0,1].$$

Let $d_j = x_0'(t_j^*)$, $b_j = \Psi(x;t_j)$, $j = 1,\ldots,n$. Then

$$P_n(x) = \frac{1}{c}\sum_{j=1}^{n} d_j(b_j - b_{j-1}) = \frac{1}{c}(b_1(d_1-d_2)+b_2(d_2-d_3)+\ldots+b_{n-1}(d_{n-1}-d_n)+b_n d_n)$$

$$= \frac{1}{c}\left(b_n d_n - \sum_{j=1}^{n-1} b_j(d_{j+1} - d_j)\right)$$

$$= \frac{1}{c}\left(x_0'(t_n^*)\Psi(x;1) - \sum_{j=1}^{n-1}\{x_0'(t_{j+1}^*) - x_0'(t_j^*)\}\Psi(x;t_j)\right).$$

Since $x_0 \in C$, there exists a constant $K_2$ such that $|x_0(t)| \leq K_2$ for all $t \in [0,1]$. Since $\Psi(x;t) = x_0(t)+2x(t)$, it follows that

$$|\Psi(x;t)| \leq |x_0(t)| + 2|x(t)|$$

$$\leq K_2 + 2K_1, \quad t \in [0,1] \text{ and } x \in B, \ \ldots\ldots\ldots\ldots\ldots (4)$$

but $x_0'$ is of bounded variation on [0,1], there exists a constant $K_3$ such that

$$\sum_{j=1}^{n-1} |x_0'(t_{j+1}^*) - x_0'(t_j^*)| \leq K_3 \qquad \text{for all } n, \ \ldots\ldots\ldots\ldots\ldots (5)$$

also by (1.48), $x_0'$ is bounded and hence there exists a constant $K_4$ such that $|x_0'(t_n^*)| \leq K_4$ . ........................(6)

From (4), (5) and (6), we have that

$$|P_n(x)| \leq \frac{1}{c}\left[|x_0'(t_n^*)|\,|\psi(x;1)| + \sum_{j=1}^{n-1}|x_0'(t_{j+1}^*)-x_0'(t_j^*)|\,|\psi(x;t_j)|\right]$$

$$\leq \frac{1}{c}\Big(K_4(K_2+2K_1)+K_3(K_2+2K_1)\Big)$$

$$= \frac{1}{c}(K_2+2K_1)(K_3+K_4)\,, \qquad \text{for all } n \text{ and } x \in B.$$

Hence the $P_n(x)$ are bounded in n and $x \in B$, since c is a positive constant. Q.E.D.

Lemma 5.8. If $L_n(y)$ is the polygonalized form of $y \in C$, then $L_n(y)$ converges uniformly to y in $[0,1]$.

Proof. Let y be any given function in C and let $\epsilon > 0$ be given. Since y is continuous on $[0,1]$ which is compact, y is uniformly continuous. Thus there exists an integer $n_0 = n_0(\epsilon)$ such that

$$|y(t') - y(t'')| < \epsilon/2 \quad \text{whenever } |t'-t''| < \frac{1}{n_0}. \;\ldots\ldots\ldots\ldots(7)$$

Since for any point $t \in [0,1]$., $t_j \leq t \leq t_{j+1}$ for some j, according to Definition 5.1 and (7) we have that for $n \geq n_0$

$$|L_n y(t)-y(t)| \leq \left|\frac{y(t_{j+1})-y(t_j)}{t_{j+1}-t_j}\right| |t-t_j| + |y(t_j)-y(t)|$$

$$< n.\frac{\epsilon}{2}.\frac{1}{n}+\frac{\epsilon}{2} = \epsilon\;.$$

Therefore $L_n(y)$ converges uniformly to y in $[0,1]$.

Q.E.D.

<u>Proof of Theorem 5.5</u>. Let M, n be any two positive integers. Define

$$C_M = \{y \in C : |y(t)| \leq M , \ t \in [0,1]\} \quad \text{and}$$

$$C_{M,n} = \{ y \in C : \ |y(j/n)| \leq M , \ j = 0,1,\dots, n\} .$$

Then $\{C_{M,2^n}\}$ is a monotone decreasing sequence of sets and hence converges to $\bigcap_{n=1}^{\infty} C_{M,2^n} = C_M$, i.e. $\lim_{n\to\infty} C_{M,2^n} = C_M$, and $C_{M_1} \subset C_{M_2}$ if $M_1 < M_2$ which implies that $\lim_{M\to\infty} C_M = C$.

<u>Step 1</u>. By letting $E_{M,n} = \{ (\xi_1,\dots,\xi_n) \in R^n : -M \leq \xi_j \leq M, j = 1,\dots,n\}$, we have $C_{M,n} = \{y \in C : (y(t_1),\dots,y(t_n)) \in E_{M,n}\}$ and

$$\chi_{C_{M,n}}(y) = \begin{cases} 1 & y \in C_{M,n} \\ 0 & y \notin C_{M,n} \end{cases} = \begin{cases} 1 & (y(t_1),\dots,y(t_n)) \in E_{M,n} \\ 0 & (y(t_1),\dots,y(t_n)) \notin E_{M,n} \end{cases}$$

$$= \chi_{E_{M,n}}(y(t_1),\dots,y(t_n)).$$

But then by Lemma 5.2, (1.30) and Theorem 4.10,

$$\int_{C_{M,n}} F(L_n(y))dW_c(y) = \int_C \chi_{C_{M,n}}(y) \cdot F(L_n(y))dW_c(y)$$

$$= \int_C \chi_{E_{M,n}}(y(t_1),\dots,y(t_n)).H(y(t_1),\dots,y(t_n))dW_c(y)$$

$$= \gamma_n \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \chi_{E_{M,n}}(\xi_1,\dots,\xi_n)H(\xi_1,\dots,\xi_n)$$

$$\cdot \exp\left\{ - \sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})^2}{ct_j - ct_{j-1}} \right\} d\xi_1 \dots d\xi_n$$

$$= \gamma_n \int_{-M}^{M} \cdots \int_{-M}^{M} H(\xi_1, \ldots, \xi_n) \exp\left[ -\sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})^2}{ct_j - ct_{j-1}} \right] d\xi_1 \ldots d\xi_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $\gamma_n = \left\{ \pi^n c^n t_1 (t_2 - t_1) \ldots (t_n - t_{n-1}) \right\}^{-\frac{1}{2}}$.

We next observe that if x is the image of y under the translation (3) and if $L_n(x)$, $L_n(x_0)$ and $L_n(y)$ are the polygonalized functions corresponding to x, $x_0$ and y respectively, then according to Definition 5.1

$$L_n y(t) = \{x(t_j) + x_0(t_j)\} + \frac{\{x(t_{j+1}) + x_0(t_{j+1}) - x(t_j) - x_0(t_j)\}}{t_{j+1} - t_j} \cdot (t - t_j)$$

$$= \left[ x(t_j) + \frac{(x(t_{j+1}) - x(t_j))}{t_{j+1} - t_j} \cdot (t - t_j) \right] + \left[ \frac{(x_0(t_{j+1}) - x_0(t_j))}{t_{j+1} - t_j} \cdot (t - t_j) + x_0(t_j) \right]$$

$$= L_n x(t) + L_n x_0(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

If we write

$$\xi_j = y(t_j), \quad \eta_j = x(t_j), \quad a_j = x_0(t_j), \; j = 0, \ldots, n, \quad \ldots\ldots\ldots (10)$$

then under (9) we have

$$\xi_j = \eta_j + a_j \;, \quad j = 0, \ldots, n. \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Since $\dfrac{\partial(\xi_1, \ldots, \xi_n)}{\partial(\eta_1, \ldots, \eta_n)} = 1$, on applying the transformation (11)

to the Lebesgue integral in (8) we find that

$$\int_{C_{M,n}} F(L_n(y))dW_c(y) = \gamma_n \int_{-M-a_1}^{M-a_1} \cdots \int_{-M-a_n}^{M-a_n} H(\eta_1 + a_1, \ldots, \eta_n + a_n)$$

$$\cdot\exp\left\{-\sum_{j=1}^{n} \frac{(\eta_j - \eta_{j-1})^2}{ct_j - ct_{j-1}}\right\} \cdot \exp\left\{-\sum_{j=1}^{n} \frac{(a_j - a_{j-1})^2}{ct_j - ct_{j-1}}\right\}$$

$$\cdot\exp\left\{-2\sum_{j=1}^{n} \frac{(a_j - a_{j-1})(\eta_j - \eta_{j-1})}{ct_j - ct_{j-1}}\right\} d\eta_1 \ldots d\eta_n .$$

By virtue of Theorem 4.10 and using (10) and the facts that $x(t_j) = L_n x(t_j)$, $x_0(t_j) = L_n x_0(t_j)$, we have that

$$\int_{C_{M,n}} F(L_n(y))dW_c(y) = \exp\left\{-\sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{ct_j - ct_{j-1}}\right\} \int_{C_{M,n}-x_0} F(L_n(x+x_0))$$

$$\cdot\exp\left\{-2\sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))(x(t_j) - x(t_{j-1}))}{ct_j - ct_{j-1}}\right\} dW_c(x).$$

............(12)

This gives us a transformation formula over $C_{M,n}$ for the polygonalized functions under the transformation (9).

Step 2. We show that $\lim_{n\to\infty} \chi_{C_{M,2^n}}(y) = \chi_{C_M}(y)$ and and $\lim_{n\to\infty} \chi_{C_{M,2^n}-x_0}(x) = \chi_{C_M-x_0}(x)$. To see this, let $\epsilon > 0$ be given.

Case 1. If $y \in C_M$ then, since $\lim_{n\to\infty} C_{M,2^n} = \bigcap_{n=1}^{\infty} C_{M,2^n} = C_M$, we have that $y \in C_{M,2^n}$ for all n. Hence

$$|\chi_{C_{M,2^n}}(y) - \chi_{C_M}(y)| < \epsilon \quad \text{for all n.}$$

<u>Case 2</u>. If $y \notin C_M$ then $y \notin C_{M,2^N}$ for some N. Since $\{C_{M,2^n}\}$ is a monotone decreasing sequence, $y \notin C_{M,2^n}$ for all $n \geq N$. Hence

$$|\chi_{C_{M,2^n}}(y) - \chi_{C_M}(y)| < \epsilon \qquad \text{for all } n \geq N.$$

It follows from both cases that

$$\lim_{n\to\infty} \chi_{C_{M,2^n}}(y) = \chi_{C_M}(y). \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Similarly, $$\lim_{n\to\infty} \chi_{C_{M,2^n}-x_0}(x) = \chi_{C_M-x_0}(x). \quad \ldots\ldots\ldots\ldots\ldots(14)$$

<u>Step 3</u>. Since $L_{2^n}(y)$ converges uniformly to y in $[0,1]$, by the continuity of F we have

$$\lim_{n\to\infty} F(L_{2^n}(y)) = F(y). \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

But then by (13),(14),(15) and the fact that $y = x+x_0$, we have

$$\left.\begin{array}{l} \lim_{n\to\infty} \chi_{C_{M,2^n}}(y).F(L_{2^n}(y)) = \chi_{C_M}(y).F(y) \, , \\ \lim_{n\to\infty} \chi_{C_{M,2^n}-x_0}(x).F(L_{2^n}(x+x_0)) = \chi_{C_M-x_0}(x).F(x+x_0) \, . \end{array}\right\} \ldots\ldots(16)$$

Also, since by hypothesis F is bounded over $C_M$ we have that $x \longmapsto F(x+x_0)$ and hence $x \longmapsto F(L_{2^n}(x+x_0))$ is bounded over $C_M - x_0$. Thus, if we let $n \to \infty$ (over the sequence $\{2^0, 2^1, \ldots\}$) in equation (12), then according to (16), Lemma 5.7 and Lebesgue's Dominated Convergence Theorem we obtain

$$\int_C \chi_{C_M}(y).F(y)dW_c(y) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_C \chi_{C_M - x_0}(x).F(x+x_0)$$

$$. \exp\left\{\frac{-2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x).$$

Since F is Wiener integrable and $\{C_M\}$ is a monotone increasing sequence of sets where $\lim_{M\to\infty} C_M = C$, according to Lebesgue's Monotone Convergence Theorem, on letting $M \longrightarrow \infty$ we obtain the desired result.

Q.E.D.

Corollary 5.9. Let $x_0 \in C$ satisfy the condition in Theorem 5.5. Then $\exp\left\{\frac{-2}{c}\int_0^1 x_0'(t)dx(t)\right\}$ is Wiener integrable and

$$\int_C \exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x) = \exp\left\{\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\}.$$

Proof. By taking $F \equiv 1$ in Theorem 5.5.

Q.E.D.

Theorem 5.10. Let $x_0 \in C$ satisfy the condition in Theorem 5.5 and let $\Gamma$ be a Wiener measurable subset of C. Then

$$W_c(\Gamma) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_{\Gamma - x_0} \exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x).$$

....................,(17)

Moreover if F is a Wiener measurable functional defined on $\Gamma$

$$\int_{\Gamma} F(y)dW_c(y) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2dt\right\}\int_{\Gamma-x_0} F(x+x_0)\exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\}dW_c(x) \qquad \text{.........................(18)}$$

in the sense that the existence of one side implies that of the other and the validity of the equality.

Proof. We begin our proof by establishing (17).

Case 1. If $\Gamma \in \mathcal{B}(C)$ then, by Theorem 5.4, $\Gamma - x_0 \in \mathcal{B}(C)$. To prove (17), we divide the proof into 3 steps :

Step 1. We consider the case $\Gamma = I^0$, where

$$I^0 = \{y \in C : \alpha_j \leq y(t_j) \leq \beta_j , j = 1,\dots,n \text{ and } 0 < t_1 < \dots < t_n \leq 1\}.$$

Let $k > 0$ be any integer. Then according to Urysohn Lemma, there exists for each j, a continuous real-valued function $\phi_{j,k}$ defined on R which equals one on $[\alpha_j, \beta_j]$, equals zero outside the interval $(\alpha_j - \frac{1}{k}, \beta_j + \frac{1}{k})$ and is linear on the remaining intervals.

Let $\chi_{I^0,k}(y) = \prod_{j=1}^{n} \phi_{j,k}(y(t_j))$ , $y \in C$.

Since each $\phi_{j,k}$ is continuous on R, it is Borel measurable. Hence, by Theorem 4.10, $\phi_{j,k}(y(t_j))$ is Wiener measurable and so is $\chi_{I^0,k}$. Since $0 \leq \phi_{j,k} \leq 1$ on R, $0 \leq \chi_{I^0,k}(y) \leq 1$ for all $y \in C$ and hence $\chi_{I^0,k}$ is uniformly bounded on C, by (1.37) $\chi_{I^0,k}$ is Wiener integrable. Finally, if $\{y_n\}$ is any sequence in C which converges uniformly in $[0,1]$ to $y \in C$, then the continuity of $\phi_{j,k}$ implies

$\lim_{n\to\infty} \phi_{j,k}(y_n(t_j)) = \phi_{j,k}(y(t_j))$. Hence

$\lim_{n\to\infty} \chi_{I^0_{,k}}(y_n) = \chi_{I^0_{,k}}(y)$. It follows that $\chi_{I^0_{,k}}$

satisfies the conditions on F in Theorem 5.5 and hence we have

$$\int_C \chi_{I^0_{,k}}(y)dW_c(y) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\}\int_C \chi_{I^0_{,k}}(x+x_0)\exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x). \quad \text{.........(19)}$$

Step 2. We show that $\lim_{k\to\infty} \chi_{I^0,k}(y) = \chi_{I^0}(y)$, $y \in C$.

Given $\epsilon > 0$. If $y \in I^0$, then $y(t_j) \in [\alpha_j, \beta_j]$ and hence

$\phi_{j,k}(y(t_j)) = 1$ for all $j = 1,\ldots,n$ and for all k, which implies

that $\chi_{I^0,k}(y) = 1$ for all k. Thus

$$|\chi_{I^0,k}(y) - \chi_{I^0}(y)| < \epsilon \qquad \text{for all k.}$$

If $y \notin I^0$, then there exists at least one j such that $y(t_j) \notin [\alpha_j, \beta_j]$.

Since $k \to \infty$, there exists a sufficiently large $k_0$ such that

$y(t_j) \in (-\infty, \alpha_j - \frac{1}{k_0}) \cup (\beta_j + \frac{1}{k_0}, \infty)$ and hence $\phi_{j,k}(y(t_j)) = 0$

for all $k \geq k_0$. Therefore $|\chi_{I^0_{,k}}(y) - \chi_{I^0}(y)| < \epsilon$ for all $k \geq k_0$.

Thus $\lim_{k\to\infty} \chi_{I^0,k}(y) = \chi_{I^0}(y)$, $y \in C$.

Step 3. Since $0 \le \chi_{I^0,k+1}(y) \le \chi_{I^0,k}(y)$ for all k, by letting $k \to \infty$ and applying Lebesgue's Monotone Convergence Theorem to (19), we have

$$W_c(I^0) = \int_C \chi_{I^0}(y)\,dW_c(y)$$

$$= \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_C \chi_{I^0}(x+x_0) \exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)\,dx(t)\right\} dW_c(x)$$

$$= \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_C \chi_{I^0-x_0}(x) \exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)\,dx(t)\right\} dW_c(x)$$

$$= \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_{I^0-x_0} \exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)\,dx(t)\right\} dW_c(x) ,$$

which is equivalent to (17) when $\Gamma = I^0$. The equality (17) holds when $\Gamma \in \mathscr{D}^0$, since both sides of it are countably additive functions of $\Gamma$. Finally, when $\Gamma$ is an arbitrary set in $\mathcal{B}(C)$, by Lemma 4.8, it can be written as $\Gamma = G - N$ where $G \in \mathscr{D}^{\sigma\downarrow}$ with $N \subseteq G$ and $W_c(N) = 0$. Since $G \in \mathscr{D}^{\sigma\downarrow}$, $G = \lim_{n\to\infty} G_n$ where $\{G_n\}$ is a decreasing sequence of members of $\mathscr{D}^0$. Applying (17) to each $G_n$ and taking the limit as $n \to \infty$, we obtain (17) for $\Gamma$ by Lebesgue's Monotone Convergence Theorem.

Case 2. If $\Gamma$ is any Wiener measurable set then, according to (1.14), $\Gamma = \Gamma_0 \sqcup A$ where $\Gamma_0 \in \mathcal{B}(C)$ and $A \subseteq B$, $B \in \mathcal{B}(C)$, $W_c(B) = 0$. But then $\Gamma - x_0 = (\Gamma_0 - x_0) \sqcup (A - x_0)$ and by substituting

$-x_0$ to $x_0$ in (17), we have $W_c(B-x_0) = 0$. Since $A-x_0 \subseteq B-x_0$, it follows from Theorem 5.4 and (1.14) that $\Gamma - x_0$ is Wiener measurable. Since Lemma 4.8 holds for any Wiener measurable set $\Gamma$, we have that steps 1-3 in case 1 also hold for $\Gamma$ in this case.

To prove (18), we first note that for any real number $\alpha_0$,

$$\{y : F(y) > \alpha_0\} - x_0 = \{x : F(x+x_0) > \alpha_0\}.$$

It follows that F is measurable on $\Gamma$ if and only if $x \longmapsto F(x+x_0)$ is measurable on $\Gamma - x_0$, but then Remark 5.6 and (1.22) imply that if and only if $x \longmapsto S(x)F(x+x_0)$ is measurable on $\Gamma - x_0$.

<u>Case 1</u>. If F is bounded and non-negative. Let M be an integer such that $0 \leq F(y) < M$ on $\Gamma$. Let n be a fixed positive integer and let

$$\Gamma_k = \left\{ y \in \Gamma : \frac{k-1}{n} \leq F(y) < \frac{k}{n} \right\},\ k = 1, \ldots, Mn. \qquad \ldots\ldots\ldots\ldots(20)$$

Since F is measurable, $\Gamma_k$'s are measurable and clearly $\Gamma = \bigsqcup_{k=1}^{Mn} \Gamma_k$.

Also, $\Gamma_k$'s are disjoint and

$$\frac{k-1}{n} W_c(\Gamma_k) \leq \int_{\Gamma_k} F(y)\,dW_c(y) \leq \frac{k}{n} W_c(\Gamma_k). \qquad \ldots\ldots\ldots\ldots(21)$$

From Remark 5.6, (17) and (20) we have

$$\frac{k-1}{n} W_c(\Gamma_k) = \frac{k-1}{n}\int_{\Gamma_k - x_0} S(x)\,dW_c(x) \leq \int_{\Gamma_k - x_0} S(x)F(x+x_0)\,dW_c(x)$$

$$\leq \frac{k}{n} \int_{\Gamma_k - x_0} S(x)\,dW_c(x) = \frac{k}{n} W_c(\Gamma_k). \qquad \ldots\ldots\ldots\ldots(22)$$

But then (21) and (22) imply that

$$\left| \int_{\Gamma_k} F(y)dW_c(y) - \int_{\Gamma_k - x_0} S(x)F(x+x_0)dW_c(x) \right| \leq \frac{1}{n} W_c(\Gamma_k) .$$

Thus

$$\left| \int_{\Gamma} F(y)dW_c(y) - \int_{\Gamma - x_0} S(x)F(x+x_0)dW_c(x) \right| = \left| \int_{\Gamma_1} F(y)dW_c(y) \right.$$

$$+ \ldots + \int_{\Gamma_{Mn}} F(y)dW_c(y) - \int_{\Gamma_1 - x_0} S(x)F(x+x_0)dW_c(x) - \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left. - \int_{\Gamma_{Mn} - x_0} S(x)F(x+x_0)dW_c(x) \right|$$

$$\leq \sum_{k=1}^{Mn} \left| \int_{\Gamma_k} F(y)dW_c(y) - \int_{\Gamma_k - x_0} S(x)F(x+x_0)dW_c(x) \right|$$

$$\leq \frac{1}{n} \sum_{k=1}^{Mn} W_c(\Gamma_k) = \frac{1}{n} W_c(\Gamma).$$

On letting $n \longrightarrow \infty$ , we obtain (18) for bounded non-negative functional.

Case 2. If F is non-negative but not bounded. Let $F_M(y) = \min(M, F(y))$. Then $F_M$ is bounded for all M. By case 1, we have (18) holds for $F_M$. Since $F_M \leq F_{M+1}$ and $\lim_{M \to \infty} F_M(y) = F(y)$, according to Lebesgue's Monotone Convergence Theorem we have (18) holds for F.

Case 3. If F is any real functional, the theorem holds for $|F|$. Then if the integrals exist for $|F|$, they exist for $F^+$ and $F^-$, and (18) holds for both and hence for F itself.

Q.E.D.

Corollary 5.10. For each $r > 0$, $W_c\{x \in C : \|x\| < r\} > C$.

Proof. We divide the proof into 2 steps :

Step 1. We show that the collection

$\mathscr{P}_0 = \{p$ : p is a polynomial in $[0,1]$ with rational coefficients and $p(0) = 0\}$ is a countable dense subset of C. To prove this, let

$$\mathscr{P} = \{ p : p \text{ is a polynomial in } [0,1] \text{ with } p(0) = 0\}.$$

Since for any $x \in C$, the sequence $\{p_n\}$ of Bernstein polynomials defined by

$$p_n(t) = \sum_{k=0}^{n} x\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}$$

converges uniformly to x. (for a proof see e.g.[2]) But $x(0) = 0$ implies that $p_n(0) = 0$ and hence $p_n \in \mathscr{P}$ for all n. This shows that $\mathscr{P}$ is dense in C. It follows that $\mathscr{P}_0$ is countable dense subset of C.

Step 2. Let $C^* = \{x \in C : x'$ exists and $x'$ is of bounded variation on $[0,1]\}$. Since for any $p \in \mathscr{P}_0$, we have $p(0) = 0$, $p'$ and $p''$ exist and p is bounded in (0,1). Thus $p'$ is of bounded variation on $[0,1]$ and hence $\mathscr{P}_0 \subseteq C^* \subseteq C$. It follows from step 1 that there exists a countable dense subset $\{z_1, z_2, \ldots, z_n, \ldots\}$ of C such that $z_n \in C^*$ for all n. Thus for any $r > 0$, we have $C = \bigcup_{n=1}^{\infty} B(z_n, r)$ where $B(z_n, r) = \{y \in C : \| y - z_n \| < r\}$. Suppose for some $r > 0$, $W_c\{x \in C : \|x\| < r\} = 0$. Then by (17), we have $W_c(B(z_n, r)) = 0$. for all n. Therefore $W_c(C) = 0$, which is a contradiction.

Q.E.D.

Remark 5.11. Every open ball has positive Wiener measure.

Proof. Let $z \in C$ be arbitrary fixed. Suppose for some $r > 0$, $W_c(B(z,r)) = 0$. According to step 1 in the proof of Corollary 5.10, there exists $z_n \in C^*$ such that $\| z_n - z \| < \frac{r}{2}$. It follows that $B(z_n, \frac{r}{2}) \subseteq B(z, r)$ and hence $W_c(B(z_n, \frac{r}{2})) = 0$. But then by (17), $W_c(B(0, \frac{r}{2})) = 0$. This contradicts Corollary 5.10.

Q.E.D.

Note that from the above remark and the fact that C is separable, we also have that every non-empty open set has positive Wiener measure.

Theorem 5.12. Let $x_0 \in C$ satisfy the condition in Theorem 5.5. Then the Wiener integral is invariant under the translation $y \longmapsto y - x_0$ if and only if $x_0 \equiv 0$.

Proof. It is clear that if $x_0 \equiv 0$ , then the Wiener integral is invariant under the translation $y \mapsto y-x_0$.

To prove the converse, let A be any Wiener measurable subset of C. Then note that the set B given by $B = A + x_0$ is Wiener measurable. Thus, if F is a Wiener integrable functional on C then according to Theorem 5.10 we have

$$\int_B F(y)dW_c(y) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_A F(x+x_0)\exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x). \quad \text{.....................(23)}$$

Therefore if the integral is invariant under the translation $y \longmapsto y-x_0$ ; i.e. if

$$\int_B F(y)dW_c(y) = \int_A F(x+x_0)\, dW_c(x) ,$$

then it follows from (23) that

$$\int_A F(x+x_0)dW_c(x) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_A F(x+x_0)\exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x).$$

for any measurable set $A \subseteq C$ and for any integrable functional F on C, in particular for $F \equiv 1$. Therefore

$$\int_A dW_c(x) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt\right\} \int_A \exp\left\{-\frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} dW_c(x),$$

for any Wiener measurable set $A \subseteq C$.

If we put $S(x) = \exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt - \frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\}$,

then for all Wiener measurable set $A \subseteq C$ we have that

$\int_A (1-S(x))dW_c(x) = 0$. It follows from Corollary 5.9 and (1.38)

that $1 - S(x) = 0$ a.e. on C. But $1 - S(x)$ is also continuous,

according to Remark 5.11 we have that $S(x) \equiv 1$ for all $x \in C$ ; i.e.

$$\exp\left\{-\frac{1}{c}\int_0^1 (x_0'(t))^2 dt - \frac{2}{c}\int_0^1 x_0'(t)dx(t)\right\} = \exp(0) \quad \text{for all } x \in C.$$

Thus $\frac{1}{c}\int_0^1 (x_0'(t))^2 dt + \frac{2}{c}\int_0^1 x_0'(t)dx(t) = 0$ for all $x \in C$,

in particular for $x \equiv 0$. Therefore $\frac{1}{c}\int_0^1 (x_0'(t))^2 dt = 0$ which

implies that $x_0 \equiv$ constant. But $x_0 \equiv$ constant implies that

$x_0 \equiv 0$, since $x_0(0) = 0$.

Q.E.D.