CHAPTER III

WARRANT PRICING MODELS

For consistency and ease of understanding, unless stated elsewhere, all notations used in all the models in this study have the following meanings.

V	is the total value of firm.
v	is the value of firm per share = V/N.
α	is instantaneous expected rate of return on the value of firm.
σ_v	is firm volatility.
dZ_t	is the Wiener process.
C	is value of a call option.
S	is stock price per share.
r	is risk-free rate of interest.
$W_{i,j}$	is value of series i warrants at time j.
n_i	is number of series i warrants.
N	is number of original outstanding shares.
λ_i	is ratio of the series i warrants to the number of shares = n_i/N.
T_i	is expiration time of series i warrants (in year).
K_i	is exercise price of series i warrants.
q	is number of outstanding warrant series.

The firm is assumed to be an all equity firm. The total value of firm is defined as

$$V = NS + \sum_{i=1}^{q} n_i W_i$$

and the value of firm per share is defined as

$$v = S + \sum_{i=1}^{q} \lambda_i W_i$$

3.1 Galai-Schneller Model

The Galai-Schneller (GS) model is very similar to the Black-Scholes model. A warrant is valued as a fraction of call option price.

$$W = \frac{C}{1+\lambda} \tag{3.1}$$

The value of the call option is determined using the Black-Scholes model. In this case, the call option is evaluated as an option on the value of firm, v, which assume to follow a geometric Brownian motion.

$$\frac{dv(t)}{v(t)} = \alpha dt + \sigma dZ_t$$

The Black and Scholes (1973) pricing model for warrants is shown below.

$$C = vN(d_1) - Ke^{-rT}N(d_2) \tag{3.2}$$

where

$$d_1 = \frac{ln(\frac{v}{K})+(r+\frac{\sigma_v^2}{2})(T)}{\sigma_v\sqrt{T}}$$

$$d_2 = d_1 - \sigma_v\sqrt{T}$$

and $N(d_1)$ and $N(d_2)$ are the probability from the cumulative standard normal distribution.

3.2 Lim-Terry Multiple Warrants Model

The Lim-Terry (LT) consider an all-equity firm that has two series of warrants outstanding, A and B. There are n_A series A warrants and n_B series B warrants. Series A and series B warrants mature at T_A and T_B and have exercise price of K_A and K_B respectively. The value of firm per share,v, is assumed to follow a geometric Brownian motion.

$$\frac{dv(t)}{v(t)} = \alpha dt + \sigma dZ_t$$

The valuation of warrants can be determined using backwards recursion by first value the warrants at time T_A and then using the risk-neutral pricing method of Cox and Ross (1976) to determine the current value.

3.2.1 Series A Warrants Pricing

The value of series A warrants at time T_A can be determined as follows.

$$W_{A,T_A} = max\left\{0, \frac{1}{1+\lambda_A}\left(v_{T_A} + \lambda_A K_A - \lambda_B W^e_{B,T_A}\right) - K_A\right\}$$

This formula implies that the series A warrants will be exercised at T_A whenever the value of firm per share exceeds v^*, which is given by

$$v^* = K_A + \lambda_B W^e_{B,T_A}(v^*)$$

where $W^e_{B,T_A}(v^*)$ is defined in the next subsection as an explicit function of v^*. Hence, this equation must be solved iteratively for v^*.

Using the risk-neutral pricing model of Cox and Ross (1976), the current value of series A warrants can be valued as follows.

$$W_{A,0} = \frac{1}{1+\lambda_A} e^{-rT_A} \int_{V^*}^{\infty} \left[v_{T_A} - K - \lambda_B W^e_{B,T_A}\right] dF\left(v_{T_A} \left| v_0\right.\right)$$

where $F\left(v_{T_A} \left| v_0\right.\right)$ denotes the distribution of the value of the firm at T_A conditional upon its current value.

Substituting for W^e_{B,T_A}as will be defined in the next subsection and taking the integrals, the current value of series A warrants can be obtained.

$$\begin{aligned} W_{A,0} = & \tfrac{1}{1+\lambda_A}\left\{v_0 N\left(d_1^*\right) - K_A e^{-rT_A} N\left(d_2^*\right)\right. \\ - & \tfrac{\lambda_B}{1+\lambda_A+\lambda_B}\left[v_0 M\left(d_1^*, d_1'; \sqrt{\tfrac{T_A}{T_B}}\right)\right. \\ + & \left.\left.\left[\lambda_A K_A e^{-rT_A} - \left(1+\lambda_A\right) K_B e^{-rT_B}\right] M\left(d_2^*, d_2'; \sqrt{\tfrac{T_A}{T_B}}\right)\right]\right\} \end{aligned} \tag{3.3}$$

where

$$d_1^* = \frac{\ln\left(\frac{v_0}{v^*}\right)+\left[r+\frac{\sigma^2}{2}\right](T_A)}{\sigma\sqrt{T_A}}$$

$$d_2^* = d_1^* - \sigma\sqrt{T_A}$$

$$d_1' = \frac{\ln\left(\frac{v_0}{(1+\lambda_A)K_B-\lambda_A K_A e^{r(T_B-T_A)}}\right)+\left[r+\frac{\sigma^2}{2}\right](T_B)}{\sigma\sqrt{T_B}}$$

$$d_2' = d_1' - \sigma\sqrt{T_B}$$

and $M(a, b; \rho)$ denotes the bivariate cumulative normal distribution with a and b as upper limits and ρ as the correlation coefficient.

3.2.2 Series B Warrants Pricing

To price series B warrants at time T_A, two cases are considered. First is the case that series A warrants are exercised at time T_A and second is the case that they are not.

- The case that series A warrants are not exercised at time T_A. The value of series B warrants can be priced as follows.

$$W^u_{B,T_A} = \frac{1}{1+\lambda_B}\left[v_{T_A} N\left(d_1^u\right) - K_B e^{-r(T_B-T_A)} N\left(d_2^u\right)\right]$$

where

$$d_1^u = \frac{\ln\left(\frac{v_{T_A}}{K_B}\right)+\left[r+\frac{\sigma^2}{2}\right](T_B-T_A)}{\sigma\sqrt{T_B-T_A}}$$

$$d_2^u = d_1^u - \sigma\sqrt{T_B-T_A}$$

- The case that series A warrants are exercised at time T_A. The value of series B warrants can be priced as follows.

$$W^e_{B,T_A} = \frac{1}{1+\lambda_A+\lambda_B}\left\{v_{T_A} N\left(d_1^e\right) + \left[\lambda_A K_A - (1+\lambda_A) K_B e^{-r(T_B-T_A)}\right] N\left(d_2^e\right)\right\}$$

where

$$d_1^e = \frac{\ln\left(\frac{v_{T_A}}{(1+\lambda_A)K_B-\lambda_A K_A e^{r(T_B-T_A)}}\right)+\left[r+\frac{\sigma^2}{2}\right](T_B-T_A)}{\sigma\sqrt{T_B-T_A}}$$

$$d_2^e = d_1^e - \sigma\sqrt{T_B - T_A}$$

Using the risk-neutral pricing model of Cox and Ross (1976), the current value of series B warrants can be obtained as follows.

$$W_{B,0} = e^{-rT_A}\left[\int_0^{v^*} W_{B,T_A}^u dF\left(v_{T_A}\left|v_{T_0}\right.\right) + \int_{v^*}^{\infty} W_{B,T_A}^e dF\left(v_{T_A}\left|v_0\right.\right)\right]$$

In a similar fashion to series A warrants valuation, the closed form of the current value of series B warrants can be determined.

$$\begin{aligned} W_{B,0} = & \tfrac{1}{1+\lambda_B}\left[v_0 M\left(-d_1^*, d_1''; -\sqrt{\tfrac{T_A}{T_B}}\right) - K_B e^{-rT_B} M\left(-d_2^*, d_2''; -\sqrt{\tfrac{T_A}{T_B}}\right)\right] \\ & + \tfrac{1}{1+\lambda_A+\lambda_B}\left[v_0 M\left(d_1^*, d_1'; \sqrt{\tfrac{T_A}{T_B}}\right)\right. \\ & + \left.\left[\lambda_A K_A e^{-rT_A} - (1+\lambda_A) K_B e^{-rT_B}\right] M\left(d_2^*, d_2'; \sqrt{\tfrac{T_A}{T_B}}\right)\right] \end{aligned} \tag{3.4}$$

where

$$d_1'' = \frac{\ln\left(\frac{v_0}{K_B}\right)+\left[r+\frac{\sigma^2}{2}\right]T_B}{\sigma\sqrt{T_B}}$$

$$d_2'' = d_1'' - \sigma\sqrt{T_B}$$

3.3 Darsinos-Satchell Multiple Series Warrants Model

The Darsinos-Satchell (DS) model can price more than two series of warrants. The first series of warrants is defined as a warrants series that expires earliest. The second series of warrants is defined as the warrants series that expires after the first series, and so on. The value of firm per share, v, is assumed to follow a geometric Brownian motion.

$$\frac{dv(t)}{v(t)} = \alpha dt + \sigma dZ_t$$

3.3.1 Series A Warrants Pricing

Series A warrants are valued the same as valuing single series.

$$W_{A,0} = \frac{C(v, T_A, K_A, \sigma, r)}{1 + \lambda_A} \tag{3.5}$$

$C(\cdot)$ can be any option pricing model. However, in this paper, $C(\cdot)$ denotes the Black-Scholes option price for an European call.

3.3.2 Series B Warrants Pricing

Series B warrants pricing can be divided into two cases, the case that series A warrants are exercised at time T_A and the case that they are not.

- The case that series A warrants are not exercised at time T_A. Since the number of shares do not change, the exercise decision is still the same (exercise when $v_{T_B} > K_B$). Therefore, if series B warrants are exercised, the payoff to the warrantholders is

$$\begin{aligned} W^u_{B,T_B} &= \tfrac{V_{T_B} + n_B K_B}{N + n_B} - K_B \\ &= \tfrac{1}{1+\lambda_B}\left(v_{T_B} - K_B\right) \end{aligned}$$

- The case that series A warrants are exercised at time T_A. In this case, the number of shares increase by the number of series A warrants due to the exercising series A warrants. If series B warrants are exercised, the payoff to the warrantholders is

$$\begin{aligned} W^e_{B,T_B} &= \tfrac{V_{T_B} + n_B K_B}{N + n_A + n_B} - K_B \\ &= \tfrac{1}{1+\lambda_A+\lambda_B}\left(v_{T_B} - (K_B + \lambda_A K_B)\right) \end{aligned}$$

Series B warrants will be exercised when the value of firm per share exceeds $K_B + \lambda_A K_B$.

Combining the two cases, the current value of series B warrants can be determined as follows.

$$W_{B,0} = (1 - p_A) \times \frac{C(v, T_B, K_B, \sigma, r)}{1 + \lambda_B} + p_A \times \frac{C(v, T_B, K_B + \lambda_A K_B, \sigma, r)}{1 + \lambda_A + \lambda_B} \tag{3.6}$$

where p_A is the probability that the value of firm per share at time T_A will be above the exercise price of the series A warrant, $p_A = Prob(v_{T_A} > K_A)$. The probability can be obtained using standard normal distribution function.

3.3.3 Other Series Pricing

A closed-form formula for series that expires after time T_B can be derived in a similar manner of series B warrants formula. The warrant pricing formula for a firm that has q outstanding warrant series is shown below.

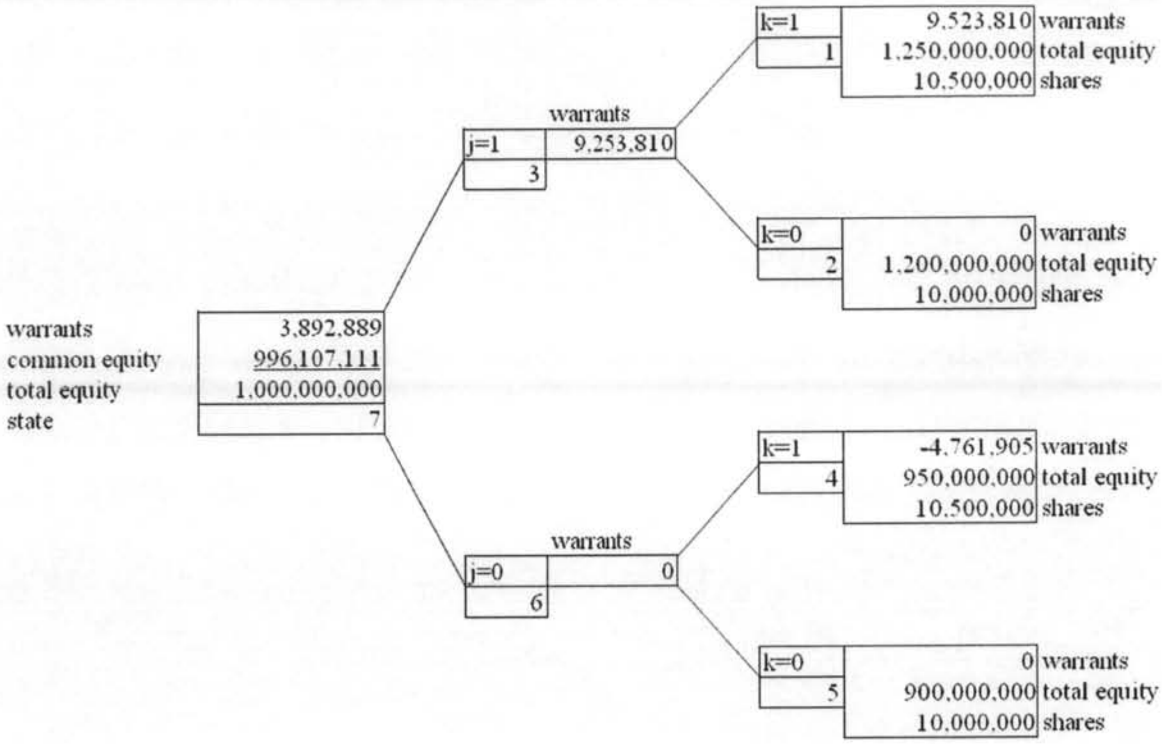
$$W_{n,0} = \sum_{S} \left[\left(\prod_{i=1}^{n-1} p_i^{\delta_i} \times (1 - p_i)^{1-\delta_i} \right) \times \frac{C\left(v, T_n, K_n + \sum_{i=1}^{n-1} \delta_i \lambda_i K_n, \sigma_v, r\right)}{1 + \sum_{i=1}^{n-1} \delta_i \lambda_i + \lambda_n} \right] \tag{3.7}$$

where $T_1 < T_2 < ... < T_n < ... < T_q$. p is the probability that the value of firm per share will be above the exercise price of the i^{th} warrant, $p_i = Prob(v_{T_i} > K_i)$. The probability can be obtained using standard normal distribution function. $C(\cdot)$ can be any option pricing model. In this paper, $C(\cdot)$ denotes the Black-Scholes option price for an European call. δ is equal to 0 if the value of firm per share is higher than the exercise price. Otherwise δ is equal to 1.

3.4 Dennis-Rendleman Multiple Series Warrants Model

The Dennis-Rendleman (DR) model uses the binomial model as the basis for multiple series of warrants valuation. The value of firm is assumed to follow a

Figure 3.1: Binomial Model Illustration



standard multiplicative binomial process with constant up and down factors of u and d, respectively. The firm issues q warrant series. Series 1 warrants expires at binomial time $t = 1$, series 2 warrants at binomial time $t = 2$ and so on. The strike price of the q warrants series are $K_i = K_1, K_2, ..., K_q$ and the number of shares associated with each is $n_i = n_1, n_2, ..., n_q$.

It should be much easier to understand how this model can be used to price multiple series of warrants by exploring through an example first. As a demonstration, we assume there is only one warrant series issued. The firm issue 500,000 warrants with the exercise price of 100 dollars per share. There are 10,000,000 shares outstanding and the firm has value of total equity, including common equity and warrants, of 1 billion dollars. The value of total equity follows a standard multiplicative binomial process with $u = 1.2$ and $d = 0.9$, and the risk-free rate of interest is 0.03 per binomial period. These parameters imply a risk-neutral probability associated with an increase in total equity value of $\pi = \frac{1+r-d}{u-d} = \frac{1+0.03-0.9}{1.2-0.9} \approx 0.4333$. The valuation process is as follows.

Figure 3.1 represents a structure of a non-recombining binomial decision tree. The first branch of the tree indicates change in the value of total equity, which can be either up with the probability of 0.4333 and down with the probability of 1-0.4333. The upper part ($j = 1$) of the branch reflects the increase in the value of total equity. In a similar fashion, the lower part of the branch ($j = 0$) reflects the decrease in the value of total equity. Each value of j is related with the value of k. If k equals to 1 (state 1 and 4), the warrants are exercised and the total value of warrants is computed from the quotient of total equity divided by shares. On the contrary, when k is zero (i.e. the warrants are not exercised in state 2 and 5), the value of warrants always equals zero.

From a set of given parameters, the current total value of warrants can be calculated, which, in this example, is 3,892,889 dollars. The value of warrants in each state can be calculated by following path in the tree. For example, the total equity in state 1 is given by $V_1 = V_0 u^{j_1} d^{1-j_1} + n_1 K_1 k_1 = 1,000,000,000 \times 1.2^1 \times 0.9^{1-1} + 500,000 \times 100 \times 1 = 1,200,000,000 + 50,000,000 = 1.25$ billion dollars. Since in the warrants are exercised ($k = 1$), the number of shares will be $10,000,000 + 500,000 = 10,500,000$. Consequently, the value of warrants in state 1 is $\left(\frac{1,250,000,000}{10,500,000} - 100\right) \times 500,000 = 9,523,810$ dollars. In state 2, the warrants are not exercised, so they have no value. The value of warrants in state 3, which is 9,253,810, can be obtained by finding the maximum value of warrants in state 1 and 2. The value of total equity, the number of shares in state 4 and 6 and the value of warrants in state 4, 5, and 6 can be calculated in the same manner. Finally, the total value of warrants in state 7 is given by $\frac{9,253,810 \times 0.4333 + 0 \times (1-0.4333)}{1.03} = 3,892,889$ dollars.

After going through the example, we now come up with the broad idea of how to value multiple series warrants using the binomial model. The key is to simultaneously keep track of binomial-based increases and decreases in total equity value and optimal exercise decisions for the q warrants series issued by the firm. For each decision process, the binary indices, applicable to binomial time t, are defined as follows.

- j_t indicates how the value of the firm changes from time $t-1$ to time t. A value of $j_t = 1$ indicates an increase in the value of total equity at time t, and a value of $j_t = 0$ indicates a decrease in value.
- k_t indicates whether or not the warrant series that matures at time t is exercised.

If $k_t = 1$, warrant t is exercised. The value of warrants will be equal to $\left(\frac{Total\ Equity}{Number\ of\ Shares} - K\right) \times Number\ of\ Issued\ Warrants$. If $k_t = 0$, warrant t is not exercised and the value of warrants is zero.

The value of the firm (V_t) is defined as $V_t = V_{t-1}u^{j_t}d^{1-j_t} + n_t K_t k_t$ for $t = 1, 2, ..., q$. V_0 is the initial value of the firm. This value reflects natural changes in equity value plus the proceed from exercising options, if there is any. Any new capital obtained though the exercise of warrants is assumed to be invested back into the firm and to follow the same binamial process as previously existing equity capital.