CHAPTER IV

THE RENORMALIZATION GROUP

Contents

4.1 Introduction (Yeomans 1992, Ma 1967, Bellac 1991)

The mean field theory does not accurately predict critical exponents. Attempts to improve mean field theory by perturbation also fails, because of the divergence of correlation length, ξ, as system close to critical points.

The idea of the "Renormalization Group" (abbreviated as RG) was formulated by Stueckelberg and Petermann (1953) and independently by Gell-Mann and Low (1954) in the context of the divergences of quantum field theory. "Renormalization" is a procedure for making a theory finite. Such procedure is not

unique and without originally. The "renormalization group" expressed the invariance of the Physics under changes in the procedure. The other version was introduced by K.G. Wilson who worked for his doctoral degree with Murray Gell-Mann at CalTech. His work revolutionized the field of both critical phenomena in classical statistical mechanics and quantum field theory, and was recognized by the 1982 Noble Prize in physics. The first RG version may be considered as the special case of the later version. The links between two version were examined in chapter 7 in "Quantum and Statistical Field Theory" by M. L. Bellac.

The basic concept of Wilson RG(Wilson 1974) originated from L.P. Kadanoff that a divergent correlation length implied that there was a relation between the coupling constants of effective Hamiltonian and the length scale over which the order parameter was defined. But his idea did not make the critical exponents calculatable, K.G. Wilson elaborated and completed Kadanoff's idea, by showing how the relationship between coupling constants at different scales could be explicitly computed. So the RG is thus capable to estimate the critical exponents and also provides a natural way to understand universality.

4.2 Block spin or real-space renormalization (Wilson 1974)

Real space renormalization technique is closer to Kadanoff block spin idea. This technique is applicable to model based on a lattice having a "discrete scaling symmetry". To clarify this means, consider taking a lattice and blocking it. This means dividing the sites of the lattice into groups or blocks and then replacing each block by just one single site which may be at the position occupied by one of the sites I that block or at some other position within the area covered by the block. The lattice has a discrete scaling symmetry if we can block it in a way that we

produce a lattice exactly like the one we started with, except for an increase in the lattice parameter $a \longrightarrow a' = la$. The process of renormalizing the lattice by a factor of l, gives us in the end exactly the same lattice as we started with.

If we group sites into blocks containing, p sites on average, then the renormalized lattice will contain fewer sites than the original one by a factor p. Since we have scaled our whole lattice by a factor l, its volume must shrink by a factor l^d, where d is dimension. If sites in the renormalized lattice are arranged in exactly the same way as those in the original one, their number must be reduced by a factor $p = l^d$.

The most common lattice displaying a discrete scaling symmetry is the square lattice. Figure 4.1 illustrates the renormalization of the square lattice in two dimensions. In this case $l = 2$ and the lattice is left with a quarter of the number of site it started with.

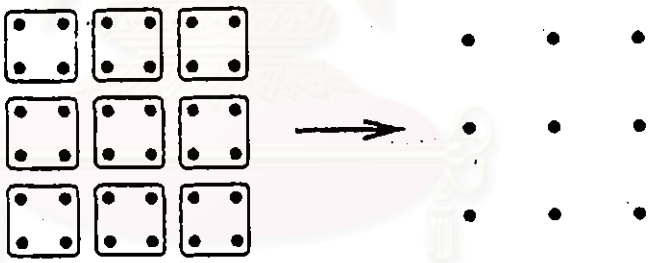

Figure 4.1 Renormalization of square lattice, the linear dimensions of the lattice on the right must be shrunk by a factor of $l = 2$ to render it similar to the original one

4.3 Basic Concept of the Renormalization Group(Wilson 1974, Bellac 1991)

The Renormalization Group consists of two principle steps. The first step is coarse-graining transform which can be characterized in real space or in momentum

space. In real space, the transform follows Kadanoff's concept of introducing block spin variables. In momentum space, the aim is to eliminate high momentum variables which correspond to short wavelength fluctuation. The second step is to identify the origin of the singular behavior.

4.3.1 Properties of Renormalization Group Transform

We consider a system described by the Hamiltonian

$$H = H(K_1, K_2, K_3, \ldots K_m, \ldots)$$

where K_m are the coupling constants. After we block spin or group together the degree of freedom in a block of linear dimension l_a, a = lattice spacing, the new system can be described by a new Hamiltonian

$$H' = R_l H \tag{4.1}$$

We call such a transformation "*renormalization group transformation*", R_l . The R_l decreases the number of degree of freedom from N to N'. The scale factor of transform, l , is defined by

$$l^d = \frac{N}{N'} \tag{4.2}$$

In general, R_l is a very complicated non-linear transformation. Since $l > 1$, there is no inverse transformation. The transformation R_l for different $l > 1$ forms a

semi-group. The two successive transformation with $l = l_1$ and $l = l_2$ should be equivalent to a combined scaled change of $l_1 l_2$.

$$\begin{aligned} R[H'] &= R_{l_1}[H] \\ [H''] &= R_{l_2}[H'] \\ &= R_{l_2} . R_{l_1}[H] \end{aligned} \tag{4.3}$$

and this

$$R_{l_1 l_2}[H] = R_{l_2} . R_{l_1}[H] \tag{4.4}$$

The essential condition to be satisfied by any renormalization group transformation is the partition function must not change.

$$Z_{N''}(H') = Z_{N'}(H) \tag{4.5}$$

Therefore the total free energy remain the same and the reduced free energy per spin, f transform as

$$f(H') = l^d f(H) \tag{4.6}$$

This is an important equation which will lead us to a scaling form for the free energy. The lengths, which are measured in term of the new lattice spacing are reduced by factor l.

$$\vec{r} = \vec{r}' = \frac{\vec{r}}{l} \tag{4.7}$$

Similarly the momenta, having the dimensions of the inverse of length are renormalized according to

$$\vec{k} \quad = \quad \vec{k}' \quad = \quad l\vec{k}$$

4.3.2 The Origin of Singular Behavior

Consider the following example of how singular behavior can arise. Let a particle move in one dimensional potential $V(x)$, see Fig 4.2. The position $x(t)$ is determined by the equation

$$\frac{dX}{dt} \quad = \quad -V'(x) \qquad (4.8)$$

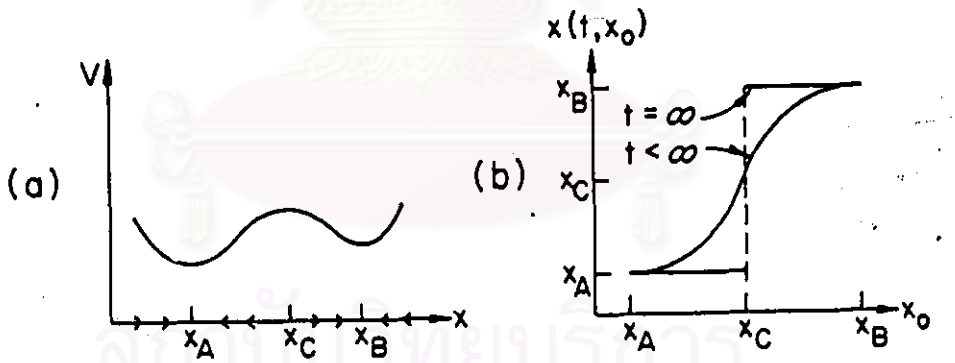

Figure 4.2 (a) The potential $V(x)$. The arrows on the x-axis indicate the direction of motion of the particle as a function of x. (b) Position of the particle after time t as a function of initial position, for finite and infinite times.

If the particle is released from any point $X < X_C$, it will roll to X_A and stop. If the particle is released from any point from $X > X_C$, it will roll to X_B and

stop. Thus, the final position of the particle is a discontinuous function of the initial position, X_0. Note that $X(t, X_0)$, the position at time t after release at X_0, is a continuous function of X_0 for finite values of t, but a discontinuous function of X_0 when $t = \infty$. The potential $V(X)$ is perfectly analytic, so the singular behavior is not due to pathologies of $V(X)$.

In fact, the origin of the singular behavior is the amplification of the initial condition due to the infinite time limit.

The points X_A, X_B and X_C are the *fixed points* of equation (4.8), if the particle is at fixed points at some time, t', then remains there for $t > t'$. There are two varieties of fixed point, repulsive and attractive. If the particle starts off near the fixed point X_C, it will always end up either X_A or X_B, but never at X_C. On the other hand, if the particle starts off near either X_A or X_B it will end up at that fixed point.

The set of initial conditions $\{X_0\}$ which flows to a given point is called "the basin of attractive of that fixed point". In the example, the basin of the attraction of X_B is $X > X_C$, while the basis of attraction of X_A is $X < X_C$, the basin of attraction of X_C is $X = X_C$.

This can be analogous to a dynamical system of a infinite number of renormalization group transformation. After n RG iteration, the system is described by coupling constants, $K_0^n, K_1^n \ldots K_m^n$.

As n varies, the system may be thought to be represented by a point moving in a space whose axes are coupling constant, $K_0, K_1 \ldots K_m$.

On iterating renormalization group transformation, a given system represented by its initial set of coupling constants, traces out a trajectory in its coupling constant space.

The set of all such trajectories, generated by different initial sets of coupling constants, generates a "*renormalization group flow*" in coupling constant space".

4.4 Fixed points(Ma 1967, Bellac 1991 Binney 1992, Yeomans 1992)

The crucial thing about renormalization group method is the recognition of the importance and the physical significance of fixed point of renormalization group transformation. More detail we will discuss below.

4.4.1 Physical Significance of Fixed Point

When we know the renormalization group transformation $R_e[H]$ or $R_e[K]$ then the fixed point of the renormalization group transformation is the point $[K^*]$in coupling constants satisfying

$$[K^*] \quad = \quad R_e[K^*] \qquad (4.9)$$

Under the Renormalization Group Transformation R_l , length scales are reduced by a scale factor l . For any particular values of the coupling constants, we can compute the correlation length, ξ, which transforms under R_l according to

$$\xi[K'] \quad = \quad \xi[K]/l \qquad (4.10)$$

This indicate that the system moves further from criticality after a renormalization group has been performed. At a fixed point,

$$\xi[K^*] \quad = \quad \xi[K^*]/l \tag{4.11}$$

implies that $\xi[K^*]$ can only be zero or infinity. We call a fixed point with $\xi = \infty$ as a "critical fixed point", and a fixed point with $\xi = 0$ as a "trivial fixed point".

In general, a renormalization group transformation have several fixed points. Each fixed point has its own basin of interaction. All points in coupling constant space, which lie within the basin of attraction of a given fixed point flow towards and ultimately reach the fixed point after an infinite number of iterations of R_l. The set of points, the basin of attraction of critical fixed points, is often called " *critical manifold* or *critical surface*".

The fact that all points on the critical manifold flow toward the same fixed point, is a basic mechanism for universality, which involves a behavior exhibited by a system close to, but not, at its critical point. So the knowledge of fixed point of renormalization group transformation enables the phase diagram to be determined, while the behavior of Renormalization Group flow near a critical fixed point determines the critical exponent.

4.4.2 Behavior near a Fixed Point, Critical Exponent

$$\text{Let} \quad K_n \quad = \quad K_n^* + \delta K_n \tag{4.12}$$

so that the starting Hamiltonian is close to the fixed point Hamiltonian i.e., the Hamiltonian with the coupling constants equal to their fixed point values.

$$H \quad = \quad H[K^*] \tag{4.13}$$

Let $H = H^* + \delta H$, now perform a renormalization group transformation

$$[K'] \quad = \quad R_l[K]$$

then $$\left[K'_n\right] \quad = \quad K'_n[K] \quad = \quad K^*_n + \delta K'_n \tag{4.14}$$

with $\delta K'_n$ given by Taylor's theories

$$K'_n\{K_1^* + \delta K_1, K_2^* + \delta K_2, \ldots\} \quad = \quad K_n^* + \sum_m \left.\frac{\partial K'_n}{\partial K_m}\right|_{K_m < K_m^*} .\delta K_m + 0\left((\delta K)^2\right) \tag{4.15}$$

So that

$$\delta K'_m \quad = \quad \sum_m M_{nm}\delta K_m \tag{4.16}$$

where $$M_{nm} \quad = \quad \left.\frac{\partial K'_n}{\partial K_m}\right|_{K=K^*} \tag{4.17}$$

is the linearized renormalization group transformation in the vicinity of the fixed point, K^*. The matrix M is real, butit is not symmetric, in general.

Consequently, M is not diagonalized and its eigenvalues are not necessarily real.

Let M^l be the linearized renormalization group transformation where the superscript l denotes the scale factor involved in the Renormalization Group Transformation R_l. For simplicity, we assume that M is symmetric. We denote the eigenvalues and eigenvector by $\lambda_l^{(\sigma)}$and $l_n^{(\sigma)}$ respectively, where σ labels the

eigenvalue and l subscript n labels the component of the vector l. Using the Einstein summation convention, we have

$$M^{l}_{nm} l^{(\sigma)}_{m} = \lambda^{(\gamma)} l^{\sigma}_{n} \tag{4.18}$$

From the semi-group properties we have

$$M^{l} M^{l'} = M^{ll'} \tag{4.19}$$

and thus

$$\lambda^{(\sigma)}_{l} \lambda^{(\sigma)}_{l'} = \lambda^{(\sigma)}_{ll'} \tag{4.20}$$

This constrains the eigenvalues to be in the form

$$\lambda^{(\sigma)}_{l} = l^{y_\sigma} \tag{4.21}$$

where y_σ are independent of the scale factor l. The y_σ are critical exponents which can be related to α, β, γ,.... .

For a Hamiltonian near fixed point K^*, the deviation from the fixed point may be expanded in term of the eigenvectors of M, $e^{(\sigma)}$

$$K = K^* + \sum_{\sigma} g^{(\sigma)} e^{(\sigma)} \tag{4.22}$$

The coefficient $g^{(\sigma)}$ are termed the linear scaling fields. Under renormalization

$$K' = K^* + \sum_{\sigma} l^{y_\sigma} g^{(\sigma)} e^{(\sigma)} \tag{4.23}$$

or, more shortly

$$g'^{(\sigma)} = l^{y_\sigma} g^{(\sigma)} \tag{4.24}$$

From (4.24) the flow of a Hamiltonian within its parameter space depends on the set of scaling fields $g^{(\sigma)}$, describing the original position of the Hamiltonian, and on the form of y_σ. For a positive y_σ the scaling fields $g^{(\sigma)}$ increase under repeated iterations of the renormalization transformation and drive the system away system from the fixed point. This is called a *relevant field.*

If y_σ is negative the corresponding scaling fields decrease under repeated iterations of the renormalization transfomation, with the system moving closer to the fixed point. These scaling fields are called *irrelevant field.*

If any y_σ is zero, the corresponding scaling fields are called marginal field. This case, which allows a continuous variation of exponents with an interaction parameter in the Hamiltonian, will not be reviewed further here. Thus the stability of a fixed point depends on the number of relevant and irrelevant fields or their eigenvalues.

4.5 The RG on Landau-Ginzburg Model (Bellac 1991)

In this section, we review the Gaussian fixed point and the linearized RG in its neighborhood. The Gaussian fixed point provides a description of critical behavior for $d > 4$, and the case $d = 4-\varepsilon$ with small $\varepsilon > 0$ is discussed. In addition to the Gaussian fixed point, there is a another fixed point which often called Wilson-Fisher fixed point (Goldenfeld 1992).

4.5.1 Transformation in Fourier Space

We begin with the Landau Ginzburg model for n component spins in D dimensional space. We shall write the Hamiltonian as

$$H = \int d^D x \left[\frac{c}{2} (\nabla S(x))^2 + \frac{r_0}{2} S(x)^2 + \frac{u_0}{4} S(x)^4 \right] \qquad (4.25)$$

The process of RGT can be done in real space by block spin transformation or in momentum space. In this case it is convenient to use the momentum representation. The RGT in Fourier space can be done by integrating over wave numbers between $\Lambda = \frac{1}{a}$ and $\Lambda' = \frac{\Lambda}{a} = \frac{1}{al}$ when a is lattice spacing, l is scaled factor. We introduce the Fourier Transform $\tilde{S}(k)$ of $S(x)$ by

$$\tilde{S}(k) = \frac{a^D}{L^{\frac{D}{2}}} \sum_x e^{ikx} S(x) = \int \frac{d^D x}{L^{\frac{D}{2}}} e^{ikx} S(x) \qquad (4.26)$$

where L is the size of the system, when u_0 is zero, (4.25) corresponding to the Gaussian model and in the momentum space, they become a simple form, By Parseval's theorem, we find

$$H = \frac{1}{2}\sum_{k\le\Lambda}(r_0 + ck^2)\tilde{S}(k)\tilde{S}(-k) \qquad (4.27)$$

Instead of writing the integration measure for the partition function in the space spanned by $S(x)$, one can write it in the space spanned by the $\tilde{S}(k)$. Because the transformation $S(x) \rightarrow \tilde{S}(k)$ is unitary to a multiplicative factor

$$\prod_x dS(x) \rightarrow \prod_{k\le\Lambda} d\tilde{S}(k)$$

$$Z = \int \prod_{k\le\Lambda} d\tilde{S}(k) \exp\left[\frac{-1}{2}\sum_k (r_0 + ck^2)\tilde{S}(k)\tilde{S}(-k)\right] \qquad (4.28)$$

We have summarized the RGT, R_l, in Fourier space which is subdivided into three steps.

1) Integration over k;

$$\frac{\Lambda}{l} \le k \le \Lambda$$

2) Dilation of unit length ;

$$x \rightarrow x' = \frac{x}{l}$$

$$k \rightarrow k' = kl$$

3) Renormalization of field;

$$S(x) \quad \rightarrow \quad S'(x') \quad = \quad l^{d_s} S(x)$$

$$\tilde{S}(k) \quad \rightarrow \quad S'(k') \quad = \quad l^{d_s - D/2} \tilde{S}(k)$$

when d_S is called *canonical dimension* of field and equals to *D/2-1* in Gaussian model. We can write down the relation for the transformation Hamiltonian $H' = R_l H$,

$$e^{-H'(S')} \quad = \quad \left[\int \prod_{\frac{\Lambda}{l} \le k \le \Lambda} d\tilde{S}(k) e^{-H(S)} \right]_{\tilde{S}(k) \to l^{(D/2)-d_s} \tilde{S}'(lk)} \tag{4.29}$$

Thus, for Gaussian model, the set of coupling constants or parameter space is two dimensional; $[K] = \{c, r_0\}$. We integrate over $d\tilde{S}(k)$ yields a constant. Since we are dealing with a product of decoupled Gaussian integrals, the new Hamiltonian is

$$H' \quad = \quad \sum_{k \le \frac{\Lambda}{l} 2} \frac{1}{2} \left(r_0 + ck^2 \right) \left| S(k) \right|^2$$

$$= \quad \sum_{k \le \Lambda} \frac{1}{2} l^{D-2d_s} \left(r_0 + cl^{-2} k'^2 \right) \left| S'(k') \right|^2 \tag{4.30}$$

Note that $S(-k) = S^*(k)$, we so write $S(k)$ instead of $\tilde{S}(k)$. The new Hamiltonian H' has the same form as its original, with the substitution the parameter space by

$$c' \quad = \quad l^{D-2-2d_s} c; \qquad r_0' \quad = \quad l^{D-2d_s} r_0 \tag{4.31}$$

These two equations allow two fixed points

(i) $D-2d_s=0$, r_0 arbitrary, $c=0$

(ii) $D-2-2d_s=0$, c arbitrary, $r_0=0$

The case (ii) is more interesting, the fixed point is defined by $[K]^*=\{c,r_0=0\}$ where c is arbitrary, on the other hand $r_0'=l^2r_0$. This shows that r_0 is a relevant field with y=2, where $\nu=\frac{1}{2}$.

4.5.2 The Gaussian Fixed Point and Wilson-Fisher Fixed Point

Now we investigate the Landau-Ginzburg Hamiltonian. The term with $S(x)^4$ is complicated in Fourier space, like

$$\int d^Dx S(x)^4 = L^{-D}\sum_{k_1,k_2,k_3} S(k_1)S(k_2)S(k_3)S(-k_1-k_2-k_3) \qquad (4.32)$$

It is impossible to find a space where all these terms are simultaneously simple. So we must use are approximation method. The standard method is the perturbation expansion in the power of u_0. Therefore the Hamiltonian H is subdivided into a Gaussian term, H_0, and an interaction term, V.

$$H = H_0+V, \qquad V = \frac{u_0}{4!}\int d^Dx S(x)^4 \qquad (4.33)$$

When we apply equation (4.29) in order to evaluate the integral over the $dS(k)$, we write

$$S(x) = S_<(x)+\bar{S}(x)$$

where $S_1(x)$ has Fourier component in the range $0 \le k \le \frac{\Lambda}{l}$ and $\bar{S}(x)$ in the range $\frac{\Lambda}{l} \le k \le \Lambda$. From equation (4.29) the integration measure is $D\bar{S}(x)$. Disregarding dilation for a moment, We find

$$e^{-H_1'} = e^{-H_0(S_1)} \frac{\int D\bar{S}(x) \exp(-H_0(\bar{S}) - V(S_1, \bar{S}))}{\int D\bar{S} \exp(-H_0(\bar{S}))} \tag{4.34}$$

If we neglect the second order and higher order in u_0, the new Hamiltonian H_1' is

$$H_1' = H_0(S_1) + \frac{\int D\bar{S} \exp(-H_0(\bar{S}))V(S_1, \bar{S})}{\int D\bar{S} \exp(-H_0(\bar{S}))} + O(u_0^2) \tag{4.35}$$

Our work in equation (4.35) is to evaluate the average of a polynomial over a Gaussian probability distribution. In fact, we have to find $<\bar{S}(x)\bar{S}(y)>_0$ where the subscript 0 means average calculated with Gaussian Hamiltonian H_0. The term $<\bar{S}(x)\bar{S}(y)>_0$ is already known. It is the correlation function of the Gaussian model.

$$<\bar{S}(x)\bar{S}(y)>_0 = \int_{k \le \Lambda} \frac{d^D k}{(2\pi)^D} \frac{e^{-ik(x-y)}}{k^2 + r_0} \tag{4.36}$$

In our case, where the integral k runs between $\frac{\Lambda}{l} \le k \le \Lambda$, the result is

$$<\bar{S}(x)\bar{S}(y)>_0 = \bar{G}_0(x-y) = \int_{\frac{\Lambda}{l} \le k \le \Lambda} \frac{d^D k}{(2\pi)^D} \frac{e^{-ik(x-y)}}{k^2 + r_0} \tag{4.37}$$

Reverting to the calculation of the $<V(S_1, \bar{S})>_0$, we have

$$<(S_1(x) + \bar{S}(x))^4>_0 = S_1^4(x) + 6S_1^2(x) <\bar{S}(x)\bar{S}(x)>_0 + <\bar{S}^4(x)>_0$$

The final term is a constant and can be dropped. The second term equals to $6S_1^2(x)\overline{G}_0(0)$. Therefore the Hamiltonian H_1' become

$$H_1' = \int d^D x \left[\frac{1}{2}(\nabla S_1)^2 + \frac{1}{2} r_0 S_1^2 + \frac{u_0}{4!} S_1^4 + \frac{u_0}{4} S_1^2(x)\overline{G}_0(0) \right]$$

In order to obtain H', it remains only to implement the dilations.

$$H' = \int d^D x' l^{D-2d_1-2} \left[\frac{1}{2}(\nabla' S')^2 + \frac{1}{2} l^2 (r_0 + \frac{u_0}{2}\overline{G}_0(0)) S'^2(x) \right.$$

$$\left. + l^{2-2d_1} \frac{u_0}{4!} S'(x)^4 \right] \qquad (4.38)$$

If we keep the kinetic term equal to 1/2, since the Gaussian model $d_S = (D/2-1)$ and $\eta = 0$, the transformation of r_0 and u_0 become

$$r_0' = l^2 \left(r_0 + \frac{u_0}{2}\overline{G}_0(0) \right)$$

$$u_0' = l^{4-D} u_0 = l^{\varepsilon} u_0 \qquad (4.39)$$

Now we evaluate $\overline{G}_0(0)$. Since when taking $r_0 \to 0$, we can take $r_0 << \Lambda / l$. Then we obtain

$$\overline{G}_0(0) = C \int_{\Lambda/l}^{\Lambda} \frac{k^{D-1}}{k^2} dk + O(r_0)$$

$$= C \frac{\Lambda^{D-2}}{D-2} (1 - l^{2-D}) + O(r_0)$$

$$= \quad 2B(1-l^{2-D})+O(r_0)$$

Equation (4.39) have a fixed point at $u_0, r_0 = 0$. When linearizing in the vicinity of fixed point, we get the matrix of transformation

$$M^l = \begin{pmatrix} l^2 & B(l^2-l^\varepsilon) \\ 0 & l^\varepsilon \end{pmatrix} \tag{4.40}$$

The eigenvalues of this matrix are $\lambda_1 = l^2$ and $\lambda_2 = l^\varepsilon$. Thus for $\varepsilon < 0$ or D > 4, we have two eigenvalues which one equal to 2 and $\varepsilon < 0$, which show the stable fixed point called Gaussian fixed point, with one relevant field and one irrelevant field.

In case $\varepsilon > 0$ or D < 4 the eigenvalues are both positive. This corresponds to the unstable fixed point called non-Gaussian fixed point or Wilson-Fisher fixed point.