CHAPTER I

WAVELET THEORY

Contents

1.1 What Are Wavelets ?

Wavelets (*Ondelettes* in French) are functions that satisfy certain requirements. The name "wavelet"(oscillatory + little) comes from the requirements that they must be oscillatory above and below the X-axis and the amplitudes quickly decay to zero in the both positive and negative directions. See Figure 1.1 for an example of a wavelet (this is a classical wavelet, often called Morlet mother wavelet).

The required oscillatory condition leads to sinusoidals as building blocks (see Figure 1.2). The fast decay condition is a windowing operation (see Figure 1.3). These two conditions must be simultaneously satisfied for a function to be wavelet.

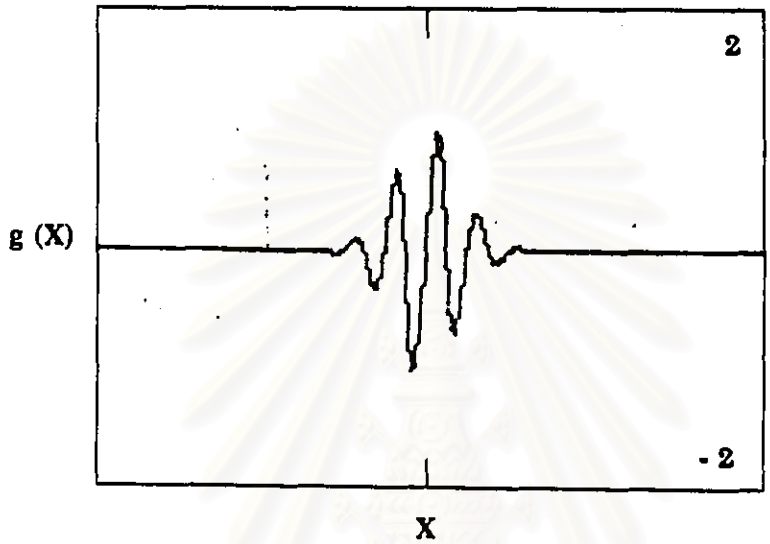

Figure 1.1 Morlet mother wavelet (Sattayatham, 1995)

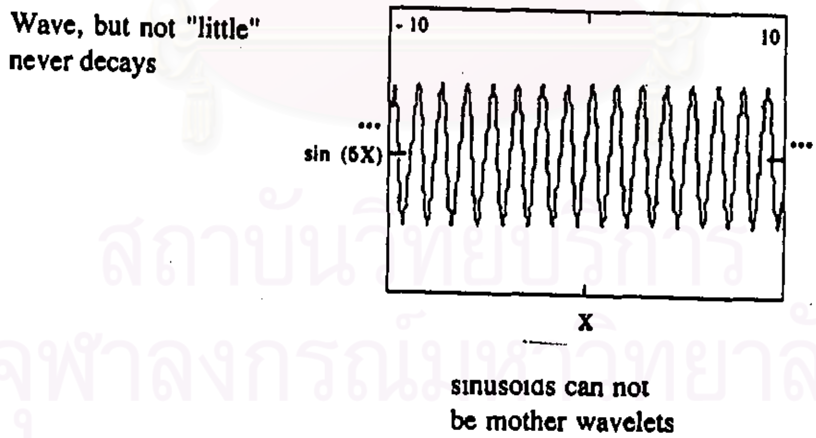

Figure 1.2 Oscillatory or wave requirement (Sattayatham, 1995)

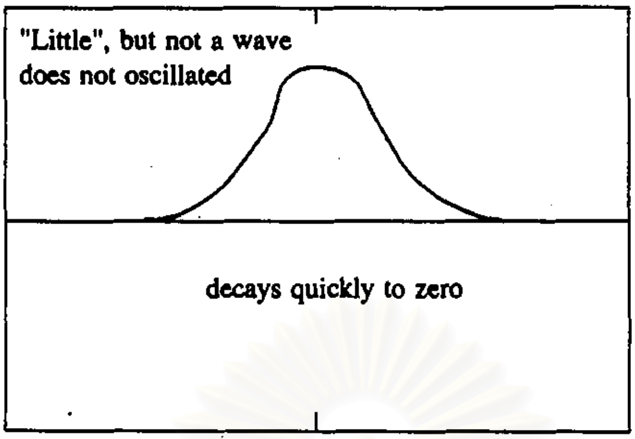

Figure 1.3 Decay requirement (Sattayatham, 1995)

Wavelets are used as basis functions in representing other functions (The original wavelet is denoted *mother wavelet*, $\psi(x)$). Each element of a wavelet set is a scaled (dilated or compressed) and translated (shifted) mother wavelet which is indexed by parameter a and b respectively.

$$\psi_{a,b}(x) = \psi((x-b)/a)$$

when $a \in \Re^{+}, b \in \Re$

Figure 1.4 and 1.5 display several elements of such a wavelet set corresponding to Morlet's wavelet in Figure 1.1. Note that the scaled wavelets include an energy normalization term, $1/\sqrt{a}$, that keep the energy of the scaled wavelets the same as the energy in the original mother wavelet. Morlet's wavelet is continuous but not orthogonal. so it is not powerful for application.

Wavelet theory can be employed in many field and applications, such as image analysis, communication system, radar, air acoustics, theoretical mathematics, etc.

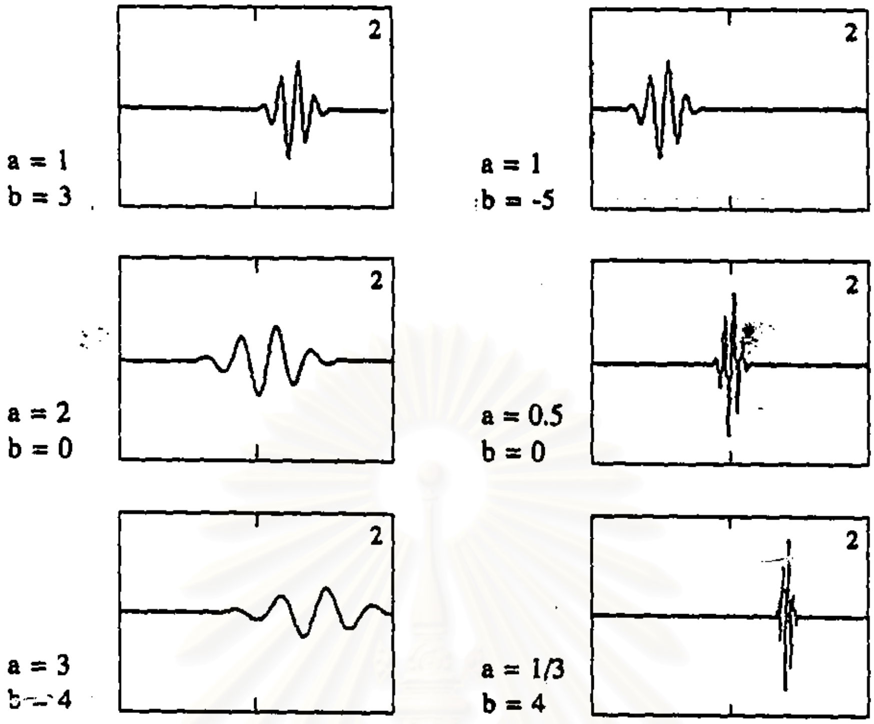

Figure 1.4 Scaled and Translated Mother wavelet (Sattayatham, 1995)

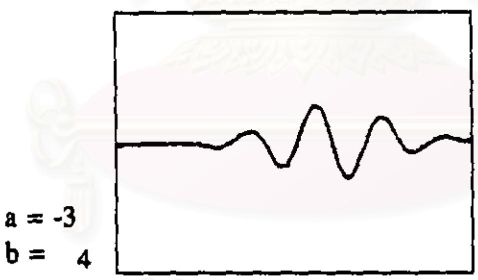

Figure 1.5 Negative Scaled Mother wavelet (Sattayatham, 1995)

1.2 History of wavelets

Wavelets were introduced in the early eighties by J. Morlet (Morlet 1983), a French geophysicist at Elf-Aquitance, as a tool for signal analysis, especially for seismic data. The numerical success of Morlet incited A. Grossmann to study the wavelet transform, whose tittle showed the name wavelets of constant shape that becomes a part of mathematic foundation (see Grossmann & Morlet, 1984), In 1985, a harmonic analyst Y. Meyer became aware of this theory and recognized many classical results in it. Meyer pointed out to Grossmann and Morlet that there were connection between their signal analysis method and existing powerful techniques in mathematical studies of singular integral operators. Then, Ingrid Daubechies became involved. All this resulted in the first construction of a special type of Frame(the concept of frame generalized the concept of basis in Hilbert space) (see Daubechies, Grossmann & Meyer, 1986) It was also the start of cross-fertilization between the signal analysis application and the purely mathematical aspect of techniques based on dilation and translation.

In 1988 Daubechies provided a major breakthrough by constructing families of orthonormal wavelets with compact support (see Daubechies, 1988). She was inspired by the work of Mallat and Meyer about multiresolution analysis, and by Mallat's algorithms for image processing (see Mallat 1988,1989).

All these activities created quite a stir among mathematicians. Apart from the application to signal analysis, the orthonormal wavelets could be useful in Physics also. The first application, to quantum field theory, can be found in Battle & Federbush, 1987. From the numerical analysis point of view interest arose in fast techniques (by analogy to fast Fourier transform) that certain integral operators can be transformed into other operators with dominant diagonals (see Beylkin, Coifman & Rokhlin, 1991).

1.3 From Fourier Transform to wavelet Transform (Kaiser 1994, Daubechies 1988)

In this section I explain about time-frequency localization and the development from Fourier analysis to wavelet analysis. Let $f(t)$ be a function depending on time. If we are interested in its "frequency content or spectrum", our first work is to compute its Fourier transform.

$$\hat{f}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\exp(-i\zeta t)dt \tag{1.1}$$

Since the integral in Fourier transform equation extends over all time, from $-\infty$ to $+\infty$, $\hat{f}(\zeta)$ arises from an average over the whole domain of $f(t)$. If some points in the lifetime of $f(t)$, there is a local oscillation representing a particular feature, this will contribute to the calculated Fourier transform $\hat{f}(\zeta)$, but its location on the time axis will be lost. There is no way to know whether the value of $\hat{f}(\zeta)$ at a particular ζ derives from frequencies representing throughout the life of $f(t)$ or during just one or a few selected periods.

This problem can be solved by the windowed Fourier transform (short time Fourier transform). Here the function f is the first "windowed" by multiplying it by a fixed $g(t)$ usually with compact support, this effectively restricts f to an interval (See Figure 1.6).

Then the Fourier coefficients of this product are computed. This process is repeated with shifted versions of g, i.e. , $g(t-nt_0)$, $n \in Z$, leading to a family of windowed Fourier coefficients,

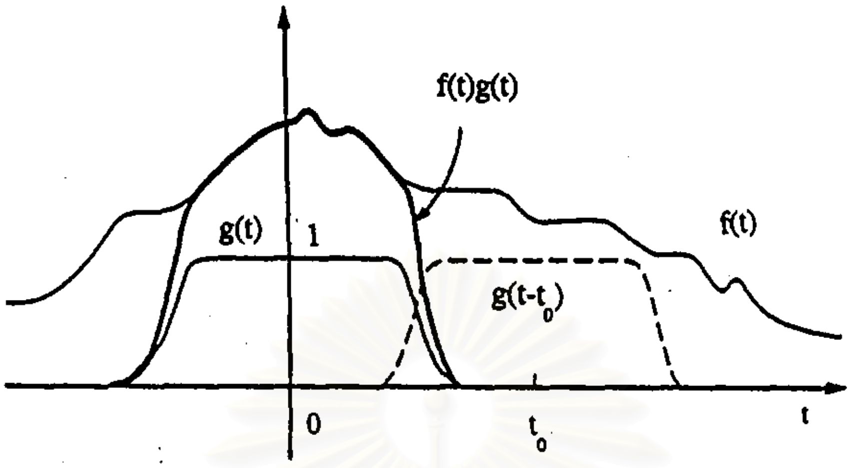

Figure 1.6 The inner product of window function with given function

$$S_{m,n} = \int f(s)g(s-nt_0)\exp(-im\zeta_0 s)ds \qquad (1.2)$$

with $m,n \in Z$. Similar coefficients also occur in a transform first proposed by Gabor for data transmission (Gabor, 1946). These coefficients can also be viewed as the inner products of the $f(t)$ with the discrete coherent states,which are well known in quamtum physics. (see Klauder-Skagerstam 1985, Ballentine 1990).

$$g_{m,n}(t) = \exp(-im\zeta_0 s)g(t-nt_0) \qquad (1.3)$$

We assume g is real. Each $g_{m,n}$ consists of an envelope function, shifted by nt_0 and then "filled in" with oscillations (see Figure 1.7); the index n gives us the time localization of $g_{m,n}$, the index m gives us its frequency.

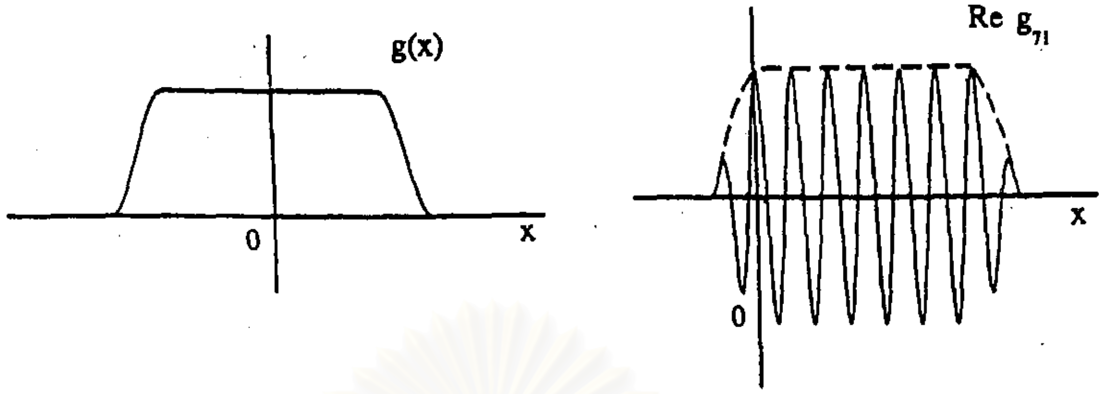

Figure 1.7 The window function

Another powerful method is the wavelet transform. The wavelet transform is similar to the windowed Fourier transform in that it also computes inner products of f with a sequence of function $\Psi_{m,n}$ with m indicating frequency localization, and n indicating time localization,

$$W_{m,n}(f) = \int f(s)\overline{\Psi_{m,n}(t)}dt \qquad (1.4)$$

but the $\Psi_{m,n}$ are generated in different way. The bar symbol means Hermitian conjugation.

The basic wavelet Ψ is typically well concentrated in time and in frequency, and has integral zero.

$$\int \Psi(t)dt = 0 \qquad (1.5)$$

This means that the basic wavelet Ψ has at least some oscillations. The $\Psi_{m,n}$ are then generated by dilation and translation (from the first section, let $a = a_0^m$; $b = nb_0a_0^m$).

$$\Psi_{m,n}(t) = a_0^{-m/2}\Psi(a_0^{-m}t - nb_0) \qquad (1.6)$$

where $a_0 > 1$ and $b_0 > 0$ are fixed parameters (similar to the ζ_0, t_0 in (1.2)) and m,n range over all of $\mathbb{Z}$). Changing m in (1.6) can compress or expand the oscillations of Ψ into a smaller ($m > 0$) or larger ($m < 0$) width i.e., to wavelets with higher or lower frequency ranges. For fixed m, the $\Psi_{m,n}$ are then $\Psi_{m,0}$ translated by $na_0^m b_0$, i.e., the wavelets are translated by the amount proportional to their width.

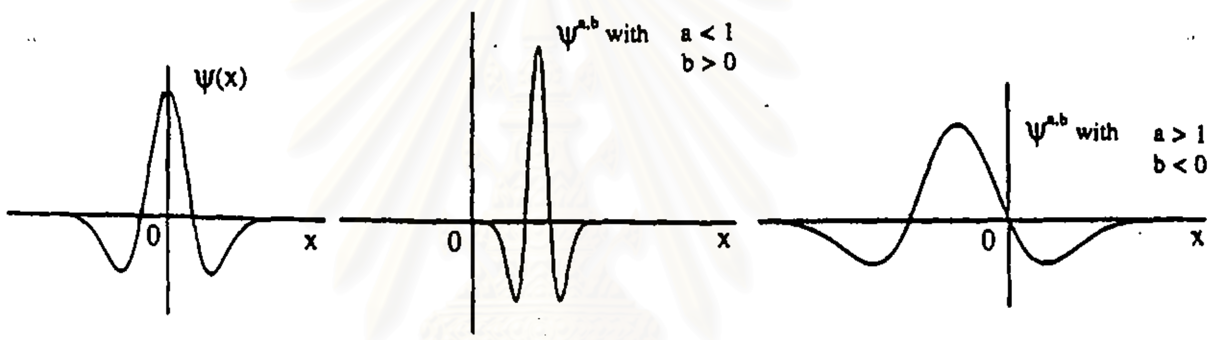

Figure 1.8 scaled and translated wavelets

A few typical wavelets are illustrated in Figure 1.8. It is clear that high frequency wavelets are narrow, and vice versa. This is the main difference between the wavelet transform and the short time Fourier transform:-- $g_{m,n}$ of windowed Fourier transform all have the same width. This is the defect of windowed Fourier transform because when we choose a wide window, we will miss high frequency (spike) or if we choose a narrow one, we will miss the periodic component of our signal. Therefore it can be expected that the wavelet transform is well adapted to functions, signals or operators with highly concentrated, high frequency components and superposed on longer lived low frequency components.

1.4 The Continuous Wavelet Transform (Daubechies 1988)

Here the dilation and translation parameter a, b vary continuously over $\Re$. That is, we define (in one dimension, for higher dimensions see later)

$$\Psi_{a,b} = \frac{1}{\sqrt{a}}\Psi\left(\frac{x-b}{a}\right) \tag{1.7}$$

with $a \in \Re^{+}, b \in \Re$, then

$$(Wf)(a,b) = \langle f, \Psi_{a,b} \rangle = \int f(x)a^{-1/2}\overline{\Psi\left(\frac{x-b}{a}\right)}$$

$$= \int \hat{f}(\zeta)a^{1/2}\overline{\hat{\Psi}(a\zeta)}e^{ib\zeta} \tag{1.8}$$

and

$$\int_0^{\infty}\int_{-\infty}^{\infty} \langle f, \Psi_{a,b} \rangle \langle \Psi_{a,b}, f \rangle \, db \frac{da}{a} = 2\pi \int_0^{\infty}\int_{-\infty}^{\infty} \hat{f}(\zeta)\overline{\hat{g}(\zeta)}|\hat{\psi}(a\zeta)|^2 d\zeta \frac{da}{a}$$

$$= 2\pi C_{\psi} \langle f, g \rangle \tag{1.9}$$

provided that

$$\int_0^{\infty} \zeta^{-1}|\hat{\Psi}(\zeta)|^2 d\zeta = \int_{-\infty}^{0} |\zeta|^{-1}|\hat{\Psi}(\zeta)|^2 d\zeta = C_{\psi} < \infty \tag{1.10}$$

Since we require that C_{ψ} is finite, the integrand defining C_{ψ} should be integrable at $\zeta = 0$. This implies that $\hat{\psi}(0) = 0$, which says that the mean value of the wavelet ψ should be zero: $\int \Psi(x)dx = 0$ as in (1.5). So ψ must change its signs on $\Re$; $\psi(x)$ will also decay to 0 as x tends to $\pm\infty$. In Figure 1.8 we take the wavelet

$$\psi(x) = (1-x^2)\exp(-\frac{1}{2}x^2)$$

the Mexican hat. The graph of ψ looks like a traverse section of a Mexican hat. Up to a constant, ψ is the second derivative of the Gaussian $\exp(-\frac{1}{2}x^2)$. We know that

$$\hat{\psi}(\zeta) = \sqrt{2\pi}\,\zeta^2\exp(-\frac{1}{2}\zeta^2)$$

and that $C_\psi = 2\pi$

Formula (1.9) can also be written as

$$f(x) = \frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty}\int_{0}^{\infty} < f, \Psi_{a,b} > \Psi_{a,b}(x)\,da\frac{db}{a^2} \qquad (1.11)$$

with weak convergence in L^2-sense. Note that (1.11) is called *the resolution of the identity* and can be read in two different ways: it can tell us how to reconstruct f from these wavelet coefficients once we know the the $< f, \Psi_{a,b} >$ and it also gives a recipe for writing any arbitrary f as a superposition of $\Psi_{a,b}$.

The *continuous wavelet transform* is useful when one wants to recognize or extract features. Scaling or translating of f leads to a shift of the $(wf)(a,b)$ in a and b, so that the whole analysis can be made to be scale and translation invariant, a desirable property in some applications. Of course, it can be cumbersome to deal with very redundant $(wf)(a,b)$.

1.5 The Discrete but Redundant Wavelet Transform: Frame

The wavelet family (1.6) and the wavelet transform (1.4) can be viewed as discretized version of the continuous wavelet transform, with *a*, *b* restricted to $a = a_0^m$; $b = nb_0a_0^m$

Generally the discrete case, there does not exist, in general, a "*resolution of the identity*" formula analogous to (1.11) as in the continuous case, These discrete wavelet transforms often provide a very redundant description of the original function. These redundancy can be exploited (for instance, it is possible to compute the wavelet transform only approximately, while still obtaining reconstruction of f with good precision) .

The choice of the wavelet Ψ used in the continuous wavelet transform or in *frame* (generalized basis see more in Daubechies (1992)) of discretely labeled families of wavelets is essentially only restricted by the requirement that C_Ψ, as defined by (1.10), is finite. For practical reasons, one usually chooses Ψ so that it is well concentrated in both time and frequency domain. For any such Ψ, one can then find threshold values such that if a_0, b_0 are chosen below these thresholds, the redundancy is eliminated (Daubechies 1988,1990). For example, if we choose $a_0 = 2,\ \ b_0 = 1$ then

$$\Psi_{m,n} = 2^{-m/2}\psi(2^{-m}x-n) \qquad (1.12)$$

constitute an orthonormal basis for $L^2(\Re)$. So any function f can be expanded in wavelet basis as

$$f(x) = \sum_m\sum_n c_{m,n}2^{-m/2}\psi(2^{-m}x-n), \quad m,n \in Z \qquad (1.13)$$

1.6 Multiresolution Analysis (Chui 1992, Daubechies 1988, Young 1992)

A multiresolution analysis consists of a sequence of successive approximation space V_j. More precisely, the closed subspaces V_j satisfy

$$...V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2}... \tag{1.14}$$

with

$$\overline{\bigcup_{j\in Z} V_j} = L^2(\Re) \tag{1.15}$$

and

$$\bigcap_{j\in Z} V_j = \{0\} \tag{1.16}$$

if we denote P_j by the orthogonal projection operator onto V_j, then (1.15) ensure that $\lim_{j\to-\infty} P_j f = f$ for all $f \in L^2(\Re)$. There exist many ladders of spaces satisfying (1.14)-(1.16) which have nothing to do with "multiresolution". The multiresolution aspect is a consequence of the addition requirement.

$$f \in V_j \Leftrightarrow f(2^j\cdot) \in V_0 \tag{1.17}$$

That is, all the spaces are scaled versions of the central space V_0

We require from a multiresolution analysis (invariance of V_0 under integer translation).

$$f(x) \in V_0 \Rightarrow f(x-n) \in V_0, \qquad \text{for all n} \in Z \tag{1.18}$$

From (1.17) it implies that if $f(x) \in V_j$, then $f(x-2^j n) \in V_j$ for all n $\in$ Z. Finally, we require also that there exits $\phi \in V_0$ such that

$$\{\phi_{0,n}; n \in Z\} \qquad \text{is orthonormal basis in } V_0 \qquad (1.19)$$

where, for all $j, n \in Z, \phi_{j,n}(x) = 2^{-j/2}\phi(2^{-j}x-n)$. Together, (1.17) and (1.19) imply that $\{\phi_{j,n}; n \in Z\}$ is an orthonormal basis for V_j for all $j \in Z$. We shall often call ϕ the "scaling function" of the multiresolution analysis.

The basic principle of multiresolution analysis is that if the collection of closed subspaces satisfies equations (1.14)-(1.19), then there exists an orthonormal wavelet basis $\{\Psi_{j,k}; j,k \in Z\}$ of $L^2(\Re)$, $\Psi_{j,k}(x) = 2^{-j/2}\Psi(2^{-j}x-k)$, such that, for all f in all $L^2(\Re)$

$$P_{j-1}f \quad = \quad P_j f + \sum_{k \in Z} \langle f, \Psi_{j,k} \rangle \Psi_{j,k} \qquad (1.20)$$

(P_j is the orthogonal projection onto V_j.). The wavelet Ψ can be constructed explicitly as below.

For every $j \in Z$, define W_j to be the orthogonal complement of V_j in V_{j-1}. We have

$$V_{j-1} \quad = \quad V_j \oplus W_j \qquad (1.21)$$

and

$W_j \perp W$ if $j \neq j'$ ($\perp$ symbol means orthogonal complement) (1.22)

(if $j > j'$, e.g., then $W_j \subset V_{j'} \perp W_{j'}$). It follows that , for $j < J$

$$V_j = V_J \oplus \bigoplus_{k=0}^{J-j-1} W_{J-k} \tag{1.23}$$

where all these subspaces are orthogonal. Equation (1.15) and (1.16) imply that

$$L^2(\Re) = \bigoplus_{j\in Z} W_j \tag{1.24}$$

a decomposition of $L^2(\Re)$ into mutually orthogonal subspaces. Furthermore, the W_j spaces inherit the scaling property (1.17) from the V_j ,

$$f(x) \in W_j \Leftrightarrow f(2^j x) \in W_0 \tag{1.25}$$

Formula (1.20) is equivalent to saying that, for fixed j, $\{\Psi_{j,k}; j,k \in Z\}$ constitutes an orthonormal basis for W_j. Because of (1.24) and (1.15), (1.16) this automatically imply that the whole collection $\{\Psi_{j,k}; j,k \in Z\}$ is an orthonormal basis for $L^2(\Re)$. On the other hand, (1.25) ensures that if $\{\Psi_{0,k}; k \in Z\}$ is an orthonormal basis for W_0 , then $\{\Psi_{j,k}; j,k \in Z\}$ will be an orthonormal basis for W_j , for any $j \in Z$. Next we have to find $\Psi \in W_0$ that the $\Psi(x-k)$ constitute an orthonormal basis for W_0.

To construct Ψ, we write out some interesting properties of ϕ and W_0.

1. Since $\phi \in V_0 \subset V_{-1}$ and the $\phi_{-1,n}$ are an orthonormal basis in V_{-1}, we have

$$\phi = \sum_n h_n \phi_{-1,n} \tag{1.26}$$

with

$$h_n = \langle \phi, \phi_{-1,n} \rangle \text{ and } \sum_{n\in Z} |h_n|^2 = 1 \tag{1.27}$$

We can rewrite (1.26) as

$$\phi(x) \quad = \quad \sqrt{2}\sum_n h_n \phi(2x-n) \tag{1.28}$$

Making Fourier Transform, we get

$$\phi(\xi) \quad = \quad \frac{1}{\sqrt{2}}\sum_n h_n e^{-in\xi/2}\hat{\phi}(\xi/2) \tag{1.29}$$

Formula (1.29) can be rewritten as

$$\hat{\phi}(\xi) \quad = \quad m_0(\xi/2)\hat{\phi}(\xi/2) \tag{1.30}$$

where

$$m_0(\xi) \quad = \quad \frac{1}{\sqrt{2}}\sum_n h_n e^{-in\xi} \tag{1.31}$$

Equation (1.31) shows, m_0 is a 2π -periodic function in $L^2([0,2\pi])$.

2. The orthogonality of the $\phi(x-k)$ leads to special properties of m_0. We have

$$\delta_{k,0} \quad = \quad \int dx\phi(x)\overline{\phi(x-k)}$$

$$= \quad \int d\xi\left|\hat{\phi}(\xi)\right|^2 e^{ik\xi} \tag{1.32}$$

$$= \quad \int_0^{2\pi} d\xi \sum_{l\in Z}\left|\hat{\phi}(\xi+2\pi l)\right|^2$$

implying that

$$\sum_l \left|\hat{\phi}(\xi + 2\pi l)\right|^2 = \frac{1}{2\pi} \tag{1.32}$$

Equation (1.30) leads to $(\varsigma = \xi / 2)$

$$\sum_l \left|m_0(\varsigma + \pi l)\right|^2 \left|\hat{\phi}(\varsigma + \pi l)\right|^2 = \frac{1}{2\pi}$$

splitting the sum into even and odd l, using the periodicity of m_0 and applying (1.32) gives

$$\left|m_0(\varsigma)\right|^2 + \left|m_0(\varsigma + \pi)\right|^2 = 1 \tag{1.33}$$

3. Now we characterize W_0: $f \in W_0$ is equivalent to $f \in V_{-1}$ and $f \perp V_0$. Since $f \in V_{-1}$, we have

$$f = \sum_n f_n \phi_{-1,n}$$

with $f_n = \langle f, \phi_{-1,n} \rangle$. From equation (1.29) it becomes

$$= \frac{1}{\sqrt{2}} \sum_n f_n e^{-in\xi/2} \hat{\phi}(\xi/2) = m_f(\xi/2)\hat{\phi}(\xi/2) \tag{1.34}$$

where

$$m_f = \frac{1}{\sqrt{2}} \sum_n f_n e^{-in\xi} \tag{1.35}$$

is a 2π -periodic function in $L^2([0,2\pi])$.

The constraint $f \perp V_0$ implies $f \perp \phi_{0,k}$ for all k, i.e.

$$\int d\xi\, \hat{f}(\xi)\overline{\hat{\phi}(\xi)}e^{ik\xi} = 0$$

or

$$\int_0^{2\pi} d\xi e^{ik\xi} \sum_l \hat{f}(\xi+2\pi l)\overline{\hat{\phi}(\xi+2\pi l)} = 0$$

therefore

$$\sum_l \hat{f}(\xi+2\pi l)\overline{\hat{\phi}(\xi+2\pi l)} = 0 \qquad (1.36)$$

Substituting (1.30) and (1.34), regrouping the sums for odd and even l and using (1.32) lead to

$$m_f(\varsigma)\overline{m_0(\varsigma)}+m_f(\varsigma+\pi)\overline{m_0(\varsigma+\pi)} = 0 \qquad (1.37)$$

Since $\overline{m_0(\varsigma)}$ and $\overline{m_0(\varsigma+\pi)}$ cannot vanish together on a set of nonzero measure (because of (1.33)), this implies the existence of a 2π-periodic function $\lambda(\varsigma)$ such that

$$m_f(\varsigma) = \lambda(\varsigma)\overline{m_0(\varsigma+\pi)} \qquad (1.38)$$

and

$$\lambda(\varsigma)+\lambda(\varsigma+\pi) = 0 \qquad (1.39)$$

The last equation can be written in the form

$$\lambda(\varsigma) \quad = \quad e^{i\varsigma}\nu(2\varsigma) \qquad (1.40)$$

where ν is 2π -periodic function. Substituting (1.40) and (1.38) into (1.34) gives

$$\hat{f}(\xi) \quad = \quad e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\,\nu(\xi)\hat{\phi}(\xi/2) \qquad (1.41)$$

4. The general form (1.41) for the Fourier Transform of $f \in W_0$ suggests that we take

$$\hat{\Psi}(\xi) \quad = \quad e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\,\hat{\phi}(\xi/2) \qquad (1.42)$$

as a candidate for our wavelet. Equation (1.41) can be written as

$$\hat{f}(\xi) \quad = \quad \left(\sum_k \nu_k e^{-ik\xi}\right)\hat{\Psi}(\xi)$$

or

$$f(x) \quad = \quad \sum_k \nu_k \psi(x-k)$$

so that the $\Psi(x-k)$ are a basis of W_0. We have to verify that the $\psi_{0,k}$ are an orthonormal basis for W_0. First, the properties of m_0 and $\hat{\phi}$ ensure that (1.42) defines an $L^2(\Re)$-function $\in V_{-1}$ and $\perp V_0$, so that $\psi \in W_0$. Orthogonality of the $\psi_{0,k}$ is easy to check :

$$\int dx\,\psi(x)\overline{\psi(x-k)} \quad = \quad \int d\xi c^{ik\xi}\left|\hat{\psi}(\xi)\right|^2$$

$$= \quad \int_0^{2\pi} d\xi e^{ik\xi}\sum_l\left|\hat{\psi}(\xi+2\pi l)\right|^2$$

Now

$$\sum_l \left|\hat{\psi}(\xi + 2\pi l)\right|^2 = \sum_l \left|m_0(\xi/2+\pi l+\pi)\right|^2 \left|\hat{\phi}(\xi/2+\pi l)\right|^2$$

$$= \left|m_0(\xi/2+\pi)\right|^2 \sum_n \left|\hat{\phi}(\xi/2+2\pi n)\right|^2$$

$$+\left|m_0(\xi/2+\pi)\right|^2 \sum_n \left|\hat{\phi}(\xi/2+\pi+2\pi n)\right|^2$$

$$= (2\pi)^{-1}\left[\left|m_0(\xi/2)\right|^2 + \left|m_0(\xi/2+\pi)\right|^2\right] \text{ by } (1.32)$$

$$= (2\pi)^{-1} \text{ by } (1.33)$$

Hence $\int dx \psi(x)\overline{\psi(x-k)} = \delta_{k,0}$. In order to check that the $\psi_{0,k}$ are a basis for all of W_0, we have to check that any $f \in W_0$ can be written as

$$f = \sum_n \gamma_n \psi_{0,n}$$

with $\sum_n |\gamma_n|^2 < \infty$, or $\hat{f}(\xi) = \gamma(\xi)\hat{\psi}(\xi)$ (1.43)

with γ 2π-periodic and $L^2([0,2\pi])$. But this is nothing but (1.41), where it is easy to check that ν is square integrable. We have therefore proved the assertation at the start of this section. There is an orthonormal wavelet basis $\{\Psi_{j,k}; j,k \in Z\}$ associated with any multiresolution analysis, the way to construct ψ is below

$$\psi(x) = \sum_n (-1)^n h_{1-n} \phi_{-1,n}$$

$$= \sqrt{2}\sum_n (-1)^n h_{1-n}\phi(2x-n) \qquad (1.44)$$

where ϕ is the scaling function of multiresolution analysis. We often call equations (1.28), (1.44) as *the two-scale difference equation* or *dilation equation* of scaling function and wavelet function, respectively. A set of the h_n coefficient is called as *filter coefficients.* The consequence of the multiresolution is that any function in $L^2(\Re)$ can be expanded in wavelet basis or combination of wavelet and scaling function

$$f(x) = \sum_m \sum_n c_{m,n} 2^{-m/2}\psi(2^{-m}x-n), \quad m,n \in Z \text{ or}$$

$$= \sum_n d_{m,n} 2^{-m/2}\phi(2^{-m}x-n) + \sum_n \sum_{m' \geq m} c_{m,n} 2^{-m'/2}\psi(2^{-m'}x-n), \quad m,m',n \in Z$$

1.7 Simple Solution of Dilation Equation and Examples

The dilation equation for scaling function, ϕ, of the previous section becomes very interesting when only a finite number of filter coefficients h_n are non-zero. This has important consequences for the construction of compactly supported orthogonal wavelet. From equation (1.28)

$$\phi(x) = \sqrt{2}\sum_n h_n \phi(2x-n), \quad x \in \Re$$

when normalized, say $\int_{-\infty}^{\infty}\phi(x)dx = 1$ then we get

$$1 = \int_{-\infty}^{\infty}\phi(x)dx = \sqrt{2}\sum_n h_n \int_{-\infty}^{\infty}\phi(2x-n)dx$$

$$= \quad \frac{1}{\sqrt{2}}\sum_n h_n \int_{-\infty}^{\infty} \phi(2x-n)\,d(2x-n)$$

$$= \quad \frac{1}{\sqrt{2}}\sum_n h_n$$

yielding

$$\sum_{n=-\infty}^{\infty} h_n \quad = \quad \sqrt{2} \tag{1.45}$$

if we take

$$c_n \quad = \quad \sqrt{2}h_n$$

so the dilation equation becomes

$$\phi(x) \quad = \quad \sum_n c_n \phi(2x-n) \tag{1.46}$$

and $$\sum_n c_n \quad = \quad 2 \tag{1.47}$$

The dilation equation for wavelet in terms of c_n becomes

$$\psi(x) \quad = \quad \sum_n (-1)^n c_{1-n} \phi(2x-n) \tag{1.48}$$

We can see the solution of the dilation equation for some set $\{c_n\}$. It appears that, when a solution of the dilation equation exists, it is unique. The following examples and pictures are taken from Strang (Strang, 1989).

Example 1

Take $c_0 = 2$ and all remaining c_n equal to zero. The Dirac delta function satisfies $\delta(x) = 2\delta(2x)$, and therefore is a solution. The Dirac function is not a regular function; the idea that we have a function with compact support (of length zero): a needle at the origin.

Example 2

Take $c_0 = c_1 = 1$, and all remaining c_n equal to zero. A solution of the dilation equation is a box function.

$$\phi(x) = \begin{cases} 1 & if \quad 0 \leq x < 1 \\ 0 & otherwise \end{cases}$$

with a support of unit length. The corresponding wavelet is

$$\psi(x) = \phi(2x) - \phi(2x-1)$$

this is the Haar wavelet and is given by

$$\psi(x) = \begin{cases} 1, & if \quad 0 \leq x < 1/2 \\ -1, & if \quad 1/2 \leq x < 1 \\ 0, & otherwise \end{cases}$$

The box function and the Haar wavelet are orthogonal with respect to their own translation.

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-n)dx = 0$$

$$\int_{-\infty}^{\infty} \psi(x)\psi(x-n)dx = 0 \qquad ,n \in Z\setminus\{0\}$$

The resulting $\psi_{m,n}$ are given by

$$\psi_{m,n} = 2^{-m/2}\psi(2^{-m}x-n), \quad m,n \in Z$$

They have the desired property that they constitute an orthogonal basis for $L^2(\Re)$. The Haar function is the original wavelet (but with poor approximation).

Example 3

Take $c_0 = 1/2, c_1 = 1, c_2 = 1/2$ and all remaining c_n equal to zero. A solution of the dilation equation is the hat function.

$$\phi(x) = \begin{cases} x, & if \quad 0 \le x \le 1 \\ 2-x, & if \quad 1 \le x \le 2 \\ 0, & otherwise \end{cases}$$

which has support of two unit length. The corresponding wavelet is given by

$$\psi(x) = \phi(2x) - \frac{1}{2}\phi(2x-1) - \frac{1}{2}\phi(2x+1)$$

The support interval is [-1, 2]. From a picture of the hat function, it is easily seen that $\phi(x)$ and $\phi(x \pm 1)$ are not orthogonal on $\Re$. Hence the translation of $\phi(x)$ cannot constitute an orthogonal set.

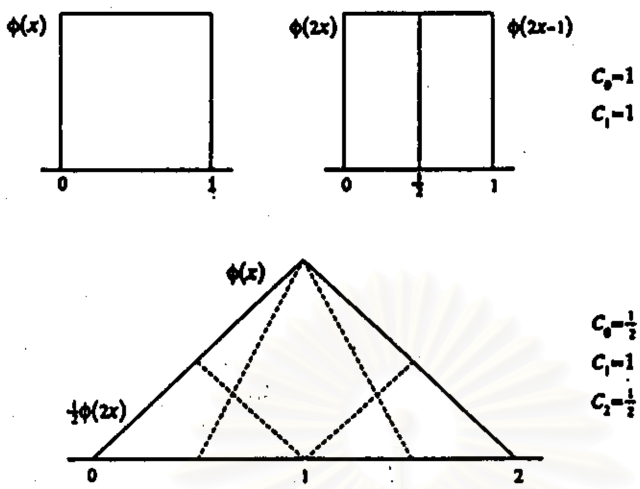

Figure 1.9 The box function and hat function

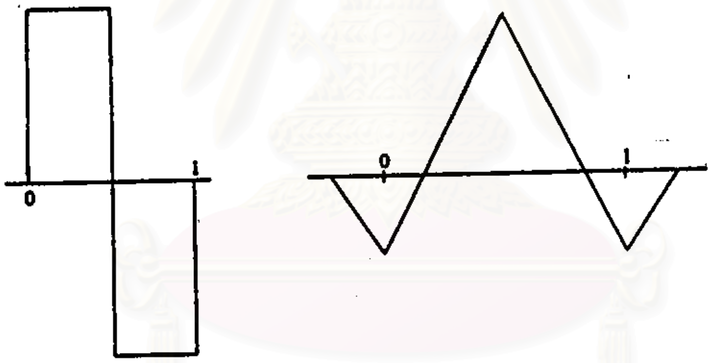

Figure 1.10 The wavelets for the box function and hat function

The interesting result from observing the above examples is that when n successive coefficients are given with the remaining equal to zero, the solution ϕ is compactly supported on an interval $[0, n-1]$ of length $n-1$.

1.8 Daubechies Wavelet (Compactly Supported Wavelet) (Newland 1994)

From the dilation equation in the previous section, we assume that N (even number) of filter coefficients are non-zero. So our set $\{c_n\}$ are $\{c_0, c_1, c_2, .., c_{N-1}\}$.

This wavelet has been introduced by Ingrid Daubechies [see Daubechies, 1988]. The prominent property is that they have compactly supported and powerful numerical procedure[Press.,W.H. et al.,1992 and Beylkin.,G et al., 1991]. Now the scaling function and the corresponding wavelet can be written in terms of N as below

$$\phi(x) = \sum_{n=0}^{N-1} c_n \phi(2x-n) \tag{1.49}$$

$$\psi(x) = \sum_{n=0}^{N-1} (-1)^n c_{N-1-n} \phi(2x-n) \tag{1.50}$$

The symbol N also represents the Daubechies basis of order N, denoted DN or DAUBN. The Daubechies wavelets are localized in the position space and approximately localized in the Fourier space.

1.8.1 Collected Filter Coefficient Condition

The filter coefficients that appear in scaling function and corresponding wavelet must satisfy the following conditions.

(1) $$\sum_{n=0}^{N-1} c_n = 2 \tag{1.51}$$

so that the scaling function is unique and retains a unit area during iteration. This become better when we generate.

$$\text{(2)} \qquad \sum_{n=0}^{N-1}(-1)^n n^k c_n \quad = \quad 0 \qquad (1.52)$$

for integer *k* = *0,1,2,...,p*-1, where *p=N/2*. This determines the approximate condition of the degree to which a polynomial $1, x, x^2, x^3, .., x^{(p-1)}$ can be reproduced exactly in the wavelet basis. This condition is equivalent to the moment condition which requires that the mother wavelet $\psi(x)$ has k vanishing moments;

$$\int x^k \psi(x)dx \quad = \quad 0, \qquad k = 0,1,2,..,p-1 \qquad (1.53)$$

A basis built from these coefficients will be able to represent exactly any polynomial of order *p-1*.

$$\text{(3)} \qquad \sum_{n=0}^{N-1} c_n c_{n+2k} \quad = \quad 0, \quad k \neq 0 \qquad (1.54)$$

for $k = 0,1,2,.., p-1$, in order to generate an orthogonal wavelet system, with the additional condition that

$$\sum_{n=0}^{N-1} c_n^2 \quad = \quad 2 \qquad (1.55)$$

which arises as the scaling function being orthogonal.

For N coefficients, the number of equations to be satisfied are as follows:

from	(1)	1
	(2)	N/2
	(3)	N/2
	Total	N+1

so that there is one more equation than the number of the coefficients.

The reason that there are $N+1$ equations for N coefficients is because the first of the accuracy condition (k = 0 in (1.53)) is redundant. It reproduces a condition derivable from the constant area and orthogonality conditions. This may be seen as follows. Putting k=0 in (1.54) gives

$$\sum_{n_{even}} c_n - \sum_{n_{odd}} c_n = 0 \qquad (1.56)$$

combining this with (1.51) gives

$$\sum_{n_{even}} c_n = \sum_{n_{odd}} c_n = 1 \qquad (1.57)$$

so that

$$\left(\sum_{n_{even}} c_n\right)^2 + \left(\sum_{n_{odd}} c_n\right)^2 = 2 \qquad (1.58)$$

After multiplying out ($k \neq 0$)

$$\sum_{n=0}^{N-1} c_n^2 + 2\sum_{n=0}^{N-1}\sum_{k=0}^{N/2-1} c_n c_{n+2k} = 2 \qquad (1.59)$$

which means that

$$\sum_{n=0}^{N-1} c_n c_{n+2k} \quad = \quad 0, \; m \neq 0 \qquad \text{the same as equation (1.54)}$$

Example 1

The Haar wavelet has only two filter coefficients, *N=2* and we have the following;

from	(1)	$c_0 + c_1$	=	2
from	(2)	$c_0 - c_1$	=	0
from	(3)	$c_0^2 - c_1^2$	=	2

whose solution is

$$c_0 = c_1 = 1$$

Example 2

For four filter coefficients, *N*=4, D4 or DAUB4, we have following;

from	(1)	$c_0 + c_1 + c_2 + c_3$	=	2
from	(2)	$c_0 - c_1 + c_2 - c_3$	=	0
		$-c_1 + 2c_2 - 3c_0$	=	0
from	(3)	$c_0 c_2 + c_1 c_3$	=	0
		$c_0^2 + c_1^2 + c_2^2 + c_3^2$	=	2

whose solution is

$$c_0 = (1+\sqrt{3})/4 \qquad c_1 = (3+\sqrt{3})/4$$

$$c_2 = (3-\sqrt{3})/4 \qquad c_3 = (1-\sqrt{3})/4$$

1.8.2 How to Generate Scaling and Wavelet Function

Consider the dilation equation for scaling function. Substituting the filter coefficient into equation, yields a scaling function that will be non zero over the range *[0,N-1]*. Hence the D4 scaling function is non zero over the range [0,3]. The boundary conditions of the dilation equation are that $\phi(x)$ should be zero outside this domain so that $\phi(0) = \phi(3) = 0$. The additional required boundary conditions can be found from the following eigenvalue equation derived from the scaling relation:

$$\sqrt{2}\begin{bmatrix} c_1 & c_0 \\ c_3 & c_2 \end{bmatrix}\begin{bmatrix} \phi(1) \\ \phi(2) \end{bmatrix} = \begin{bmatrix} \phi(1) \\ \phi(2) \end{bmatrix} \tag{1.60}$$

When this eigensystem is solved, we find that, after applying the normalization condition, the components of the eigenvector corresponding to eigenvalue 1 are

$$\phi(1) = \frac{1}{2}(1+\sqrt{3}), \qquad \phi(2) = \frac{1}{2}(1-\sqrt{3})$$

This provides all the required boundary conditions on the scaling relation, and $\phi(x)$ in the D4 case can be found recursively (see detail in Appendix B). Use of equation (1.50) yields the corresponding mother wavelet.

There is the other to solve the dilation equation of the scaling function. One may iterate

$$\phi_j(x) = \sum_n c_n \phi_{j-1}(2x-n), \qquad j = 1,2,..,n \tag{1.61}$$

with the box function as the staring function $\phi_0(x)$. Then $\phi_j(x) \to \phi(x)$ as $j \to \infty$.

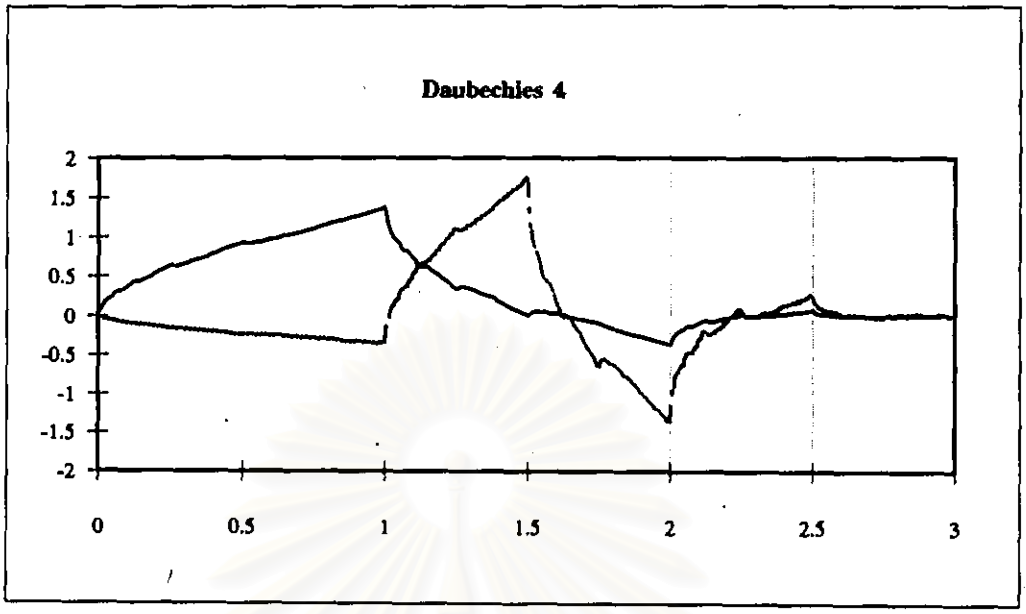

Figure 1.11 (a) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 4

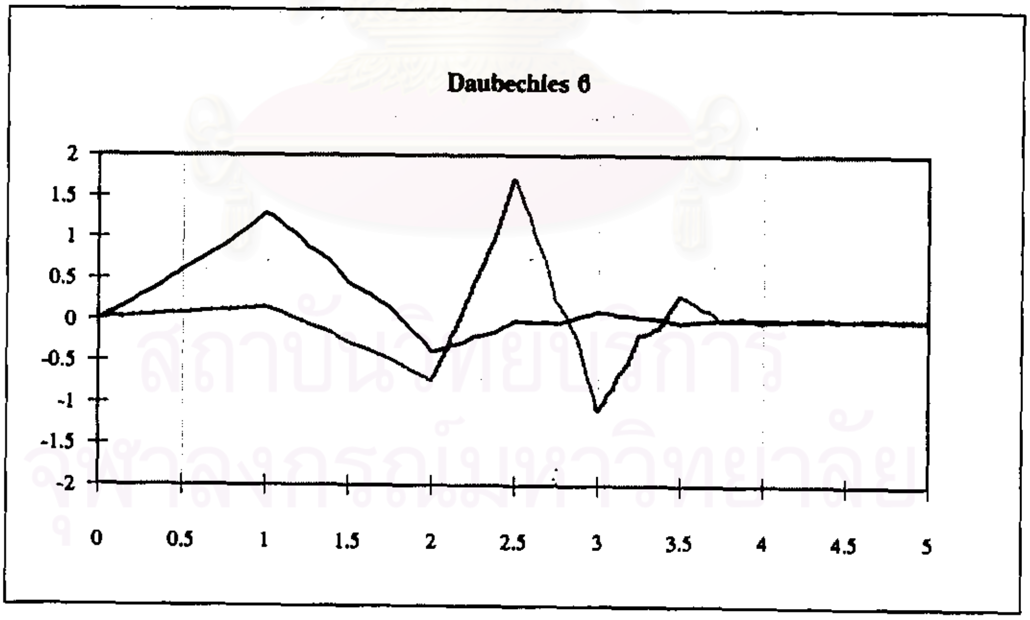

Figure 1.11 (b) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 6

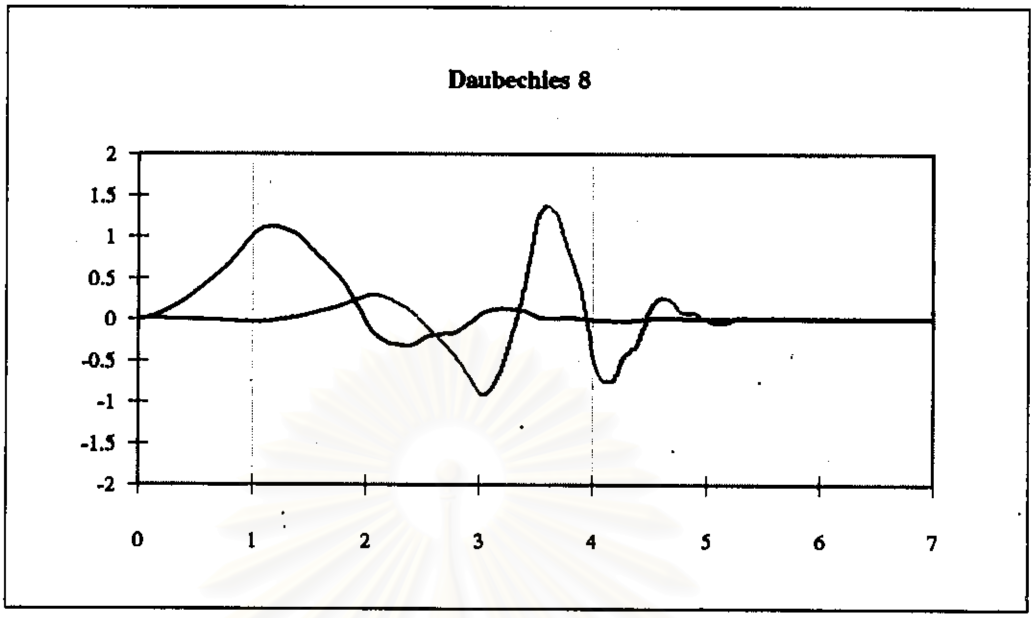

Figure 1.11 (c) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 8

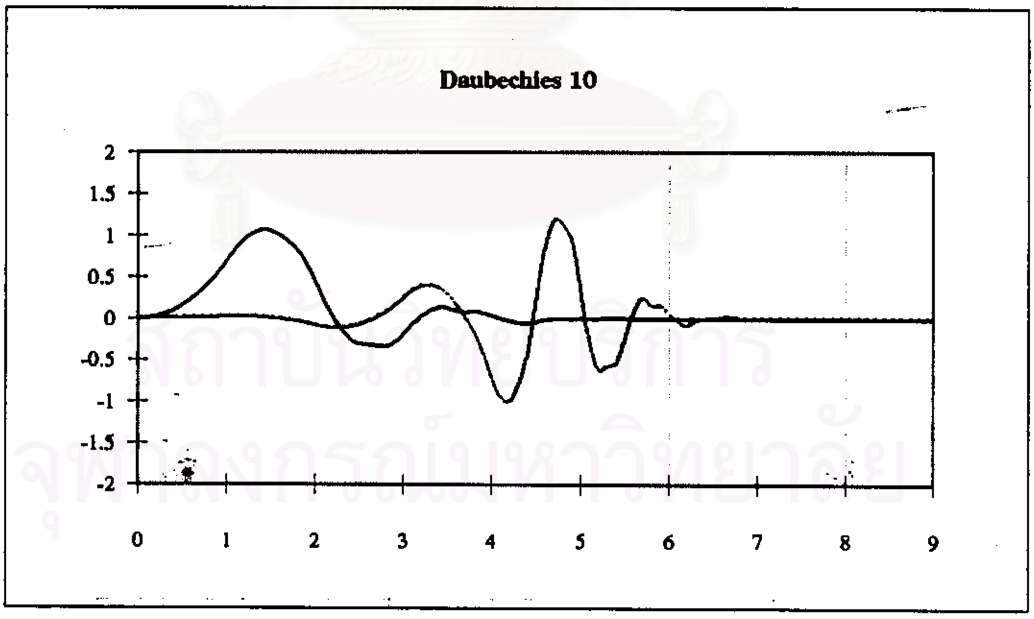

Figure 1.11 (d) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 10

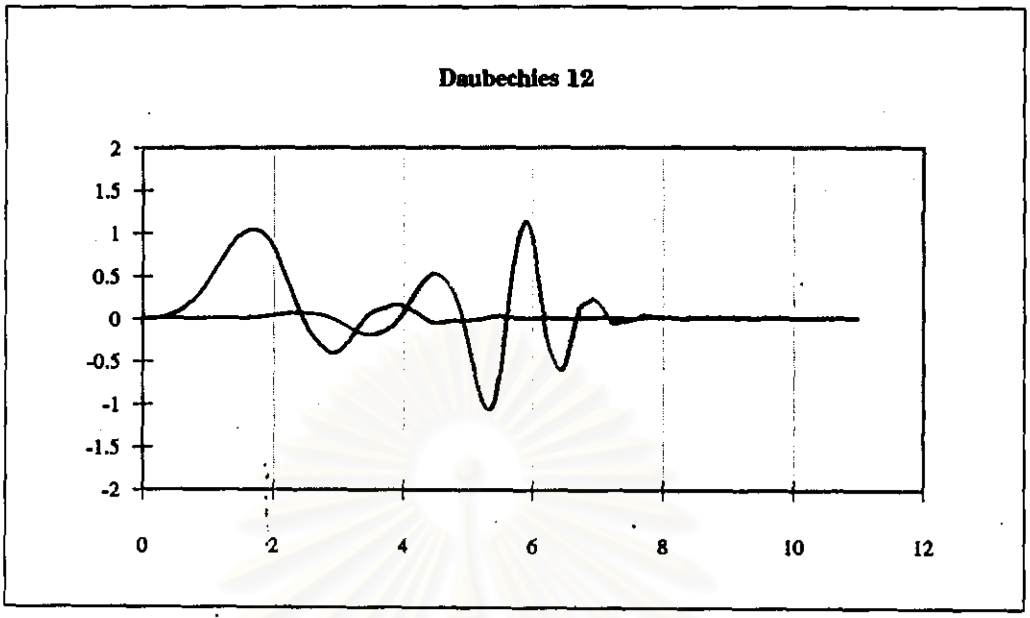

Figure 1.11 (e) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 12

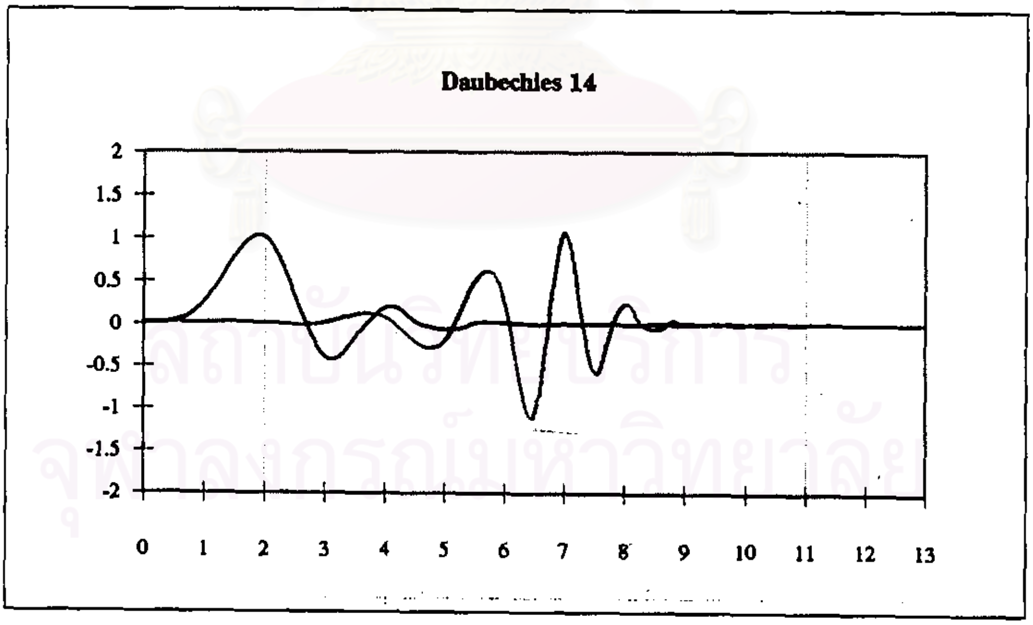

Figure 1.11 (f) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 14

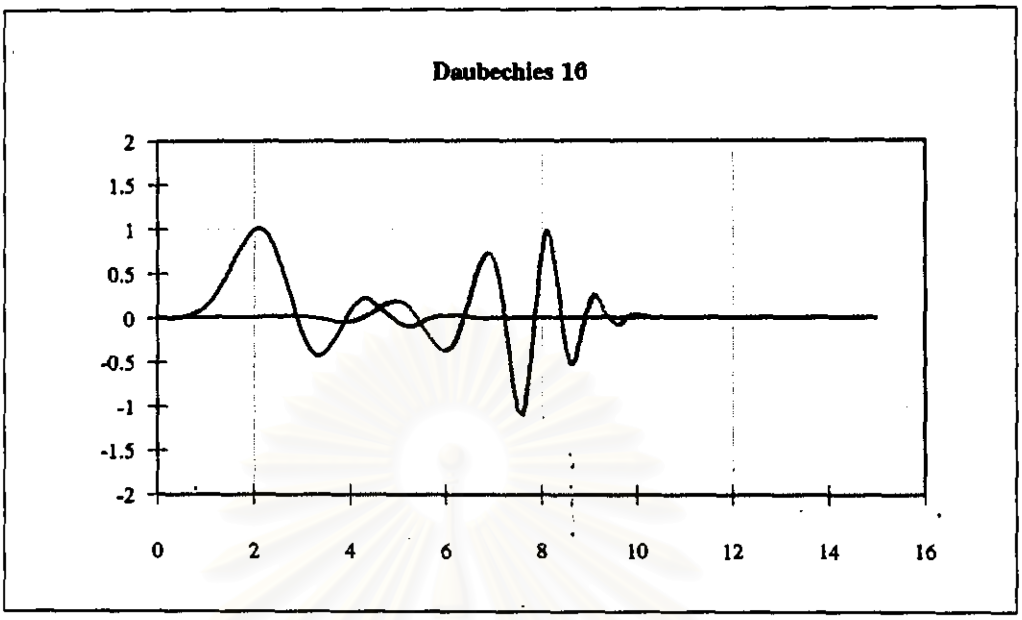

Figure 1.11 (g) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 16

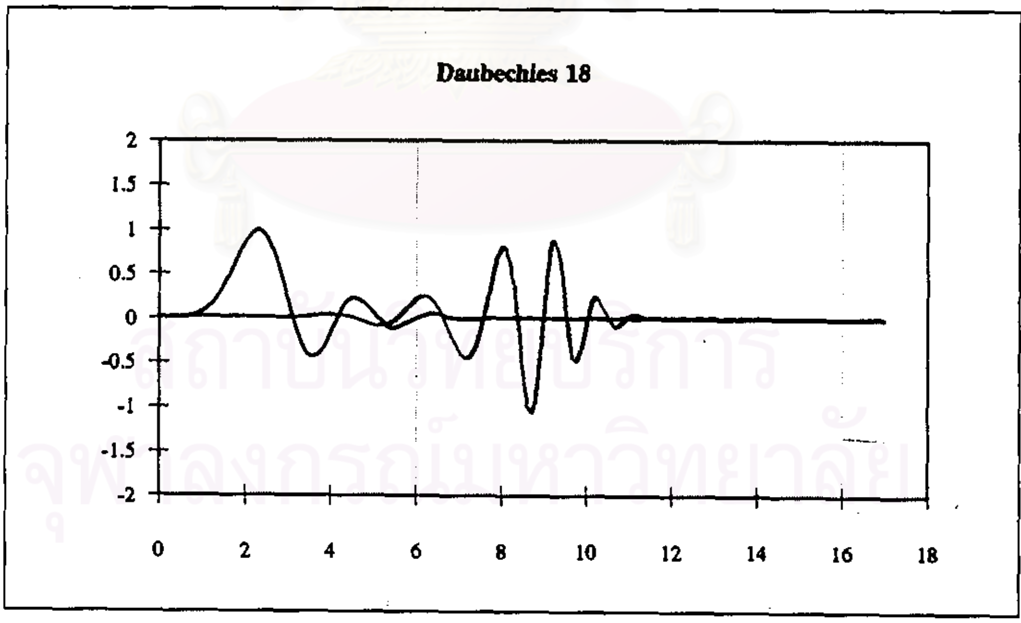

Figure 1.11 (h) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 18

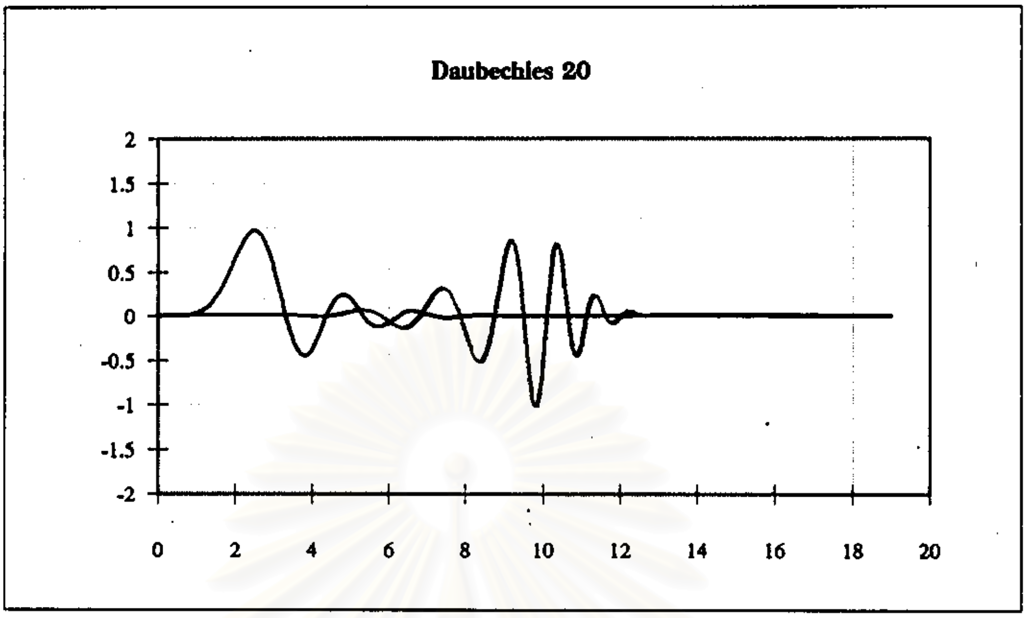

Figure 1.11 (i) The scaling function (solid curve) and wavelet (dashed curve) of Daubechies 20

1.9. Two Dimensional Wavelets

Up to now we have restricted ourselves to one dimension. It is very easy to extend the multiresolution analysis to more dimensions. Let us consider two dimensions. The case of n dimensions is completely similar. Assume that we dispose of a one dimensional multiresolution analysis, i.e., we have a successive spaces $\overline{V}_m$, and function ϕ,ψ satisfying equation (1.14-1.25).

Define

$$\overline{V}_m \quad = \quad V_m \oplus V_m$$

when $\overline{V}_m$ is a successive subspace of $L^2(\Re^2)$ satisfying (1.14). and the equivalent, for $\Re^2$, of (1.15-1.16). Moreover (1.17) holds. We define

$$\bar{\phi}(x_1,x_2) = \phi(x_1)\phi(x_2)$$

Then this two-dimensional function can be analogous to (1.19) by

$$\{\bar{\phi}_{m,n}; n \in Z^2\} \quad \text{is orthonormal basis in} \quad \bar{V}_m$$

where
$$\bar{\phi}_{m,n}(x_1,x_2) = 2^{-m/2}\bar{\phi}(2^{-m}x_1 - n_1, 2^{-m}x_2 - n_2)$$

$$= \phi_{mn_1}(x_1)\phi_{mn_2}(x_2)$$

because of the properties of W_m, we find that

$$\bar{V}_{m-1} = \bar{V}_m[(V_m \oplus W_m)(W_m \oplus V_m)(W_m \oplus W_m)]$$

This implies that an orthonormal basis for the orthogonal complement $\bar{W}_m$ of $\bar{V}_m$ in $\bar{V}_{m-1}$ is given by the function $\phi_{mn_1}\psi_{mn_2}, \psi_{mn_1}\phi_{mn_2}, \psi_{mn_1}\psi_{mn_2}$ with $n_1, n_2 \in Z$ or equivalently, by the two dimensional wavelets $\psi^k_{m,n}$

$$\psi^k_{mn}(x_1,x_2) = 2^{-m/2}\psi^k(2^{-m}x_1 - n_1, 2^{-m}x_2 - n_2),$$

where $k = 1,2,3$, $n \in Z^2$ and

$$\psi^1(x_1,x_2) = \phi(x_1)\psi(x_2)$$
$$\psi^2(x_1,x_2) = \psi(x_1)\phi(x_2)$$
$$\psi^3(x_1,x_2) = \psi(x_1)\psi(x_2)$$

It follow that the $\psi^k_{m,n}$ where k =1,2,3, $m \in Z$, $n \in Z^2$ constitute an orthonormal basis of wavelets for $L^2(\Re^2)$