CHAPTER I

COMPLETE THEORIES

According to a well-known theorem of A. Lindenbaum, every consistent theory can be extended to form a consistent and complete theory. The question arises how many such extensions are there ? We state a general theorem from the meta-sentential calculus (due to Tarski) by which under certain assumptions only a single extension exists in the domain of a sentential logic.

We begin by saying what the symbols of a sentential logic are. These fall into three categories :

(i) A denumerable set of sentence variables :

$p, q, r, s, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \ldots\ldots$

(ii) Logical connectives : $\rightarrow$ and a subset of $\{\sim, \vee, \wedge, \leftrightarrow\}$.

(iii) Parentheses : (,).

1.1 Definition. For a sentential logic, the intersection of all those sets which contain all sentence variables and are closed under every connectives of this sentential logic is called the set of all sentences of this sentential logic, and denote this set by S. Call the elements of S sentences.

1.2 Rules of Inference. Let $\phi, \psi \in S$.

(i) Modus Ponen (or MP.) : From ϕ and $\phi \rightarrow \psi$ infer ψ.

(ii) Substitution (or Subs.) : From ϕ, b is a sentence variable in ϕ, infer the sentence $S^{b}_{\psi}\phi|$ where $S^{b}_{\psi}\phi|$ is the sentence resulted by substitution for each occurrence of b throughout ϕ by ψ.

1.3 Definition. Let X and Y be sets of sentences. $Sb_Y(X)$ is the set of all sentences which are obtained by replacing all sentence variables in the sentences of the set X by sentences of the set Y in such a way that variables of the same shape which occur in a given sentence are replaced by sentences of the same shape. Denote the sentence obtained from a sentence ϕ in the set X by substituting all distinct sentence variables $a_1, \ldots, a_n$ in ϕ by sentences $\phi_1, \ldots, \phi_n$ in the set Y by $S^{a_1, \ldots, a_n}_{\phi_1, \ldots, \phi_n}\phi|$.

If Y = S, we write Sb(X) instead of $Sb_S(X)$.

1.4 Definition. A proof from a set X of sentences is a finite sequence of sentences $\psi_1, \ldots, \psi_n$ such that each ψ_i, $1 \leq i \leq n$, is

(i) a sentence in X, or

(ii) a conclusion from ψ_j $(j < i)$ by Subs., or

(iii) a conclusion from ψ_j, ψ_k $(j,k < i)$ by MP..

1.5 Definition. A sentence ϕ is a theorem of a set X of sentences in notation $\vdash_X \phi$, if and only if ϕ is the last sentence of a proof from X.

1.6 Remark. Let X be a set of sentences. Then the set of all theorems of X is the intersection of all sets of sentences which include the set Sb(X) and are closed under MP. Denote this set by Cn(X).

1.7 Definition. A set X of sentences is said to be a (deductive) theory if and only if Cn(X) = X.

1.8 Definition. A set X of sentences is said to be inconsistent if and only if Cn(X) = S. Otherwise X is consistent.

1.9 Definition. X and Y are sets of sentences. X is said to be complete with respect to Y if and only if for every sentence ϕ in Y either $\phi \in Cn(X)$ or the set $X \cup \{\phi\}$ is inconsistent.

If Y = S, we say that X is complete.

1.10 Theorem. (Lindenbaum's Theorem.) Every consistent theory Σ can be extended to a complete and consistent theory.

Proof. Let us arrange all the sentences in a list,

$$\phi_0, \phi_1, \phi_2, \ldots\ldots\ldots\ldots\ldots\ldots$$

Define a sequence of theories E_0, E_1, ... as follows :

(i) $E_0 = \Sigma$

(ii) $E_{n+1} = \begin{cases} Cn(E_n \cup \{\phi_n\}, & \text{if } E_n \cup \{\phi_n\} \text{ is consistent} \\ E_n & \text{, otherwise.} \end{cases}$

Let $E = Cn(\bigcup_{n \geq 0} E_n)$. Clearly E is an extension of Σ and E_n is a consistent theory for all n.

Claim that E is consistent. Suppose not. Then $p \in Cn(E)$, and so there is a finite sequence of sentences in E which is a proof of p, say $\psi_1, \ldots, \psi_n$. Therefore there exists an m such that $\psi_1, \ldots, \psi_n \in E_m$, and hence $p \in Cn(E_m)$. Consequently E_m is inconsistent which is a contradiction. Therefore E is consistent.

Next, we claim that E is a complete theory. Let ϕ be any sentence. Then $\phi = \phi_n$ for some n. Suppose $\phi \notin Cn(E)$. If $E \cup \{\phi_n\}$ is consistent, $E_n \cup \{\phi_n\}$ which is contained in $E \cup \{\phi_n\}$ is also consistent, so $E_{n+1} = Cn(E_n \cup \{\phi_n\})$, hence $\phi = \phi_n \in Cn(E_{n+1}) \subseteq Cn(E)$ which is a contradiction. Therefore $E \cup \{\phi_n\}$ is inconsistent. This proves that E is complete.

1.11 <u>Theorem</u>. For every set X of sentences, $Sb_X(S)$ is the smallest set of sentences which includes X and is closed under every connective.

<u>Proof</u>. (In [3] p. 395.)

1.12 <u>Theorem</u>. (Tarski's Theorem) Let X be a consistent theory which satisfies the following condition : there is a set Z of sentences such that X is complete with respect to the set $Sb_Z(S)$ and the set $X \cup Sb_Z(S)$ is inconsistent. Then there exists exactly one consistent and complete theory Y, which includes the set X.

<u>Proof</u>. (In [3] pp. 395-397.)