CHAPTER I

PRELIMINARIES

In this thesis, we assume a basis knowledge of real analysis. However, we will recall, without proof, some notions and facts from measure and integration theory.

The materials of this chapter are drawn from references [2], [3], [4], [5], [6] and [7].

A. General Measure and Integration Theory, The Lebesque Measure.

(1) A collection $\mathscr{S}$ of subsets of a set X is called a semialgebra of sets if,

i) the intersection of any two sets in $\mathscr{S}$ is again in $\mathscr{S}$; and

ii) the complement of any set in $\mathscr{S}$ is a finite disjoint union of sets in $\mathscr{S}$.

(2) A non-empty collection $\mathscr{A}$ of subsets of set X is called an algebra of sets if,

i) $E \in \mathscr{A}$ and $F \in \mathscr{A}$, then $E \cup F \in \mathscr{A}$, and

ii) $E \in \mathscr{A}$, then $E' \in \mathscr{A}$ where $E' = X \sim E$.

It follows that an algebra $\mathscr{A}$ of sets is also closed under the formation of intersections and differences, and $\emptyset \in \mathscr{A}$, $X \in \mathscr{A}$.

(3) If $\mathscr{S}$ is any semialgebra of sets, then the collection $\mathscr{A}$ consisting of the empty set and all finite disjoint union of sets in $\mathscr{S}$ is an algebra of sets which is called the algebra generated by $\mathscr{S}$.

(4) If $\mathcal{A}$ is an algebra of subsets of a set X such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$ whenever $A_k \in \mathcal{A}$, then $\mathcal{A}$ is called a <u>σ-algebra of sets</u> in X.

(5) Let $\mathcal{F}$ be any collection of subsets of a set X and let $\mathcal{A}$ be a σ-algebra of subsets of X such that

i) $\mathcal{F} \subseteq \mathcal{A}$,

ii) if $\mathcal{A}^*$ is any other σ-algebra containing $\mathcal{F}$ then $\mathcal{A} \subseteq \mathcal{A}^*$.

The σ-algebra $\mathcal{A}$, <u>the smallest σ-algebra containing $\mathcal{F}$</u>, is called <u>the σ-algebra generated by $\mathcal{F}$</u> and will be denoted by $\sigma[\mathcal{F}]$.

The collection $\mathcal{B}(X)$ of <u>Borel sets</u> of a topological space X is the smallest σ-algebra which contains all of the open sets in X. It is also the smallest σ-algebra which contains all closed sets in X.

(6) A <u>set function</u> is a function whose domain of definition is a collection of sets.

(7) An extended real-valued set function μ defined on a collection $\mathcal{A}$ of subsets of a set X is <u>additive</u> if, for any $E \in \mathcal{A}$, $F \in \mathcal{A}$, $E \cup F \in \mathcal{A}$ and $E \cap F = \emptyset$ we have $\mu(E \cup F) = \mu(E) + \mu(F)$.

μ is countably additive if, for every disjoint sequence $\{A_k\}$ of sets in $\mathcal{A}$ whose union is also in $\mathcal{A}$ we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

(8) An extended real-valued set function μ defined on an algebra $\mathcal{A}$ of subsets of a set X is called a <u>measure</u> if it is non-negative, countably additive and such that $\mu(\emptyset) = 0$. μ is said to be <u>finite</u> if $\mu(X) < \infty$ and <u>σ-finite</u> if there exists a sequence $\{A_n\}$ of sets in $\mathcal{A}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$, $n = 1,2,\ldots$.

μ is called <u>a probability measure</u> if $\mu(X) = 1$. μ is called <u>complete</u> if the conditions $E \in \mathcal{A}$, $F \subset E$ and $\mu(E) = 0$ imply that $F \in \mathcal{A}$.

(9) If μ is a non-negative set function defined on a semialgebra of sets $\mathcal{S}$ with $\mu(\emptyset) = 0$ (if $\emptyset \in \mathcal{S}$).

Then μ has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{S}$ if the following conditions are satisfied :

i). If a set A in $\mathcal{S}$ is the union of a finite disjoint collection $\{A_i\}$ of sets in $\mathcal{S}$, then $\mu(A) = \sum \mu(A_i)$.

ii). If a set A in $\mathcal{S}$ is the union of a countable disjoint collection $\{A_i\}$ of sets in $\mathcal{S}$, then $\mu(A) \leq \sum \mu(A_i)$.

Condition i) implies that, if A is the union of each of two finite disjoint collection $\{A_i\}$ and $\{B_i\}$ of sets in $\mathcal{S}$, then $\sum \mu(B_i) = \sum \mu(A_i)$. Condition ii) implies that μ is countably additive on $\mathcal{A}$.

(10) Let μ^* be an extended real-valued set function whose domain is the class $\mathcal{P}X$ of all subsets of X. We say that μ^* is an <u>outer measure</u> if it satisfies the following properties :

i) $\mu^*(\emptyset) = 0$,

ii) If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.

iii) $\mu^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$.

(11) Let μ be a measure on an algebra $\mathcal{A}$ of sets. If we define

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(A_k), \quad E \in \mathcal{P}X$$

where $\{A_k\}$ ranges over all sequences from $\mathcal{A}$ such that $E \subseteq \bigcup_{k=1}^{\infty} A_k$. Then μ^* is an outer measure induced by μ, it is an extension of μ to $\mathcal{P}X$; if μ is (finite) σ-finite, then so is μ^*. Moreover, if $E \in \mathcal{A}$ then $\mu^*(E) = \mu(E)$.

(12) A set E in X is said to be <u>measurable</u> with respect to μ^* if for every set A in X we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E).$$

The collection $\mathcal{M}$ of all μ^*-measurable sets is a σ-algebra of sets.

While outer measure has the advantage that it is defined for all sets, it is not countably additive. It becomes countably additive, however, if we suitably reduce the family of sets on which it is defined. Perhaps, the best way of doing this is due to Carathéodory :

(13) <u>Carathéodory's Theorem</u> : Let μ be a measure on an algebra $\mathcal{A}$, and μ^* the outer measure induced by μ. Then the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a

σ-algebra $\mathcal{M}$ containing $\mathcal{A}$. If μ is finite (or σ-finite) so is $\bar{\mu}$. If μ is σ-finite, then $\bar{\mu}$ is the only measure on the smallest σ-algebra containing $\mathcal{A}$ which is an extension of μ. Moreover, $\bar{\mu}$ is a complete measure on $\mathcal{M}$.

(14) If μ is a σ-finite measure on an algebra $\mathcal{A}$, and if μ^* is the outer measure induced by μ, then the collection $\mathcal{M}$ of all μ^*-measurable sets is identical with the collection of all sets of the form $E \cup M$, where $E \in \sigma[\mathcal{A}]$ and M is a subset of a set of measure zero in $\sigma[\mathcal{A}]$.

(15) By a <u>measurable space</u> we mean a couple $(X, \mathcal{M})$ consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X.

By a <u>measure space</u> $(X, \mathcal{M}, \mu)$ we mean a measurable space $(X, \mathcal{M})$ together with a measure μ defined on $\mathcal{M}$.

(16) If E_1, E_2 are measurable, $E_1 \subseteq E_2$ and E_1 has finite measure, then $E_2 - E_1$ is measurable and $\mu(E_2 - E_1) = \mu(E_2) - \mu(E_1)$.

(17) A set $E \subseteq R^n$, the n-dimensional Euclidean space, is called a <u>rectangle</u> if there exist intervals (open, closed, half-open) $A_1, \ldots$ $\ldots, A_n$ in R^n such that $E = A_1 \times \ldots \times A_n$ where $A_1 \times \ldots \times A_n$ $= \{(\alpha_1, \ldots, \alpha_n) : \alpha_i \in A_i,\ i = 1, \ldots, n\}$.

For any rectangle E in R^n, we define

$$m(E) = \prod_{i=1}^{n} (b_i - a_i)$$

where $(b_i - a_i)$ is the length of the interval A_i.

(18) By the outer measure of a set $A \subseteq R^n$ is meant the number $m^*(A) = \inf \sum_{k=1}^{\infty} m(E_k)$ where $\{E_k\}$ ranges over all sequences from the collection $\mathcal{E}$ of rectangles such that $A \subseteq \bigcup_{k=1}^{\infty} E_k$. The restriction $\bar{m}$ of m^* to the m^*-measurable sets is an extension of m to a σ-algebra $\mathcal{L}$ containing the collection $\mathcal{E}$ of rectangles in R^n. $\bar{m}$ is called the <u>Lebesgue measure</u> on R^n and the members of $\mathcal{L}$ are the <u>Lebesgue measurable sets</u> which contain all Borel sets in R^n (Note that by (5), we also have $\mathcal{B}(R^n) = \sigma[\mathcal{E}]$).

(19) If $E \subseteq R^n$ is a Lebesgue measurable set, then given $\epsilon > 0$ there exists a closed set $F \subseteq E$ such that $\bar{m}(E \sim F) < \epsilon$.

(20) Let $(X, \mathcal{M}, \mu)$ be a measure space, a function $f : X \longrightarrow R^*$ (the set of extended real numbers) is said to be <u>measurable</u> if the set $\{a : f(a) < \alpha\} \in \mathcal{M}$, for each $\alpha \in R$. The proposition remains true if $<$ is replaced by $\leqslant$, $\geqslant$ or $>$.

(21) Let $(X, \mathcal{M}, \mu)$ be a measure space, with X a topological space such that each open set belongs to $\mathcal{M}$. Then $f : X \longrightarrow R^*$ is measurable if f is continuous.

(22) If f and g are measurable functions on X, then so is fg.

(23) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \longrightarrow R^*$. We define $f^+(a) = \max\{f(a), 0\}$ and $f^-(a) = \max\{-f(a), 0\}$, so that $f = f^+ - f^-$. Then f is measurable if and only if both f^+ and f^- are measurable.

(24) Let $(X, \mathcal{M}_1, \mu)$ and $(Y, \mathcal{M}_2, \eta)$ be measure spaces, a transformation T from X into Y is said to be measurable if the inverse image of every measurable set is measurable.

(25) The characteristic function χ_A of a set A is defined by

$$\chi_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases} .$$

Then χ_A is measurable if and only if A is measurable.

(26) Let $E_1, \ldots, E_n$ be any finite collection of disjoint measurable sets ; and let $\alpha_1, \ldots, \alpha_n$ be a set of finite real numbers. A function f defined by

$$f(a) = \sum_{k=1}^{n} \alpha_k \chi_{E_k}(a)$$

is called a simple function, i.e. every simple function is a finite linear combination of characteristic functions of disjoint measurable sets and a simple function is bounded and measurable.

(27) Every extended real-valued measurable function f is the limit of a sequence $\{f_n\}$ of simple functions; if f is non-negative, then each f_n may be taken non-negative and the sequence $\{f_n\}$ may be assumed non-decreasing.

(28) Let $(X, \mathcal{M}, \mu)$ be a measure space. If s is a simple function on X, of the form

$$s = \sum_{k=1}^{n} \alpha_k \chi_{E_k} ,$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s, and if $E \in \mathcal{M}$, we define

$$\int_E s \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(E_k \cap E). \qquad \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

The convention $0. \infty = 0$ is used here ; it may happen that $\alpha_k = 0$ for some k and that $\mu(E_k \cap E) = \infty$. If $\alpha_j \neq \alpha_k$ for $j \neq k$, then any other description of s as a linear combination of characteristic functions amounts to a partitioning of the sets E_k. From this comment and the additivity of μ , it follows at once that $\int_E s \, d\mu$ is independent of the description of s in terms of characteristic functions.

(29) Let f be a nonnegative extended real-valued measurable function on the measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, we define

$$\int_E f \, d\mu = \sup \int_E s \, d\mu , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (**)$$

the supremum being taken over all simple functions s such that $0 \leq s \leq f$.

Observe that we apparently have two definitions for $\int_E f \, d\mu$ if f is simple, namely, (*) and (**). However, these assign the same value to the integral, since f is, in this case, the largest of the functions s which occur on the right of (**).

The left member of (**) is called <u>the integral of f over E</u>, with respect to the measure μ. It is a number in $[0,\infty]$. If in (**), $E \subseteq R^n$ and μ is the Lebesgue measure then the integral is called <u>the Lebesgue integral of f over E</u> which we will usually denote by $\int_E f(t)dt$. If $E = [a,b]$, we write $\int_a^b f(t)dt$ instead of $\int_E f(t)dt$.

(30) If $f \geqslant 0$, then $\int_E f\,d\mu = \int_X \chi_E f d\mu$.

This result shows that we could have restricted our definition of integration to integrals over all of X, without losing any generality. If we wanted to integrate over subsets, we could then use (30) as the definition.

(31) If $\mu(E) = 0$, then $\int_E f\,d\mu = 0$ even if $f(a) = \infty$ for every $a \in E$.

(32) If $0 \leq f \leq g$, then $0 \leq \int_E f\,d\mu \leq \int_E g\,d\mu$.

It follows that the definition of the integral of a non-negative measurable function in (**) is somewhat awkward to apply, since we are taking a supremum over a large collection of simple functions. Consequently, we begin our treatment of the integral by establishing the convergence theorem. This then enables us to determine the value of $\int_E f\,d\mu$ by taking the limit of $\int_E s_n\,d\mu$ for any increasing sequence $\{s_n\}$ of simple functions which converges to f.

(33) Lebesgue's Monotone Convergence Theorem :

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a nondecreasing sequence of nonnegative measurable functions which converges pointwise to a function f over X. Then $\lim\limits_{n\to\infty} \int_X f_n d\mu = \int_X f\, d\mu$.

Note that according to Lebesgue's Monotone Convergence Theorem, we also have that if $\{f_n\}$ is a nonincreasing sequence of nonnegative measurable functions which converges pointwise to a function f on X and $\int_X f_1\, d\mu < \infty$, then $\lim\limits_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$.

(34) A non-negative function f is called integrable (over a measurable set E with respect to μ) if it is measurable and $\int_E f\, d\mu < \infty$.

If f is any measurable function and at least one of the functions f^+ and f^- is integrable on E, we define

$$\int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu .$$

We say that f is integrable on E if both f^+ and f^- are integrable on E.

(35) Let $E_1, \ldots, E_n$ be disjoint measurable sets with $E_0 = \bigcup\limits_{k=1}^{n} E_k$, and let f be a function which is integrable on each set E_k, then f is integrable on E_0 and

$$\int_{E_0} f\, d\mu = \sum_{k=1}^{n} \int_{E_k} f\, d\mu .$$

(36) Lebesgue's Dominated Convergence Theorem :

Let $(X, \mathcal{M}, \mu)$ be a measure space and let g be integrable on X. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on X and for almost everywhere on X we have $\lim_{n\to\infty} f_n(t) = f(t)$. Then

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n d\mu .$$

(A property is said to hold almost everywhere abbreviated a.e., if the set of points where it fails to hold is a set of measure zero.)

(37) If f is bounded and measurable on E, then f is integrable on E, provided $\mu(E) < \infty$.

(38) Let $(X, \mathcal{M}, \mu)$ be a measure space. If f is an integrable function such that $\int_E f\, d\mu = 0$ for every $E \in \mathcal{M}$, then $f = 0$ a.e. on X.

B. Relation Between Riemann and Lebesgue Integrals

(39) If f is Riemann integrable on $[a,b]$, then it is also Lebesgue integrable on $[a,b]$ and the two integrals are equal, i.e.,

$$\int_a^b f(t)\, dt = (R) \int_a^b f(t) dt$$

where the right hand side is the Riemann integral.

(40) A bounded function f is Riemann integrable on [a,b] if and only if the set of points at which f is discontinuous has measure zero. Hence, if f is continuous a.e. on [a,b] then f is Lebesgue integrable on [a,b] .

C. Integration on Product Spaces

(41) Let $(X, \mathcal{M}_1)$ and $(Y, \mathcal{M}_2)$ are measurable spaces. A measurable rectangle is any set of the form $A \times B$, where $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$.

$\mathcal{M}_1 \times \mathcal{M}_2$ is defined to be the smallest σ-algebra in $X \times Y$ which contains every measurable rectangle.

(42) If $E \subseteq X \times Y$, $u \in X$, $v \in Y$, we define

$$E_u = \{ v : (u,v) \in E \}, \qquad E^v = \{ u : (u,v) \in E \}.$$

We call E_u and E^v the u-section and v-section, respectively, of E.

If $E \in \mathcal{M}_1 \times \mathcal{M}_2$, then $E_u \in \mathcal{M}_2$ and $E^v \in \mathcal{M}_1$ for every $u \in X$ and $v \in Y$.

(43) Let f be a function on $X \times Y$. With each $u \in X$, we associate a function f_u defined on Y by $f_u(v) = f(u, v)$. Similarly, if $v \in Y$, f^v is the function defined on X by $f^v(u) = f(u,v)$. If f is an $(\mathcal{M}_1 \times \mathcal{M}_2)$-measurable function on $X \times Y$. Then f_u is a $\mathcal{M}_2$-measurable function on Y and f^v is a $\mathcal{M}_1$-measurable function on X.

(44) Let $(X, \mathcal{M}_1, \mu)$ and $(Y, \mathcal{M}_2, \pi)$ be σ-finite measure spaces. Suppose $Q \in \mathcal{M}_1 \times \mathcal{M}_2$. If

$$\varphi(u) = \pi(Q_u), \qquad \psi(v) = \mu(Q^v)$$

for every $u \in X$ and $v \in Y$, then φ is $\mathcal{M}_1$-measureable, ψ is $\mathcal{M}_2$-measurable and $\int_X \varphi \, d\mu = \int_Y \psi \, d\pi$.

i.e. $$\int_X d\mu(u) \int_Y \chi_Q(u,v) d\pi(v) = \int_Y d\pi(v) \int_X \chi_Q(u,v) d\mu(u).$$

(45) If $(X, \mathcal{M}_1, \mu)$ and $(Y, \mathcal{M}_2, \pi)$ are as in (44) and if $Q \in \mathcal{M}_1 \times \mathcal{M}_2$, we define

$$(\mu \times \pi)(Q) = \int_X \pi(Q_u) d\mu(u) = \int_Y \mu(Q^v) d\pi(v).$$

We call $\mu \times \pi$ the product of the measures μ and π, and $\mu \times \pi$ is a σ-finite measure on $\mathcal{M}_1 \times \mathcal{M}_2$.

(46) <u>The Fubini Theorem</u> : Let $(X, \mathcal{M}_1, \mu)$ and $(Y, \mathcal{M}_2, \pi)$ be two σ-finite measure spaces, and let f be a non-negative measurable function on $X \times Y$. Then

a.) $$\varphi(u) = \int_Y f_u \, d\pi \quad \text{is } \mathcal{M}_1\text{-measurable,}$$

$$\psi(v) = \int_X f^v \, d\mu \quad \text{is } \mathcal{M}_2\text{-measurable, } (u \in X,\ v \in Y).$$

b.) $$\int_X \varphi \, d\mu = \int_{X\times Y} f \, d(\mu \times \nu) = \int_Y \psi \, d\nu.$$

i.e. $$\int_X d\mu(u) \int_Y f(u,v) d\nu(v) = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y d\nu(v) \int_X f(u,v) d\mu(u)$$

The first and last integrals are called " iterated integrals " of f and the middle integral is often referred to as a " double integral ".

D. Riemann-Stieltjes Integral

(47) A real-valued function f defined on an interval $[a,b]$ is said to be of bounded variation if there is a constant $M > 0$ such that $\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \leq M$ for every partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$ by points of subdivisions $t_0, t_1, \ldots, t_n$.

(48) Every function of bounded variation on $[a,b]$ must be bounded on $[a,b]$.

(49) Let $f,g : [a,b] \longrightarrow R$ and subdivide $[a,b]$ by points $t_0, \ldots, t_n$ such that $a = t_0 < t_1 < \ldots < t_n = b$. Choose a point t_k^* in each $[t_{k-1}, t_k]$ and form the sum $\sigma = \sum_{k=1}^{n} f(t_k^*)[g(t_k) - g(t_{k-1})]$. If the sum σ approaches a limit as $l \to 0$, where $l = \max_k \{t_k - t_{k-1}\}$, independent of the choice of both the points of subdivisions t_k and the choice of t_k^*, then this limit is called the Riemann-Stieltjes integral of f with respect to g and is denoted by $\int_a^b f(t)dg(t)$.

(50) If f is continuous on [a,b] and g is of bounded variation on [a,b], then $\int_a^b f(t)dg(t)$ exists.

(51) If $\int_a^b f(t)dg(t)$ exists, then $\int_a^b g(t)df(t)$ exists

and $$\int_a^b f(t)dg(t) + \int_a^b g(t)df(t) = f(b)g(b) - f(a)g(a).$$

(52) $$\int_a^b f(t)d(g_1(t)+\alpha g_2(t)) = \int_a^b f(t)dg_1(t) + \alpha\int_a^b f(t)dg_2(t),$$

where α is a real number.

(53) If f is continuous on [a,b] and g is a differentiable function such that g' is Riemann integrable, then

$$\int_a^b f(t)dg(t) = (R)\int_a^b f(t)g'(t)dt.$$

(54) A function of bounded variation on [a,b] is Riemann integrable on [a,b].

(55) If g is a function of bounded variation on [a,b] and $\{f_n\}$ is a sequence of continuous functions which converges uniformly on [a,b] to f, then

$$\lim_{n\to\infty} \int_a^b f_n(t)dg(t) = \int_a^b f(t)dg(t).$$

(56) If f is continuous on [a,b] and if f' exists and is bounded in the interior, then f is of bounded variation on [a,b].

E. Limits of a Sequence of Sets

(57) Let $\{\Gamma_n\}$ be a sequence of sets. We define the superior limit and the inferior limit of the sequence as follows :

$$\overline{\lim_{n\to\infty}}\, \Gamma_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \Gamma_n \; ; \quad \underline{\lim_{n\to\infty}}\, \Gamma_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \Gamma_n \, .$$

(58) A sequence $\{\Gamma_n\}$ of sets is said to be <u>convergent</u> if

$$\overline{\lim_{n\to\infty}}\, \Gamma_n = \underline{\lim_{n\to\infty}}\, \Gamma_n \, .$$

For a convergent sequence, the set which appears as both $\overline{\lim_{n\to\infty}}\, \Gamma_n$ and $\underline{\lim_{n\to\infty}}\, \Gamma_n$ is called $\lim_{n\to\infty} \Gamma_n$.

(59) A monotone sequence of sets is one which is either an increasing sequence or a decreasing sequence, Every monotone sequence $\{\Gamma_n\}$ of sets is convergent and $\lim_{n\to\infty} \Gamma_n$ is equal to $\bigcup_n \Gamma_n$ or $\bigcap_n \Gamma_n$ according as the sequence is increasing or decreasing.