CHAPTER IV

THE POISSON INTEGRAL OF A MEASURE

The aim of this chapter is to represent a temperature on a half-space in the form of the Poisson integral of a measure. In the sequel we shall construct a sequence of bounded measure which converges to a bounded measure in a w^*- topology.

A sequence $\{\mu_j\}$ of bounded measures is said to converge to a measure μ in w^*- topology if for all $f \in C_0(\mathbb{R}^n)$, the class of all real valued continuous functions (in $\mathbb{R}^n$) vanishing at infinity, then

$$\lim_{j \to \infty} \int f d\mu_j = \int f d\mu .$$

Denote $\mathrm{supp}(f) = \{ x \,/\, f(x) \neq 0 \}$.

By $C(\mathbb{R}^n)$ we denote the class of all real-valued functions (in $\mathbb{R}^n$), and furnish it with the topology of convergence uniform on compact subsets of $\mathbb{R}^n$. This topology $\mathcal{X}$ may be defined by the seminorms

$$\mathcal{P}_K(f) = \sup \{ |f(y)| \,/ f \in C(\mathbb{R}^n),\ y \in K \},$$

K ranging over any preassigned base for the compact subset of $\mathbb{R}^n$. Moreover, f_n is said to converge to f if and only if there exists a compact subset K such that

$$\mathrm{supp}(f_n) \subseteq K$$

and

$$f_n \longrightarrow f \quad \text{uniformly in } K.$$

A real linear functional β on $C(\mathbb{R}^n)$ is said to be bounded if for each compact set K, there exists $m_K > 0$ such that

$$|\beta f| \leq m_K \rho_K(f) \quad (f \in C(\mathbb{R}^n) \text{ with } \operatorname{supp}(f) \subseteq K).$$

For each $f \in C_0(\mathbb{R}^n)$, we define $\|f\|_\infty$ by

$$\|f\|_\infty = \sup\{|f(y)| / y \in \mathbb{R}^n\}.$$

We notice that $\|f\|_\infty$ is finite, since $f \in C_0(\mathbb{R}^n)$.

A set $P \subset X$ is said to be convex if

$$tx + (1 - t)y \in P \quad (x, y \in P,\ 0 \leq t \leq 1).$$

A neighborhood of a point $x \in X$ is an open set that contains x.

A collection $\mathfrak{B}$ of neighborhoods of a point $x \in X$ is a local base at x if every neighborhood of x contains a member of $\mathfrak{B}$. In the vector space context, the term local base will always mean a local base at 0.

A topological vector space is locally convex if there is a local base $\mathfrak{B}$ whose members are convex.

Note that $C(\mathbb{R}^n)$ with the family of seminorms ρ_K on $C(\mathbb{R}^n)$ is a locally convex topological vector space (see Edwards, R.E., Functional Analysis, [3], §1.10.1, p 78).

Theorem 4.1.1. (Hahn-Banach Theorem). If Λ is a bounded linear functional on a subspace M of a locally convex space X, then there exists a bounded linear functional β on X, such that

$$\beta = \Lambda \qquad \text{on } M.$$

(See Rudin. W., Functional Analysis,[5], McGraw-Hill Book Company).

Theorem 4.1.2. (Riesz Reppressentation Theorem). If β be a bounded linear functional on $C(\mathbb{R}^n)$. Then there exists a uniquely determined real Radon measure μ on $\mathbb{R}^n$ having a compact support and such that

$$\beta f = \int_{\mathbb{R}^n} f d\mu \qquad (f \in C(\mathbb{R}^n)).$$

(See Edwards, R.E., Functional Analysis,[3], Theorem 4.10.1, p 203).

Before proving Theorem 4.1.5., we first notice that $C_o(\mathbb{R}^n)$ is separable (see Bourbaki,[2], Chapter X, § 3.3, Corollary of Theorem 1). Thus there exists a countable set

$$E = \left\{ f_i \in C_o(\mathbb{R}^n) \;/\; i = 1,2,\ldots \right\}, \text{ say,}$$

is dense in $C_o(\mathbb{R}^n)$. Later on, we need two lemmas (Lemma 4.1.3. and Lemma 4.1.4.).

Lemma 4.1.3. Let $\{\mu_i\}$ be a sequence of Radon measures defined on the Borel subsets of $\mathbb{R}^n$ with $\mu_i(\mathbb{R}^n) < k$, for all i, and for some positive number k. Then there is a Radon measure μ defined on the Borel subsets of $\mathbb{R}^n$ with

$$\mu(\mathbb{R}^n) \leq k,$$

and a subsequence $\{\mu_{i_j}\}$ which converges to μ in the w^*- topology.

Proof: Since $C_o(\mathbb{R}^n)$ is separable, there exists $E = \{f_1, f_2, \ldots\}$ a countable dense subset of $C_o(\mathbb{R}^n)$.

The sequence of real numbers

$$\int f_1 d\mu_i$$

is bounded by $k\|f_1\|_\infty$. There is a subsequence $\{\mu_{i_1}\}$ of $\{\mu_i\}$ such that

$$\lim_{i\to\infty} \int f_1 d\mu_{i_1} \quad \text{exists.}$$

Now consider the sequence of real number

$$\int f_2 d\mu_{i_1}$$

which is bounded by $k\|f_2\|_\infty$. There is a subsequence $\{\mu_{i_2}\}$ of $\{\mu_{i_1}\}$ such that

$$\lim_{i\to\infty} \int f_2 d\mu_{i_2} \quad \text{exists.}$$

Continue this precess, we get a subsequence $\{\mu_{i_j}\}$ of $\{\mu_{i_{(j-1)}}\}$, $j \geq 2$,

$$\lim_{i \to \infty} \int f_j d\mu_{i_j} \quad \text{exists for } j \geq 2 .$$

Now consider any $f \in C_o(\mathbb{R}^n)$ and any $\varepsilon > 0$. Let $f_{m_o} \in E$ be such that

$$\| f - f_{m_o} \| < \frac{\varepsilon}{2k} .$$

Then

$$\left| \int f d\mu_{i_i} - \int f d\mu_{j_j} \right| < \left| \int (f - f_{m_o}) d\mu_{i_i} \right| + \left| \int f_{m_o} d\mu_{i_i} - \int f_{m_o} d\mu_{j_j} \right| + \left| \int f_{m_o} d\mu_{j_j} - \int f d\mu_{j_j} \right|$$
$$< \varepsilon + \left| \int f_{m_o} d\mu_{i_i} - \int f_{m_o} d\mu_{j_j} \right| .$$

Since $f_{m_o} \in E$, the sequence of absolute values on the right approaches zero as $i, j \to \infty$ and

$$\overline{\lim_{i,j \to \infty}} \left| \int f d\mu_{i_i} - \int f d\mu_{j_j} \right| < \varepsilon .$$

This shows that the sequence

$$\int f d\mu_{i_i}, \quad i \geq 1$$

is a Cauchy sequence and that

$$\Lambda f = \lim_{i \to \infty} \int f d\mu_{i_i} \quad \text{exists for each } f \in C_o(\mathbb{R}^n).$$

To check that Λ is bounded on $C_o(\mathbb{R}^n)$.

For any K compact subset of $\mathbb{R}^n$, for any $f \in C_o(\mathbb{R}^n)$ such that supp $(f) \subseteq K$,

$$\int_{\mathbb{R}^n} f d\mu_{i_i} \leq \int_{\mathbb{R}^n} |f| d\mu_{i_i}$$
$$\leq k\, \mathcal{P}_K(f),$$

therefore Λ is bounded linear functional on $C_o(\mathbb{R}^n)$.

By Theorem 4.1.1., there exists a bounded linear functional β defined on $C(\mathbb{R}^n)$ such that

$$\beta f = \Lambda f \qquad \text{for all } f \in C_o(\mathbb{R}^n).$$

Apply Theorem 4.1.2. to β , we have that there exists a uniquely determined real Radon measure μ on $\mathbb{R}^n$ having a compact support and such that

$$\beta f = \int_{\mathbb{R}^n} f d\mu \qquad \text{for all } f \in C(\mathbb{R}^n).$$

Thus $$\Lambda f = \int_{\mathbb{R}^n} f d\mu \qquad \text{for all } f \in C_o(\mathbb{R}^n).$$

$$\int f d\mu = \lim_{i \to \infty} \int f d\mu_{i_i} \qquad \text{for all } f \in C_o(\mathbb{R}^n).$$

and this is just the definition of w^*- convergence.

Next, we shall show that a necessary and sufficient for a temperature u to be positive on the strip $H_{(0,c)}$ is that

$$u(x,t) = \int_{\mathbb{R}^n} K(y-x,t)\, d\mu(y),$$

where μ is a Radon measure. Note that the proof of the sufficiency is given in Theorem 2.1.7.

Lemma 4.1.4. $\lim_{|y|\to\infty} \frac{K(y-x,t)}{K(y,t_o)} = 0, \quad 0 < t < t_o$.

Proof: This lemma follows immediately for the following identities :

$$\frac{k(y_i-x_i,t)}{k(y_i,t_o)} = \sqrt{\frac{t_o}{t}}\ \frac{\exp(-(y_i-x_i)^2/4t)}{\exp(-y_i^{\,2}/4t_o)}$$

$$= \sqrt{\frac{t_o}{t}}\ \exp\left\{\frac{(t-t_o)}{4tt_o}\left[y_i+\frac{x_i t_o}{t-t_o}\right]^2 - \frac{x_i^{\,2}}{2t} - \frac{x_i^{\,2}t_o}{4t(t-t_o)}\right\}$$

$$= c\ \exp\left\{\frac{(t-t_o)}{4tt_o}\left[y_i+\frac{x_i t_o}{t-t_o}\right]^2\right\},\quad c = \sqrt{\frac{t_o}{t}}\ \exp\left\{-\frac{x_i^{\,2}}{2t} - \frac{x_i^{\,2}t_o}{4t(t-t_o)}\right\}.$$

Theorem 4.1.5. If $u(x,t) \in \mathcal{H}$ and $u(x,t) \geqslant 0$, on the strip $H_{(0,c)}$, then

$$u(x,t) = \int_{\mathbb{R}^n} K(y-x,t)\,d\mu ,$$

where μ is a Radon measure on $\mathbb{R}^n$.

Proof: For a fixed $t_o > 0$, we set

$$\alpha_\delta(E) = \int_E K(y,t_o)u(y,\delta)\,dy,$$

where $0 < \delta < c$ is such that $t_o < c-\delta$ and E is a measurable set of $\mathbb{R}^n$.

By Theorem 3.1.4., we get

$$0 \leq \alpha_\delta(E) \leq u(0,t_0+\delta) < \infty, \text{ for all } \delta,$$

and for all measurable sets E.

Therefore α_δ is a Radon measure. By Lemma 4.1.3., there is a Radon measure α such that $\{\alpha_\delta\}$ converges to α in w^*- topology as $\delta \to 0$, i.e.,

$$\lim_{\delta \to 0} \int f d\alpha_\delta = \int f d\alpha, \quad \text{for all } f \in C_0(\mathbb{R}^n),$$

and

$$\alpha(E) \leq u(0,t_0+\delta).$$

Since $u(x_0,\delta) = \lim_{(x,t) \to (x_0,0^+)} \int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)dy$

$$= \lim_{(x,t) \to (x_0,0^+)} \int_{\mathbb{R}^n} \frac{K(y-x,t)d}{K(y,t_0)} \alpha_\delta(y),$$

$$\int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)dy \leq u(x,t+\delta), \text{ for } 0 < t < c-\delta,$$

and $\lim_{(x,t+\delta) \to (x_0,\delta)} u(x,t+\delta) = u(x_0,\delta)$, we have that

$$\lim_{(x,t+\delta) \to (x_0,\delta)} \left\{ u(x,t+\delta) - \int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)dy \right. = 0.$$

Since (4.1) $u(x,t+\delta) - \int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)dy$,

belongs to $\mathcal{H}$ in the strip $H_{(0,c-\delta)}$,

$$u(x_0,\delta) - \int_{\mathbb{R}^n} K(y-x_0,t)u(y,\delta)dy = 0.$$

Hence, apply the Theorem 3.1.6. to the function (4.1), we get

$$u(x,t+\delta) - \int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)dy = 0$$

in the strip $H_{(0,c-\delta)}$.

Therefore
$$\begin{aligned} u(x,t) &= \lim_{\delta\to 0} u(x,t+\delta) \\ &= \lim_{\delta\to 0}\int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)dy \\ &= \lim_{\delta\to 0}\int_{\mathbb{R}^n} \frac{K(y-x,t)}{K(y,t_o)}d\alpha_\delta(y). \end{aligned}$$

Since $\frac{K(y-x,t)}{K(y,t_o)} \in C_o(\mathbb{R}^n)$ (by Lemma 4.1.4.), we have, by Lemma 4.1.3., that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^n} \frac{k(y-x,t)}{k(y,t_o)}d\alpha_\delta(y) = \int_{\mathbb{R}^n} \frac{K(y-x,t)}{K(y,t_o)}d\alpha(y), \quad t < t_o.$$

Define

$$\mu(E) = \int_E \frac{1}{K(y,t_o)}d\alpha(y),$$

where E is a measurable subset of $\mathbb{R}^n$.

We can see that $\mu(E) \geqslant 0$ for all E measurable set, and $\mu(E)$ is finite whenever E is compact. Hence μ is a Radon measure, and

$$u(x,t) = \int_{\mathbb{R}^n} K(y-x,t)d\mu(y).$$