CHAPTER III

CONSTRUCTION OF WIENER MEASURE

The purpose of this chapter is to define, for each $c > 0$, a set function W_c on the algebra $\mathcal{J}$ of subsets of C and then prove that this set function is indeed a measure. The main theorem in this chapter is about the countably additivity of the set function.

Definition of Wiener measure for C

Definition 3.1. Let c be a positive constant and let $W_c : \mathcal{J} \longrightarrow R^*$. If $I \in \mathcal{J}$ is defined by (2.1), then $W_c(I)$ is given by

$$W_c(I) = \int \cdots \int_E K\left\{t_1,\ldots,t_n, \xi_1,\ldots, \xi_n\right\} \, d\xi_1 \ldots d\xi_n \quad \ldots\ldots\ldots(1)$$

where

$$K\left\{t_1,\ldots,t_n, \xi_1,\ldots, \xi_n\right\} = \frac{1}{\sqrt{\pi^n c^n t_1(t_2-t_1)\ldots(t_n-t_{n-1})}}$$

$$\cdot\exp\left\{-\sum_{j=1}^{n}\frac{(\xi_j-\xi_{j-1})^2}{ct_j-ct_{j-1}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

with the understanding that $\xi_0 = 0 = t_0$.

(Throughout this thesis, let c be an arbitrary positive number).

In view of the fact that $W_c(I)$ for the interval I as given by (2.1) is defined by means of the restriction points and the restricting set that appear in (2.1) and since, by Remark 2.6, the expression of an interval by restriction points and restricting set is not unique, so it is necessary to prove that the value of $W_c(I)$ is independent of the choice of restriction points and restricting set that describe I.

Uniqueness of the definition of the set function W_c.

To prove the uniqueness of the definition of $W_c(I)$ we show that when a finite number of the points of the interval $(0,1]$ are added and trivial restrictions are added at the additional restriction points, the value of $W_c(I)$ remains unchanged. By induction, it suffices to consider in the case of only one additional point t^* satisfying $t_{k-1} < t^* < t_k$ together with trivial restriction at t^*.

Proof. With the partition points $t_1, \ldots, t_{k-1}, t^*, t_k, \ldots, t_n$,

$W_c(I)$ is given according to (1) and (2) by

$$W_c(I) = \int_E \cdots \int K\left\{t_1, \ldots, t_{k-1}, t_k, \ldots, t_n, \xi_1, \ldots, \xi_{k-1}, \xi_k, \ldots, \xi_n\right\} \cdot L\left\{t_{k-1}, t^*, t_k, \xi_{k-1}, v, \xi_k\right\} d\xi_1 \ldots d\xi_n$$

where $$L\left\{t_{k-1}, t^*, t_k, \xi_{k-1}, v, \xi_k\right\} = \sqrt{\frac{(t_k - t_{k-1})}{\pi c(t^* - t_{k-1})(t_k - t^*)}} \cdot \int_{-\infty}^{\infty} \exp\left\{\frac{-(v - \xi_{k-1})^2}{ct^* - ct_{k-1}} - \frac{(\xi_k - v)^2}{ct_k - ct^*} + \frac{(\xi_k - \xi_{k-1})^2}{ct_k - ct_{k-1}}\right\} dv$$

$$= \sqrt{\frac{(t_k - t_{k-1})}{\pi c(t^* - t_{k-1})(t_k - t^*)}} \cdot \exp\left\{\frac{(\xi_k - \xi_{k-1})^2}{ct_k - ct_{k-1}}\right\} \cdot \int_{-\infty}^{\infty} \exp\left\{\frac{-(v - \xi_{k-1})^2}{ct^* - ct_{k-1}} - \frac{(\xi_k - v)^2}{ct_k - ct^*}\right\} dv$$

...........(3)

Thus, for the uniqueness of W_c we only have to show that

$$L\left\{t_{k-1},\ t^*,\ t_k,\ \xi_{k-1},\ v,\ \xi_k\right\} = 1.$$

For the integrands in (3) we write

$$\frac{(v-\xi_{k-1})^2}{ct^* - ct_{k-1}} + \frac{(\xi_k - v)^2}{ct_k - ct^*} = \frac{v^2 - 2\xi_{k-1}v + \xi_{k-1}^2}{ct^* - ct_{k-1}} + \frac{\xi_k^2 - 2\xi_k v + v^2}{ct_k - ct^*}$$

$$= Av^2 + 2Bv + D \quad = \quad A\left(v + \frac{B}{A}\right)^2 + \left(D - \frac{B}{A}^{2}\right)$$

where

$$A = \frac{1}{ct^* - ct_{k-1}} + \frac{1}{ct_k - ct^*} = \frac{ct_k - ct_{k-1}}{(ct^* - ct_{k-1})(ct_k - ct^*)},$$

$$B = \frac{-\xi_{k-1}}{ct^* - ct_{k-1}} - \frac{\xi_k}{ct_k - ct^*},$$

$$D = \frac{\xi_{k-1}^2}{ct^* - ct_{k-1}} + \frac{\xi_k^2}{ct_k - ct^*}.$$

Hence

$$\frac{B^2}{A} = \left[\frac{\xi_{k-1}^2}{(ct^* - ct_{k-1})^2} + \frac{\xi_k^2}{(ct_k - ct^*)^2} + \frac{2\xi_{k-1}\xi_k}{(ct^* - ct_{k-1})(ct_k - ct^*)}\right] \cdot \frac{(ct^* - ct_{k-1})(ct_k - ct^*)}{(ct_k - ct_{k-1})}$$

$$= \frac{\xi_{k-1}^2(ct_k - ct^*)}{(ct^* - ct_{k-1})(ct_k - ct_{k-1})} + \frac{\xi_k^2(ct^* - ct_{k-1})}{(ct_k - ct^*)(ct_k - ct_{k-1})} + \frac{2\xi_{k-1}\xi_k}{(ct_k - ct_{k-1})}$$

$$D - \frac{B^2}{A} = \frac{\xi_{k-1}^2(ct_k - ct_{k-1}) - \xi_{k-1}^2(ct_k - ct^*)}{(ct^* - ct_{k-1})(ct_k - ct_{k-1})}$$

$$+ \frac{\xi_k^2\,(ct_k - ct_{k-1}) - \xi_k^2\,(ct^* - ct_{k-1})}{(ct_k - ct^*)(ct_k - ct_{k-1})} - \frac{2\,\xi_{k-1}\,\xi_k}{(ct_k - ct_{k-1})}$$

$$= \frac{(\xi_{k-1} - \xi_k)^2}{ct_k - ct_{k-1}} \quad .$$

Thus

$$\int_{-\infty}^{\infty} \exp\left\{\frac{-(v - \xi_{k-1})^2}{ct^* - ct_{k-1}} - \frac{(\xi_k - v)^2}{ct_k - ct^*}\right\} dv = \exp\left\{\frac{-(\xi_{k-1} - \xi_k)^2}{ct_k - ct_{k-1}}\right\}$$

$$\cdot \int_{-\infty}^{\infty} \exp\left\{- A\,(v + \frac{B}{A})^2\right\} dv. \quad \ldots\ldots\ldots\ldots(4)$$

Since $\int_{-\infty}^{\infty} \exp\,(-\xi^2)d\xi = \sqrt{\pi}$, we have

$$\int_{-\infty}^{\infty} \exp\left\{- A(v + \frac{B}{A})^2\right\} dv = \int_{-\infty}^{\infty} \exp\left\{-\left(\sqrt{A}\left(v + \frac{B}{A}\right)\right)^2\right\}.\ \frac{1}{\sqrt{A}}\, d\left(\sqrt{A}\left(v + \frac{B}{A}\right)\right)$$

$$= \sqrt{\frac{\pi}{A}}$$

$$= \sqrt{\frac{\pi(ct^* - ct_{k-1})(ct_k - ct^*)}{ct_k - ct_{k-1}}} \quad \ldots\ldots\ldots\ldots(5)$$

It follows from (3), (4) and (5) that

$$L\left\{t_{k-1},\ t^*,\ t_k,\ \xi_{k-1},\ v,\ \xi_k\right\} = 1.$$

Q.E.D.

Properties of W_c

(i). From the definition of $W_c(I)$, the value of $W_c(I)$ is non-negative for any $I \in \mathcal{J}$.

(ii). Since $\emptyset = \{x \in C : x(t_1) \in \emptyset,\ 0 < t_1 \leq 1\}$, by (1),(2) and (1.31) we have

$$W_c(\emptyset) = \frac{1}{\sqrt{\pi c t_1}} \int_{\emptyset} \exp\left\{\frac{-\xi_1^2}{ct_1}\right\} d\xi_1 = 0.$$

(iii). Since $C = \{x \in C : x(t_1) \in R,\ 0 < t_1 \leq 1\}$ and $\int_{-\infty}^{\infty} \exp\{-\xi^2\} d\xi = \sqrt{\pi}$, by (1) and (2) we have

$$W_c(C) = \frac{1}{\sqrt{\pi c t_1}} \int_{R} \exp\left\{\frac{-\xi_1^2}{ct_1}\right\} d\xi_1 = 1.$$

(iv). W_c is additive, i.e. if $I_1, I_2 \in \mathcal{J}$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 \in \mathcal{J}$ then $W_c(I_1 \cup I_2) = W_c(I_1) + W_c(I_2)$.

Proof. By Remark 2.9, we may assume that

$$I_1 = \{x \in C : (x(t_1), \ldots, x(t_n)) \in E_1\} \quad \text{and}$$

$$I_2 = \{x \in C : (x(t_1), \ldots, x(t_n)) \in E_2\}.$$

Since $I_1 \cap I_2 = \emptyset$, by Lemma 2.10 (ii) we have $E_1 \cap E_2 = \emptyset$ and according to Lemma 2.10 (i), $I_1 \cup I_2 = \{x \in C : (x(t_1), \ldots, x(t_n)) \in E_1 \cup E_2\}$. Hence, by Definition 3.1 and (1.35), we obtain

$$W_c(I_1 \sqcup I_2) = \int_{E_1 \sqcup E_2} \cdots \int K\left\{t_1, \ldots, t_n, \xi_1, \ldots, \xi_n\right\} d\xi_1 \ldots d\xi_n$$

$$= \int_{E_1} \cdots \int K\left\{t_1, \ldots, t_n, \xi_1, \ldots, \xi_n\right\} d\xi_1 \ldots d\xi_n +$$

$$\int_{E_2} \cdots \int K\left\{t_1, \ldots, t_n, \xi_1, \ldots, \xi_n\right\} d\xi_1 \ldots d\xi_n$$

$$= W_c(I_1) + W_c(I_2).$$

<u>Q.E.D.</u>

By induction, we can show that W_c is <u>finite additive</u> ; i.e. if $I_1, \ldots, I_n \in \mathcal{J}$, $I_i \cap I_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} I_i \in \mathcal{J}$, then

$$W_c(I_1 \cup \ldots \cup I_n) = W_c(I_1) + \ldots + W_c(I_n).$$

(v). W_c is <u>countably additive</u>.

<u>Theorem 3.2</u>. Let γ be a fixed constant satisfying $0 < \gamma < \frac{1}{2}$ and a be an arbitrary positive number. Let a subset of C, A_a be defined by

$$A_a = \left\{ x \in C : |x(t_2) - x(t_1)| \leq a c^{\gamma} (t_2 - t_1)^{\gamma}, \quad 0 \leq t_1 < t_2 \leq 1 \right\}.$$

Then for any quasi-interval I disjoint from A_a

$$W_c(I) < c^{*} a^{-4/(1-2\gamma)}$$

where c^{*} is a positive number and is independent from a.

The proof of countably additive of W_c is based on this estimate and is given in Theorem 3.13. We prove this theorem by showing that if an element x of C does not belong to A_a then it belongs to an element of a finite collection of quasi-intervals $\{J_n\}$ so that if $I \cap A_a = \emptyset$ then $I \subset \bigcup_n J_n$ and $W_c(I) \leq \sum_n W_c(J_n)$. The quasi-intervals J_n are so chosen that $\sum_n W_c(J_n) < c^* a^{-4/(1-2\gamma)}$. The above is done by means of Lemma 3.5-3.9.

<u>Notation</u>. Let n be a positive integer and consider a sequence of numbers $0 = u_0 < u_1 < \ldots < u_{2^n} = 1$ where $u_j = \frac{j}{2^n}$ for $j = 0,2,4,\ldots, 2^n$. Let $\mathcal{P}^k$ be the collection of points of $[0,1]$ defined by

$$\mathcal{P}^k = \left\{ P^k\{l\} = u_{2^k l} \;,\quad l = 0,1,2,\ldots, 2^{n-k} \right\},\ k = 0,1,\ldots,n.$$

<u>For example</u>, $\mathcal{P}^0 = \{ P^0\{l\} = u_l \;,\quad l = 0,1,\ldots, 2^n \}$

$$= \{u_0, u_1, \ldots, u_{2^n}\}.$$

$$\mathcal{P}^1 = \{ P^1\{l\} = u_{2l} \;,\quad l = 0, 1, \ldots, 2^{n-1} \}$$

$$= \{u_0, u_2, u_4, \ldots, u_{2^n}\}.$$

We shall often write P^k to mean a member of $\mathcal{P}^k$.

Let $X^k\{l\}$ be the straight segment connecting the two points $P^k\{l-1\}$ and $P^k\{l\}$ for $l = 1,2,\ldots,2^{n-k}$, $k = 0,1,2,\ldots,n$. When we are not particularly concerned about their positions we will write merely X^k for $X^k\{l\}$.

The length of $X^k\{\iota\}$ will be denoted by λ_k. Then $\lambda_k = \frac{2^k}{2^n} = 2^{k-n}$ independently of ι in case $k = 1,\ldots,n$, but when $k = 0$, λ_k depends on ι and we only have the estimate $0 < \lambda_0 < \lambda_1 = 2^{1-n}$.

In connection with the above definitions we make a few remarks.

Remark 3.3. From the definition of $\mathcal{P}^k$ it follows that $\mathcal{P}^k \supset \mathcal{P}^{k+1}$, $k = 0,1,\ldots, n-1$. Thus for any point on $[0,1]$ which is also an element of $\mathcal{P}^0$, the superscript k is not unique. However the largest superscript for the point is unique and will be called the index of the point. For example, the index of u_1 is 0 and the index of any point u_ι where ι is an odd number in $\{1,2,\ldots, 2^n\}$ will also be 0, the index of u_2 is 1, the index of u_4 is 2, the index of u_6 is 1, the index of u_8 is 3.

Then it follows that if k_0 is the index of $P \in \mathcal{P}^0$, then $P \in \mathcal{P}^k$ for $k \leq k_0$ and $P \notin \mathcal{P}^k$ for $k > k_0$. On the other hand for a segment on $[0,1]$ which is an X^k the superscript k is unique and such a segment will be referred to as an edge of index k.

Remark 3.4. If $P = P^{k_0}\{\iota\}$ where k_0 is the index of P and $P \in X^k$ then either

(i) P is an end point of X^k and hence $P \in \mathcal{P}^k$ and $k \leq k_0$, or

(ii) P is not an end point of X^k and hence $P \notin \mathcal{P}^k$ and $k > k_0$.

For example, let $P = u_6 = P^1\{3\}$. Then u_6 is an end point of X^k for $k \leq 1$ and u_6 is not an end point of X^k for $k > 1$.

Lemma 3.5. Let f be a real-valued function defined on $\mathcal{P}^0$. Suppose f is such that if P_1', P_2' are the end points of some X^k then

$$\left| f(P_2') - f(P_1') \right| \leq bac^{\gamma} \left| P_2' - P_1' \right|^{\gamma}$$

where $a > 0$, $b = \dfrac{(1 - 2^{-\gamma})}{6}$ and $0 < \gamma < \dfrac{1}{2}$.

Then for any P_1'', $P_2'' \in \mathcal{P}^0$,

$$\left| f(P_2'') - f(P_1'') \right| \leq ac^{\gamma} \left| P_2'' - P_1'' \right|^{\gamma} .$$

Proof. Let P_1'', $P_2'' \in \mathcal{P}^0$ and $P_1'' = u_p$, $P_2'' = u_q$. If $p = q$, then $P_1'' = P_2''$ and hence $f(P_1'') - f(P_2'') = 0$. Therefore, without loss of generality, we may assume that $p < q$.

Step 1. Consider the set of integers $\{p, p+1, \ldots, q\}$. Each of these integers except possibly for p which may be 0, can be written as $2^k l$ with an odd integer l. For an odd integer the exponent k is 0 and for an even integer k can not be 0 and is the largest possible integer for that number because l is an odd integer. Let k^* be the greatest of all the unique exponents of $p, p+1, \ldots, q$. Let r be a member of the set $\{p, p+1, \ldots, q\}$ with this greatest exponent k^*. Then r is unique for the existence of two such numbers would contradict the choice of k^*. Let $r = 2^{k^*} l^*$. Then $p \leq r \leq q$ with $p < q$. Let $P_3 = u_r$ and consider $f(P_2'') - f(P_3)$. If $r = q$ then $P_2'' = P_3$ and $f(P_2'') - f(P_3) = 0$. Therefore, without loss of generality, we may assume that $r < q$ and express $q - r$ in the binary scale :

$$q - r = 2^{k_\alpha} + 2^{k_{\alpha-1}} + \ldots + 2^{k_1} + 2^{k_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

with $k_\alpha > k_{\alpha-1} > \ldots > k_1 > k_0 \geqslant 0$.

Step 2. We want to show that $k^* > k_\alpha$. Suppose on the contrary. Since $r = 2^{k^*}\ell^*$, by (6) we have $q - 2^{k_{\alpha-1}} - \ldots - 2^{k_1} - 2^{k_0} = 2^{k_\alpha} + 2^{k^*}\ell^*$.

$$\text{Let} \quad d = 2^{k_\alpha} + 2^{k^*}\ell^* . \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$\text{Then} \quad r < d < q . \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Case 1. If $k_\alpha = k^*$, then $d = 2^{k^*}(\ell^* + 1)$. ....(9)

Since ℓ^* is an odd integer, ℓ^*+1 is an even integer and hence $\ell^*+1 = 2^m\ell$ where ℓ is an odd integer and $m \geqslant 1$. But then, by (9), $d = 2^{k^*+m}\ell$. Since $q < 2^{k^*}(\ell^*+2)$, for otherwise the choice of r would be contradicted, it follows from (8) that

$$r < d = 2^{k^*+m}\ell = 2^{k^*}(\ell^*+1) < q < 2^{k^*}(\ell^*+2) < 2^{k^*+m}(\ell+2) ;$$

i.e. d is the only integer in the set $\{d,\ldots,q\}$ with the unique exponent

$$k^* + m > k^* \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

We also note that $p > 2^{k^*}(\ell^*-2)$, for otherwise the choice of r would be contradicted. Thus

$$2^{k^*+m}(\ell-2) < 2^{k^*}(\ell^*-2) < p < r < d = 2^{k^*+m}\ell ;$$

i.e. d is the only integer in the set $\{p,\ldots, d\}$ with the unique exponent

$$k^* + m > k^* \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

It follows from (10) and (11) that d is the only integer in the set $\{p,\ldots, q\}$ with the unique exponent $k^* + m > k^*$ which contradicts the choice of r. Therefore $k_\alpha \neq k^*$.

<u>Case 2</u>. If $k_\alpha > k^*$, then $k_\alpha - k^* > 0$ and hence $2^{k_\alpha - k^*} + \ell^*$ is an odd integer. But then by (7), $d = 2^{k^*}(2^{k_\alpha - k^*} + \ell^*)$, we have that d is the integer such that $r < d < q$ and has the exponent k^*. This contradicts the choice of r.

It follows from both cases that $k^* > k_\alpha$.

<u>Step 3</u>. Since $P_3 = u_r = u_{2^{k^*}\ell^*}$ and by (6) $P_2'' = u_q = u_{2^{k_0}\ell_0}$, we have P_3 is an end point of some X^{k^*} and P_2'' is an end point of some X^{k_0}. Since $P_2'' = u_{2^{k_0}\ell_0} = P^{k_0}\{\ell_0\}$, by (6) we have

$$P^{k_0}\{\ell_0 - 1\} = u_{2^{k_0}(\ell_0 - 1)} = u_{2^{k_0}\ell_0 - 2^{k_0}} = u_{2^{k_1}\ell_1} = P^{k_1}\{\ell_1\}.$$

Thus $P^{k_0}\{\ell_0 - 1\}$ is an end point of some X^{k_1} and has the index k_1. Also by (6), $P^{k_1}\{\ell_1 - 1\} = u_{2^{k_1}\ell_1 - 2^{k_1}} = u_{2^{k_2}\ell_2} = P^{k_2}\{\ell_2\}$ which is an end point of some X^{k_2} and has the index k_2. Thus, repeating the process we reach a point $P^{k_\alpha}\{\ell_\alpha\}$ which is an end point of some X^{k_α} and has the index k_α. But then by (6), $P^{k_\alpha}\{\ell_\alpha - 1\} = u_{2^{k_\alpha}\ell_\alpha - 2^{k_\alpha}}$ $= u_{2^{k^*}\ell^*} = u_r = P_3$ with the index k^*. In this manner we construct a sequence of points $\{P_2'' = P^{k_0}\{\ell_0\}, P^{k_1}\{\ell_1\}, \ldots, P^{k_\alpha}\{\ell_\alpha\}, P_3\}$ which have $k_0, k_1, \ldots, k_\alpha, k^*$ as their indices. Furthermore, any two successive points in the sequence are the end points of an edge whose index is equal to the index of the first of the two points by means of the definition of an edge. .... (12)

Let us use the sequence of points $P''_2, P^{k_1}\{l_1\}, \ldots, P^{k_\alpha}\{l_\alpha\}, P_3$ as intermediate points between P''_2 and P_3 in estimating $f(P''_2) - f(P_3)$. Then by the hypothesis and (12) and the fact that $\lambda_k = 2^{k-n}$ for $k \neq 0$, $0 < \lambda_0 < 2^{1-n}$,

$$\left|f(P''_2)-f(P_3)\right| \leq \left|f(P''_2)-f(P^{k_1}\{l_1\})\right| + \left|f(P^{k_1}\{l_1\})-f(P^{k_2}\{l_2\})\right|$$

$$+\ldots+\left|f(P^{k_\alpha}\{l_\alpha\}) - f(P_3)\right|$$

$$\leq bac^\gamma \left(\lambda^\gamma_{k_0} + \lambda^\gamma_{k_1} + \ldots + \lambda^\gamma_{k_\alpha}\right)$$

$$< bac^\gamma \sum_{i=0}^{k_\alpha} \lambda_i^\gamma$$

$$< bac^\gamma \left(2^{(1-n)\gamma} + \sum_{i=1}^{k_\alpha} 2^{(i-n)\gamma}\right). \quad \ldots\ldots\ldots\ldots\ldots (13)$$

Since $0 < \gamma < \frac{1}{2}$, $0 < 2^\gamma - 1 < 2^{\frac{1}{2}} - 1 < 1$, so that

$$2^\gamma < \frac{2^\gamma}{2^\gamma - 1} = \frac{1}{1-2^{-\gamma}}$$

and hence $2^{(1-n)\gamma} < \dfrac{2^{-n\gamma}}{1-2^{-\gamma}} = 2^{-n\gamma}\left(1+2^{-\gamma}+2^{-2\gamma}+\ldots..\right)$

$$= \sum_{i=0}^{-\infty} 2^{(i-n)\gamma} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

Thus, by (13) and (14) we have

$$|f(P_2'')- f(P_3)| < bac^\gamma \sum_{i=k_\alpha}^{-\infty} 2^{(i-n)\gamma}$$

$$= bac^\gamma \; 2^{(k_\alpha -n)\gamma} (1+2^{-\gamma}+2^{-2\gamma} + ...)$$

$$= bac^\gamma \; 2^{(k_\alpha -n)\gamma} \cdot \frac{1}{1-2^{-\gamma}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Step 4. We want to estimate $f(P_2'')-f(P_3)$ in term of $|P_2''- P_3|$.

Case 1. If $k_\alpha = 0$ or 1, then it follows from (6) that $q-r \leq 3$. But then by hypothesis

$$|f(P_2'')-f(P_3)| < 3\; bac^\gamma |P_2''- P_3|^\gamma < \frac{3bac^\gamma}{1-2^{-\gamma}} |P_2''- P_3|^\gamma .$$

Case 2. For $k_\alpha \geq 2$. Since if q is even, we have $u_q = \frac{q}{2^n} > \frac{(q-1)}{2^n}$ and if q is odd, $\frac{(q-1)}{2^n} < u_q < \frac{(q+1)}{2^n}$. Thus, $u_q > \frac{(q-1)}{2^n}$ whether q is even or odd and similarly $u_r < \frac{(r+1)}{2^n}$. Therefore according to (6) and the fact that $k_\alpha - 1 \geq 1$, we have

$$|P_2'' - P_3| = u_q - u_r \geq \frac{(q-1)-(r+1)}{2^n} = \frac{(q-r)- 2}{2^n}$$

$$\geq \frac{2^{k_\alpha} - 2}{2^n} \geq 2^{k_\alpha -n} - \frac{2}{2}^{k_\alpha -1} \cdot 2^{1-n}$$

$$= \frac{1}{2} \cdot 2^{(k_\alpha -n)}$$

Thus $$|P_2'' - P_3|^\gamma \geq \frac{1}{2} \cdot 2^{(k_\alpha -n)\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

By (15) and (16),

$$|f(P_2'') - f(P_3)| < \frac{2\,bac^{\gamma}}{1-2^{-\gamma}} |P_2''- P_3|^{\gamma} .$$

It follows from both cases that

$$|f(P_2'')- f(P_3)| < \frac{3\,bac^{\gamma}}{1-2^{-\gamma}} |P_2'' - P_3|^{\gamma} \quad . \ldots\ldots\ldots\ldots\ldots(17)$$

Step 5. Exactly the same estimate as (17) holds for the pair P_1'' and P_3 and from the facts that $|P_1''- P_2''| \geq \max\{|P_2''- P_3| , |P_1''- P_3|\}$ and $b = (1-2^{-\gamma})/6$ we have

$$\begin{aligned} |f(P_2'')-f(P_1'')| &\leq |f(P_2'')-f(P_3)| + |f(P_3)-f(P_1'')| \\ &< \frac{6\,bac^{\gamma}}{1-2^{-\gamma}} |P_1''- P_2''|^{\gamma} \\ &= ac^{\gamma} |P_1''- P_2''|^{\gamma} , \end{aligned}$$

and the equality holds when $P_1'' = P_2''$. Therefore

$$|f(P_2'') - f(P_1'')| \leq ac^{\gamma} |P_1''- P_2''|^{\gamma} .$$

Q.E.D.

Lemma 3.6. Let a real number be assigned to each element of $\mathscr{P}^{\circ}$. Let f be a function defined on [0,1] in such a way that f takes on the preassigned values on $\mathscr{P}^{\circ}$, is linear on each X^0. Then for any P,Q on the X^0 with the end points Q_1 and Q_2,

$$|f(P) - f(Q)| = \frac{|f(Q_2)- f(Q_1)|}{|Q_1- Q_2|} \cdot |P-Q| .$$

Proof. Let P and Q be any two points on the X° with the end points Q_1 and Q_2. Since f is linear on the X°, all of the values of f on the X° must lie on the same line L which has the equation

$$\frac{f(P^*)-f(Q_1)}{P^*-Q_1} = \frac{f(Q_2)-f(Q_1)}{Q_2-Q_1}$$

where $(P^*, f(P^*))$ is any point on L.

Thus, $$f(P^*) = \frac{f(Q_2)-f(Q_1)}{Q_2-Q_1} \cdot (P^*-Q_1)+f(Q_1).$$

Since $(P, f(P))$ and $(Q, f(Q))$ are any points on L, we have

$$f(P) = \frac{f(Q_2)-f(Q_1)}{Q_2-Q_1} \cdot (P-Q_1)+f(Q_1)$$

and $$f(Q) = \frac{f(Q_2)-f(Q_1)}{Q_2-Q_1} \cdot (Q-Q_1)+f(Q_1) \quad .$$

Therefore,

$$|f(P)-f(Q)| = \frac{|f(Q_2)-f(Q_1)|}{|Q_2-Q_1|} \cdot |P-Q| \, .$$

Q.E.D.

Lemma 3.7. Let a real number be assigned to each element of $\mathscr{P}^{\circ}$ and let f be a function defined as in Lemma 3.6. Assume more that for any P''_1, $P''_2 \in \mathscr{P}^{\circ}$

$$|f(P''_2)-f(P''_1)| \leqslant ac^{\gamma} \, |P''_1-P''_2|^{\gamma} \, . \quad \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

Then for any P_1, $P_2 \in [0,1]$,

$$|f(P_2)-f(P_1)| \leq 3ac^{\gamma} |P_1 - P_2|^{\gamma} .$$

Proof. Let P_1, $P_2 \in [0,1]$ be arbitrary given and consider $f(P_2)-f(P_1)$ where $P_2 \neq P_1$.

Case 1. If the segment P_1P_2 is contained in one X°.

Let P_3, P_4 be the end points of that X°. Then by Lemma 3.6 and (18), we have

$$\begin{aligned} |f(P_2)-f(P_1)| &= \frac{|P_1 - P_2|}{|P_3 - P_4|} |f(P_4)-f(P_3)| \\ &\leq \frac{|P_1 - P_2|^{\gamma}}{|P_3 - P_4|^{\gamma}} |f(P_4)-f(P_3)| \\ &\leq ac^{\gamma} |P_1 - P_2|^{\gamma} . \end{aligned}$$

Case 2. If the segment P_1P_2 is contained in two adjacent X°.

Using the intersection of the segment P_1P_2 with the boundaries of the two X° as intermediate point between P_1 and P_2. By case 1, we have

$$|f(P_2)-f(P_1)| \leq 2ac^{\gamma} |P_1 - P_2|^{\gamma} .$$

Case 3. If the segment P_1P_2 is not contained in one or two adjacent X°.

Let X_1° and X_2° be the two X° which contain P_1 and P_2 respectively. Let the intersections of the segment P_1P_2 with the boundaries of X_1° and X_2° be P_{11} and P_{21} respectively. Using P_{11} and P_{21} as intermediate points between P_1 and P_2. Then by case 1,

$$|f(P_1)-f(P_{11})| \leq ac^{\gamma}|P_1-P_{11}|^{\gamma} \quad \text{and}$$

$$|f(P_2)-f(P_{21})| \leq ac^{\gamma}|P_2-P_{21}|^{\gamma} \quad \text{and by hypothesis}$$

$$|f(P_{11})-f(P_{21})| \leq ac^{\gamma}|P_{11}-P_{21}|^{\gamma} \quad . \text{ Furthermore}$$

$$|P_1-P_2| > |P_{11}-P_{21}| \text{ and } |P_{11}-P_{21}| \geq \max\left\{|P_1-P_{11}|, |P_{21}-P_2|\right\}.$$

Therefore

$$\begin{aligned}|f(P_2)-f(P_1)| &\leq |f(P_2)-f(P_{21})| + |f(P_{21})-f(P_{11})| + |f(P_{11})-f(P_1)| \\ &\leq ac^{\gamma}|P_{21}-P_2|^{\gamma} + ac^{\gamma}|P_{11}-P_{21}|^{\gamma} + ac^{\gamma}|P_{11}-P_1|^{\gamma} \\ &< 3ac^{\gamma}|P_1-P_2|^{\gamma}.\end{aligned}$$

It follows from all cases that

$$|f(P_2)-f(P_1)| < 3ac^{\gamma}|P_1-P_2|^{\gamma}$$

where $P_1 \neq P_2$ are any points in $[0,1]$ and the equality holds when $P_1 = P_2$. Thus

$$|f(P_2)-f(P_1)| \leq 3ac^{\gamma}|P_1-P_2|^{\gamma} \quad ; \; P_1, P_2 \in [0,1].$$

Q.E.D.

Lemma 3.8. Let f be a real-valued function defined on $[0,1]$ and linear on each X°. Furthermore let f be such that if P_1', P_2' are the end points of some X^k

$$|f(P_2')-f(P_1')| \leq \frac{1}{3 \times 6}(1-2^{-\gamma})\, ac^{\gamma}|P_1'-P_2'|^{\gamma} \quad \text{holds.}$$

Then for any $P_1, P_2 \in [0,1]$

$$|f(P_2)-f(P_1)| \leq ac^{\gamma}|P_1-P_2|^{\gamma}.$$

Proof. Let P_1', P_2' are the end points of some X^k.

Then by Lemma 3.5,

$$|f(P_2'')-f(P_1'')| \leq \frac{1}{3} ac^{\gamma} |P_1''-P_2''|^{\gamma} \quad \text{for any } P_1'', P_2'' \in \mathcal{P}^{*}.$$

By Lemma 3.7,

$$|f(P_2)-f(P_1)| \leq ac^{\gamma} |P_1-P_2|^{\gamma} \quad \text{for any } P_1, P_2 \in [0,1].$$

Q.E.D.

Lemma 3.9. Let $0 \leq t_1 < t_2 \leq 1$. Then for the quasi-interval

$$I = \{x \in C : |x(t_2)-x(t_1)| > ac^{\gamma}(t_2-t_1)^{\gamma}\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

$$W_c(I) \leq \exp\left\{\frac{-a^2}{(ct_2-ct_1)^{1-2\gamma}}\right\}.$$

Proof. Let $0 \leq t_1 < t_2 \leq 1$.

Case 1. If $t_1 = 0$, then the set I given by (19) is a quasi-interval with restriction point t_2 and restricting set $E_1 = \{\xi \in R : |\xi| > b_1\}$ where $b_1 = ac^{\gamma}t_2^{\gamma}$, which is an open set and hence a Borel set in R, i.e. $I = \{x \in C : x(t_2) \in E_1\}$. According to (1) and (2),

$$W_c(I) = \frac{1}{\sqrt{\pi c t_2}} \int_{|\xi|>b_1} \exp\left\{\frac{-\xi^2}{ct_2}\right\} d\xi = \frac{2}{\sqrt{\pi c t_2}} \int_{b_1}^{\infty} \exp\left\{\frac{-\xi^2}{ct_2}\right\} d\xi .$$

But then by letting $v_1 = \xi - b_1$, we have

$$W_c(I) = \frac{2}{\sqrt{\pi c t_2}} \int_0^{\infty} \exp\left\{\frac{-(v_1+b_1)^2}{ct_2}\right\} dv_1$$

$$= \frac{2}{\sqrt{\pi c t_2}} \exp\left\{\frac{-b_1^2}{ct_2}\right\} \int_0^{\infty} \exp\left\{\frac{-v_1^2-2b_1v_1}{ct_2}\right\} dv_1$$

$$\leqslant \frac{2}{\sqrt{\pi c t_2}} \exp\left\{\frac{-b_1^2}{ct_2}\right\} \int_0^\infty \exp\left\{\frac{-v_1^2}{ct_2}\right\} dv_1 \ . \quad \text{...........(20)}$$

Since $\displaystyle\int_0^\infty \exp\left\{\frac{-v_1^2}{ct_2}\right\} dv_1 = \frac{\sqrt{\pi c t_2}}{2}$ and $b_1 = ac^{\gamma} t_2^{\gamma}$,

(20) becomes

$$W_c\ (I) \leqslant \exp\left\{\frac{-b_1^2}{ct_2}\right\} = \exp\left\{\frac{-a^2}{(ct_2 - ct_1)^{1-2\gamma}}\right\} .$$

<u>Case 2</u>. If $t_1 \neq 0$, then the set I given by (19) is a quasi-interval with restriction points t_1, t_2 and restricting set

$E = \{ (\xi_1, \xi_2) \in R^2 : |\xi_2 - \xi_1| > ac^{\gamma}(t_2 - t_1)^{\gamma} \}$ which is an open set and hence a Borel set in R^2, i.e. $I = \{x \in C : (x(t_1), x(t_2)) \in E \}$.
According to (1) and (2),

$$W_c\ (I) = \frac{1}{\sqrt{\pi^2 c^2 t_1 (t_2 - t_1)}} \iint_E \exp\left\{\frac{-\xi_1^2}{ct_1} - \frac{(\xi_2 - \xi_1)^2}{ct_2 - ct_1}\right\} d\xi_1\, d\xi_2 .$$

With the transformation $v_2 = \xi_1$, $v_3 = \xi_2 - \xi_1$, the restricting set E of I becomes $E_2 = \{(v_2, v_3) \in R^2 : -\infty < v_2 < \infty,\ |v_3| > b_2 \}$ where $b_2 = ac^{\gamma}(t_2 - t_1)^{\gamma}$. Since $\dfrac{\partial(\xi_1, \xi_2)}{\partial(v_1, v_2)} = 1$, we have

$$W_c\ (I) = \frac{1}{\sqrt{\pi^2 c^2 t_1 (t_2 - t_1)}} \iint_{E_2} \exp\left\{\frac{-v_2^2}{ct_1} - \frac{v_3^2}{ct_2 - ct_1}\right\} dv_2\, dv_3$$

$$= \frac{1}{\sqrt{\pi^2 c^2 t_1 (t_2 - t_1)}} \int_{-\infty}^{\infty} \exp\left\{\frac{-v_2^2}{ct_1}\right\} dv_2 \cdot \int_{|v_3| > b_2} \exp\left\{\frac{-v_3^2}{ct_2 - ct_1}\right\} dv_3 .$$

$$\text{.................(21)}$$

To consider $\int_{|v_3| > b_2} \exp\left\{\frac{-v_3^2}{ct_2-ct_1}\right\} dv_3$, let $v_4 = v_3 - b_2$.

Then

$$\int_{|v_3| > b_2} \exp\left\{\frac{-v_3^2}{ct_2-ct_1}\right\} dv_3 = 2 \int_0^{\infty} \exp\left\{\frac{-(v_4+b_2)^2}{ct_2-ct_1}\right\} dv_4$$

$$= 2\exp\left\{\frac{-b_2^2}{ct_2-ct_1}\right\} \int_0^{\infty} \exp\left\{\frac{-v_4^2-2b_2v_4}{ct_2-ct_1}\right\} dv_4$$

$$\leq 2\exp\left\{\frac{-b_2^2}{ct_2-ct_1}\right\} \int_0^{\infty} \exp\left\{\frac{-v_4^2}{ct_2-ct_1}\right\} dv_4$$

$$= \exp\left\{\frac{-b_2^2}{ct_2-ct_1}\right\} \cdot \sqrt{\pi c(t_2-t_1)} ,$$

and since $\int_{-\infty}^{\infty} \exp\left\{\frac{-v_2^2}{ct_1}\right\} dv_2 = \sqrt{\pi ct_1}$, we have that (21) becomes

$$W_c(I) \leq \exp\left\{\frac{-b_2^2}{ct_2-ct_1}\right\} = \exp\left\{\frac{-a^2}{(ct_2-ct_1)^{1-2\gamma}}\right\} .$$

<u>Q.E.D.</u>

<u>Proof of Theorem 3.2.</u> Let $I \in \mathcal{J}$ be such that $I \cap A_a = \emptyset$. Since $I \in \mathcal{J}$, there exists a collection of points $0 < s_1 < \ldots < s_m \leq 1$ and a set $E \in \mathcal{B}(R^m)$ such that $I = \{x \in C : (x(s_1),\ldots,x(s_m)) \in E\}$. Let n be so large that

$$2^{1-n} < \min \{s_1, s_2-s_1, \ldots, s_m-s_{m-1}\}$$

and $2^{1-n} < 1-s_m$ in case $s_m \neq 1$.

Partition the interval $[0,1]$ into the subintervals of length 2^{1-n} by the partition points $u_{2\ell} = \frac{2\ell}{2^n}$, $\ell = 0,1,\ldots,2^{n-1}$. So each interval $(u_{2(\ell-1)}, u_{2\ell})$ contains at most one s_i, $i = 1,\ldots,m$. If it contains one s_i let $u_{2\ell-1} = s_i$; otherwise let $u_{2\ell-1} = \frac{u_{2(\ell-1)} + u_{2\ell}}{2}$. Then

$$\{s_0, s_1,\ldots, s_m\} \subset \{u_0, u_1,\ldots, u_{2^n}\} = \mathscr{P}^\circ.$$

According to Remark 2.6, let $\{u_1,\ldots, u_{2^n}\}$ be the restriction points of I by adding trivial restrictions at the additional restriction points and extending E to a Borel set E^* in R^{2^n}.

For any $x \in I$, let x^* be such that $x^* = x$ on $\mathscr{P}^\circ$ and x^* is linear on each X^ℓ. Since x^* agrees with x on the collection of restriction points $\{u_1,\ldots, u_{2^n}\}$ of I, $x^* \in I$ and hence $x^* \notin A_a$. Thus, by Lemma 3.8, there exist P_1^*, P_2^* such that P_1^* and P_2^* are the end points of some X^k and

$$|x^*(P_2^*) - x^*(P_1^*)| > \frac{1}{3\times 6}(1-2^{-\gamma})ac^\gamma |P_1^* - P_2^*|^\gamma .$$

Since x agree with x^* on $\mathscr{P}^\circ$ which contains endpoints of X^k,

$$|x(P_2^*) - x(P_1^*)| > \frac{1}{3\times 6}(1-2^{-\gamma})\, ac^\gamma\, |P_1^* - P_2^*|^\gamma .$$

Therefore $x \in J\{k,\ell\}$ for some $k = 0,1,\ldots,n$; $\ell = 1,\ldots,2^{n-k}$ where $J\{k,\ell\} = \{x \in C : |x(P^k\{\ell\}) - x(P^k\{\ell-1\})| > bc^\gamma(P^k\{\ell\} - P^k\{\ell-1\})^\gamma\}$ and $b = \frac{1}{3\times 6}(1-2^{-\gamma})\, a$. Since x is an arbitrary element in I,

$$I \subset \bigcup_{k=0}^{n} \bigcup_{\ell=1}^{2^{n-k}} J\{k, \ell\}$$

and from the finite additivity of W_c

$$W_c\,(I) \leq \sum_{k=0}^{n} \sum_{\ell=1}^{2^{n-k}} W_c(J\{k,\ell\}).$$

But then, by Lemma 3.9, we have

$$W_c\,(I) \leq \sum_{k=0}^{n} \sum_{\ell=1}^{2^{n-k}} \exp\left\{\frac{-b^2}{(cu_{2^k\ell} - cu_{2^k(\ell-1)})^{1-2\gamma}}\right\}.$$

<u>Case 1</u>. For $k = 0$, $u_\ell - u_{\ell-1} = \eta_0 < 2^{1-n}$ and hence

$$\frac{-b^2}{(cu_\ell - cu_{\ell-1})^{1-2\gamma}} < -\,b^2c^{2\gamma-1}2^{(n-1)(1-2\gamma)}\ .$$

Since the exponential function is increasing, we have

$$\sum_{\ell=1}^{2^n} \exp\left\{\frac{-b^2}{(cu_\ell - cu_{\ell-1})^{1-2\gamma}}\right\} < \sum_{\ell=1}^{2^n} \exp\left\{-b^2c^{2\gamma-1}\,2^{(n-1)(1-2\gamma)}\right\}$$

$$= 2^n \exp\left\{-b^2c^{2\gamma-1}\,2^{(n-1)(1-2\gamma)}\right\}.$$

<u>Case 2</u>. For $k \geq 1$, $u_{2^k\ell} - u_{2^k(\ell-1)} = \eta_k = 2^{k-n}$ and hence

$$\sum_{k=1}^{n} \sum_{\ell=1}^{2^{n-k}} \exp\left\{\frac{-b^2}{(cu_{2^k\ell} - cu_{2^k(\ell-1)})^{1-2\gamma}}\right\} = \sum_{k=1}^{n} \sum_{\ell=1}^{2^{n-k}} \exp\left\{-b^2c^{2\gamma-1}2^{(n-k)(1-2\gamma)}\right\}$$

$$= \sum_{k=1}^{n} 2^{n-k} \exp\left\{-b^2c^{2\gamma-1}2^{(n-k)(1-2\gamma)}\right\}.$$

It follows from both cases that

$$W_c(I) < 2^n \exp\left\{-b^2c^{2\gamma-1}2^{(n-1)(1-2\gamma)}\right\} + \sum_{k=1}^{n} 2^{n-k} \exp\left\{-b^2c^{2\gamma-1}2^{(n-k)(1-2\gamma)}\right\}.$$

Since $\sum_{k=1}^{n} 2^{n-k} \exp\left\{-b^2c^{2\gamma-1}2^{(n-k)(1-2\gamma)}\right\} = 2^{n-1}\exp\left\{-b^2c^{2\gamma-1}2^{(n-1)(1-2\gamma)}\right\}$

$+2^{n-2} \exp\left\{-b^2c^{2\gamma-1}\, 2^{(n-2)(1-2\gamma)}\right\} + \ldots + 2^{o}\exp\left\{-b^2c^{2\gamma-1}\, 2^{o}\right\}$

$= 2^{o} \exp\left\{-b^2c^{2\gamma-1}2^{o}\right\} +\ldots+ 2^{n-2} \exp\left\{-b^2c^{2\gamma-1}\, 2^{(n-2)(1-2\gamma)}\right\} +$
$2^{n-1} \exp\left\{-b^2c^{2\gamma-1}\, 2^{(n-1)(1-2\gamma)}\right\}$

$= \sum_{k=1}^{n} 2^{k-1} \exp\left\{-b^2 c^{2\gamma-1}\, 2^{(k-1)(1-2\gamma)}\right\}$ and we notice that

$2^{n} \exp\left\{-b^2 c^{2\gamma-1}\, 2^{(n-1)(1-2\gamma)}\right\}$ is two times the n^{th} member

of $\sum_{k=1}^{n} 2^{k-1} \exp\left\{-b^2 c^{2\gamma-1}\, 2^{(k-1)(1-2\gamma)}\right\}$, it follows that

$$W_c(I) < 2 \times 2^{n-1}\exp\left\{-b^2c^{2\gamma-1}2^{(n-1)(1-2\gamma)}\right\} + \sum_{k=1}^{n} 2^{k-1}\exp\left\{-b^2c^{2\gamma-1}2^{(k-1)(1-2\gamma)}\right\}$$

$$< 2\sum_{k=1}^{n} 2^{k-1} \exp\left\{-b^2c^{2\gamma-1}2^{(k-1)(1-2\gamma)}\right\} + \sum_{k=1}^{n} 2^{k-1} \exp\left\{-b^2c^{2\gamma-1}2^{(k-1)(1-2\gamma)}\right\}$$

$$< 3\sum_{k=1}^{\infty} 2^{k-1} \exp\left\{-b^2c^{2\gamma-1}\, 2^{(k-1)(1-2\gamma)}\right\}$$

$$= 3\sum_{k=0}^{\infty} 2^{-k}\, 2^{2k} \exp\left\{-b^2c^{2\gamma-1}\, 2^{k(1-2\gamma)}\right\} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

To estimate the last series let us consider the function

$$\psi(s) = s^2 \exp\left\{-b^2c^{-\lambda}s^{\lambda}\right\} \quad \text{for } s \geqslant 0 \text{ with } \lambda > 0,$$

$$\psi'(s) = (2s - \lambda b^2c^{-\lambda}s^{\lambda+1}) \exp\left\{-b^2 c^{-\lambda}s^{\lambda}\right\}.$$

Let $\psi'(s) = 0$, then $2s - \lambda b^2 c^{-\lambda} s^{\lambda+1} = 0$ which implies that $s = 0$ or $2 - \lambda b^2 c^{-\lambda} s^{\lambda} = 0$.

For $s = 0$, $\psi(s) = 0$ and for $2 - \lambda b^2 c^{-\lambda} s^{\lambda} = 0$ we have

$s = c\left(\frac{2}{\lambda b^2}\right)^{\frac{1}{\lambda}}$ which implies that

$$\psi(s) = c^2 \left(\frac{2}{\lambda b^2}\right)^{\frac{2}{\lambda}} \exp\left\{\frac{-2}{\lambda}\right\} = c^2(\lambda b^2 e/2)^{-\frac{2}{\lambda}}.$$ Thus

we have $s\text{-max} = c\left(\frac{2}{\lambda b^2}\right)^{\frac{1}{\lambda}}$ and $\psi(s\text{-max}) = c^2(\lambda b^2 e/2)^{-\frac{2}{\lambda}}$.

Therefore, if we let $s = 2^k$ and $\lambda = 1-2\gamma$ then

$$2^{2k} \exp\left\{-b^2 c^{2\gamma-1} 2^{k(1-2\gamma)}\right\} \leq c^2 \left\{\frac{(1-2\gamma)\, b^2 e}{2}\right\}^{-2/(1-2\gamma)} \quad \text{.........(23)}$$

By (22) and (23)

$$W_c(I) < 3c^2 \left\{\frac{(1-2\gamma) b^2 e}{2}\right\}^{-2/(1-2\gamma)} \sum_{k=0}^{\infty} 2^{-k}$$

$$= 6c^2 \left\{\frac{(1-2\gamma) b^2 e}{2}\right\}^{-2/(1-2\gamma)}$$

$$= 6c^2 \left\{\frac{(1-2\gamma)(1-2^{-\gamma})^2 e}{2 \times (3 \times 6)^2}\right\}^{-2/(1-2\gamma)} \cdot a^{-4/(1-2\gamma)}$$

$$= c^* a^{-4/(1-2\gamma)},$$

where $c^* = 6c^2 \left\{\frac{(1-2\gamma)(1-2^{-\gamma})^2 e}{2 \times (3 \times 6)^2}\right\}^{-2/(1-2\gamma)}$ is independent of a.

<u>Q.E.D.</u>

Lemma 3.10. The subset A_a of C defined by

$$A_a = \{x \in C : |x(P_2) - x(P_1)| \leq ac^{\gamma}|P_2 - P_1|^{\gamma} \ ; P_1, P_2 \in [0,1]\}$$

is compact.

Proof. First, we want to show that A_a is uniformly bounded.

Let x be any element in A_a. Then

$$|x(P_2) - x(P_1)| \leq ac^{\gamma}|P_2 - P_1|^{\gamma}, \quad \text{for any } P_1, P_2 \in [0,1].$$

In particular, $|x(P)-x(0)| \leq ac^{\gamma}P^{\gamma} \leq ac^{\gamma}$, for any $P \in [0,1]$. Thus $|x(P)| \leq ac^{\gamma}$, for all $x \in A_a$ and for all $P \in [0,1]$. Therefore A_a is uniformly bounded.

Second, to show that A_a is equicontinuous. Given $\epsilon > 0$ and choose $\delta = \frac{1}{c}\left(\epsilon/a\right)^{1/\gamma}$. Then for all $x \in A_a$,

$$|x(P_2)-x(P_1)| \leq ac^{\gamma}|P_2 - P_1|^{\gamma} < ac^{\gamma}\delta^{\gamma} = ac^{\gamma}\frac{\epsilon}{ac^{\gamma}} = \epsilon \quad \text{whenever}$$

$|P_1 - P_2| < \delta$.

Hence A_a is equicontinuous. Since $A_a \subset \mathscr{C}^{\circ}[0,1]$, by Arzéla-Ascoli Theorem, A_a is relatively compact.

It remains to show that A_a is a closed subset of $\mathscr{C}^{\circ}[0,1]$. Let $\{x_n\}$ be any sequence in A_a such that $x_n \longrightarrow x$ where $x \in \mathscr{C}^{\circ}[0,1]$ is a limit point of A_a, i.e. $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that $\{x_n\}$ converges uniformly to x on $[0,1]$. Hence x is continuous and $x(0) = 0$. So $x \in C$. Since $\{x_n\}$ converges uniformly to x and $\epsilon > 0$ is given, there exists an integer n_0 such that

$$|x_n(P) - x(P)| < \epsilon/2 \quad \text{for all } P \in [0,1] \text{ and for all } n \geqslant n_0 . \quad \ldots\ldots\ldots\ldots\ldots(24)$$

Since $x_n \in A_a$,

$$|x_n(P_2) - x_n(P_1)| \leqslant ac^\gamma |P_2 - P_1|^\gamma \text{ for any } P_1, P_2 \in [0,1]. \quad \ldots\ldots\ldots(25)$$

From (24) and (25) we have

$$\begin{aligned} |x(P_2)-x(P_1)| &\leqslant |x(P_1)-x_n(P_1)| + |x_n(P_1)-x_n(P_2)| + |x_n(P_2)-x(P_2)| \\ &< \frac{\epsilon}{2} + ac^\gamma |P_1 - P_2|^\gamma + \frac{\epsilon}{2} \\ &= ac^\gamma |P_1 - P_2|^\gamma + \epsilon \qquad \text{for any } P_1, P_2 \in [0,1]. \end{aligned}$$

Since ϵ is arbitrary, $x \in A_a$ and hence A_a is closed. Therefore $\bar{A}_a = A_a$ and hence A_a is compact .

Q.E.D.

Corollary 3.11. Let $J \in \mathcal{J}$ be defined by s restriction points $t_1, \ldots, t_s$ and a set G closed in the usual topology of R^s. Then $J \cap A_a$ is compact.

Proof. Since a set closed in the usual topology of a Euclidean space is a Borel set, J is indeed a quasi-interval. Let $\{x_n\}$ be any sequence in $J \cap A_a$. Then $\{x_n\}$ is a sequence in A_a and by Lemma 3.10 there exists a sub-sequence $\{y_n\}$ of $\{x_n\}$ which converges uniformly to an element $x_0 \in A_a$. It remains to show that $x_0 \in J$ also. Since $y_n \longrightarrow x_0$ uniformly on $[0,1]$, $y_n(t) \longrightarrow x_0(t)$ for all $t \in [0,1]$ and in particular on the restriction points ; i.e. $y_n(t_i) \longrightarrow x_0(t_i)$ for $i = 1, \ldots, s$.

Hence $(y_n(t_1),\ldots,y_n(t_s)) \longrightarrow (x_0(t_1),\ldots,x_0(t_s))$. Since $\{x_n\}$ is a sequence in J and $\{y_n\}$ is a subsequence of $\{x_n\}$, $\{y_n\}$ is a sequence in J. Thus $(y_n(t_1),\ldots, y_n(t_s)) \in G$ and since G is closed, $(x_0(t_1),\ldots, x_0(t_s)) \in G$ which implies that $x_0 \in J$ and hence $x_0 \in J \cap A_a$. Therefore $J \cap A_a$ is compact.

Q.E.D.

Lemma 3.12. Let $I \in \mathcal{I}$ be defined by s restriction points $t_1,\ldots,t_s$ and a Borel set E of R^s as restricting set. Then for any $\epsilon > 0$ there exists a closed set $G \subseteq E$ such that the quasi-interval J defined by the same s restriction points and with G as restricting set satisfies

$$W_c(I - J) < \epsilon .$$

Proof. Given $\epsilon > 0$ and let $\epsilon_0 = \epsilon\{\pi^s c^s t_1(t_2-t_1)\ldots(t_s-t_{s-1})\}^{\frac{1}{2}}$. Since $E \in \mathcal{B}(R^s)$, E is a Lebesgue measurable set and hence there exists a closed set $G \subseteq E$ such that (Leb.) $\bar{m}(E - G) < \epsilon_0$. Let

$$J = \{x \in C : (x(t_1),\ldots, x(t_s)) \in G\}.$$

Since by hypothesis, $I = \{x \in C : (x(t_1),\ldots,x(t_s)) \in E\}$, it follows from Lemma 2.10 (iv) that

$$I - J = \{x \in C : (x(t_1),\ldots,x(t_s)) \in E - G\} .$$

By (1) and (2),

$$W_c(I - J) = \int\cdots\int_{E - G} K\{t_1,\ldots,t_s, \xi_1,\ldots, \xi_s\}\; d\xi_1\ldots d\xi_s$$

where $K\{t_1,\ldots,t_s, \xi_1,\ldots, \xi_s\} = \dfrac{1}{\{\pi^s c^s t_1(t_2-t_1)\ldots(t_s-t_{s-1})\}^{1/2}}$

$$\cdot \exp\left\{\frac{-\xi_1^2}{ct_1} \cdots\cdots \frac{(\xi_s-\xi_{s-1})^2}{ct_s - ct_{s-1}}\right\}.$$

Since $K\{t_1,\ldots,t_s, \xi_1,\ldots, \xi_s\} \leq \dfrac{1}{\{\pi^s c^s t_1(t_2-t_1)\ldots(t_s-t_{s-1})\}^{1/2}}$

for all $(\xi_1,\ldots,\xi_s) \in E - G$,

it follows that

$$W_c(I - J) < \frac{1}{\{\pi^s c^s t_1(t_2-t_1)\ldots(t_s-t_{s-1})\}^{1/2}} \cdot \epsilon_0 = \epsilon .$$

Q.E.D.

Theorem 3.13. The set function W_c is countably additive on $\mathcal{J}$; i.e. if a sequence of quasi-intervals $\{I_n\}$ in $\mathcal{J}$ is such that $I_i \cap I_j = \emptyset$ for $i \neq j$ and $I = \bigcup_{n=1}^{\infty} I_n \in \mathcal{J}$, then $W_c(I) = \sum_{n=1}^{\infty} W_c(I_n)$.

Proof. We divide the proof into three steps :

Step 1. Since $\mathcal{J}$ is an algebra of sets, we have $I - \bigcup_{j=1}^{n} I_j \in \mathcal{J}$.

Let $J_n = (I - \bigcup_{j=1}^{n} I_j)$. ....(26)

Then $\bigcap_{n=1}^{\infty} J_n = \emptyset$, for otherwise if $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$ then there exists an element $x \in \bigcap_{n=1}^{\infty} J_n$. But then $x \in J_n$ for all $n = 1,2,\ldots$. It follows that $x \in I$ but $x \notin \bigcup_{j=1}^{n} I_j$ for all $n = 1,2,\ldots$. Therefore $x \in I$ but $x \notin I_j$ for all $j = 1,2,\ldots$, which is a contradiction .

By (26), $\{J_n\}$ is a monotonically decreasing sequence. Thus $\{J_n\}$ is a convergent sequence and $\lim_{n\to\infty} J_n = \bigcap_{n=1}^{\infty} J_n = \emptyset$. From (26) we also have that

$$\left(\bigcup_{j=1}^{n} I_j\right) \cap J_n = \emptyset \quad \text{and} \quad I = \left(\bigcup_{j=1}^{n} I_j\right) \cup J_n$$

and since W_c is finite additive,

$$W_c(I) = W_c\left(\bigcup_{j=1}^{n} I_j\right) + W_c(J_n) = \sum_{j=1}^{n} W_c(I_j) + W_c(J_n), \quad \text{for all } n .$$

Thus for each n, $0 \leq \sum_{j=1}^{n} W_c(I_j) \leq W_c(I)$; i.e. for each n ,

$\sum_{j=1}^{n} W_c(I_j)$ is a bounded (non-negative) monotically increasing sequence of real numbers and hence converges. Therefore for the proof of the countably additivity of W_c we only have to show that $\lim_{n\to\infty} W_c(J_n) = 0$, i.e. given $\epsilon > 0$ there exists an integer n_0 such that $W_c(J_n) < \epsilon$ for all $n \geq n_0$.

Step 2. Since $J_n \in \mathcal{J}$, there exists a sequence of points $0 = t_0^n < t_1^n < \ldots < t_{s_n}^n \leq 1$ and a Borel set E_n of R^{s_n} such that $J_n = \{x \in C : (x(t_1^n),\ldots,x(t_{s_n}^n)) \in E_n\}$. By Lemma 3.12, given $\epsilon > 0$ there exists a closed set $G_n \subseteq E_n$ such that the quasi-interval K_n defined by the same s_n restriction points and the restricting set G_n satisfies $W_c(J_n - K_n) < \epsilon/2^{n+1}$.

Let $L_n = \bigcap_{j=1}^{n} K_j$. Then $L_n \in \mathcal{J}$ and by Lemma 2.10 (iii), L_n has the closed set G which is the intersection of the closed sets $G_1, \ldots, G_n$ (all being raised to the same Euclidean dimensionality) as its restricting set. Hence $J_n - L_n \in \mathcal{J}$ and $L_n \subseteq K_n \subseteq J_n$........(27)

Since $J_n = (J_n - L_n) \cup L_n$ and W_c is finite additive,

$$W_c(J_n) = W_c(J_n - L_n) + W_c(L_n). \qquad \text{..........................(28)}$$

Since

$$J_n - L_n = J_n - \bigcap_{j=1}^{n} K_j \subset \bigcup_{j=1}^{n} J_j - \bigcap_{j=1}^{n} K_j$$

$$= \bigcup_{j=1}^{n} J_j \cap \bigcup_{j=1}^{n} (C - K_j)$$

$$= \bigcup_{j=1}^{n} (J_j - K_j),$$

and from the finite additivity of W_c, we have

$$W_c(J_n - L_n) \leqslant \sum_{j=1}^{n} W_c(J_j - K_j) < \sum_{j=1}^{n} \frac{\epsilon}{2^{j+1}} < \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2} \text{ for all n.}$$

................(29)

Step 3. It remains to show that there exists an integer n_0 such that $W_c(L_n) < \frac{\epsilon}{2}$ for all $n \geqslant n_0$ by means of Theorem 3.2.

First, we want to show that there exists an integer n_0 such that $L_n \cap A_a = \emptyset$ for all $n \geqslant n_0$. Let $M_n = L_n \cap A_a$. Since $\{L_n\}$ is monotonically decreasing, so is $\{M_n\}$ and hence $\{M_n\}$ is a convergent sequence and $\lim_{n\to\infty} M_n = \bigcap_{n=1}^{\infty} M_n$. Since $M_n \subseteq L_n$ and by (27) $L_n \subseteq J_n$, $M_n \subseteq J_n$. Thus, $\lim_{n\to\infty} M_n = \bigcap_{n=1}^{\infty} M_n \subseteq \bigcap_{n=1}^{\infty} J_n = \emptyset$ . ...(30)

Claim that there exists an integer n_0 such that $M_n = \emptyset$ for all $n \geqslant n_0$. Suppose on the contrary, then since $\{M_n\}$ is monotonically decreasing, $M_n \neq \emptyset$ for all n. Let $x_n \in M_n$ and consider the sequence $\{x_n\}$. Since $M_n = L_n \cap A_a$, $\{x_n\}$ is in A_a and by Lemma 3.10 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow x_0$ uniformly in [0,1] where $x_0 \in A_a$. Hence

$$x_{n_k}(P) \longrightarrow x_0(P) \quad \text{for all } P \in [0,1]. \quad \ldots\ldots\ldots\ldots\ldots\ldots(31)$$

To show that $x_0 \in M_{n_k}$ for all n_k, let n_{k_0} be arbitrarily fixed. Since $\{M_{n_k}\}$ is monotonically decreasing, $x_{n_k} \in M_{n_{k_0}}$ for all $n_k \geqslant n_{k_0}$. Since $M_{n_{k_0}} = L_{n_{k_0}} \cap A_a$, by Corollary 3.11 we have that $M_{n_{k_0}}$ is compact and hence there exists a subsequence $\{x_{n_{k_\ell}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_\ell}} \longrightarrow x_0^*$ uniformly on [0,1] and $x_0^* \in M_{n_{k_0}}$. Thus

$$x_{n_{k_\ell}}(P) \longrightarrow x_0^*(P) \quad \text{for all } P \in [0,1]. \quad \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

By virtue of (31) and the fact that $\{x_{n_{k_\ell}}\}$ is a subsequence of $\{x_{n_k}\}$ we have that the sequence of numbers $\{x_{n_{k_\ell}}(P)\}$ being a subsequence of $\{x_{n_k}(P)\}$ converges to $x_0(P)$ for all $P \in [0,1]$. ....(33)

From (32) and (33) we have $x_0^*(P) = x_0(P)$ for all $P \in [0,1]$ and hence $x_0 \in M_{n_{k_0}}$. Since n_{k_0} is arbitrary, $x_0 \in M_{n_k}$ for all n_k. Thus x_0 belongs to infinitely many M_n and hence $x_0 \in \overline{\lim_{n\to\infty}} M_n$. Since $\{M_n\}$ is a convergent sequence, it follows from (30) that $x_0 \in \overline{\lim_{n\to\infty}} M_n = \lim_{n\to\infty} M_n = \emptyset$, which is a contradiction.

Therefore there must exists an integer n_0 such that $M_n = \emptyset$ for all $n \geqslant n_0$ and hence by Theorem 3.2, $W_c(L_n) < c^* a^{-4/(1-2\gamma)}$ for all $n \geqslant n_0$, where c^* is a positive number and is independent of a. Choose $a > 0$ be so large that

$$W_c (L_n) < \frac{\epsilon}{2} \quad \text{for all} \quad n \geqslant n_0 \quad \text{.........................(34)}$$

It follows from (28),(29) and (34) that

$$W_c (J_n) < \epsilon \quad \text{for all} \quad n \geqslant n_0 \text{ .}$$

Q.E.D.

According to (1.8), properties (i), (ii) and (v) imply that W_c is a measure on $\mathcal{J}$ and together with property (iii) it is a probability measure. Since c is an arbitrary positive number, W_c is a measure on $\mathcal{J}$ for all $c > 0$.