CHAPTER II

TRANSFORMATION SEMIGROUPS

Schein has shown in [8] that the full transformation semigroup on any set X is quasi-inverse. In this chapter, we prove that the partial transformation semigroup on any set X is also quasi-inverse. It is shown that the full transformation semigroup on a set X is orthodox if and only if the cardinality of X, $|X| \leq 2$, and the partial transformation semigroup on a set X is orthodox if and only if the cardinality of X, $|X| \leq 1$. Right-inverse transformation semigroups and generalized inverse transformation semigroups are also studied.

Let X be a set. Let $\mathcal{T}_X$ and T_X denote the full transformation semigroup on the set X and the partial transformation semigroup on the set X; respectively. Note that for any set X, T_X is a regular semigroup and $\mathcal{T}_X$ is a regular subsemigroup of T_X.

The following theorem has been proved by Schein in [8] :

2.1 <u>Theorem</u> [8]. For any set X, the full transformation semigroup on the set X, $\mathcal{T}_X$, is quasi-inverse.

Using Theorem 2.1, it can be shown that for any set X, T_X is also quasi-inverse.

2.2 Proposition. For any set X, the partial transformation semigroup on the set X, T_X, is quasi-inverse.

Proof : Let $\alpha \in T_X$. Then $\alpha \in \mathcal{T}_{\Delta\alpha}$. Since $\mathcal{T}_{\Delta\alpha}$ is quasi-inverse and $\alpha \in \mathcal{T}_{\Delta\alpha}$, there exists an inverse subsemigroup T of $\mathcal{T}_{\Delta\alpha}$ such that $\alpha \in T$. Because $\mathcal{T}_{\Delta\alpha}$ is a subsemigroup of T_X and T is an inverse subsemigroup of $\mathcal{T}_{\Delta\alpha}$ containing α, T is an inverse subsemigroup of T_X containing α. This proves that T_X is quasi-inverse as required. #

For any set X, let $|X|$ denote the cardinality of X. If X is a finite set and assume $X = \{1, 2, \ldots, n\}$, then for $k_1, k_2, \ldots, k_n \in X$, the notation $\alpha = (k_1, k_2, \ldots, k_n)$ denotes the map on X defined by $1\alpha = k_1$, $2\alpha = k_2$, ... , $n\alpha = k_n$. For convenience, for any set X, 0 and 1 are denoted for the zero and the identity of T_X; respectively.

In Chapter I, we show that if $X = \{1, 2, 3\}$, then $\mathcal{T}_X$ is not orthodox. The two following theorems show that for any set X, $\mathcal{T}_X$ is orthodox if and only if $|X| \leq 2$ and T_X is orthodox if and only if $|X| \leq 1$.

2.3 Theorem. For any set X, the full transformation semigroup on X, $\mathcal{T}_X$, is orthodox if and only if $|X| \leq 2$.

Proof : Assume $|X| \leq 2$. If $|X| = 0$ or $|X| = 1$, then $E(\mathcal{T}_X) = \mathcal{T}_X$, so $\mathcal{T}_X$ is a band and hence $\mathcal{T}_X$ is orthodox.

Assume $|X| = 2$, say $X = \{1, 2\}$. Then $\mathcal{T}_X = \{1, \alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (2\ 1)$, $\alpha_2 = (1\ 1)$ and $\alpha_3 = (2\ 2)$. Therefore $E(\mathcal{T}_X) =$

$\{1, \alpha_2, \alpha_3\}$. Because $\alpha_2\alpha_3 = \alpha_3$ and $\alpha_3\alpha_2 = \alpha_2$, $E(\mathcal{T}_X)$ is a subsemigroup of $\mathcal{T}_X$, so $\mathcal{T}_X$ is orthodox.

Conversely, suppose $|X| > 2$. Let a, b, c be three distinct elements in X. Define α and $\beta \in \mathcal{T}_X$ by

$$x\alpha = \begin{cases} c, & \text{if } x \neq b \\ b, & \text{if } x = b \end{cases}$$

for all $x \in X$ and

$$x\beta = \begin{cases} b, & \text{if } x \neq a \\ a, & \text{if } x = a \end{cases}$$

for all $x \in X$. Since $x\alpha^2 = (x\alpha)\alpha = c\alpha = c = x\alpha$ if $x \neq b$ and $b\alpha^2 = (b\alpha)\alpha = b\alpha$, $\alpha^2 = \alpha$ and so $\alpha \in E(\mathcal{T}_X)$. Because $x\beta^2 = (x\beta)\beta = b\beta = b = x\beta$ if $x \neq a$ and $a\beta^2 = (a\beta)\beta = a\beta$, $\beta^2 = \beta$, so $\beta \in E(\mathcal{T}_X)$. Since $a(\beta\alpha) = (a\beta)\alpha = a\alpha = c$ and $a(\beta\alpha)^2 = (a\beta\alpha)\beta\alpha = c(\beta\alpha) = (c\beta)\alpha = b\alpha = b$, $(\beta\alpha)^2 \neq \beta\alpha$, so $\beta\alpha \notin E(\mathcal{T}_X)$, hence $\mathcal{T}_X$ is not orthodox. This proves that if $\mathcal{T}_X$ is orthodox, then $|X| \leq 2$. #

2.4 <u>Theorem</u>. For any set X, the partial transformation semigroup, T_X, is orthodox if and only if $|X| \leq 1$.

<u>Proof</u> : **Assume** $|X| \leq 1$. If $|X| = 0$, then $T_X = \{0\}$ and so T_X is orthodox. If $|X| = 1$, then $T_X = \{0, 1\}$ and therefore it is a band so that T_X is orthodox.

Conversely, suppose $|X| > 1$. Let $X' = \{a, b\}$ where a, b are two distinct elements of X. Then $T_{X'} = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ where

$$\Delta\alpha_1 = \{a\} = \nabla\alpha_1, \quad \Delta\alpha_2 = \{b\} = \nabla\alpha_2$$

$\Delta\alpha_3 = \{a\}$, $\nabla\alpha_3 = \{b\}$, $\Delta\alpha_4 = \{b\}$, $\nabla\alpha_4 = \{a\}$

$\Delta\alpha_5 = X'$, $\nabla\alpha_5 = \{a\}$, $\Delta\alpha_6 = X'$, $\nabla\alpha_6 = \{b\}$

$\Delta\alpha_7 = X' = \nabla\alpha_7$ and $a\alpha_7 = b$ and. $b\alpha_7 = a$.

Under the composition of maps, we have the following table :

·	0	1	α_1	α_2	α_3	α_4	α_5	α_6	α_7
0	0	0	0	0	0	0	0	0	0
1	0	1	α_1	α_2	α_3	α_4	α_5	α_6	α_7
α_1	0	α_1	α_1	0	α_3	0	α_1	α_3	α_3
α_2	0	α_2	0	α_2	0	α_4	α_4	α_2	α_4
α_3	0	α_3	0	α_3	0	α_1	α_1	α_3	α_1
α_4	0	α_4	α_4	0	α_2	0	α_4	α_2	α_2
α_5	0	α_5	α_5	0	α_6	0	α_5	α_6	α_6
α_6	0	α_6	0	α_6	0	α_5	α_5	α_6	α_5
α_7	0	α_7	α_4	α_3	α_2	α_1	α_5	α_6	1

From the table, $E(T_{X'}) = \{0, 1, \alpha_1, \alpha_2, \alpha_5, \alpha_6\}$ but $\alpha_1\alpha_6 = \alpha_3 \notin E(T_{X'})$. Then $T_{X'}$ is not orthodox. Because $T_{X'}$ is a subsemigroup of T_X and $\alpha_1, \alpha_6 \in E(T_{X'}) \subseteq E(T_X)$ and $\alpha_1\alpha_6 = \alpha_3 \notin E(T_{X'})$, $\alpha_1, \alpha_6 \in E(T_X)$ and $\alpha_1\alpha_6 \notin E(T_X)$. Hence T_X is not orthodox. This shows that T_X is orthodox implies $|X| \leq 1$. #

The next two propositions give the necessary and sufficient conditions for a full transformation semigroup and a partial transformation semigroup to be right-inverse or to be generalized inverse.

2.5 <u>Proposition</u>. For any set X,

(i) $\mathcal{T}_X$ is right-inverse if and only if $|X| \leq 2$,

and (ii) T_X is right-inverse if and only if $|X| \leq 1$.

<u>Proof</u> : (i) Assume $\mathcal{T}_X$ is right-inverse. Then $\mathcal{T}_X$ is orthodox, so $|X| \leq 2$ by Theorem 2.3.

Conversely, assume $|X| \leq 2$. If $|X| = 0$, then $\mathcal{T}_X = \{0\}$. If $|X| = 1$, then $\mathcal{T}_X = \{1\}$. Hence if $|X| \leq 1$, then $\mathcal{T}_X$ is clearly a right-inverse semigroup. Assume $|X| = 2$. Then $\mathcal{T}_X = \{1, \alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1, \alpha_2, \alpha_3$ are defined as in the proof of Theorem 2.3. Because $E(\mathcal{T}_X) = \{1, \alpha_2, \alpha_3\}$, $\alpha_2\alpha_3\alpha_2 = \alpha_2 = \alpha_3\alpha_2$ and $\alpha_3\alpha_2\alpha_3 = \alpha_3 = \alpha_2\alpha_3$, it then follows that $\mathcal{T}_X$ is right-inverse.

(ii) Assume T_X is right-inverse. Then T_X is orthodox, so $|X| \leq 1$ by Theorem 2.4.

Conversely, assume $|X| \leq 1$. If $|X| = 0$, then $T_X = \{0\}$, so T_X is right-inverse. If $|X| = 1$, then $T_X = \{0, 1\}$, so T_X is clearly right-inverse. #

2.6 <u>Proposition</u>. For any set X,

(i) $\mathcal{T}_X$ is generalized inverse if and only if $|X| \leq 2$,

and (ii) T_X is generalized inverse if and only if $|X| \leq 1$.

<u>Proof</u> : (i) Assume $\mathcal{T}_X$ is generalized inverse. Then $\mathcal{T}_X$ is orthodox, so $|X| \leq 2$ by Theorem 2.3.

Conversely, assume $|X| \leq 2$. If $|X| = 0$, then $\mathcal{T}_X = \{0\}$. If $|X| = 1$, then $\mathcal{T}_X = \{1\}$. Hence if $|X| \leq 1$, then $\mathcal{T}_X$ is clearly a generalized inverse semigroup. Assume $|X| = 2$. Then $\mathcal{T}_X = \{1, \alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1, \alpha_2, \alpha_3$ are defined as in the proof of Theorem 2.3. Because $E(\mathcal{T}_X) = \{1, \alpha_2, \alpha_3\}$ and $\alpha_2\alpha_3 = \alpha_3$ and $\alpha_3\alpha_2 = \alpha_2$, $\mathcal{T}_X$ is a generalized inverse semigroup.

(ii) Assume T_X is generalized inverse. Then T_X is orthodox, so $|X| \leq 1$ by Theorem 2.4.

Conversely, assume $|X| \leq 1$. If $|X| = 0$, then $T_X = \{0\}$. If $|X| = 1$, then $T_X = \{0, 1\}$. Hence T_X is a generalized inverse semigroup. #

In Chapter I, it is shown that a regular subsemigroup of a quasi-inverse semigroup need not be quasi-inverse but its ideals are.

Let X be a set. For each cardinal c, let

$$D_c = \{\alpha \in T_X \mid |\Delta\alpha| \leq c\}.$$

Then $0 \in D_c$. It is clearly seen that D_c is a subsemigroup of T_X for each cardinal c. Let c be a given cardinal number. To show D_c is regular, let $\alpha \in D_c$. Because $T_{\Delta\alpha}$ is regular, there is $\beta \in T_{\Delta\alpha}$ such that $\alpha = \alpha\beta\alpha$. Then $\beta \in T_X$ and $|\Delta\beta| \leq |\Delta\alpha| \leq c$. Therefore $\beta \in D_c$ and hence α is a regular element of D_c. This proves that D_c is a regular subsemigroup of T_X. In general, D_c need not be an ideal of T_X. An example is given as follows :

<u>Example</u>. Let $X = \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers and let n be a positive integer. Then $D_n = \{\alpha \in T_{\mathbb{R}} \mid |\Delta\alpha| \leq n\}$. Let $\alpha, \beta \in T_{\mathbb{R}}$

be defined by $\Delta\alpha = \{0\} = \nabla\alpha$ and $\Delta\beta = \mathbb{R}$ and $\nabla\beta = \{0\}$. Then $\alpha \in D_n$, but $\beta\alpha \notin D_n$ because $\Delta\beta\alpha = \mathbb{R}$ which is uncountable. Hence D_n is not an ideal of $T_{\mathbb{R}}$. #

The previous example shows that for any set X, D_c, defined as before, need not be an ideal of T_X. However, D_c is always a quasi-inverse subsemigroup of T_X for any set X and for any cardinal c.

2.7 Proposition. Let X be a set. For any cardinal c,

$$D_c = \{\alpha \in T_X \mid |\Delta\alpha| \leq c\}$$

is a quasi-inverse subsemigroup of T_X.

Proof : Let c be any cardinal number. D_c is a subsemigroup of T_X as the previous mention. Let $\alpha \in D_c$. Then $\alpha \in T_{\Delta\alpha}$ which is a quasi-inverse semigroup [Proposition 2.2], there exists an inverse subsemigroup B of $T_{\Delta\alpha}$ such that $\alpha \in B$. Because $|\Delta\alpha| \leq c$, $T_{\Delta\alpha}$ is a subsemigroup of D_c. Therefore B is an inverse subsemigroup of D_c containing α. Hence D_c is a quasi-inverse subsemigroup of T_X. #

Let X be a set. For each cardinal c, let

$$R_c = \{\alpha \in T_X \mid |\nabla\alpha| \leq c\},$$

and

$$F_c = \{\alpha \in \mathcal{T}_X \mid |\nabla\alpha| \leq c\} \quad \text{if } c > 0.$$

The following proposition shows that R_c is an ideal of T_X for any cardinal c and F_c is an ideal of $\mathcal{T}_X$ for each cardinal c such that $c > 0$.

2.8 Proposition. Let X be a set. For any cardinal c,

$$R_c = \{\alpha \in T_X \mid |\nabla\alpha| \leq c\}$$

is an ideal of T_X and for $c > 0$,

$$F_c = \{\alpha \in \mathcal{T}_X \mid |\nabla\alpha| \leq c\}$$

is an ideal of $\mathcal{T}_X$. Hence R_c is a quasi-inverse subsemigroup of T_X for any cardinal c, and for each cardinal c such that $c > 0$, F_c is a quasi-inverse subsemigroup of $\mathcal{T}_X$.

Proof : Let c be a given cardinal number. Since $0 \in R_c$, $R_c \neq \phi$. Let $\alpha \in R_c$, $\beta \in T_X$. Then $|\nabla\alpha| \leq c$. Because $\nabla\beta\alpha \subseteq \nabla\alpha$, $|\nabla\beta\alpha| \leq |\nabla\alpha| \leq c$, so $\beta\alpha \in R_c$. Because $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta$ and β is a map, $|\nabla\alpha\beta| = |(\nabla\alpha \cap \Delta\beta)\beta| \leq |\nabla\alpha \cap \Delta\beta| \leq |\nabla\alpha| \leq c$, so $\alpha\beta \in R_c$. Hence R_c is an ideal of T_X.

Next, let c be a cardinal such that $c > 0$. Then $F_c \neq \phi$. By the same argument as above, we have F_c is an ideal of $\mathcal{T}_X$. #

From the proof of Proposition 2.7 and the proof of Proposition 2.8, the following remark follows easily :

2.9 Remark. Let X be a set. For any cardinal c such that $c > 0$,

$$\bar{D}_c = \{\alpha \in T_X \mid |\Delta\alpha| < c\}$$

and

$$\bar{R}_c = \{\alpha \in T_X \mid |\nabla\alpha| < c\}$$

are quasi-inverse subsemigroup of T_X, and if $c > 1$, then

$$\bar{F}_c = \{\alpha \in \mathcal{T}_X \mid |\nabla\alpha| < c\}$$

is a quasi-inverse subsemigroup of $\mathcal{T}_X$ and also an ideal of $\mathcal{T}_X$.