CHAPTER II

POISSON INTEGRAL OF A FUNCTION

The object of this chapter is to construct the Poisson integral which represents a temperature on a half-space H, $\mathbb{R}^n \times \{t/t \geqslant 0\}$, with prescribed initial condition.

The class of all temperatures on the half-space will be denoted by $\mathcal{H}_e$.

We denote $H_{[a,b]}$, the strip $a \leqslant t \leqslant b$, the set of all (x,t) such that $x \in \mathbb{R}^n$ and $a \leqslant t \leqslant b$. In the same way, we define $H_{(a,b)}$, $H_{(a,b]}$, $H_{[a,b)}$.

Lemma 2.1.1. Let $0 < \rho < \delta$. If $0 < t < \dfrac{\delta - \rho}{4a|y_{o,i}+\delta|}$, $a > 0$, and $t < 1/4a$,

(2.1) $$-(1-4at)y_i+(y_{o,i}+\rho) < 0,$$

whenever $y_i \geqslant y_{o,i}+\delta$, y_i, $y_{o,i} \in \mathbb{R}$.

Proof: Case 1. If $y_{o,i}+\delta < 0$, $0 < t < \dfrac{\delta - \rho}{-4a(y_{o,i}+\delta)}$,

$$-4at(y_{o,i}+\delta) < \delta - \rho$$

$$4at(y_{o,i}+\delta) < -4at(y_{o,i}+\delta) < \delta - \rho$$

$$4at(y_{o,i}+\delta)+(y_{o,i}+\delta)-(y_{o,i}+\delta) < \delta - \rho$$

$$-(1-4at)(y_{o,i}+\delta)+ y_{o,i}+\rho < 0.$$

Since $y_i \geqslant y_{o,i}+\delta$, $t < 1/4a$, we have that

$$-(1-4at)y_i \leqslant -(1-4at)(y_{o,i}+\delta).$$

Hence $-(1-4at)y_i+(y_{o,i}+\rho) < -(1-4at)(y_{o,i}+\delta)+(y_{o,i}+\rho) < 0.$

Case 2. If $y_{o,i}+\delta > 0$, $0 < t < \frac{\delta-\rho}{4a(y_{o,i}+\delta)}$,

$$4at(y_{o,i}+\delta) < \delta-\rho .$$

By following the same argrument in case 1, we get

$$-(1-4at)y_i+(y_{o,i}+\rho) < 0 .$$

Case 3. $y_{o,i}+\delta = 0$, $y_i \geq y_{o,i}+\delta = 0.$

Since $y_{o,i}+\rho < y_{o,i}+\delta = 0,$

$$-(1-4at)(y_{o,i}+\delta) + (y_{o,i}+\rho) < 0 .$$

Since $-(1-4at)y_i < 0 = -(1-4at)(y_{o,i}+\delta)$, we get

$$-(1-4at)y_i+(y_{o,i}+\rho) < 0.$$

The proof is complete.

Remark 2.1.2. If $0 < t < \frac{1}{4a|y_{o,i}-\delta|}$, $a > 0$, and

$0 < t < 1/4a$,

(2.2) $-(1-4at)y_i+(y_{o,i}-\rho) > 0$ whenever $y_i \leq y_{o,i}-\delta$

y_i , $y_{o,i} \in \mathbb{R}$.

Lemma 2.1.3. If $0 < t < \frac{\delta-\rho}{4a|y_{o,i}+\delta|}$, $0 < \rho < \delta$

(2.3) $k(y_i-y_{o,i}-\rho,t)\exp(ay_i^2) \leq k(\delta-\rho,t)\exp(a(y_{o,i}+\delta)^2)$,

$y_i \geq y_{o,i}+\delta$.

Proof: $k(y_i-y_{o,i}-\rho,t)\exp(ay_i^2)$

$$= (4\pi t)^{-\frac{1}{2}}\exp\left(-\frac{(y_i-y_{o,i}-\rho)^2+4aty_i^2}{4t}\right) .$$

Differentiate with respect to y_i,

$$\frac{\partial}{\partial y_i} k(y_i-y_{o,i}-\rho,t)\exp(ay_i^2)$$
$$= k(y_i-y_{o,i}-\rho,t)\left[\frac{-(1-4at)y_i+(y_{o,i}+\rho)}{2t}\right].$$

By Lemma 2.1.1. , if $0<t<\frac{\delta-\rho}{4a|y_{o,i}+\delta|}$,

$-(1-4at)y_i+y_{o,i}+\rho<0$, whenever $y_i \geqslant y_{o,i}+\delta$, and

$$\frac{\partial}{\partial y_i} k(y_i-y_{o,i}-\rho,t)\exp(ay_i^2)<0,\ y_i \geqslant y_{o,i}+\delta .$$

That is, $k(y_i-y_{o,i}-\rho,t)\exp(ay_i^2)$ decreases, for $y_i \geqslant y_{o,i}+\delta$.

$k(y_i-y_{o,i}-\rho,t)\exp(ay_i^2) \leqslant k(\delta-\rho,t)\exp(a(y_{o,i}+\delta)^2)$,

whenever $y_i \geqslant y_{o,i}+\delta$.

Remark 2.1.4. If $0<t<\frac{\delta-\rho}{4a|y_{o,i}-\delta|}$, $0<\rho<\delta$,

$$k(y_i-y_{o,i}+\rho,t)\exp(ay_i^2) \leqslant k(-\delta+\rho,t)\exp(a(y_{o,i}-\delta)^2)$$
$$= k(\delta-\rho,t)\exp(a(y_{o,i}-\delta)^2) ,$$

$y_i \leqslant y_{o,i}-\delta$.

Lemma 2.1.5. Let $x_{o,i} \in \mathbb{R}$, $t_o>0$. Let $\delta>0$ be such that $|x_i-x_{o,i}|<\delta$, $0<t_o-\delta<t<t_o+\delta$. Then

(2.4) $\quad k(y_i-x_i,t)<A_i\exp(-ay_i^2)$,

(2.5) $\quad \frac{1}{2t} k(y_i-x_i,t)<B_i\exp(-ay_i^2)$,

for all $y_i \in \mathbb{R}$, for some positive constants A_i, B_i .

Proof: For $y_i \geqslant b$, $b>x_{o,i}+\delta$.

$$k(y_i-x_i,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{(y_i-x_i)^2}{4t})$$

$$\leqslant \frac{1}{\sqrt{4\pi(t_o-\delta)}} \exp\left(-\frac{(y_i-x_i)^2}{4t}\right)$$

$$= \sqrt{\frac{t_o+\delta}{t_o-\delta}} \frac{1}{\sqrt{4\pi(t_o+\delta)}} \exp\left(-\frac{(y_i-x_i)^2}{4t}\right)$$

$$\leqslant \sqrt{\frac{t_o+\delta}{t_o-\delta}} \frac{1}{\sqrt{4\pi(t_o+\delta)}} \exp\left(-\frac{(y_i-x_{o,i}-\delta)^2}{4t}\right)$$

$$\leqslant \sqrt{\frac{t_o+\delta}{t_o-\delta}} \frac{1}{\sqrt{4\pi(t_o+\delta)}} \exp\left(-\frac{(y_i-x_{o,i}-\delta)^2}{4(t_o+\delta)}\right),$$

$y_i > x_{o,i}+\delta$, $x_i < x_{o,i}+\delta$, $t < t_o+\delta$. That is, we get

$$(2.6) \qquad k(y_i-x_i,t) \leqslant \sqrt{\frac{t_o+\delta}{t_o-\delta}}\, k(y_i-x_{o,i}-\delta,\ t_o+\delta)\ .$$

The function $k(y_i-x_{o,i}-\delta, t_o+\delta) \exp(ay_i^2)$ has the absolute maximum value at $y_i = \dfrac{x_{o,i}+\delta}{1-4a(t_o+\delta)}$, since

$$\frac{\partial}{\partial y_i} k(y_i-x_{o,i}-\delta, t_o+\delta) \exp(ay_i^2)$$

$$= \frac{k(y_i-x_{o,i}-\delta, t_o+\delta)}{4(t_o+\delta)} \exp(ay_i^2)\Big[-(1-4a(t_o+\delta))2y_i + 2(x_{o,i}+\delta)\Big]$$

is positive if $y_i < \dfrac{x_{o,i}+\delta}{1-4a(t_o+\delta)}$, and it is negative if

if $y_i > \dfrac{x_{o,i}+\delta}{1-4a(t_o+\delta)}$. That is, $k(y_i-x_{o,i}-\delta, t_o+\delta)$

is dominated by a constant multiple of $\exp(-ay_i^2)$, $y_i \geq b$. Since $t_o+\delta < \frac{1}{4a}$,

$$\frac{1}{2t} k(y_i-x_{o,i}-\delta,\ t_o+\delta) \exp(ay_i^2)$$

is bounded by a constant multiple of $\exp(-ay_i^2)$. Hence for $y_i \geq b$, $b > x_{o,i}+\delta$, $k(y_i-x_i,t)\exp(ay_i^2)$ and

$\frac{1}{2t} k(y_i-x_i,t)\exp(ay_i^2)$ are dominated by a constant multiple of $\exp(-ay_i^2)$. Similary, for $y_i \leq d$, $d < x_{o,i}-\delta$, they are still dominated by a constant multiple of $\exp(-ay_i^2)$. Since $k(y_i-x_i,t)\exp(ay_i^2)$ and $\frac{1}{2t} k(y_i-x_i,t)\exp(ay_i^2)$ are continuous functions of y_i, $y_i \in [d,b]$, $|x_i-x_{o,i}| < \delta$, $0 < t_o-\delta < t < t_o+\delta$, so they are bounded.

Hence $k(y_i-x_i,t)$ and $\frac{1}{2t} k(y_i-x_i,t)$ are dominated by a constant multiple of $\exp(-ay_i^2)$, say,

$$k(y_i-x_i,t) < A_i \exp(-ay_i^2)$$

$$\frac{1}{2t} k(y_i-x_i,t) < B_i \exp(-ay_i^2),$$

for all $y_i \in \mathbb{R}$, $|x_i-x_{o,i}| < \delta$, $0 < t_o-\delta < t < t_o+\delta$, for some positive constants A_i, B_i.

Lemma 2.1.6. Let $x_{o,i} \in \mathbb{R}$, $t_o > 0$. Let $\delta > 0$ be such that $|x_i-x_{o,i}| < \delta$, $0 < t_o-\delta < t < t_o+\delta$. Then

$$\frac{(y_i-x_i)^2}{4t^2} k(y_i-x_i,t) < C_i \exp(-ay_i^2) ,$$

for all $y_i \in \mathbb{R}$, for some positive constant C_i .

Proof: If $|x_i - x_{o,i}| < \delta$, $0 < t_o - \delta < t < t_o + \delta$, we can get (2.6) in the same way whenever $y_i \geq A > x_{o,i} + \delta$. Consider $(y_i - x_i)^2\ k(y_i - x_i, t) \leq (y_i - x_i)^2 k(y_i - x_{o,i} - \delta, t_o + \delta)$

$$\leq (y_i - x_{o,i} + \delta)^2 k(y_i - x_{o,i} - \delta, t_o + \delta),$$

$x_{o,i} - \delta < x_i < y_i$, $y_i \geq A$. Consider,

$$0 \leq (y_i - x_{o,i} + \delta)^2 k(y_i - x_{o,i} - \delta,\ t_o + \delta)\ \exp(\, a y_i^2)$$

$$= \frac{(y_i - x_{o,i} + \delta)^2}{\sqrt{4\pi(t_o + \delta)}} \exp\left\{ -\frac{1-4a(t_o+\delta)}{4(t_o+\delta)}\left[y_i - \frac{x_{o,i}+\delta}{1-4a(t_o+\delta)}\right]^2 + \frac{(x_{o,i}+\delta)^2}{4(t_o+\delta)} + \frac{(x_{o,i}+\delta)^2}{4a(t_o+\delta)(1-4a(t_o+\delta))}\right\}$$

$$= C(y_i - x_{o,i} + \delta)^2 \exp\left\{ -\frac{(1-4a(t_o+\delta))}{4(t_o+\delta)}\left[y_i - \frac{x_{o,i}+\delta}{1-4a(t_o+\delta)}\right]^2\right\},$$

where $C = \dfrac{1}{\sqrt{4\pi(t_o+\delta)}} \exp\left\{\dfrac{(x_{o,i}+\delta)^2}{4a(t_o+\delta)} \times \dfrac{(2-4a(t_o+\delta))}{(1-4a(t_o+\delta))}\right\}$.

By using L'Hospital's rule twice, the last quantity tends to zero as y_i approaches ∞ or $-\infty$. Therefore

$(y_i - x_{o,i} + \delta)^2 \exp(a y_i^2)\ k(y_i - x_{o,i} - \delta, t_o + \delta)$ approaches to zero as y_i approaches ∞ or $-\infty$. But it is continuous function. So it is a bounded function of y_i, $y_i \in (-\infty, \infty)$. Hence

$\dfrac{(y_i - x_i)^2}{4t^2}\ k(y_i - x_i, t)$ is dominated by a constant multiple of

$\exp(-a y_i^2)$, $y_i \geq A$. Similarly, $\dfrac{1}{4t^2}\ (y_i - x_i)^2 k(y_i - x_i, t)$

is bounded by a constant multiple of $\exp(-ay_i^2)$, $y_i \leq B <$ $x_{o,i} - \delta$. On $[B,A]$, $(y_i-x_i)^2k(y_i-x_i,t)\exp(ay_i^2)/4t^2$ is continuous, so it is bounded. On the other hand, on $[B,A]$, $(y_i-x_i)^2k(y_i-x_i,t)/4t^2$ is bounded by a constant multiple of $\exp(-ay_i^2)$. Hence

$$\frac{(y_i-x_i)^2k(y_i-x_i,t)}{4t^2} \leq C_i\exp(-ay_i^2), \text{ for}$$

$y_i \in \mathbb{R}$, $|x_i-x_{o,i}| < \delta$, $0 < t_o - \delta < t < t_o + \delta$, and for some positive constants C_i .

Theorem 2.1.7. If

1. $\phi(y)\exp(-a|y|^2) \in L(\mathbb{R}^n)$ for some $a > 0$,

2. $F(x,t) = \int_{\mathbb{R}^n} K(y-x,t)\phi(y)dy$,

then the Poisson integral $F(x,t)$ is defined and belongs to $\mathcal{H}$ in $H_{(0,4a)}$.

Proof: Let $(x_o,t_o) = (x_{o,1},x_{o,2},\ldots,x_{o,n},t_o)$ in the strip $H_{(0,4a)}$.

Choose δ so small that $\bar{B}((x_o,t_o),\delta) \subset H_{(0,4a)}$

Let $(x,t) \in B((x_o,t_o),\delta)$. It follows that

$$|x_i-x_{o,i}| < \delta \ , 0 < t_o - \delta < t < t_o + \delta, \ i = 1,2,3,\ldots,n.$$

By Lemma 2.1.5 and 2.1.6, for i = 1 or 2 or ... or n,

$$k(y_i-x_j,t) \leq A_i \exp(-ay_i^2)$$

$$\frac{1}{2t} k(y_i - x_i, t) \leqslant B_i \cdot \exp(-ay_i^2) ,$$

and $\frac{(y_i - x_i)^2}{4t^2} k(y_i - x_i, t) \leqslant C_i \exp(-ay_i^2)$,

for $y_i \in \mathbb{R}$, and for some positive constants A_i, B_i, C_i .

Hence $$\frac{1}{2t} K(y-x,t) = \frac{1}{2t} \prod_{i=1}^{n} k(y_i - x_i, t)$$

$$\leqslant K \exp(-a|y|^2) , K = B_j \prod_{j \neq i=1}^{n} A_i$$

i.e., $$\frac{1}{2t} K(y-x,t)|\phi(y)| \leqslant K|\phi(y)|\exp(-a|y|^2) ,$$

and $\frac{|y - x|^2}{4t^2} K(y-x,t) \leqslant L \exp(-a|y|^2)$, $L = \sum_{j=1}^{n} C_j \prod_{j \neq i=1}^{n} A_i$,

i.e., $$\frac{|y - x|^2}{4t^2} K(y-x,t)\left|\phi(y)\right| \leqslant L\left|\phi(y)\right|\exp(-a|y|^2).$$

Since

$$\frac{\partial}{\partial t} K(y-x,t)\phi(y) = \frac{|y-x|^2}{4t^2} K(y-x,t)\phi(y) - \frac{n}{2t} K(y-x,t)\phi(y), \tag{2.7}$$

there is a nonnegative function $|\phi(y)|\exp(-a|y|^2) \in L(\mathbb{R}^n)$ such that

$$\left|\frac{\partial}{\partial t} K(y-x,t)\,\phi(y)\right| \leq M\,|\phi(y)|\exp(-a|y|^2), \; M = L+nK,$$

for all $y \in \mathbb{R}^n$ and $(x,t) \in B((x_o, t_o),\delta)$.

For a fixed point $(x,t) \in B((x_0,t_0),\delta)$, define f by

$$f(y) = K(y-x,t)\phi(y) .$$

Since $K(y-x,t)$ is a continuous function of y, f (y) is a measurable function on $\mathbb{R}^n$. And since it is dominated by $|\phi(y)|\exp(-a|y|^2) \in L(\mathbb{R}^n)$, $f(y) \in L(\mathbb{R}^n)$.

Lastly, (2.7), the partial differentiate with respect to t, exists for all $(x,t) \in B((x_0, t_0),\delta)$ and $y \in \mathbb{R}^n$.

Therefore, $K(y-x,t)\phi(y)$ satisfies all conditions in Theorem 1.2.2.. Hence, the integral

$$\int_{\mathbb{R}^n} K(y-x,t)\,\phi(y)dy$$

exists for all $(x,t) \in B((x_0,t_0),\delta)$. And $F(x,t)$ is differentiable with respect to t at each interior point of the ball, and moreover,

$$\frac{\partial}{\partial t} F(x,t) = \int_{\mathbb{R}^n} K(y-x,t)\phi(y)\left[\frac{|y-x|^2}{4t^2} - \frac{n}{2t}\right] dy .$$

In the same way, we can show that

$$\frac{\partial^2}{\partial x_i^2} F(x,t) = \int_{\mathbb{R}^n} K(y-x,t)\phi(y)\left[\frac{(y_i-x_i)^2}{4t^2} - \frac{1}{2t}\right] dy,$$

for all i=1 or 2 or ... or n.

$$\Delta F = \frac{\partial}{\partial t} F.$$

That is, $F(x,t) \in \mathcal{H}$ in the ball. Since (x_0, t_0) is arbitrary point in $H_{(0,4a)}$, $F(x,t) \in \mathcal{H}$ in the strip $H_{(0,4a)}$.

Lemma 2.1.8. Let a, b be any two extended real numbers such that $\lambda \geq b$ whenever $\lambda > a$. Then $a \geq b$.

Proof: Assume that $a < b$. And assume the Hypothesis.

Then there is α such that $a < \alpha < b$.

i.e., $a < \alpha$

By Hypothesis, $\alpha \geq b$.

It contradicts to the fact that $\alpha < b$.

Hence $a \geq b$.

Theorem 2.1.9. If $F(x,t)$ and $\phi(y)$ are defined as in Theorem 2.1.7, then

$$\overline{\lim_{(x,t)\to(y_0,0^+)}} \left| F(x,t) \right| \leq \overline{\lim_{y\to y_0}} \left|\phi(y)\right|.$$

And, if $\lim_{y\to y_0} \phi(y)$ exists, $\lim_{(x,t)\to(y_0,0^+)} F(x,t) = \lim_{y\to y_0} \phi(y)$.

Proof: If $\overline{\lim_{y\to y_0}} |\phi(y)| = \infty$, there is nothing to prove.

We assume $\overline{\lim\limits_{y \to y_0}} |\phi(y)| < \infty$. Let a real number λ be such that

$\overline{\lim\limits_{y \to y_0}} |\phi(y)| < \lambda$. By definition of limit supremum,

$$\overline{\lim_{y \to y_0}} |\phi(y)| = \inf_{\delta > 0} \sup_{y \in B(y_0, \varepsilon)} |\phi(y)|,$$

there exists a $\delta > 0$ such that $\sup\limits_{y \in B(y_0, \delta)} |\phi(y)| < \lambda$.

$$F(x,t) = \int_{\mathbb{R}^n} K(y-x,t)\phi(y)\,dy$$

$$(2.8) \quad |F(x,t)| \leq \int_{B(y_0,\delta)} K(y-x,t)|\phi(y)|\,dy$$

$$+ \int_{B'(y_0,\delta)} K(y-x,t)|\phi(y)|\,dy.$$

$$\leq \sup_{y \in B(y_0,\varepsilon)} |\phi(y)| \int_{B(y_0,\delta)} K(y-x,t)\,dy$$

$$+ \int_{B'(y_0,\delta)} K(y-x,t)|\phi(y)|\,dy.$$

$$\leq \lambda + \int_{B'(y_0,\delta)} K(y-x,t)|\phi(y)|\,dy.$$

Let $S(y_o,\delta')$ denote the set $\{y\in\mathbb{R}^n / \ y_i \geq y_{o,i}+\delta'$ or $y_i \leq y_{o,i}-\delta', \ y_o = (y_{o,1}, y_{o,2}, \ldots, y_{o,n})\}$, $0<\delta'<\delta$, such that $B(y_o,\delta)$ is containing $S(y_o,\delta')$. Therefore,

$$(2.9) \quad \int_{S'(y_o,\delta')} K(y-x,t)|\phi(y)|\,dy \geq \int_{B'(y_o,\delta)} K(y-x,t)|\phi(y)|\,dy.$$

Claim that $\int_{S'(y_o,\delta')} K(y-x,t)|\phi(y)|\,dy$ tends to zero as t tends to zero, whenever $|x-y_o|<\rho<\delta'$.

For $y\in S'(y_o,\delta')$, $y_i \geq y_{o,i}+\delta'$. If $|x-y_o|<\rho<\delta'$

$$k(y_i-x_i,t) < k(y_i-y_{o,i}-\delta',t).$$

By Lemma 2.1.3, if $0<t<\dfrac{\delta'-\rho}{4a|y_{o,i}+\delta'|}$,

$$k(y_i-x_i,t) < k(\delta'-\rho,t)\exp(a(y_{o,i}+\delta')^2 - ay_i^2),$$

$y_i \geq y_{o,i}+\delta'$.

For $y\in S'(y_o,\delta')$, $\quad y_i \leq y_{o,i}-\delta'$. If $|x-y_o|<\rho<\delta'$,

$$k(y_i-x_i,t) < k(y_i-y_{o,i}+\delta',t).$$

By Remark 2.1.4., if $0<t<\dfrac{\delta'-\rho}{4a|y_{o,i}-\delta'|}$,

$$k(y_i-x_i,t) < k(\delta'-\rho,t)\exp(a(y_{o,i}-\delta')^2-ay_i^2),$$

$y_i \leq y_{o,i}-\delta'$. Therefore,

if $M_i = \max((y_{o,i}+\delta')^2, (y_{o,i}-\delta')^2) \neq 0$,

$$k(y_i - x_i, t) < k(\delta' - \rho, t)\exp(aM_i - ay_i^2),$$

$y_i \geqslant y_{o,i} + \delta'$ or $y_i \leqslant y_{o,i} - \delta'$.

Hence $K(y-x,t)|\phi(y)| \leq [k(\delta - \rho, t)]^n \exp(a\sum_{i=1}^{n} M_i - a|y|^2) \times |\phi(y)|$

where $y \in S'(y_o, \delta')$, $|x - y_0| < \delta'$, $0 < t < \dfrac{\delta' - e}{4a \max_i(M_i)}$.

Therefore $\displaystyle\int_{S'(y_o,\delta')} K(y-x,t)|\phi(y)|dy$

$$< \prod_{i=1}^{n} k(\delta' - \rho, t)\exp(a\sum_{i=1}^{n} M_i) \int_{S'(y_o,\delta')} \exp(-a|y|^2)|\phi(y)|dy.$$

By Proposition 1.1.3.,

$\displaystyle\int_{S'(y_o,\delta')} K(y-x,t)|\phi(y)|$ tends to 0 as t tends to 0^+.

By (2.9), $\displaystyle\int_{B'(y_o,\delta)} K(y-x,t)|\phi(y)|dy$ tends to 0 as t tends to 0^+.

i.e., given $\varepsilon > 0$, there is a $\delta_1 > 0$ such that

(2.10) $\displaystyle\int_{B'(y_o,\delta)} K(y-x,t)|\phi(y)|dy < \varepsilon$, for all

$$(x,t) \in B((y_0,0),\delta_1) \cap \left\{(x,t)/t>0\right\} = B^+((y_0,0),\delta_1).$$

By (2.8) and (2.10), $|F(x,t)| < \lambda + \epsilon$, $(x,t) \in B^+((y_0,0),\delta_1)$

$$\sup_{(x,t)\in B^+((y_0,0),\delta_1)} |F(x,t)| < \lambda + \varepsilon$$

Thus $\inf_{\delta > 0} \sup_{(x,t)\ B^+((y_0,0),\delta)} |F(x,t)| < \lambda + \varepsilon.$

Since ε is arbitrary, $\overline{\lim}_{(x,t)\to(y_0,0^+)} |F(x,t)| \leq \lambda.$

By Lemma 2.1.8., we get

$$\overline{\lim}_{(x,t)\to(y_0,0^+)} |F(x,t)| \leq \overline{\lim}_{y\to y_0} |\phi(y)|. \tag{2.11}$$

If $\lim_{y\to y_0} \phi(y)$ exists and is equal to M, then

$\lim_{y\to y_0} (\phi(y)-M) = 0 = \overline{\lim}_{y\to y_0} (\phi(y)-M).$ Since

$$F(x,t) - M = \int_{\mathbb{R}^n} K(y-x,t)\left[\phi(y) - M\right]dy$$

Applying (2.11) by replacing $\phi(y)$ with $\phi(y) - M$,

$$\overline{\lim}_{(x,t)\to(y_0,0^+)} |F(x,t) - M| \leq \overline{\lim}_{y\to y_0} |\phi(y) - M|. \tag{2.12}$$

Since the limit of the right hand of (2.12) is zero,

$$\overline{\lim_{(x,t)\to(y_0,0^+)}} \left| F(x,t) - M \right| = 0.$$

i.e., $$\overline{\lim_{(x,t)\to(y_0,0^+)}} (F(x,t) - M)$$

$$= \underline{\lim}_{(x,t)\to(y_0,0^+)} (F(x,t) - M) = 0.$$

Hence $$\lim_{(x,t)\to(y_0,0^+)} F(x,t) = M = \lim_{y\to y_0} \Phi(y).$$