ผลตอบสนองสภาวะชั่วครู่ของทรงกระบอกเชิงประกอบเทอร์โมไพอิโซอิเล็กทริกขนาดจำกัด

นายภาณุพงศ์ บุญเป็นนิมิต

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิศวกรรมศาสตรมหาบัณฑิต

สาขาวิชาวิศวกรรมโยธา ภาควิชาวิศวกรรมโยธา

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ปีการศึกษา 2555

TRANSIENT RESPONSE OF THERMOPIEZOELECTRIC FINITE COMPOSITE CYLINDER

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Engineering Program in Civil Engineering
Department of Civil Engineering
Faculty of Engineering
Chulalongkorn University
Academic Year 2012

ภาณุพงศ์ บุญเป็นนิมิต : ผลตอบสนองสภาวะชั่วครู่ของทรงกระบอกเชิงประกอบเทอร์โมไพอิโซอิเล็กทริกขนาดจำกัด (TRANSIENT RESPONSE OF THERMO-PIEZOELECTRIC FINITE COMPOSITE CYLINDER) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: รศ. ดร.จรูญ รุ่งอมรรัตน์, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม: ศ. ดร.ธีรพงศ์ เสนจันทร์ฒิไชย, 82 หน้า.

งานวิจัยนี้ เสนอผลเฉลยสมบูรณ์ของทรงกระบอกเชิงประกอบเทอร์โมไพอิโซอิเล็กทริกขนาดจำกัด ที่มีคุณสมบัติเหมือนกันตามขวาง ภายใต้การเปลี่ยนแปลงอุณหภูมิและแรงกระทำแบบสมมาตรกับแนวแกน ผลเฉลยทั่วไปสำหรับทรงกระบอกนี้หาได้จากการใช้ฟังก์ชันของการเปลี่ยนแปลงตำแหน่งเพื่อแก้สมการเชิงอนุพันธ์ย่อย โดยที่ผลเฉลยทั่วไปจะแสดงในรูปของอนุกรมฟูเรียร์เบสเซล ซึ่งประกอบด้วยฟังก์ชันตรีโกณมิติและฟังก์ชันไฮเพอร์โปลิกในทิศทางตามความยาวของทรงกระบอก และฟังก์ชันเบสเซลในทิศทางรัศมีของทรงกระบอก ในการหาค่าคงที่ต่าง ๆ ที่ปรากฏในผลเฉลยทั่วไป สามารถหาได้จากการแก้ปัญหาค่าขอบของชิ้นส่วนทรงกระบอกเชิงประกอบไพโซอิเล็กทริกภายใต้การเปลี่ยนแปลงอุณหภูมิและแรงกระทำที่ผิวนอกของทรงกระบอก ในงานวิจัยนี้ ได้เสนอหน่วยแรงที่เกิดขึ้นในทรงกระบอกสำหรับกรณีที่มีการเปลี่ยนแปลงอุณหภูมิแบบสม่ำเสมอและแรงกระทำเชิงกลแบบราบที่ผิวนอกของทรงกระบอก การวิเคราะห์หน่วยแรงที่เกิดขึ้นในชิ้นส่วนไพโซอิเล็กทริกทรงกระบอกจะใช้วิธีการคำนวณเชิงตัวเลข เพื่อศึกษาอิทธิพลของพารามิเตอร์ต่าง ๆ ที่ส่งผลต่อหน่วยแรงที่เกิดขึ้นในชิ้นส่วนเชิงประกอบไพโซอิเล็กทริกทรงกระบอก

ภาควิชา ....วิศวกรรมโยธา............ ลายมือชื่อนิสิต ....
สาขาวิชา ....วิศวกรรมโยธา........... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....
ปีการศึกษา ....2555................ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

5370564821: MAJOR CIVIL ENGINEERING

KEYWORDS: THERMOPIEZOELECTRIC / FINITE COMPOSITE CYLINDER / TRANSIENT RESPONSE / LAPLACE TRANSFORM / POTENTIAL FUNCTION

PANUPONG BOONPHENNIMIT: TRANSIENT RESPONSE OF THERMOPIEZOELECTRIC FINITE COMPOSITE CYLINDER. ADVISOR: ASSOC. PROFESSOR JAROON RUNGAMORNRAT, Ph.D., CO-ADVISOR: PROF. TEERAPONG SENJUNTICHAI, Ph.D., 82 pp.

In this thesis, a complete solution of transient responses of a transversely isotropic thermopiezoelectric finite composite cylinder under axisymmetric boundary conditions is presented. In the formulation, the temperature field is governed by Fourier heat conduction and assumed to be fully uncoupled. A series solution of the temperature field is obtained first in a Laplace transform space. A potential function technique is then applied to decouple the thermoelectromechanical governing equations and their general series solutions are obtained via standard procedure. The general solution is presented in terms of Fourier-Bessel series of trigonometric and hyperbolic functions. The boundary value problems corresponding to uniform temperature and mechanical loading at the outer curve surface of the cylinder are presented. Two types of mechanicals loading are considered in the present study, i.e., constant and banded loadings. A selected numerical technique is adopted to efficiently perform the Laplace transform inversion of all quantities. Selected numerical results for thermopiezoelectric filed of a cylinder are presented for both transient and steady states. Influence of various governing parameters on the response of piezoelectric cylinder are investigated and discussed.

Department: ……Civil Engineering……	Student 's Signature: ……………………..
Field of Study: ……Civil Engineering……	Advisor 's Signature: ……………..……...
Academic Year: ……2012……....	Co-advisor 's Signature: …………….……...

Thesis Title	TRANSIENT RESPONSE OF THERMOPIEZOELECTRIC FINITE COMPOSITE CYLINDER
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ACKNOWLEDGEMENTS

The author wishes to express his gratitude to everyone who advised and supported his to complete this thesis. Firstly, he would like to gratefully acknowledge the Department of Civil Engineering, Faculty of Engineering, Chulalongkorn University, who allowed his to study for Master's degree and also provided the support for all three years. Next, he would like to express his greatest appreciation to his thesis advisor, Associate Professor Dr. Jaroon Rungamornrat, and his thesis co-advisor, Professor Dr. Teerapong Senjentichai, who gave his valuable guidance of this work, continuous support and patience throughout this study. He also would like to express his sincere thanks to all of his thesis committees who spent time and energy reading and commenting on his thesis. Furthermore, special thanks to everyone who helped directly and indirectly in the preparation of this thesis, namely, thanks to Mr. Yasothorn Sapsathiarn for their good advice on running the computer program and thanks to his close friends, Mr. Supanat Woonprasert, for helping preparing his thesis report. Last but not least, the author would like to express his deepest gratitude to his family for their constant support, encouragement and endless love.

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CHAPTER I

INTRODUCTION

1.1 General

In recent years, there is a growing interest among researchers in engineering mechanics to study behaviors of smart materials due to their useful applications in various disciplines. Due to their inherent coupling phenomena (electro-mechanical or magneto-mechanical), smart materials have been extensively used in aerospace structures, intelligent or smart structures, nondestructive testing devices, medical devices, and sensing and actuation applications. Several types of smart materials have already been developed, namely piezoelectric materials, shape memory alloys, electrostrictive materials, magenetostrictive or piezomagnetic materials, electroactive polymers and electro/magneto–rheological fluids, etc. Among all materials mentioned above, the most widely used smart materials in practical applications are piezoelectric materials. A few examples of piezoelectric materials are Barium Titanate (BaTiO3), Lead Zirconate Titanate (PZT) and Polyvinylidene Fluoride (PVDF), etc. Piezoelectric materials exhibit electro-mechanical coupling phenomenon that they can produce electric field when deformed under a mechanical stress (direct piezoelectric effect), and conversely they can deform when subjected to an electric field (converse piezoelectric effect), which are very useful for sensing and actuation applications.

Even though smart materials are applicable in various fields, they also have some drawbacks in practical situations due to their fracture behaviors. Therefore, composite materials, which are composed of two or more different materials to achieve desire performance, have been developed. The examples of traditional composite materials are fiber-reinforced concrete and metal matrix composite (MMC), whereas 1-3 piezocomposite is a smart composite material, which is composed of piezoelectric fiber in one direction through the thickness embedded in a passive non-piezoelectric polymer (see Figure 1.1). The behavior of composite materials is very complicated and depends on several factors such as the volume fraction of the piezoelectric fiber, the material properties of each component, the aspect ratio, permeable and impermeable conditions at the interface. In addition, temperature range under working condition is also an important factor. For example, piezocomposites that are employed for sensing and actuation applications in aerospace structures could be subjected to extreme environmental conditions, in which the temperature varies from –500°C to +1000°C.

To develop suitable piezocomposites for practical applications under extreme temperature range, fundamental understanding of mechanics and effective properties of 1-3 piezocomposites is important. In order to achieve that it is ordinary to focus on a typical unit cell of piezocomposite. Therefore, this research is concerned with the development of accurate model and analytical solution of a thermopiezoelectric finite composite cylinder subjected to axisymmetric loading with considering temperature effects. Finally, a computer program has been developed to investigate transient behavior of a unit cell of 1-3 piezocomposite under axisymmetric mechanical, electric and thermal loading.

1.2 Background and Review

In this section, a background in this field and the literature review including the previous work relevant to the current work are provided. In order to know overview, background and literature review are separated into three parts. The first part is focus on the models, technique to solve the problem of circular cylinder subjected to arbitrary axisymmetric loading. The governing equation concerns only mechanical field. The second part is focus on model, technique to solve the problem and solution from piezoelectric composite cylinder subjected to axisymmetric load, both mechanical load and electric load. And the governing equation in this part not related to entropy equation. The third part is focus on model, technique to solve the problem and solution thermopiezoelectric composite cylinder in term of axisymmetric load by using governing equation and entropy equation.

1.2.1 Review on Circular Cylinders

One of the most fundamental problems in the theory of elasticity is stress analysis of finite cylinder subjected to arbitrary boundary conditions. Filon (1902) investigate the case of a finite solid cylinder subjected to uniaxial compression with end friction. Saito (1952) presented a Fourier Bessel solution for a circular cylinder by using Love's stress function to satisfy axisymmetric boundary conditions. Wei et al. (1999) presented a new analytical solution for an elastic solid finite cylinder subjected to the axial point load strength test (PLST) by using the displacement potential technique to uncouple equilibrium equations. They found that the maximum tensile stress increases with increasing Young's modulus but it decreases when Poison's ratio and area loading are decreased. Wei and Chau (2000) later derived a general solution for finite elastic isotropic solid circular cylinders subjected to arbitrary surface load. They said, this approach is most compatible with stress analysis of finite solid isotropic elastic cylinder. Shao (2005) investigated the case of a multi-layered circular hollow cylinder subjected to axisymmetric loading including steady state temperature. By using the solution from Shao (2005), Shao and Ma (2008) obtained thermal and mechanical stresses in a hollow cylinder by employing Laplace transform technique and series expansion method. Wei and Chau (2009) derived an analytical solution in the form of Fourier-Bessel series for a finite transversely isotropic elastic cylinder subjected to non-uniform compression with the end boundary conditions constraint by friction. They proposed general solution from Lekhnitskii's stress function and the result is useful in the analysis of fiber – reinforced coposites. In addition, Ying and Wang (2010) presented an analytical solution for a finite hollow cylinder under plane strain condition subjected to non-uniform thermal shock by using trigonometric series expansion and the separation of variable techniques.

1.2.2 Review on Piezoelectric and Piezocomposite Materials

The theoretical foundation and electroelastic governing equations of linear piezoelectric materials are presented by Parton and Kudryavtsev (1988). Sottos and Li (1994) investigated the influence of matrix stiffness, inter layer stiffness, rod aspect ratio and rod volume fraction on 1-3 piezocomposites under hydrostatic response.

Rajapakse and Zhou (1997) studied an infinite piezoelectric composite cylinder subjected to axisymmetric electromechanical loading by using Fourier integral transform. Hou et al. (2003) presented plane strain solution of a non-homogenous piezoelectric hollow cylinder subjected to dynamic loading by using the separation of variable technique. Rajapakse et al. (2004) developed a general solution for a finite annular piezoelectric cylinder subjected to axisymmetric end loading. Senjuntichai et al. (2008) presented analytical solution for piezoelectric cylinder subjected to electric voltage and mechanical axial loading applied at the end. Rajapakse and Chen (2008) presented a fully coupled analytical model for hydrostatic response of 1-3 piezocomposites to determine the effective properties of 1-3 piezocomposites. This problem considered linear quasi-static. Recently, Wu and Tsai (2012) presented analytical solutions of circular hollow sandwich cylinders, made of functionally graded piezoelectric materials (FGPM), subjected to electro-mechanical loading to investigate the influence of various parameters such as aspect ratio, open and closed-circuit surface conditions, and materials properties to the solutions.

1.2.3 Review on Thermopiezoelectricity

Kapuria et al (1996) used a potential function technique to obtain an analytical solution of a finite transversely isotropic piezoelectric cylindrical shell subjected to axisymmetric thermal, pressure and electrostatic loading. Fulin et al (1996) proposed potential functions and Fourier-Hankel transforms to develop axisymmetric solutions of transversely isotropic thermopiezoelastic materials. By using potential functions, Ashida and Tauchert (1998) investigated temperature, displacement, stress and electric fields of a finite circular piezoelectric disk subjected to axisymmetric loading. In addition, they also presented general solutions for a three dimensional thermopiezoelectric solid of class 6 mm, and for an infinite plate of class 2 mm with the same boundary conditions. Ding et al. (2000) proposed a general solution of dynamic problems for piezothermoelastic of transversely isotropic piezoelectric materials, and showed that the solution can be degenerated for quasi-static problems by ignoring the inertia terms. Zheng et al. (2002) employed potential function and integral transfrom techniques to investigate thermopiezoelectric response of a piezoelectric thin film subjected to laser heating. In addition, they also presented a prediction of a failure mechanism under heating environment.

Wang et al. (2001) presented an analytical solution for piezothermoelastic solids of crystal class 6 mm by using potential functions that satisfy thermal, mechanical and electrical boundary conditions with coupling effects, and showed that their result agree with those given by Ashida et al. (1994). Wang (2006) investigated transient thermal fracture of a piezoelectric cylinder subjected to transient thermal environment by considering two types of boundary conditions. The first type is set up based on the classical theory of thermal conduction, whereas the second type involves the stress and electric displacement intensity factor at the crack tip in the cylinder. Both types of boundary conditions include electrically permeable and impermeable conditions. Tanigiwa and Ootao (2007) proposed the exact solution for transient temperature of piezothermoelastic with two-layered hollow cylinder, which consisted of isotropic elastic and piezoelectric layers, subjected to axisymmetric heating by employing Airy's stress function and the Laplace transforms.

1.3 Research Objective

The key objective of this research is to solve an exact solution for thermopiezoelectric finite composite cylinder and consider temperature effect in transient state.

1.4 Research Scopes

1) Studies the thermopiezoelectric finite cylinder, in which compose of piezoelectric fiber embedded in finite transversely isotropic peizoelectric matrix.
2) The problem is subjected to axisymmetric loading; mechanic load, electric load, thermal load.
3) Loading conditions are assumed to be symmetry along plane x-axis.

1.5 Research Methodology

1) Formulate boundary value problem due to axisymmetric problem and reduce the field equations into equilibrium equations in term of elastic displacement, electric potential and temperature field.
2) Using Laplace transform technique to solve temperature equation separately.
3) Take Laplace transform to equilibrium equation and uncouple equilibrium equations by using potential functions technique into homogenous part and non-homogenous part.
4) Get the general solution in term of arbitrary constants in time domain by solving differential equations of homogenous and non-homogenous equations.
5) Match boundary conditions to get arbitrary constants for complete the general solution in time domain.
6) Take inverse Laplace transform to complete general solution in time domain for get complete general solution.

1.6 Research Significance

The exact solution for thermopiezoelectric finite composite cylinder is used to be a benchmark solution for unit cell. In addition, this solution can used to investigate an effective properties of composite materials such as modulus, behavior under loading with consider temperature effect.

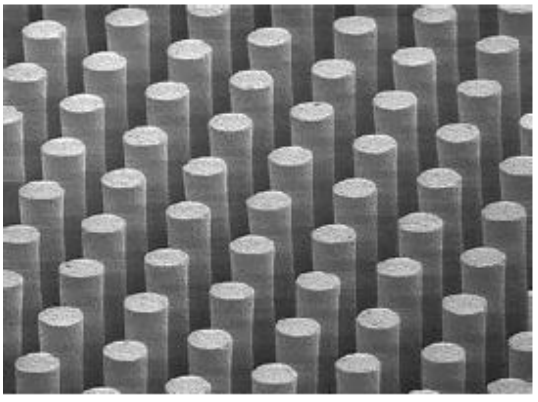

Figure 1.1 1-3 Piezocomposite

CHAPTER II

BASIC EQUATIONS AND GENERAL SOLUTIONS

In this chapter, the formulation of boundary value problem associated with an axisymmetric boundary condition of thermopiezoelectric finite composite cylinder is presented.

2.1 Problem Statement

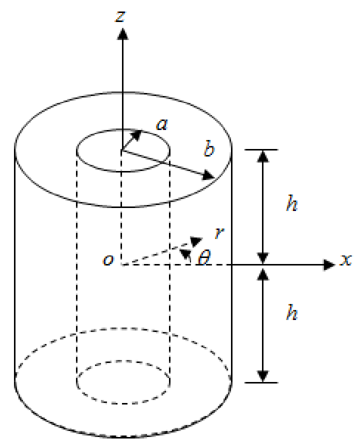

Figure 2.1 A thermopiezoelectric finite composite cylinder under consideration

Consider a linear thermopiezoelectric finite composite cylinder of length 2*h* as shown in Fig. 2.1. It consists of an embedded fiber of radius *a* with the outer radius of the matrix denoted by *b*. Both fiber and matrix are made of a linear, transversely isotropic thermopiezoelectric material of a special class 6mm. A Cartesian reference coordinate system $\{x, y, z\}$ and the cylindrical coordinate system $\{r, \theta, z\}$ are chosen, for convenience, such that the origin is located at the center of the cylinder, and the z-axis directs along the axis of the cylinder. The composite cylinder is subjected to axisymmetric boundary conditions (i.e. axisymmetric mechanical, electrical and thermal boundary conditions) that are even functions (or, equivalently, symmetric) with respect to the coordinate *z*.

2.2 Basic Field Equations

Basic field equations for a three-dimensional linear thermopiezoelectric material presented in this section follow directly from Mindlin (1974) and Parton and Kudryavtsev (1988). The Cauchy stress tensor σ_{ij}, the electric induction vector D_i and the heat flux vector h_i, in the absence of body forces, free charges and heat sources, are governed by equilibrium equations and conservation laws as,

$$\sigma_{ij,j} = 0 \tag{2.1}$$

$$D_{i,i} = 0 \tag{2.2}$$

$$h_{i,i} = -T_0 \dot{S} \tag{2.3}$$

where T_0 and S denote the constant positive reference temperature at which natural state of zero stress and strain exist, and the entropy density respectively. In addition, $f_{,i}$ and $\dot{f}$ denote the spatial and time derivatives of a function *f*, respectively. Hereafter, the lower-case indices range from 1 to 3 and the repeated indices imply the summation over their range.

The infinitesimal strain tensor ε_{ij}, the electric field vector E_i, and the temperature gradient e_i can be expressed in terms of the elastic displacement vector u_i, the electric potential φ and the temperature change θ (from the reference temperature T_0) respectively as,

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2.4}$$

$$E_i = -\varphi_{,i} \tag{2.5}$$

$$e_i = -\theta_{,i} \tag{2.6}$$

The Cauchy stress tensor σ_{ij}, the electric induction vector D_i, the entropy density S and the heat flux vector h_i are related to the infinitesimal strain tensor ε_{ij}, the electric field vector E_i, the temperature change θ and the temperature gradient according to the following linear constitutive laws,

$$\sigma_{\mathrm{ij}} = c_{ijkl}\varepsilon_{kl} - e_{kil}E_k - \lambda_{ij}\theta \tag{2.5}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \in_{ik} E_k + p_i\theta \tag{2.6}$$

$$S = \lambda_{kl}\varepsilon_{kl} + p_k E_k + \alpha\theta \tag{2.5}$$

$$h_i = K_{ij}e_j \tag{2.6}$$

where $c_{ijkl}, e_{ikl}, \lambda_{kl}, K_{ij}, \in_{ik}, p_k, \alpha$ denote the elastic constants, piezoelectric constants, temperature-stress coefficients, coefficients of heat conduction, dielectric constants, pyroelectric constants, and the thermal expansion coefficient. It is noted that the thermal expansion coefficient is written in term of the mass density ρ, the heat capacity per unit volume at constant strain C_v and the absolute temperature T as $\alpha = \rho C_v / T$.

The above field equations can be written for a special case of a transversely isotropic thermopiezoelectric medium undergoing axisymmetric deformation along the z-axis. Equations (2.1) – (2.3) for this particular case, based on the cylindrical coordinate systems, are given by

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\partial\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{2.11}$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{2.12}$$

$$\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0 \tag{2.13}$$

$$\frac{\partial h_r}{\partial r}+\frac{\partial h_z}{\partial z}+\frac{h_r}{r}=-T_0\dot{S} \tag{2.14}$$

Similarly, the equations (2.4) – (2.6) then become

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r},\varepsilon_{\theta\theta}=\frac{u_r}{r},\varepsilon_{zz}=\frac{\partial u_z}{\partial z},\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \tag{2.15a – 2.15d}$$

$$E_r=-\frac{\partial\varphi}{\partial r},E_z=-\frac{\partial\varphi}{\partial z} \tag{2.16a – 2.16b}$$

$$e_r=-\frac{\partial\theta}{\partial r},e_z=-\frac{\partial\theta}{\partial z} \tag{2.17a – 2.17b}$$

In addition, the constitutive relations (2.7) – (2.10) for this case are given by

$$\sigma_{rr}=c_{11}\varepsilon_{rr}+c_{12}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz}-e_{31}E_z-\lambda_{11}\theta \tag{2.18a}$$

$$\sigma_{\theta\theta}=c_{12}\varepsilon_{rr}+c_{11}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz}-e_{31}E_z-\lambda_{11}\theta \tag{2.18b}$$

$$\sigma_{zz}=c_{13}\varepsilon_{rr}+c_{13}\varepsilon_{\theta\theta}+c_{33}\varepsilon_{zz}-e_{33}E_z-\lambda_{33}\theta \tag{2.18c}$$

$$\sigma_{rz}=2c_{44}\varepsilon_{rz}-e_{15}E_r \tag{2.18d}$$

$$D_r=2e_{15}\varepsilon_{rz}+\in_{11}E_r+p_1\theta \tag{2.19a}$$

$$D_z=e_{31}\varepsilon_{rr}+e_{31}\varepsilon_{\theta\theta}+e_{33}\varepsilon_{zz}+\in_{33}E_z+p_3\theta \tag{2.19d}$$

$$S=\lambda_{11}\varepsilon_{rr}+\lambda_{11}\varepsilon_{\theta\theta}+\lambda_{33}\varepsilon_{zz}+p_1E_r+p_3E_z+\alpha\theta \tag{2.20}$$

$$h_r=K_{11}e_r\ ;\quad h_z=K_{33}e_z \tag{2.21a – 2.21b}$$

By combining (2.14), (2.17), (2.20) and (2.21) along with additional assumptions that the velocity gradient is negligible and the electric field is quasi-static, it leads to an uncoupled Fourier heat conduction equation governing the temperature change θ ,

$$\left[\delta^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}\right]\theta=\frac{\partial\theta}{\partial t} \tag{2.22}$$

where $\delta=\sqrt{K_{33}/K_{11}}$ is the ratio of the heat conduction coefficient, κ denotes the thermal diffusivity and t denotes the time variable.

Similarly, by combining (2.11) – (2.12), (2.15) and (2.18), it yields two equilibrium equations in terms of the elastic displacement and the electric potential and the temperature change θ :

$$c_{11}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)u_r-c_{11}\frac{u_r}{r^2}+c_{44}\frac{\partial^2 u_r}{\partial z^2}+\left(c_{13}+c_{44}\right)\frac{\partial^2 u_z}{\partial r\partial z}+\left(e_{15}+e_{31}\right)\frac{\partial^2\varphi}{\partial r\partial z}-\lambda_{11}\frac{\partial\theta}{\partial r}=0 \tag{2.23}$$

$$\left(c_{13}+c_{44}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+c_{44}\frac{\partial^2 u_z}{\partial r^2}+c_{33}\frac{\partial^2 u_z}{\partial z^2}+c_{44}\frac{1}{r}\frac{\partial u_z}{\partial r}+e_{15}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+e_{33}\frac{\partial^2\varphi}{\partial z^2}-\lambda_{33}\frac{\partial\theta}{\partial z}=0 \tag{2.24}$$

Finally, combining (2.13), (2.16) and (2.19) leads to the governing field equation.

$$\begin{aligned}&\left(e_{15}+e_{31}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+e_{15}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+e_{33}\frac{\partial^2 u_z}{\partial z^2}\\&\quad-\in_{11}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)-\in_{33}\frac{\partial^2\varphi}{\partial z^2}+p_3\frac{\partial\theta}{\partial z}+p_1\left(\frac{\partial\theta}{\partial r}+\frac{\theta}{r}\right)=0\end{aligned} \tag{2.25}$$

It is evident that the three governing field equations (2.23) – (2.25) are fully coupled whereas the governing equation for the temperature change (2.22) is independent of the elastic displacement and the electric potential.

2.3 General Solution for Temperature and Potential Functions

Before solving for the general solution, the following non-dimensional parameters are introduced and shown in appendix. For convenient notation, all quantities have been used the same as previous notation. The equations (2.22) – (2.25) become

$$\left[\delta^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}\right]\theta=\frac{\partial\theta}{\partial t} \tag{2.26}$$

$$c_{11}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)u_r-c_{11}\frac{u_r}{r^2}+c_{44}\frac{\partial^2 u_r}{\partial z^2}+\left(c_{13}+c_{44}\right)\frac{\partial^2 u_z}{\partial r\partial z}+\left(e_{15}+e_{31}\right)\frac{\partial^2\varphi}{\partial r\partial z}-\lambda_{11}\frac{\partial\theta}{\partial r}=0 \tag{2.27}$$

$$\left(c_{13}+c_{44}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+c_{44}\frac{\partial^2 u_z}{\partial r^2}+c_{33}\frac{\partial^2 u_z}{\partial z^2}+c_{44}\frac{1}{r}\frac{\partial u_z}{\partial r}+e_{15}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+e_{33}\frac{\partial^2\varphi}{\partial z^2}-\lambda_{33}\frac{\partial\theta}{\partial z}=0 \tag{2.28}$$

$$\left(e_{15}+e_{31}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+e_{15}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+e_{33}\frac{\partial^2 u_z}{\partial z^2}-\in_{11}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)-\in_{33}\frac{\partial^2\varphi}{\partial z^2}+p_3\frac{\partial\theta}{\partial z}=0 \tag{2.29}$$

The general solution to a system of the above governing differential equations (2.26) – (2.29) is constructed as follows. First, the Fourier heat conduction equation (2.26) is solved separately in a Laplace transform domain using a separation of variable technique. Once the general solution for the temperature change is obtained, the three coupled equations (2.27) – (2.29) are solved simultaneously in the Laplace transform domain using the potential function approach along with the separation of variable technique. Details of such solution procedure are outlined below.

2.3.1) Solution for temperature

The Laplace transform of any function $\phi(r,z,t)$ with respect to a time t can be expressed as (Sneddon, 1951)

$$\bar{\phi}\left(r,z,s\right)=\int_0^\infty\phi\left(r,z,t\right)e^{-st}dt \tag{2.30}$$

The inverse Laplace transform of $\bar{\phi}(\bar{r},\bar{z},s)$ with respect to Laplace transform parameter is given by (Sneddon, 1951)

$$\phi(r,z,s)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\bar{\phi}(r,z,s)e^{st}ds \tag{2.31}$$

where $i=\sqrt{-1}$ and α is a sufficiently large real number

By taking the Laplace transform of Eq. (2.26), it leads to

$$\left(\frac{\partial^2}{\partial \Upsilon^2}+\frac{1}{\Upsilon}\frac{\partial}{\partial \Upsilon}+\frac{\partial^2}{\partial z^2}\right)\bar{\theta}=s\bar{\theta} \tag{2.32}$$

where $\Upsilon=\frac{r}{\delta}$. The partial differential equation (2.28) can be solved by using a standard separation of variable technique, i.e. $\bar{\theta}=\bar{R}(r,s)\bar{Z}(z,s)$, and the resulting general solution for $\bar{\theta}$ is given by

$$\bar{\theta}(r,z,s)=\sum_{p=0}^{\infty}\left[A_p{}'I_0\left(\frac{\chi_p{}'r}{\delta}\right)+B_p{}'K_0\left(\frac{\chi_p{}'r}{\delta}\right)\right]\left[C_p{}'\cos\left(\vartheta_p{}'z\right)+D_p\sin\left(\vartheta_p{}'z\right)\right]$$

$$+\sum_{q=0}^{\infty}\left[E_q{}'J_0\left(\frac{\mu_q{}'r}{\delta}\right)+F_q{}'Y_0\left(\frac{\mu_q{}'r}{\delta}\right)\right]\left[G_q{}'\cosh\left(\eta_q{}'z\right)+H_q{}'\sinh\left(\eta_q{}'z\right)\right] \tag{2.33}$$

where $\chi_p{}'=\sqrt{\left(\vartheta_p{}'\right)^2+s}$; $\eta_q{}'=\sqrt{\left(\mu_q{}'\right)^2+s}$; $\vartheta_p{}', \chi_p{}', \eta_q{}', \mu_q{}', A_p{}', B_p{}', C_p{}', D_p{}'$, $E_q{}', F_q{}', G_q{}'$ and $H_q{}'$ $(p,q=0,1,2,...,\infty)$ are arbitrary functions; I_n and K_n are modified Bessel functions of the first kind and second kind of the n^{th} order, respectively; and J_n and Y_n are Bessel functions of the first kind and second kind of the n^{th} order, respectively (see Watson, 1962).

2.3.2) Potential functions for displacements and electric potential

To solve a coupled system of equations (2.27) – (2.29), the potential function technique (Ashida, 1994) is utilized. In this technique, the elastic displacement u_r, u_z and the electric potential φ are represented by four potential functions $\psi_1, \psi_{2,} \psi_3$ and ψ_4 in the following forms.

$$u_r = \frac{\partial}{\partial r}\left(\psi_1 + \psi_2 + \psi_3 + \psi_4\right) \tag{2.34a}$$

$$u_z = \frac{\partial}{\partial z}\left(l_{11}\psi_1 + l_{12}\psi_2 + l_{13}\psi_3 + l_{14}\psi_4\right) \tag{2.34b}$$

$$\varphi = \frac{\partial}{\partial z}\left(l_{21}\psi_1 + l_{22}\psi_2 + l_{23}\psi_3 + l_{24}\psi_4\right) \tag{2.35}$$

where l_{1i} and l_{2i} $\left(i = 1, 2, 3, 4\right)$ are unknown constant to be determined. By inserting (2.34)–(2.35) into (2.27) – (2.29), it leads to the following three equations.

$$\frac{\partial}{\partial r}\sum_{i=1}^{4}\left(c_{11}\psi_{i,rr} + \frac{1}{r}c_{11}\psi_{i,r} + M_j\psi_{i,zz}\right) = \lambda_{11}\frac{\partial\theta}{\partial r} \tag{2.36}$$

$$\frac{\partial}{\partial z}\sum_{i=1}^{4}\left(N_i\psi_{i,rr} + N_i\frac{1}{r}\psi_{i,r} + P_i\psi_{i,zz}\right) = \lambda_{33}\frac{\partial\theta}{\partial z} \tag{2.37}$$

$$\frac{\partial}{\partial z}\sum_{i=1}^{4}\left(F_i\psi_{i,rr} + F_i\frac{1}{r}\psi_{i,r} + G_i\psi_{i,zz}\right) = -p_3\frac{\partial\theta}{\partial z} - p_1\left(\frac{\partial\theta}{\partial r} - \frac{\theta}{r}\right) \tag{2.38}$$

where M_i, N_i, P_i, F_i, and G_i are arbitrary constants defined in appendix.

To obtain the solution of (2.36) – (2.38), it is sufficient to find the potential functions such that $\psi_i\left(i = 1, 2, 3\right)$ satisfy a system of homogenous differential equations (2.39) and ψ_4 satisfy a system of non-homogenous differential equations (2.40).

$$\frac{\partial}{\partial r}\sum_{i=1}^{3}\left(c_{11}\psi_{i,rr}+\frac{1}{r}c_{11}\psi_{i,r}+M_j\psi_{i,zz}\right)=0 \tag{2.39a}$$

$$\frac{\partial}{\partial z}\sum_{i=1}^{3}\left(N_i\psi_{i,rr}+N_i\frac{1}{r}\psi_{i,r}+P_i\psi_{i,zz}\right)=0 \tag{2.39b}$$

$$\frac{\partial}{\partial z}\sum_{i=1}^{3}\left(F_i\psi_{i,rr}+F_i\frac{1}{r}\psi_{i,r}+G_i\psi_{i,zz}\right)=0 \tag{2.39c}$$

$$\frac{\partial}{\partial r}\left(c_{11}\psi_{4,rr}+c_{11}\frac{1}{r}\psi_{4,r}+M_4\psi_{4,zz}\right)=\lambda_{11}\frac{\partial\theta}{\partial r} \tag{2.40a}$$

$$\frac{\partial}{\partial z}\left(N_4\psi_{4,rr}+N_4\frac{1}{r}\psi_{4,r}+P_4\psi_{4,zz}\right)=\lambda_{33}\frac{\partial\theta}{\partial z} \quad 2.40b)$$

$$\frac{\partial}{\partial z}\left(F_4\psi_{4,rr}+F_4\frac{1}{r}\psi_{4,r}+G_4\psi_{4,zz}\right)=-p_3\frac{\partial\theta}{\partial z}-p_1\left(\frac{\partial\theta}{\partial r}+\frac{\theta}{r}\right) \tag{2.40c}$$

To avoid directly solving a system of fully coupled homogeneous equations (2.39), it is customary to obtain the solution for each $\psi_i\left(i=1,2,3\right)$ from the following system of fully uncoupled homogeneous equations.

$$c_{11}\psi_{i,rr}+\frac{1}{r}c_{11}\psi_{i,r}+M_j\psi_{i,zz}=0 \tag{2.41a}$$

$$N_i\psi_{i,rr}+N_i\frac{1}{r}\psi_{i,r}+P_i\psi_{i,zz}=0 \tag{2.41b}$$

$$F_i\psi_{i,rr}+F_i\frac{1}{r}\psi_{i,r}+G_i\psi_{i,zz}=0 \tag{2.41c}$$

In order to obtain a non-trivial solution for ψ_i of the above system, M_i, N_i, P_i, G_i and F_i must satisfy the following relation,

$$\frac{M_i}{\hat{c}_{11}} = \frac{P_i}{N_i} = \frac{G_i}{F_i} = \gamma_i^2, i = 1, 2, 3 \tag{2.42}$$

The equation (2.42) is used to determine l_{1i}, l_{2i} and $\gamma_i^2 \left(i = 1, 2, 3\right)$. In particular, γ_i^2 are roots of the following cubic equation.

$$\bar{A}\left(\gamma_i^2\right)^3 + \bar{B}\left(\gamma_i^2\right)^2 + \bar{C}\left(\gamma_i^2\right) + \bar{D} = 0 \tag{2.43}$$

where $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are constants expressed explicitly in terms of materials properties (see appendix). Upon exploiting the relation (2.42), the system of three equations simply reduces to

$$\frac{\partial^2 \psi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_i}{\partial r} + \gamma_i^2 \frac{\partial^2 \psi_i}{\partial z^2} = 0 \tag{2.44}$$

By taking Laplace transform of Eq. (2.44), it results in

$$\frac{\partial^2 \bar{\psi}_i}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\psi}_i}{\partial r} + \frac{\partial^2 \bar{\psi}_i}{\partial z_i^2} = 0 \tag{2.45}$$

where $z_i = \frac{z}{\gamma_i}$. The general solution ψ_i in the Laplace transform domain can readily be obtained by using the separation of variable technique, i.e. $\bar{\psi}_i = \bar{R}\left(r, s\right)\bar{Z}\left(z, s\right)$, $\bar{\psi}_i = \bar{R}\left(r, s\right)$ and $\bar{\psi}_i = \bar{Z}\left(\bar{z}, s\right)$. The final result is given by

$$\bar{\psi}_i\left(r, z\right) = \sum_{m=1}^{\infty}\left[A_{im} I_0\left(\upsilon_{im} r\right) + B_{im} K_0\left(\upsilon_{im} r\right)\right]\left[C_{im}\cos\left(\frac{\upsilon_{im} z}{\gamma_i}\right) + D_{im}\sin\left(\frac{\upsilon_{im} z}{\gamma_i}\right)\right]$$

$$+\sum_{n=1}^{\infty}\left[E_{in} J_0\left(\xi_{in} r\right) + F_{in} Y_0\left(\xi_{in} r\right)\right]\left[G_{in}\cosh\left(\frac{\xi_{in} z}{\gamma_i}\right) + H_{in}\sinh\left(\frac{\xi_{in} z}{\gamma_i}\right)\right] + A_{0i}\left[r^2 - 2\left(\frac{z}{\gamma_i}\right)^2\right] + B_i \ln r$$

(2.46)

where υ_m, ξ_n are constant; $A_{0i}, A_{im}, B_{im}, C_{im}, D_{im}, E_{in}, F_{in}, G_{in}$ and $H_{in}\ (i=1,2,3)$ are arbitrary constants.

A particular solution of a system of non-homogenous equations (2.40) can be obtained explicitly for the thermopiezoelectric crystal class 6 mm (i.e., $p_1 = 0$). By first taking Laplace transform of (2.40), it leads to

$$\frac{\partial}{\partial r}\left(c_{11}\psi_{4,rr} + c_{11}\frac{1}{r}\psi_{4,r} + M_4\psi_{4,zz} \right) = \lambda_{11}\frac{\partial \theta}{\partial r} \tag{2.47a}$$

$$\frac{\partial}{\partial z}\left(N_4\psi_{4,rr} + N_4\frac{1}{r}\psi_{4,r} + P_4\psi_{4,zz} \right) = \lambda_{33}\frac{\partial \theta}{\partial z} \tag{2.47b}$$

$$\frac{\partial}{\partial z}\left(F_4\psi_{4,rr} + F_4\frac{1}{r}\psi_{4,r} + G_4\psi_{4,zz} \right) = -p_3\frac{\partial \theta}{\partial z} \tag{2.47c}$$

By substituting the general solution for the temperature change given by (2.33) into (2.47) and choosing the constants M_4, N_4, P_4, F_4, and G_4 such that

$$\frac{\lambda_{11}}{\left(\frac{\chi_m}{\delta}\right)^2 c_{11} - \vartheta_m^2 M_4} = \frac{\lambda_{33}}{\left(\frac{\chi_m}{\delta}\right)^2 N_4 - \vartheta_m^2 P_4} = \frac{-p_3}{\left(\frac{\chi_m}{\delta}\right)^2 F_4 - \vartheta_m^2 G_4} = \varpi \tag{2.48a}$$

$$\frac{\lambda_{11}}{\left(\frac{\mu_m}{\delta}\right)^2 c_{11} - \eta_m^2 M_4} = \frac{\lambda_{33}}{\left(\frac{\mu_m}{\delta}\right)^2 N_4 - \eta_m^2 P_4} = \frac{-p_3}{\left(\frac{\mu_m}{\delta}\right)^2 F_4 - \eta_m^2 G_4} = \Psi_m \tag{2.48b}$$

The particular solution $\bar{\psi}_4$ can then be obtained in the Laplace transform domain as

$$\bar{\psi}_4(r,z,s) = \sum_{p=0}^{\infty}\left[A_{4p} I_0\left(\frac{\chi_p' r}{\delta}\right) + B_{4p} K_0\left(\frac{\chi_p' r}{\delta}\right) \right]\left[C_{4p}\cos\left(\vartheta_p' z\right) + D_{4p}\sin\left(\vartheta_p' z\right) \right]$$

$$+\sum_{q=0}^{\infty}\left[E_{4q}J_0\left(\frac{\mu_q'r}{\delta}\right)+F_{4q}Y_0\left(\frac{\mu_q'r}{\delta}\right)\right]\left[G_{4q}\cosh\left(\eta_q'z\right)+H_{4q}\sinh\left(\eta_q'z\right)\right] \qquad (2.49)$$

where $\chi_p', \vartheta_p', \mu_p', \eta_p'$ and $A_{4p}, B_{4p}, C_{4p}, D_{4p}, E_{4q}, F_{4q}, G_{4q}$ and H_{4q} are arbitrary functions. It is remarked that the conditions (2.48) are employed to determine the constants l_{14} and l_{24} (see the appendix).

Since the current research focuses only on axisymmetric boundary conditions that are symmetric with respect to the plane $z=0$, the potential functions must be even functions with respect to the z-coordinate. For completeness of the general solution, the potential functions given by (2.46) must contain a solution of a radially symmetric plane problem of an annular cylinder. As a result, the general solution for the potential function and the temperature change in the Laplace transform domain are given by

$$\bar{\psi}_i(r,z)=A_{0i}\left[r^2-2\left(\frac{z}{\gamma_i}\right)^2\right]+B_{00}\ln r+\sum_{m=1}^{\infty}\left[A_{im}I_0\left(\upsilon_{im}r\right)+B_{im}K_0\left(\upsilon_{im}r\right)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right)$$

$$+\sum_{n=1}^{\infty}\left[E_{in}J_0\left(\xi_{in}r\right)+F_{in}Y_0\left(\xi_{in}r\right)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \qquad \text{for } i=1,2,3 \qquad (2.50a)$$

$$\bar{\psi}_4(r,z,s)=\sum_{p=0}^{\infty}\left[A_{4p}I_0\left(\frac{\chi_p'r}{\delta}\right)+B_{4p}K_0\left(\frac{\chi_p'r}{\delta}\right)\right]\cos\left(\vartheta_p'z\right)$$

$$+\sum_{q=0}^{\infty}\left[E_{4q}J_0\left(\frac{\mu_q'r}{\delta}\right)+F_{4q}Y_0\left(\frac{\mu_q'r}{\delta}\right)\right]\cosh\left(\eta_q'z\right) \qquad (2.50b)$$

$$\bar{\theta}(r,z,s)=\sum_{p=0}^{\infty}\left[A_p'I_0\left(\frac{\chi_p'r}{\delta}\right)+B_p'K_0\left(\frac{\chi_p'r}{\delta}\right)\right]\cos\left(\vartheta_p'z\right)$$

$$+\sum_{q=0}^{\infty}\left[E_q{}'J_0\left(\frac{\mu_q{}'r}{\delta}\right)+F_q{}'Y_0\left(\frac{\mu_q{}'r}{\delta}\right)\right]\cosh\left(\eta_q{}'z\right) \tag{2.51}$$

2.4 General Solutions for Field Quantities

The general solution of the elastic displacements and electric potential $\bar{u}_r, \bar{u}_z, \bar{\varphi}$ in the Laplace transform domain can readily be obtained by substituting (2.50) – (2.51) into (2.34) – (2.35), and this results in

$$\bar{u}_r = 2\sum_{i=1}^{3} A_{0i}r + B_{00}\frac{1}{r} + \sum_{i=1}^{3}\sum_{m=1}^{\infty}\upsilon_{im}\left[A_{im}I_1\left(\upsilon_{im}r\right) - B_{im}K_1\left(\upsilon_{im}r\right)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right)$$

$$-\sum_{i=1}^{3}\sum_{n=1}^{\infty}\xi_{in}\left[E_{in}J_1\left(\xi_{in}r\right)+F_{in}Y_1\left(\xi_{in}r\right)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right)+\sum_{p=0}^{\infty}\frac{\chi_p{}'}{\delta}\left[A_{4p}I_1\left(\frac{\chi_p{}'r}{\delta}\right)-B_{4p}K_1\left(\frac{\chi_p{}'r}{\delta}\right)\right]\cos\left(\vartheta_p{}'z\right)$$

$$-\sum_{q=0}^{\infty}\frac{\mu_q{}'}{\delta}\left[E_{4q}J_1\left(\frac{\mu_q{}'r}{\delta}\right)+F_{4q}Y_1\left(\frac{\mu_q{}'r}{\delta}\right)\right]\cosh\left(\eta_q{}'z\right) \tag{2.52}$$

$$\bar{u}_z = -4\sum_{i=1}^{3}\frac{l_{1i}}{\gamma_i^2}A_{0i}z + \sum_{i=1}^{3}l_{1i}\sum_{m=1}^{\infty}\frac{\upsilon_{im}}{\gamma_i}\left[-A_{im}I_0\left(\upsilon_{im}r\right)-B_{im}K_0\left(\upsilon_{im}r\right)\right]\sin\left(\frac{\upsilon_{im}z}{\gamma_i}\right)$$

$$+\sum_{i=1}^{3}l_{1i}\sum_{n=1}^{\infty}\frac{\xi_{in}}{\gamma_i}\left[E_{in}J_0\left(\xi_{in}r\right)+F_{in}Y_0\left(\xi_{in}r\right)\right]\sinh\left(\frac{\xi_{in}z}{\gamma_i}\right)-l_{14}\sum_{p=0}^{\infty}\vartheta_p{}'\left[A_{4p}I_0\left(\frac{\chi_p{}'r}{\delta}\right)+B_{4p}K_0\left(\frac{\chi_p{}'r}{\delta}\right)\right]\sin\left(\vartheta_p{}'z\right)$$

$$+l_{14}\sum_{q=0}^{\infty}\eta_q{}'\left[E_{4q}J_0\left(\frac{\mu_q{}'r}{\delta}\right)+F_{4q}Y_0\left(\frac{\mu_q{}'r}{\delta}\right)\right]\sinh\left(\eta_q{}'z\right) \tag{2.53}$$

$$\bar{\varphi} = -4\sum_{i=1}^{3}\frac{l_{2i}}{\gamma_i^2}A_{0i}z + \sum_{i=1}^{3}l_{2i}\sum_{m=1}^{\infty}\frac{\upsilon_{im}}{\gamma_i}\left[-A_{im}I_0\left(\upsilon_m r\right)-B_{im}K_0\left(\upsilon_{im}r\right)\right]\sin\left(\frac{\upsilon_{im}z}{\gamma_i}\right)$$

$$+\sum_{i=1}^{3}l_{2i}\sum_{n=1}^{\infty}\frac{\xi_{in}}{\gamma_i}\left[E_{in}J_0\left(\xi_{in}r\right)+F_{in}Y_0\left(\xi_{in}r\right)\right]\sinh\left(\frac{\xi_{in}z}{\gamma_i}\right)-l_{14}\sum_{p=0}^{\infty}\vartheta_p{}'\left[A_{4p}I_0\left(\frac{\chi_p{}'r}{\delta}\right)+B_{4p}K_0\left(\frac{\chi_p{}'r}{\delta}\right)\right]\sin\left(\vartheta_p{}'z\right)$$

$$+l_{24}\sum_{q=0}^{\infty}\eta_q'\left[E_{4q}J_0\left(\frac{\mu_q' r}{\delta}\right)+F_{4q}Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\sinh\left(\eta_q' z\right) \tag{2.54}$$

The general solutions for the strain, the electric field, the temperature gradient, the stress and electric induction can be obtained from (2.15) – (2.19). For convenience, such solution is separated into five parts. The first part corresponds to the non-series term, and is denoted by a superscript '0'. The second part corresponds to the series of the modified Bessel function associated with l_{1i} and l_{2i} $\left(i=1,2,3\right)$, and it is denoted by a superscript '1'. The third part corresponds to the series of the Bessel function associated with l_{1i} and l_{2i} $\left(i=1,2,3\right)$, and it is denoted by a superscript '2'. The fourth part corresponds to the series of the modified Bessel function associated with l_{14}, l_{24} and the temperature field, and it is denoted by a superscript '3'. The last part corresponds to the series of the Bessel function associated with l_{14} ,l_{24} and the temperature field, and it is denoted by superscript '4'.

$$\bar{\varepsilon}_{rr}^{0}=2\sum_{i=1}^{3}A_{0i}-B_{00}\frac{1}{r^2} \tag{2.55a}$$

$$\bar{\varepsilon}_{rr}^{1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\left\{\left[\left(\upsilon_{im}\right)^2 I_0\left(\upsilon_{im}r\right)-\frac{\upsilon_{im}}{r}I_1\left(\upsilon_{im}r\right)\right]A_{im}+\left[\left(\upsilon_{im}\right)^2 K_0\left(\upsilon_{im}r\right)+\frac{\upsilon_{im}}{r}K_1\left(\upsilon_{im}r\right)\right]B_{im}\right\}\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.55b}$$

$$\bar{\varepsilon}_{rr}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left\{\left[-\left(\xi_{in}\right)^2 J_0\left(\xi_{in}r\right)+\frac{\xi_{in}}{r}J_1\left(\xi_{in}r\right)\right]E_{in}+\left[-\left(\xi_{in}\right)^2 Y_0\left(\xi_{in}r\right)+\frac{\xi_{in}}{r}Y_1\left(\xi_{in}r\right)\right]F_{in}\right\}\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.55c}$$

$$\bar{\varepsilon}_{rr}^{3}=\sum_{p=0}^{\infty}\left[\left(\frac{\chi_p'}{\delta}\right)^2 I_0\left(\frac{\chi_p' r}{\delta}\right)-\frac{\chi_p'}{\delta r}I_1\left(\frac{\chi_p' r}{\delta}\right)\right]A_{4p}\cos\left(\vartheta_p' z\right)$$

$$+\sum_{p=0}^{\infty}\left[\left(\frac{\chi_p'}{\delta}\right)^2 K_0\left(\frac{\chi_p' r}{\delta}\right)+\frac{\chi_p'}{\delta r}K_1\left(\frac{\chi_p' r}{\delta}\right)\right]B_{4p}\cos\left(\vartheta_p' z\right) \tag{2.55d}$$

$$\bar{\varepsilon}_{rr}^{4}=\sum_{q=0}^{\infty}\left[-\left(\frac{\mu_q'}{\delta}\right)^2 J_0\left(\frac{\mu_q' r}{\delta}\right)+\frac{\mu_q'}{\delta r}J_1\left(\frac{\mu_q' r}{\delta}\right)\right]E_{4q}\cosh\left(\eta_q' z\right)$$

$$+\sum_{q=0}^{\infty}\left[-\left(\frac{\mu_q{}'}{\delta}\right)^2 Y_0\left(\frac{\mu_q{}' r}{\delta}\right)+\frac{\mu_q{}'}{\delta r}Y_1\left(\frac{\mu_q{}' r}{\delta}\right)\right]F_{4q}\cosh\left(\eta_q{}' z\right) \tag{2.55e}$$

$$\bar{\varepsilon}_{\theta\theta}^{0}=2\sum_{i=1}^{3}A_{0i}+B_{00}\frac{1}{r^2} \tag{2.56a}$$

$$\bar{\varepsilon}_{\theta\theta}^{1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\frac{\upsilon_{im}}{r}\left[A_{im}I_1\left(\upsilon_{im}r\right)-B_{im}K_1\left(\upsilon_{im}r\right)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.56b}$$

$$\bar{\varepsilon}_{\theta\theta}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{\xi_{in}}{r}\left[-E_{in}J_1\left(\xi_{in}r\right)-F_{in}Y_1\left(\xi_{in}r\right)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.56c}$$

$$\bar{\varepsilon}_{\theta\theta}^{3}=\sum_{p=0}^{\infty}\frac{\chi_p{}'}{\delta r}\left[A_{4p}I_1\left(\frac{\chi_p{}' r}{\delta}\right)-B_{4p}K_1\left(\frac{\chi_p{}' r}{\delta}\right)\right]\cos\left(\vartheta_p{}' z\right) \tag{2.56d}$$

$$\bar{\varepsilon}_{\theta\theta}^{4}=\sum_{q=0}^{\infty}\frac{\mu_q{}'}{\delta r}\left[-E_qJ_1\left(\frac{\mu_{4q}{}' r}{\delta}\right)-F_qY_1\left(\frac{\mu_{4q}{}' r}{\delta}\right)\right]\cosh\left(\eta_q{}' z\right) \tag{2.56e}$$

$$\bar{\varepsilon}_{zz}^{0}=-4\sum_{i=1}^{\infty}\frac{l_{1i}}{\gamma_i^2}A_{0i} \tag{2.57a}$$

$$\bar{\varepsilon}_{zz}^{1}=\sum_{i=1}^{3}l_{1i}\sum_{m=1}^{\infty}\left(\frac{\upsilon_{im}}{\gamma_i}\right)^2\left[-A_{im}I_0\left(\upsilon_{im}r\right)-B_{im}K_0\left(\upsilon_{im}r\right)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.57b}$$

$$\bar{\varepsilon}_{zz}^{2}=\sum_{i=1}^{3}l_{1i}\sum_{n=1}^{\infty}\left(\frac{\xi_{in}}{\gamma_{in}}\right)^2\left[E_{in}J_0\left(\xi_{in}r\right)+F_{in}Y_0\left(\xi_{in}r\right)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.57c}$$

$$\bar{\varepsilon}_{zz}^{3}=l_{14}\sum_{p=0}^{\infty}\left(\vartheta_p{}'\right)^2\left[-A_{4p}I_0\left(\frac{\chi_p{}' r}{\delta}\right)-B_{4p}K_0\left(\frac{\chi_p{}' r}{\delta}\right)\right]\cos(\vartheta_p{}' z) \tag{2.57d}$$

$$\hat{\bar{\varepsilon}}_{zz}^{4}=l_{14}\sum_{q=0}^{\infty}\left(\eta_q{}'\right)^2\left[E_{4q}J_0\left(\frac{\mu_q{}' r}{\delta}\right)+F_{4q}Y_0\left(\frac{\mu_q{}' r}{\delta}\right)\right]\cosh\left(\eta_q{}' z\right) \tag{2.57e}$$

$$\bar{\varepsilon}_{zr}^{1}=\frac{1}{2}\sum_{i=1}^{3}\sum_{m=1}^{\infty}\frac{\upsilon_{im}^2}{\gamma_i}\left[\left(-l_{1i}-1\right)A_{im}I_1\left(\upsilon_{im}r\right)+\left(l_{1i}+1\right)B_{im}K_1\left(\upsilon_{im}r\right)\right]\sin\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.58a}$$

$$\bar{\varepsilon}_{zr}^{2}=\frac{1}{2}\sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{\xi_n^2}{\gamma_i}\left[\left(-l_{1i}-1\right)E_{in}J_1\left(\xi_{in}r\right)+\left(-l_{1i}-1\right)F_{in}Y_1\left(\xi_{in}r\right)\right]\sinh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.58b}$$

$$\bar{\varepsilon}_{zr}^{3}=\frac{1}{2}\sum_{p=0}^{\infty}\frac{\vartheta_p{}'\chi_p{}'}{\delta}\left[(-l_{14}-1)A_{4p}I_1\left(\frac{\chi_p{}'r}{\delta}\right)+(l_{14}+1)B_{4p}K_1\left(\frac{\chi_p{}'r}{\delta}\right)\right]\sin\left(\vartheta_p{}'z\right) \tag{2.58c}$$

$$\bar{\varepsilon}_{zr}^{4}=\frac{1}{2}\sum_{q=0}^{\infty}\frac{\mu_q{}'\eta_q{}'}{\delta}\left[(-l_{14}-1)E_{4q}J_1\left(\frac{\mu_q{}'r}{\delta}\right)+(-l_{14}-1)F_{4q}Y_1\left(\frac{\mu_q{}'r}{\delta}\right)\right]\sinh\left(\eta_q{}'z\right) \tag{2.58d}$$

$$\bar{E}_{r}^{1}=\sum_{i=1}^{3}l_{2i}\sum_{m=1}^{\infty}\frac{\upsilon_{im}^{2}}{\gamma_i}\left[A_{im}I_1(\upsilon_{im}r)-B_{im}K_1(\upsilon_{im}r)\right]\sin\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.59a}$$

$$\bar{E}_{r}^{2}=\sum_{i=1}^{3}l_{2i}\sum_{n=1}^{\infty}\frac{\xi_{in}^{2}}{\gamma_i}\left[E_{in}J_1(\xi_{in}r)+F_{in}Y_1(\xi_{in}r)\right]\sinh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.59b}$$

$$\bar{E}_{r}^{3}=l_{24}\sum_{p=0}^{\infty}\frac{\vartheta_p{}'\chi_p{}'}{\delta}\left[A_{4p}I_1\left(\frac{\chi_p{}'r}{\delta}\right)-B_{4p}K_1\left(\frac{\chi_p{}'r}{\delta}\right)\right]\sin\left(\vartheta_p{}'z\right) \tag{2.59c}$$

$$\bar{E}_{r}^{4}=l_{24}\sum_{q=0}^{\infty}\frac{\mu_q{}'\eta_q{}'}{\delta}\left[E_{4q}J_1\left(\frac{\mu_q{}'r}{\delta}\right)+F_{4q}Y_1\left(\frac{\mu_q{}'r}{\delta}\right)\right]\sinh\left(\eta_q{}'z\right) \tag{2.59d}$$

$$\bar{E}_{z}^{0}=4\sum_{i=1}^{\infty}\frac{l_{2i}}{\gamma_i^{2}}A_{0i} \tag{2.60a}$$

$$\bar{E}_{z}^{1}=\sum_{i=1}^{3}l_{2i}\sum_{m=1}^{\infty}\left(\frac{\upsilon_{im}}{\gamma_i}\right)^{2}\left[A_{im}I_0(\upsilon_{im}r)+B_{im}K_0(\upsilon_{im}r)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.60b}$$

$$\bar{E}_{z}^{2}=\sum_{i=1}^{3}l_{2i}\sum_{n=1}^{\infty}\left(\frac{\xi_{in}}{\gamma_i}\right)^{2}\left[-E_{in}J_0(\xi_{in}r)-F_{in}Y_0(\xi_{in}r)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.60c}$$

$$\bar{E}_{z}^{3}=l_{24}\sum_{p=0}^{\infty}\left(\vartheta_p{}'\right)^{2}\left[A_{4p}I_0\left(\frac{\chi_p{}'r}{\delta}\right)+B_{4p}K_0\left(\frac{\chi_p{}'r}{\delta}\right)\right]\cos\left(\vartheta_p{}'z\right) \tag{2.60d}$$

$$\bar{E}_{z}^{4}=l_{24}\sum_{q=0}^{\infty}\left(\eta_q{}'\right)^{2}\left[-E_{4q}J_0\left(\frac{\mu_q{}'r}{\delta}\right)-F_{4q}Y_0\left(\frac{\mu_q{}'r}{\delta}\right)\right]\cosh\left(\eta_q{}'z\right) \tag{2.60e}$$

$$\bar{e}_{r}^{3}=\sum_{p=0}^{\infty}\frac{\chi_p{}'}{\delta}\left[-A_p{}'I_1\left(\frac{\chi_p{}'r}{\delta}\right)+B_p{}'K_1\left(\frac{\chi_p{}'r}{\delta}\right)\right]\cos\left(\vartheta_p{}'z\right) \tag{2.61a}$$

$$\bar{e}_{r}^{4}=\sum_{q=0}^{\infty}\frac{\mu_q{}'}{\delta}\left[E_q{}'J_1\left(\frac{\mu_q{}'r}{\delta}\right)+F_q{}'Y_1\left(\frac{\mu_q{}'r}{\delta}\right)\right]\cosh\left(\eta_q{}'z\right) \tag{2.61b}$$

$$\bar{e}_z^3 = \sum_{p=0}^{\infty} \vartheta_p{}' \left[A_p{}' I_0\left(\frac{\chi_p r}{\delta}\right) + B_p{}' K_0\left(\frac{\chi_p r}{\delta}\right)\right] \sin\left(\vartheta_p{}' z\right) \tag{2.62a}$$

$$\bar{e}_z^4 = \sum_{q=1}^{\infty} \eta_q{}' \left[-E_q{}' J_0\left(\frac{\mu_q{}' r}{\delta}\right) - F_p{}' Y_0\left(\frac{\mu_q{}' r}{\delta}\right)\right] \sinh\left(\eta_q{}' z\right) \tag{2.62b}$$

$$\bar{\sigma}_{rr}^0 = \sum_{i=1}^{3} \left(2c_{11} + 2c_{12} - 4c_{13}\frac{l_{1i}}{\gamma_i^2} - 4e_{31}\frac{l_{2i}}{\gamma_i^2}\right) A_{0i} + \left(-\frac{2c_{11}}{r^2} + \frac{2c_{12}}{r^2}\right) B_{00} \tag{2.63a}$$

$$\bar{\sigma}_{rr}^1 = \sum_{i=1}^{3}\sum_{m=1}^{\infty} \left[\begin{array}{l} \left(\upsilon_{im}\right)^2 \left(c_{11} - c_{13}\frac{l_{1i}}{\gamma_i^2} - e_{31}\frac{l_{2i}}{\gamma_i^2}\right) I_0\left(\upsilon_{im} r\right) \\ +\upsilon_{im}\left(-\frac{c_{11}}{r} + \frac{c_{12}}{r}\right) I_1\left(\upsilon_{im} r\right) \end{array}\right] A_{im} \cos\left(\frac{\upsilon_{im} z}{\gamma_i}\right)$$

$$+\sum_{i=1}^{3}\sum_{m=1}^{\infty} \left[\begin{array}{l} \left(\upsilon_{im}\right)^2 \left(c_{11} - c_{13}\frac{l_{1i}}{\gamma_i^2} - e_{31}\frac{l_{2i}}{\gamma_i^2}\right) K_0\left(\upsilon_{im} r\right) \\ +\upsilon_{im}\left(+\frac{c_{11}}{r} - \frac{c_{12}}{r}\right) K_1\left(\upsilon_{im} r\right) \end{array}\right] B_{im} \cos\left(\frac{\upsilon_{im} z}{\gamma_i}\right) \tag{2.63b}$$

$$\bar{\sigma}_{rr}^2 = \sum_{i=1}^{3}\sum_{n=1}^{\infty} \left[\begin{array}{l} \left(\xi_{in}\right)^2 \left(-c_{11} + c_{13}\frac{l_{1i}}{\gamma_i^2} + e_{31}\frac{l_{2i}}{\gamma_i^2}\right) J_0\left(\xi_{in} r\right) \\ +\xi_{in}\left(\frac{c_{11}}{r} - \frac{c_{12}}{r}\right) J_1\left(\xi_{in} r\right) \end{array}\right] E_{in} \cosh\left(\frac{\xi_{in} z}{\gamma_i}\right)$$

$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty} \left[\begin{array}{l} \left(\xi_{in}\right)^2 \left(-c_{11} + c_{13}\frac{l_{1i}}{\gamma_i^2} + e_{31}\frac{l_{2i}}{\gamma_i^2}\right) Y_0\left(\xi_{in} r\right) \\ +\xi_{in}\left(\frac{c_{11}}{r} - \frac{c_{12}}{r}\right) Y_1\left(\xi_{in} r\right) \end{array}\right] F_{in} \cosh\left(\frac{\xi_{in} z}{\gamma_i}\right) \tag{2.63c}$$

$$\bar{\sigma}_{rr}^3 = \sum_{p=0}^{\infty} \left[\begin{array}{l} \left(c_{11}\left(\frac{\chi_p{}'}{\delta}\right)^2 - c_{13} l_{14}\left(\vartheta_p{}'\right)^2 - e_{31} l_{24}\left(\vartheta_p{}'\right)^2 \right) I_0\left(\frac{\chi_p{}' r}{\delta}\right) \\ +\frac{\chi_p{}'}{\delta}\left(-\frac{c_{11}}{r} + \frac{c_{12}}{r}\right) I_1\left(\frac{\chi_p{}' r}{\delta}\right) \end{array}\right] A_{4p} \cos\left(\vartheta_p{}' z\right)$$

$$+\sum_{p=0}^{\infty}\left[\begin{array}{l}\left(c_{11}\left(\frac{\chi_p'}{\delta}\right)^2-c_{13}l_{14}\left(\vartheta_p'\right)^2-e_{31}l_{24}\left(\vartheta_p'\right)^2\right)K_0\left(\frac{\chi_p' r}{\delta}\right)\\+\frac{\chi_p'}{\delta r}\left(\frac{c_{11}}{r}-\frac{c_{12}}{r}\right)K_1\left(\frac{\chi_p' r}{\delta}\right)\end{array}\right]B_{4p}\cos\left(\vartheta_p' z\right)$$

$$-\lambda_{11}\sum_{p=0}^{\infty}\left[A_p' I_0\left(\frac{\chi_p' r}{\delta}\right)+B_p' K_0\left(\frac{\chi_p' r}{\delta}\right)\right]\cos\left(\vartheta_p' z\right) \tag{2.63d}$$

$$\bar{\sigma}_{rr}^4=\sum_{q=0}^{\infty}\left[\begin{array}{l}\left(-c_{11}\left(\frac{\mu_q'}{\delta}\right)^2+c_{13}l_{14}\left(\eta_q'\right)^2+e_{31}l_{24}\left(\eta_q'\right)^2\right)J_0\left(\frac{\mu_q' r}{\delta}\right)\\+\frac{\mu_q'}{\delta r}\left(\frac{c_{11}}{r}-\frac{c_{12}}{r}\right)J_1\left(\frac{\mu_q' r}{\delta}\right)\end{array}\right]E_{4q}\cosh\left(\eta_q' z\right)$$

$$+\sum_{q=0}^{\infty}\left[\begin{array}{l}\left(-c_{11}\left(\frac{\mu_q'}{\delta}\right)^2+c_{13}l_{14}\left(\eta_q'\right)^2+e_{31}l_{24}\left(\eta_q'\right)^2\right)Y_0\left(\frac{\mu_q' r}{\delta}\right)\\+\frac{\mu_q'}{\delta}\left(\frac{c_{11}}{r}-\frac{c_{12}}{r}\right)Y_1\left(\frac{\mu_q' r}{\delta}\right)\end{array}\right]F_{4q}\cosh\left(\eta_q' z\right)$$

$$-\lambda_{11}\sum_{q=0}^{3}\left[E_q' J_0\left(\frac{\mu_q' r}{\delta}\right)+F_q' Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right) \tag{2.63e}$$

$$\bar{\sigma}_{\theta\theta}^0=\sum_{i=1}^{3}\left(2c_{12}+2c_{11}-4c_{13}\frac{l_{1i}}{\gamma_i^2}-4e_{31}\frac{l_{2i}}{\gamma_i^2}\right)A_{0i}+\left(-c_{12}\frac{1}{r^2}B_{00}+\frac{c_{11}}{r^2}\right)B_{00} \tag{2.64a}$$

$$\bar{\sigma}_{\theta\theta}^1=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\left[\begin{array}{l}\left(\upsilon_{im}\right)^2\left(c_{12}-c_{13}\frac{l_{1i}}{\gamma_i^2}-e_{31}\frac{l_{2i}}{\gamma_i^2}\right)I_0\left(\upsilon_{im} r\right)\\+\upsilon_{im}\left(-\frac{c_{12}}{r}+\frac{c_{11}}{r}\right)I_1\left(\upsilon_{im} r\right)\end{array}\right]A_{im}\cos\left(\frac{\upsilon_{im} z}{\gamma_i}\right)$$

$$+\sum_{i=1}^{3}\sum_{m=1}^{\infty}\left[\begin{array}{l}\left(\upsilon_{im}\right)^2\left(c_{12}-c_{13}\frac{l_{1i}}{\gamma_i^2}-e_{31}\frac{l_{2i}}{\gamma_i^2}\right)K_0\left(\upsilon_{im} r\right)\\+\upsilon_{im}\left(\frac{c_{12}}{r}-\frac{c_{11}}{r}\right)K_1\left(\upsilon_{im} r\right)\end{array}\right]B_{im}\cos\left(\frac{\upsilon_{im} z}{\gamma_i}\right) \tag{2.64b}$$

$$\bar{\sigma}_{\theta\theta}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left[\begin{array}{l}\left(\xi_{in}\right)^{2}\left(-c_{12}+c_{13}\dfrac{l_{1i}}{\gamma_i^2}+e_{31}\dfrac{l_{2i}}{\gamma_i^2}\right)J_0\left(\xi_{in}r\right)\\ +\xi_{in}\left(\dfrac{c_{12}}{r}-\dfrac{c_{11}}{r}\right)J_1\left(\xi_{in}r\right)\end{array}\right]E_{in}\cosh\left(\frac{\xi_n z}{\gamma_i}\right)$$

$$+\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left[\begin{array}{l}\left(\xi_{in}\right)^{2}\left(-c_{12}+c_{13}\dfrac{l_{1i}}{\gamma_i^2}+e_{31}\dfrac{l_{2i}}{\gamma_i^2}\right)Y_0\left(\xi_{in}r\right)\\ +\xi_{in}\left(\dfrac{c_{12}}{r}-\dfrac{c_{11}}{r}\right)Y_1\left(\xi_{in}r\right)\end{array}\right]F_{in}\cosh\left(\frac{\xi_n z}{\gamma_i}\right) \qquad (2.64c)$$

$$\bar{\sigma}_{\theta\theta}^{3}=\sum_{p=0}^{\infty}\left[\begin{array}{l}\left(c_{12}\left(\dfrac{\chi_p'}{\delta}\right)^{2}-c_{13}l_{14}\left(\vartheta_p'\right)^{2}-e_{31}l_{24}\left(\vartheta_p'\right)^{2}\right)I_0\left(\dfrac{\chi_p' r}{\delta}\right)\\ +\dfrac{\chi_p'}{\delta}\left(-\dfrac{c_{12}}{r}+\dfrac{c_{11}}{r}\right)I_1\left(\dfrac{\chi_p' r}{\delta}\right)\end{array}\right]A_{4p}\cos\left(\vartheta_p' z\right)$$

$$+\sum_{p=0}^{\infty}\left[\begin{array}{l}\left(c_{12}\left(\dfrac{\chi_p'}{\delta}\right)^{2}-c_{13}l_{14}\left(\vartheta_p'\right)^{2}-e_{31}l_{24}\left(\vartheta_p'\right)^{2}\right)K_0\left(\dfrac{\chi_p' r}{\delta}\right)\\ +\dfrac{\chi_p'}{\delta}\left(\dfrac{c_{12}}{r}-\dfrac{c_{11}}{r}\right)K_1\left(\dfrac{\chi_p' r}{\delta}\right)\end{array}\right]B_{4p}\cos\left(\vartheta_p' z\right)$$

$$-\lambda_{11}\sum_{p=0}^{3}\left[A_p' I_0\left(\frac{\chi_p' r}{\delta}\right)+B_p' K_0\left(\frac{\chi_p' r}{\delta}\right)\right]\cos\left(\vartheta_p' z\right) \qquad (2.64d)$$

$$\bar{\sigma}_{\theta\theta}^{4}=\sum_{q=0}^{\infty}\left[\begin{array}{l}\left(-c_{12}\left(\frac{\mu_q'}{\delta}\right)^2+c_{13}l_{14}\left(\eta_q'\right)^2+e_{31}l_{24}\left(\eta_q'\right)^2\right)J_0\left(\frac{\mu_q' r}{\delta}\right)\\+\frac{\mu_q'}{\delta}\left(\frac{c_{12}}{r}-\frac{c_{11}}{r}\right)J_1\left(\frac{\mu_q' r}{\delta}\right)\end{array}\right]E_{4q}\cosh\left(\eta_q' z\right)$$

$$+\sum_{q=0}^{\infty}\left[\begin{array}{l}\left(-c_{12}\left(\frac{\mu_q'}{\delta}\right)^2+c_{13}l_{14}\left(\eta_q'\right)^2+e_{31}l_{24}\left(\eta_q'\right)^2\right)Y_0\left(\frac{\mu_q' r}{\delta}\right)\\+\frac{\mu_q'}{\delta}\left(\frac{c_{12}}{r}-\frac{c_{11}}{r}\right)Y_1\left(\frac{\mu_q' r}{\delta}\right)\end{array}\right]F_{4q}\cosh\left(\eta_q' z\right)$$

$$-\lambda_{11}\sum_{q=0}^{3}\left[E_q' J_0\left(\frac{\mu_q' r}{\delta}\right)+F_q' Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right) \tag{2.64e}$$

$$\bar{\sigma}_{zz}^{0}=4\sum_{i=1}^{3}\left(c_{13}-c_{33}\frac{l_{1i}}{\gamma_i^2}-e_{33}\frac{l_{2i}}{\gamma_i^2}\right)A_{0i} \tag{2.65a}$$

$$\bar{\sigma}_{zz}^{1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\left(\upsilon_{im}\right)^2\left[c_{13}-\frac{c_{33}l_{1i}}{\gamma_i^2}-\frac{e_{33}l_{2i}}{\gamma_i^2}\right]\left[A_{im}I_0\left(\upsilon_{im}r\right)+B_{im}K_0\left(\upsilon_{im}r\right)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.65b}$$

$$\bar{\sigma}_{zz}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left(\xi_{in}\right)^2\left[-c_{13}+\frac{c_{33}l_{1i}}{\gamma_i^2}+\frac{e_{33}l_{2i}}{\gamma_i^2}\right]\left[E_{in}J_0\left(\xi_{in}r\right)+F_{in}Y_0\left(\xi_{in}r\right)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.65c}$$

$$\bar{\sigma}_{zz}^{3}=\sum_{p=0}^{\infty}\left[c_{13}\left(\frac{\chi_p'}{\delta}\right)^2-c_{33}l_{14}\left(\vartheta_p'\right)^2-e_{33}l_{24}\left(\vartheta_p'\right)^2\right]\left[A_{4p}I_0\left(\frac{\chi_p' r}{\delta}\right)+B_{4p}K_0\left(\frac{\chi_p' r}{\delta}\right)\right]\cos\left(\vartheta_p' z\right)$$

$$-\lambda_{33}\sum_{p=0}^{\infty}\left[A_p' I_0\left(\frac{\chi_p' r}{\delta}\right)+B_p' K_0\left(\frac{\chi_p' r}{\delta}\right)\right]\cos\left(\vartheta_p' z\right) \tag{2.65d}$$

$$\bar{\sigma}_{zz}^{4}=\sum_{q=0}^{\infty}\left[-c_{13}\left(\frac{\mu_q'}{\delta}\right)^2+c_{33}l_{14}\left(\eta_q'\right)^2+e_{33}l_{24}\left(\eta_q'\right)^2\right]\left[E_{4q}J_0\left(\frac{\mu_q' r}{\delta}\right)+F_{4q}Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right)$$

$$-\lambda_{33}\sum_{q=1}^{3}\left[E_q' J_0\left(\frac{\mu_q' r}{\delta}\right)+F_q' Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right) \tag{2.65e}$$

$$\bar{\sigma}_{zr}^{1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\frac{\upsilon_{im}^2}{\gamma_i}\left(c_{44}+c_{44}l_{1i}+e_{15}l_{2i}\right)\left[-A_{im}I_1\left(\upsilon_{im}r\right)+B_{im}K_1\left(\upsilon_{im}r\right)\right]\sin\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.66a}$$

$$\bar{\sigma}_{zr}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{\xi_{in}^{2}}{\gamma_i}\left(c_{44}+c_{44}l_{1i}+e_{15}l_{2i}\right)\left[-E_{in}J_1\left(\xi_{in}r\right)-F_{in}Y_1\left(\xi_{in}r\right)\right]\sinh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.66b}$$

$$\bar{\sigma}_{zr}^{3}=\sum_{p=0}^{\infty}\frac{\chi_p{}'\vartheta_p{}'}{\delta}\left(c_{44}+c_{44}l_{14}+e_{15}l_{24}\right)\left[-A_{4p}I_1\left(\frac{\chi_p{}'r}{\delta}\right)+B_{4p}K_1\left(\frac{\chi_p{}'r}{\delta}\right)\right]\sin\left(\vartheta_p{}'z\right) \tag{2.66c}$$

$$\hat{\bar{\sigma}}_{zr}^{4}=\sum_{q=0}^{\infty}\frac{\mu_q{}'\eta_q{}'}{\delta}\left(-c_{44}-c_{44}l_{14}-e_{15}l_{24}\right)\left[E_{4q}J_1\left(\frac{\mu_q{}'r}{\delta}\right)-F_{4q}Y_1\left(\frac{\mu_q{}'r}{\delta}\right)\right]\sinh\left(\eta_q{}'z\right) \tag{2.66d}$$

$$\bar{D}_{r}^{1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\frac{\upsilon_{im}^{2}}{\gamma_i}\left(e_{15}+e_{15}l_{1i}-\in_{11}l_{2i}\right)\left[-A_{im}I_1\left(\upsilon_{im}r\right)+B_{im}K_1\left(\upsilon_{im}r\right)\right]\sin\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.67a}$$

$$\bar{D}_{r}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{\xi_{in}^{2}}{\gamma_i}\left(e_{15}+e_{15}l_{1i}-\in_{11}l_{2i}\right)\left[-E_{in}J_1\left(\xi_{in}r\right)-F_{in}Y_1\left(\xi_{in}r\right)\right]\sinh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.67b}$$

$$\bar{D}_{r}^{3}=\sum_{p=0}^{\infty}\frac{\chi_p{}'\vartheta_p{}'}{\delta}\left(e_{15}+e_{15}l_{14}-\in_{11}l_{24}\right)\left[-A_{4p}I_1\left(\frac{\chi_p{}'r}{\delta}\right)+B_{4p}K_1\left(\frac{\chi_p{}'r}{\delta}\right)\right]\sin\left(\vartheta_p{}'z\right) \tag{2.67c}$$

$$\bar{D}_{r}^{4}=\sum_{q=0}^{\infty}\frac{\mu_q{}'\eta_q{}'}{\delta}\left(e_{15}+e_{15}l_{14}-\in_{11}l_{24}\right)\left[-E_{4q}J_1\left(\frac{\mu_q{}'r}{\delta}\right)-F_{4q}Y_1\left(\frac{\mu_q{}'r}{\delta}\right)\right]\sinh\left(\eta_q{}'z\right) \tag{2.67d}$$

$$\bar{D}_{z}^{0}=4\sum_{i=1}^{3}\left(e_{31}-e_{33}\frac{l_{1i}}{\gamma_i^2}+\in_{33}\frac{l_{2i}}{\gamma_i^2}\right)A_{0i} \tag{2.68a}$$

$$\bar{D}_{z}^{1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\left(\upsilon_{im}\right)^2\left(e_{31}-e_{33}\frac{l_{1i}}{\gamma_i^2}+\in_{33}\frac{l_{2i}}{\gamma_i^2}\right)\left[A_{im}I_0\left(\upsilon_{im}r\right)+B_{im}K_0\left(\upsilon_{im}r\right)\right]\cos\left(\frac{\upsilon_{im}z}{\gamma_i}\right) \tag{2.68b}$$

$$\bar{D}_{z}^{2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\left(\xi_{in}\right)^2\left(-e_{31}+e_{33}\frac{l_{1i}}{\gamma_i^2}-\in_{33}\frac{l_{2i}}{\gamma_i^2}\right)\left[E_{in}J_0\left(\xi_{in}r\right)+F_{in}Y_0\left(\xi_{in}r\right)\right]\cosh\left(\frac{\xi_{in}z}{\gamma_i}\right) \tag{2.68c}$$

$$\bar{D}_{z}^{3}=\sum_{p=0}^{\infty}\left[e_{31}\left(\frac{\chi_p{}'}{\delta}\right)-e_{33}l_{14}\left(\vartheta_p{}'\right)^2+\in_{33}l_{24}\left(\vartheta_p{}'\right)^2\right]\left[A_{4p}I_0\left(\frac{\chi_p{}'r}{\delta}\right)+B_{4p}K_0\left(\frac{\chi_p{}'r}{\delta}\right)\right]\cos\left(\vartheta_p{}'z\right)$$
$$+\hat{p}_3\sum_{p=0}^{\infty}\left[A_p{}'I_0\left(\frac{\chi_p{}'r}{\delta}\right)+B_p{}'K_0\left(\frac{\chi_p{}'r}{\delta}\right)\right]\cos\left(\vartheta_p{}'z\right) \tag{2.68d}$$

$$\bar{D}_z^4 = \sum_{q=0}^{\infty}\left[-e_{31}\left(\frac{\mu_q'}{\delta}\right)^2 + e_{33}l_{14}\left(\eta_q'\right)^2 - \in_{33} l_{24}\left(\eta_q'\right)^2\right]\left[E_{4q}J_0\left(\frac{\mu_q' r}{\delta}\right) + F_{4q}Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right)$$

$$+ p_3\sum_{q=0}^{3}\left[E_q' J_0\left(\frac{\mu_q' r}{\delta}\right) + F_q' Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right) \tag{2.68e}$$

$$\bar{h}_r = -\frac{K_{11}}{\delta}\sum_{p=0}^{\infty}\left(\chi_p'\right)\left[A_p' I_1\left(\frac{\chi_p' r}{\delta}\right) - B_p' K_1\left(\frac{\chi_p' r}{\delta}\right)\right]\cos\left(\vartheta_p' z\right)$$

$$-\frac{K_{11}}{\delta}\sum_{q=0}^{\infty}\left(\mu_q'\right)\left[-E_q' J_1\left(\frac{\mu_q' r}{\delta}\right) - F_q' Y_1\left(\frac{\mu_q' r}{\delta}\right)\right]\cosh\left(\eta_q' z\right) \tag{2.69}$$

$$\bar{h}_z = K_{11}\sum_{p=0}^{\infty}\left(\vartheta_p'\right)\left[A_p' I_0\left(\frac{\chi_p' r}{\delta}\right) - B_p' K_0\left(\frac{\chi_p' r}{\delta}\right)\right]\sin\left(\vartheta_p' z\right)$$

$$-K_{11}\sum_{q=0}^{\infty}\left(\eta_q'\right)\left[E_q' J_0\left(\frac{\mu_q' r}{\delta}\right) + F_q' Y_0\left(\frac{\mu_q' r}{\delta}\right)\right]\sin\left(\eta_q' z\right) \tag{2.70}$$

It is worth noting that all unknown constants appearing in the general solution presented above need to be determined once the boundary value problem is formulated.

CHAPTER III

FORMULATION OF PIEZOCOMPOSITE CYLINDER

3.1 Piezocomposite Cylinders Subjected to Thermal Loading

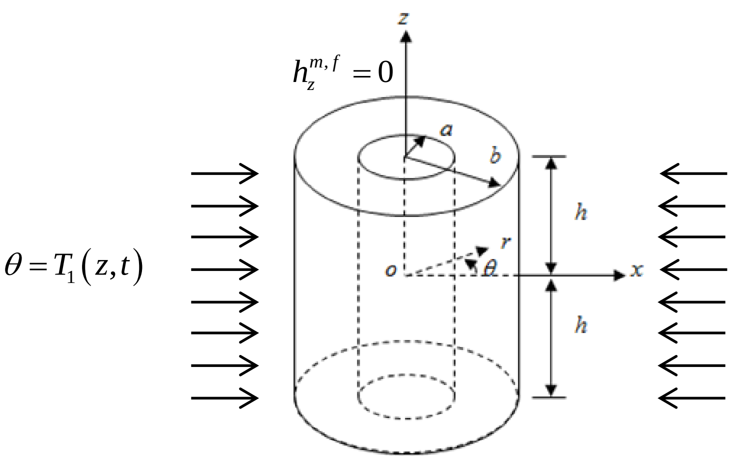

Fig. 3.1 Piezocomposite cylinder subjected to zero heat flux at both ends and prescribed temperature at curve surface.

In this section, arbitrary constants $\vartheta_p', \chi_p', \eta_q', \mu_q', A_p', B_p', E_q', F_q'$ defined previously are solved separately for the thermal boundary conditions. Consider the case of a piezocomposite cylinder as shown in Fig. 2.2, which is subjected to zero heat flux at both ends and a prescribed temperature $T_1(z,t)$ at its curved surface. The interface at $r = a$ is assumed to be perfectly bonded and permeable. Because the general solutions of temperature fields in previous section are in the Laplace domain, the Laplace transform is applied to the boundary conditions. To identify the domains of fiber and matrix, the superscript m represents the matrix domain and the fiber domain is identified by the superscript f. The boundary condition in the Laplace domain can be expressed as

$$\bar{h}_z^m(r, \pm 1) = 0 \qquad \text{for} \qquad a \le r \le b \qquad (3.1a)$$

$$\bar{h}_z^f(r,\pm 1)=0 \qquad \text{for} \qquad 0\le r\le a \qquad (3.1b)$$

$$\bar{\theta}^m(a,z)=\bar{\theta}^f(a,z),\ \bar{h}_r^m(a,z)=\bar{h}_r^f(a,z) \qquad \text{for} \qquad -1\le z\le 1 \qquad (3.2a - 3.22b)$$

$$\bar{\theta}^m(b,z)=\bar{T}_1(z,s) \qquad \text{for} \qquad -1\le z\le 1 \qquad (3.33)$$

where $\bar{T}_1(z,s)=\dfrac{\bar{T}_1(z,s)}{\bar{T}_0}$ denotes the non-dimensional temperature.

First consider the Eq. (3.1), the acceptable eigenvalue and the eigenfunction must be in terms of cosine functions. Therefore, the general solutions of temperature field are

$$\bar{\theta}^m=\sum_{p=0}^{\infty}\left[A_p^{m\prime}I_0\left(\frac{\chi_p^m r}{\delta^m}\right)+B_p^{m\prime}K_0\left(\frac{\chi_p^m r}{\delta^m}\right)\right]\cos\left(\vartheta_p^m z\right) \qquad \text{for} \qquad a\le r\le b \qquad (3.4a)$$

$$\bar{\theta}^f=\sum_{p=0}^{\infty}\left[A_p^{f\prime}I_0\left(\frac{\chi_p^f r}{\delta^f}\right)\right]\cos\left(\vartheta_p^f z\right) \qquad \text{for} \qquad 0\le r\le a \qquad (3.4b)$$

Where $\vartheta_p^m=\vartheta_p^m=p\pi$ for $p=1,2,3,...$

In view of Eq. (3.2), $B_p^{m\prime}$ and $A_p^{f\prime}$ can be expressed in terms of $A_p^{m\prime}$ as

$$B_p^{m\prime}=\left[\frac{\Gamma_p I_1\left(\frac{\chi_p^m a}{\delta^m}\right)I_0\left(\frac{\chi_p^f a}{\delta^f}\right)-I_0\left(\frac{\chi_p^m a}{\delta^m}\right)I_1\left(\frac{\chi_p^f a}{\delta^f}\right)}{\Gamma_p K_1\left(\frac{\chi_p^m a}{\delta^m}\right)I_0\left(\frac{\chi_p^f a}{\delta^f}\right)+K_0\left(\frac{\chi_p^m a}{\delta^m}\right)I_1\left(\frac{\chi_p^m a}{\delta^m}\right)}\right]A_p^{m\prime} \qquad (3.5a)$$

$$A_p^{f\prime}=\Gamma_p\left[\frac{I_1\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)K_0\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)+K_1\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)I_0\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)}{\Gamma K_1\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)I_0\left(\frac{\chi_p^f}{\delta^f}\hat{a}\right)+K_0\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)I_1\left(\frac{\chi_p^m}{\delta^m}\hat{a}\right)}\right]A_p^{m\prime} \qquad (3.5b)$$

where $\Gamma_p=\dfrac{\hat{K}_{11}^m\delta^f}{\hat{K}_{11}^f\delta^m}\dfrac{\chi_p^m}{\chi_p^f}$

Substitution $B_p^{m\prime}$ from Eq. (3.5a) into Eq. (3.3). The unknown $A_p^{m\prime}$ can be determined by introduced Fourier cosine series expansion into the right hand side terms

$$\bar{T}_1(z,s) = \frac{\bar{T}_{10}}{2} + \sum_{p=1}^{\infty} \bar{T}_{1p} \cos\left(\vartheta_p^m z\right) \tag{3.6a}$$

where $\bar{T}_{10}(z,s) = 2\int_0^1 \cos\left(\vartheta_p^m z\right) dz$ and $\bar{T}_{1p} = 2\int_0^1 \bar{T}_1(z,s)\cos\left(\vartheta_p^m z\right) dz$ (3.6b - 3.6c)

Then, the unknown $A_p^{m\prime}$ can be determined from

$$\left[I_0\left(\frac{\chi_0^m r}{\delta^m}\right) + \left[\frac{\Gamma_0 I_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^f a}{\delta^f}\right) - I_0\left(\frac{\chi_0^m a}{\delta^m}\right) I_1\left(\frac{\chi_0^f a}{\delta^f}\right)}{\Gamma_0 K_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^f a}{\delta^f}\right) + K_0\left(\frac{\chi_0^m a}{\delta^m}\right) I_1\left(\frac{\chi_0^m a}{\delta^m}\right)} \right] K_0\left(\frac{\chi_0^m r}{\delta^m}\right) \right] A_0^{m\prime} = \frac{\bar{T}_{10}}{2} \tag{3.7a}$$

$$\sum_{p=1}^{\infty} \left[I_0\left(\frac{\chi_p^m r}{\delta^m}\right) + \left[\frac{\Gamma I_1\left(\frac{\chi_p^m a}{\delta^m}\right) I_0\left(\frac{\chi_p^f a}{\delta^f}\right) - I_0\left(\frac{\chi_p^m a}{\delta^m}\right) I_1\left(\frac{\chi_p^f a}{\delta^f}\right)}{\Gamma K_1\left(\frac{\chi_p^m a}{\delta^m}\right) I_0\left(\frac{\chi_p^f a}{\delta^f}\right) + K_0\left(\frac{\chi_p^m a}{\delta^m}\right) I_1\left(\frac{\chi_p^m a}{\delta^m}\right)} \right] K_0\left(\frac{\chi_p^m r}{\delta^m}\right) \right] A_p^{m\prime} \cos\left(\vartheta_p^m z\right) = \sum_{p=1}^{\infty} \bar{T}_{1p} \cos\left(\vartheta_p^m z\right) \tag{3.7b}$$

The constants $B_p^{m\prime}$ and $A_p^{f\prime}$ can be taken from Eq. (3.5). In addition, the potential functions $\bar{\psi}_4^m$ and $\bar{\psi}_4^f$ in Eq. (2.49) can be expressed as

$$\bar{\psi}_4^m = \sum_{p=0}^{\infty} \left[A_{4p}^m I_0\left(\frac{\chi_p^{m\prime} r}{\delta^m}\right) + B_{4p}^m K_0\left(\frac{\chi_p^{m\prime} r}{\delta^m}\right) \right] \cos\left(\vartheta_p^{m\prime} z\right) \tag{3.8a}$$

$$\bar{\psi}_4^f = \sum_{p=1}^{\infty} A_{4p}^f I_0\left(\frac{\chi_p^{f\prime} r}{\delta^f}\right) \cos\left(\vartheta_p^{f\prime} z\right) \tag{3.8b}$$

where $A_{4p}^{m}=\dfrac{\lambda_{11}^{m}}{\left[c_{11}^{m}\left(\dfrac{\chi_{p}^{m\prime}}{\delta}\right)^{2}-M_{4}^{m}\left(\vartheta_{p}^{m\prime}\right)^{2}\right]}A_{p}^{m\prime}=\varpi^{m}A_{p}^{m\prime}$ (3.8c)

$$B_{4p}^{m}=\frac{\lambda_{11}^{m}}{\left[c_{11}^{m}\left(\dfrac{\chi_{p}^{m\prime}}{\delta}\right)^{2}-M_{4}^{m}\left(\vartheta_{p}^{m\prime}\right)^{2}\right]}B_{p}^{m\prime}=\varpi^{m}B_{p}^{m\prime} \qquad (3.8d)$$

$$A_{4p}^{f}=\frac{\lambda_{11}^{f}}{\left[c_{11}^{f}\left(\dfrac{\chi_{p}^{f\prime}}{\delta^{f}}\right)^{2}-M_{4}^{f}\left(\vartheta_{p}^{f\prime}\right)^{2}\right]}A_{p}^{f\prime}=\varpi^{f}A_{p}^{f\prime} \qquad (3.8e)$$

3.2 Piezocomposite Cylinder Subjected to Mechanical and Electrical Loading

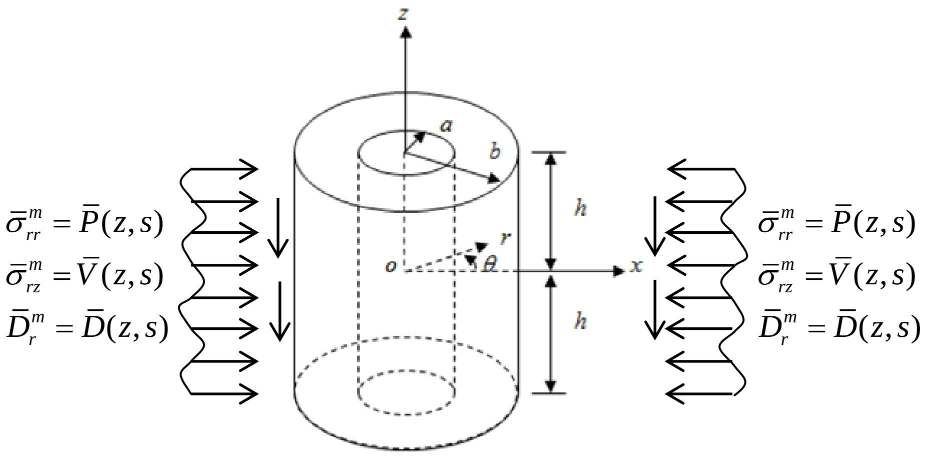

Fig 3.2 Piezocomposite cylinder subjected to prescribed traction and electric displacement at curve surface

In this section, all arbitrary constants A_{0i},B_{00}, A_{im},B_{im}, E_{in},F_{in}, A_{4p},B_{4p}, E_{4q},F_{4q}, υ_{im},ξ_{in} in the previous section are solved for the applied mechanical and electrical loading. Consider the case of a piezocomposite cylinder as shown in Fig. 3.2 This composite cylinder is subjected to prescribed traction and electric displacement at its curve surface. The boundary condition in Laplace domain can be expressed as

$$\bar{\sigma}_{zz}^{m}(r,\pm 1,s)=\bar{\sigma}_{zz}^{m-0}+\bar{\sigma}_{zz}^{m-1}+\bar{\sigma}_{zz}^{m-2}+\bar{\sigma}_{zz}^{m-3}=0 \qquad \text{for} \quad a\le r\le b \qquad (3.9a)$$

$$\bar{\sigma}_{zz}^{f}(r,\pm 1,s)=\bar{\sigma}_{zz}^{f-0}+\bar{\sigma}_{zz}^{f-1}+\bar{\sigma}_{zz}^{f-2}+\bar{\sigma}_{zz}^{f-3}=0 \qquad \text{for} \quad 0\le r\le a \qquad (3.9b)$$

$$\bar{\sigma}_{zr}^{m}(r,\pm 1,s)=\bar{\sigma}_{zr}^{1-m}+\bar{\sigma}_{zr}^{2-m}+\bar{\sigma}_{zr}^{3-m}=0 \qquad \text{for} \quad a\le r\le b \qquad (3.10a)$$

$$\bar{\sigma}_{zr}^{f}(r,\pm 1,s)=\bar{\sigma}_{zr}^{1-f}+\bar{\sigma}_{zr}^{2-f}+\bar{\sigma}_{zr}^{3-f}=0 \qquad \text{for} \quad 0\le r\le a \qquad (3.10b)$$

$$\bar{D}_{z}^{m}(r,\pm 1,s)=\bar{D}_{z}^{m-0}+\bar{D}_{z}^{m-1}+\bar{D}_{z}^{m-2}+\bar{D}_{z}^{m-3}=0 \qquad \text{for} \quad a\le r\le b \qquad (3.11a)$$

$$\bar{D}_{z}^{f}(r,\pm 1,s)=\bar{D}_{z}^{f-0}+\bar{D}_{z}^{f-1}+\bar{D}_{z}^{f-2}+\bar{D}_{z}^{f-3}=0 \qquad \text{for} \quad 0\le r\le a \qquad (3.11b)$$

$$\bar{\sigma}_{rr}^{m}\left(b,z,s\right)=\bar{\sigma}_{rr}^{m-0}+\bar{\sigma}_{rr}^{m-1}+\bar{\sigma}_{rr}^{m-2}+\bar{\sigma}_{rr}^{m-3}=\bar{P}\left(z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.12)$$

$$\bar{\sigma}_{rz}^{m}\left(b,z,s\right)=\bar{\sigma}_{rz}^{m-1}+\bar{\sigma}_{rz}^{m-2}+\bar{\sigma}_{rz}^{m-3}=\bar{V}\left(z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.13)$$

$$\bar{D}_{r}^{m}\left(b,z,s\right)=\bar{D}_{r}^{m-1}+\bar{D}_{r}^{m-2}+\bar{D}_{r}^{m-3}=\bar{D}\left(z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.14)$$

$$\bar{\sigma}_{rr}^{m}\left(a,z,s\right)=\bar{\sigma}_{rr}^{f}\left(a,z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.15a)$$

$$\bar{\sigma}_{rz}^{m}\left(a,z,s\right)=\bar{\sigma}_{rz}^{f}\left(a,z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.15b)$$

$$\bar{u}_{r}^{m}\left(a,z,s\right)=\bar{u}_{r}^{f}\left(a,z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.16a)$$

$$\bar{u}_{z}^{m}\left(a,z,s\right)=\bar{u}_{z}^{f}\left(a,z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.16b)$$

$$\bar{\varphi}^{m}\left(a,z,s\right)=\bar{\varphi}^{f}\left(a,z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.17)$$

$$\bar{D}_{r}^{m}\left(a,z,s\right)=\bar{D}_{r}^{f}\left(a,z,s\right) \qquad \text{for} \quad -1\le z\le 1 \qquad (3.18)$$

Let

$$E_{in}^{m}J_{1}\left(\xi_{in}^{m}a\right)+F_{in}^{m}Y_{1}\left(\xi_{in}^{m}a\right)=0 \tag{3.19a}$$

$$E_{in}^{m}J_{1}\left(\xi_{in}^{m}b\right)+F_{in}^{m}Y_{1}\left(\xi_{in}^{m}b\right)=0 \tag{3.19b}$$

To satisfy the eigenvalue problem of Eq. (3.19), the eigenvalue must be the same value for each i $(i=1,2,3)$ and

$$E_{in}^{m}=-\frac{Y_{1}\left(\xi_{n}^{m}a\right)}{J_{1}\left(\xi_{n}^{m}a\right)}F_{in}^{m} \tag{3.20}$$

First, consider the boundary condition in Eq. (3.10). To satisfy the boundary condition, let choose $\upsilon_{im}^{m,f}=m\pi\gamma_{i}^{m,f}$. Therefore, $\bar{\sigma}_{zr}^{m,f}(\bar{r},\pm1,s)$ can be reduced to $\hat{\bar{\sigma}}_{zr}^{m,f}=\hat{\bar{\sigma}}_{zr}^{m,f2}+\hat{\bar{\sigma}}_{zr}^{m,f3}$. Then, using Eq. (2.65) with boundary condition in Eq. (3.9a), the following equations can be obtained

$$\bar{\sigma}_{zz}^{m0}=\sum_{i=1}^{3}4\Lambda_{zzi}^{m0}A_{0i}^{m} \tag{3.21a}$$

$$\bar{\sigma}_{zz}^{m1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\Lambda_{zzmi}^{m1}\left[A_{im}^{m}I_{0}\left(m\pi\gamma_{i}^{m}r\right)+B_{im}^{m}K_{0}\left(m\pi\gamma_{i}^{m}r\right)\right]\cos\left(m\pi z\right) \tag{3.21b}$$

$$\bar{\sigma}_{zz}^{m2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\Lambda_{zzni}^{m2}\left[E_{in}^{m}J_{0}\left(\xi_{n}^{m}r\right)+F_{in}^{m}Y_{0}\left(\xi_{n}^{m}r\right)\right]\cosh\left(\frac{\xi_{n}^{m}z}{\gamma_{i}^{m}}\right) \tag{3.21c}$$

$$\begin{aligned}\bar{\sigma}_{zz}^{m3}=&\sum_{p=0}^{\infty}\Lambda_{zzp}^{mT}\left[A_{4p}^{m}I_{0}\left(\frac{\chi_{p}^{m\prime}r}{\delta^{m}}\right)+B_{4p}^{m}K_{0}\left(\frac{\chi_{p}^{m\prime}r}{\delta^{m}}\right)\right]\cos\left(p\pi z\right)\\&-\lambda_{33}^{m}\sum_{p=0}^{\infty}\left[A_{p}^{m\prime}I_{0}\left(\frac{\chi_{p}^{m\prime}r}{\delta^{m}}\right)+B_{p}^{m\prime}K_{0}\left(\frac{\chi_{p}^{m\prime}r}{\delta^{m}}\right)\right]\cos\left(p\pi z\right)\end{aligned} \tag{3.21d}$$

Where Λ_{zzi}^{m0}, Λ_{zzmi}^{m1}, Λ_{zzni}^{m2} and Λ_{zzp}^{mT} see appendix

Substituting Eq. (3.20) into Eq. (3.21c), and let

$$H_0\left(\xi_n^m r\right) = -\frac{Y_1\left(\xi_n^m a\right)}{J_1\left(\xi_n^m a\right)} J_0\left(\xi_n^m r\right) + Y_0\left(\xi_n^m r\right) \tag{3.22}$$

Eq. (3.22) then becomes

$$\bar{\sigma}_{zz}^{m2} = \sum_{n=1}^{\infty}\sum_{i=1}^{3} \Lambda_{zzni}^{m2} \cosh\left(\frac{\xi_n^m}{\gamma_i^m}\right) F_{in}^m H_0\left(\xi_n^m r\right) \tag{3.23}$$

Then, use basic function $H_0\left(\xi_n^m r\right)$ to expanding the Bessel and modified Bessel function in radial direction of matrix.

$$I_n\left(m\pi\gamma_i^m r\right) = c_{I0mi}^m + \sum_{j=1}^{\infty} c_{Ijmi}^m H_0\left(\xi_j^m r\right) \tag{3.24a}$$

$$K_n\left(m\pi\gamma_i^m r\right) = c_{K0ni}^m + \sum_{j=1}^{\infty} c_{Ijmi}^m H_0\left(\xi_j^m r\right) \tag{3.24b}$$

Where c_{I0mi}^m, c_{Ijmi}^m, c_{K0ni}^m and c_{Kjni}^m see appendix

n is order of Bessel function or modified Bessel function. (3.24g)

Then, Eq. (3.21) becomes

$$\bar{\sigma}_{zz}^{m1} = \sum_{n=1}^{\infty}\sum_{i=1}^{3}(-1)^n \Lambda_{zzni}^{m1} c_{I0ni}^{m1} A_{in}^m + \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3}(-1)^n \Lambda_{zzni}^{m1} c_{Ijni}^{m1} A_{in}^m H_0\left(\xi_j^m r\right)$$

$$+\sum_{n=1}^{\infty}\sum_{i=1}^{3}(-1)^n \Lambda_{zzni}^{m1} c_{K0ni}^{m1} B_{in}^m + \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3}(-1)^n \Lambda_{zzni}^{m1} c_{Kjni}^{m1} B_{in}^m H_0\left(\xi_j^m r\right) \tag{3.25a}$$

$$\bar{\sigma}_{zz}^{m3} = \sum_{n=0}^{\infty}\left[(-1)^n \Lambda_{zzp}^{mT} \varpi^m - \lambda_{33}^m\right] c_{I0ni}^{m3} A_n^{m\prime} + \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\left[(-1)^n \Lambda_{zzp}^{mT} \varpi^m - \lambda_{33}^m\right] c_{Ijni}^{m3} A_n^{m\prime} H_0\left(\xi_j^m r\right)$$

$$+\sum_{n=0}^{\infty}\left[\left(-1\right)^{n}\Lambda_{zzp}^{mT}\varpi^{m}-\lambda_{33}^{m}\right]c_{K0ni}^{m3}B_{n}^{m\prime}+\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\left[\left(-1\right)^{n}\Lambda_{zzp}^{mT}\varpi^{m}-\lambda_{33}^{m}\right]c_{Kjni}^{m3}B_{n}^{m\prime}H_{0}\left(\xi_{j}^{m}r\right)$$

(3.25b)

By using linearly dependent vectors, the sum of coefficient in front of vectors $H_0\left(\xi_j^m r\right)$ is must be equal to zero, yields

$$\sum_{i=1}^{3}4\Lambda_{zzi}^{m0}A_{0i}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^{n}\Lambda_{zzni}^{m1}c_{I0ni}^{m1}A_{in}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^{n}\Lambda_{zzni}^{m1}c_{K0ni}^{m1}B_{in}^{m}$$

$$+\sum_{n=0}^{\infty}\left[\left(-1\right)^{n}\Lambda_{zzp}^{mT}\varpi^{m}-\lambda_{33}^{m}\right]c_{I0ni}^{m3}A_{n}^{m\prime}+\sum_{n=0}^{\infty}\left[\left(-1\right)^{n}\Lambda_{zzp}^{mT}\varpi^{m}-\lambda_{33}^{m}\right]c_{K0ni}^{m3}B_{n}^{m\prime}=0 \qquad (3.26a)$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^{n}\Lambda_{zzni}^{m1}c_{Ijni}^{m1}A_{in}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^{n}\Lambda_{zzni}^{m1}c_{Kjni}^{m1}B_{in}^{m}+\sum_{i=1}^{3}\Lambda_{zzni}^{m2}\cosh\left(\frac{\xi_{n}^{m}}{\gamma_{i}^{m}}\right)F_{in}^{m}$$

$$+\sum_{n=0}^{\infty}\left[\left(-1\right)^{n}\Lambda_{zzp}^{mT}\varpi^{m}-\lambda_{33}^{m}\right]c_{Ijni}^{m3}A_{n}^{m\prime}+\sum_{n=0}^{\infty}\left[\left(-1\right)^{n}\Lambda_{zzp}^{mT}\varpi^{m}-\lambda_{33}^{m}\right]c_{Kjni}^{m3}B_{n}^{m\prime}=0 \qquad (3.26b)$$

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{zz}^{m}=0$ is $1+Q$.

Then, using Eq. (3.65) with boundary condition in Eq. (3.9b), the following equations can be obtained

$$\bar{\sigma}_{zz}^{f0}=\sum_{i=1}^{3}4\Lambda_{zzi}^{f0}A_{0i}^{f} \qquad (3.27a)$$

$$\bar{\sigma}_{zz}^{f1}=\sum_{i=1}^{3}\sum_{m=1}^{\infty}\Lambda_{zzmi}^{f1}A_{im}^{f}I_{0}\left(m\pi\gamma_{i}^{f}r\right)\cos\left(m\pi z\right) \qquad (3.27b)$$

$$\bar{\sigma}_{zz}^{f2}=\sum_{i=1}^{3}\sum_{n=1}^{\infty}\Lambda_{zzni}^{f2}E_{in}^{f}J_{0}\left(\xi_{n}^{f}r\right)\cosh\left(\frac{\xi_{n}^{f}z}{\gamma_{i}^{f}}\right) \qquad (3.27c)$$

$$\bar{\sigma}_{zz}^{f3} = \sum_{p=0}^{\infty} \Lambda_{zzp}^{fT} A_{4p}^{f} I_0\left(\frac{\chi_p^{f\prime} r}{\delta^f}\right)\cos\left(p\pi z\right) - \lambda_{33}^{f} \sum_{p=0}^{\infty} A_p^{f\prime} I_0\left(\frac{\chi_p^{f\prime} r}{\delta^f}\right)\cos\left(p\pi z\right) \tag{3.27d}$$

Where Λ_{zzi}^{f0}, Λ_{zzmi}^{f1}, Λ_{zzni}^{f2} and Λ_{zzp}^{fT} see appendix

Then, use basic function $J_0\left(\xi_n^f r\right)$ due to $J_1\left(\xi_n^f a\right) = 0$ for expanding the Bessel and modified Bessel function in radial direction of fiber.

$$I_n\left(m\pi\gamma_i^f r\right) = c_{I0mi}^{f} + \sum_{j=1}^{\infty} c_{Ijmi}^{f} J_0\left(\xi_j^f r\right) \tag{3.28}$$

Where c_{I0mi}^{f} and c_{Ijmi}^{f} see appendix

n is order of Bessel function or modified Bessel function. Eq. (3.27) becomes

$$\bar{\sigma}_{zz}^{f1} = \sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^n \Lambda_{zzni}^{f1} c_{I0ni}^{f1} A_{in}^{f} + \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^n \Lambda_{zzni}^{f1} c_{Ijni}^{f1} A_{in}^{f} J_0\left(\xi_j^m r\right) \tag{3.29a}$$

$$\bar{\sigma}_{zz}^{f3} = \sum_{n=0}^{\infty}\left[\left(-1\right)^n \Lambda_{zzp}^{fT}\varpi^f - \lambda_{33}^{f}\right] c_{I0ni}^{f3} A_n^{f\prime} + \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\left[\left(-1\right)^n \Lambda_{zzp}^{fT}\varpi^f - \lambda_{33}^{f}\right] c_{Ijni}^{f3} A_n^{f\prime} J_0\left(\xi_j^m r\right) \tag{3.29b}$$

By using linearly dependent vectors, the sum of coefficient in front of vectors $J_0\left(\xi_j^m r\right)$ is must be equal to zero, yields

$$\sum_{i=1}^{3} 4\Lambda_{zzi}^{f0} A_{0i}^{f} + \sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^n \Lambda_{zzni}^{f1} c_{I0ni}^{f1} A_{in}^{f} + \sum_{n=0}^{\infty}\left[\left(-1\right)^n \Lambda_{zzp}^{fT}\varpi^f - \lambda_{33}^{f}\right] c_{I0ni}^{f3} A_n^{f\prime} = 0 \tag{3.30a}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{3}\left(-1\right)^n \Lambda_{zzni}^{f1} c_{Ijni}^{f1} A_{in}^{f} + \sum_{i=1}^{3}\Lambda_{zzni}^{f2}\cosh\left(\frac{\xi_n^f}{\gamma_i^f}\right) E_{in}^{f} + \sum_{n=0}^{\infty}\left[\left(-1\right)^n \Lambda_{zzp}^{fT}\varpi^f - \lambda_{33}^{f}\right] c_{Ijni}^{f3} A_n^{f\prime} = 0 \tag{3.30b}$$

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{zz}^{f}=0$ is $1+Q$.

By following this method, the boundary conditions at both end (3.10), $\bar{\sigma}_{zr}^{m}=\bar{\sigma}_{zr}^{f}=0$ can be reduced to

$$\sum_{i=1}^{3}\Lambda_{zrni}^{m2}F_{in}^{m}=0 \tag{3.31a}$$

$$\sum_{i=1}^{3}\Lambda_{zrni}^{f2}E_{ni}^{f}=0 \tag{3.31b}$$

Where Λ_{zrni}^{m2} and Λ_{zrni}^{f2} see appendix

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{zr}^{m}=0$ is Q and $\bar{\sigma}_{zr}^{f}=0$ is Q.

The boundary conditions at both end (3.11), $\bar{D}_{z}^{m}=\bar{D}_{z}^{f}=0$ can be reduced to

$$\sum_{i=1}^{3}\Lambda_{zi}^{m0}A_{0i}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}c_{I0ni}^{m1}\Lambda_{zni}^{m1}A_{ni}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}c_{K0ni}^{m1}\Lambda_{zni}^{m1}B_{ni}^{m}+\sum_{n=0}^{\infty}c_{I0n}^{m3}\Lambda_{zn}^{m3}A_{n}^{m\prime}+\sum_{n=0}^{\infty}c_{K0n}^{m3}\Lambda_{zn}^{m3}B_{n}^{m\prime}=0 \tag{3.32a}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{3}c_{Ijni}^{m1}\Lambda_{zni}^{m1}A_{ni}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}c_{Kjni}^{m1}\Lambda_{zni}^{m1}B_{ni}^{m}+\sum_{j=1}^{3}\Lambda_{zji}^{m2}F_{ji}^{m}+\sum_{n=0}^{\infty}c_{Ijn}^{f3}\Lambda_{zn}^{m3}A_{n}^{m\prime}+\sum_{n=0}^{\infty}c_{Kjn}^{f3}\Lambda_{zn}^{m3}B_{n}^{m\prime}=0 \tag{3.32b}$$

Where Λ_{zi}^{m0}, Λ_{zni}^{m1}, Λ_{zni}^{m1}, Λ_{zrni}^{m2} and Λ_{zji}^{m3} see appendix

$$\sum_{i=1}^{3}\Lambda_{zi}^{f0}A_{0i}^{f}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}c_{I0ni}^{f1}\Lambda_{zni}^{f1}A_{ni}^{f}+\sum_{n=0}^{\infty}c_{I0n}^{f3}\Lambda_{zn}^{f3}A_{n}^{f\prime}=0 \tag{3.33a}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{3}c_{Ijni}^{f1}\Lambda_{zni}^{f1}A_{ni}^{f}+\sum_{j=1}^{3}\Lambda_{zji}^{f2}E_{ji}^{f}+\sum_{n=0}^{\infty}c_{Ijn}^{f3}\Lambda_{zn}^{f3}A_{n}^{f\prime}=0 \quad (3.33b)$$

Where Λ_{zi}^{f0}, Λ_{zni}^{f1}, Λ_{zni}^{f1}, Λ_{zji}^{f2} and Λ_{zn}^{f3} see appendix

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{D}_z^m = 0$ is $1+Q$ and $\bar{D}_z^f = 0$ is $1+Q$.

Then consider boundary conditions at outer surface (3.12) – (3.14) and at the interface (3.15) – (3.18). To formulate linear equations by using linearly independent vector method, Fourier cosine – sine series expansion in z direction is needed.

Fourier cosine series expansion

$$f(z)=\frac{f_{c0}(z)}{2}+\sum_{n=1}^{\infty}f_{cn}(z)\cos(n\pi z) \quad (3.34a)$$

Where $f_{c0}(z)=2\int_0^1 f(z)dz$ (3.34b)

$$f_{cn}(z)=2\int_0^1 f(z)\cos(n\pi z)dz \quad (3.34c)$$

Fourier sine series expansion

$$f(z)=\sum_{n=1}^{\infty}f_{sn}\sin(n\pi z) \quad (3.35a)$$

Where $f_{sn}=2\int_0^1 f(z)\sin(n\pi z)dz$ (3.35b)

Therefore, linear equations due to the boundary condition at outer surface (3.12), $\bar{\sigma}_{rr}^m = \bar{P}(z,s)$ can be expressed as

$$\sum_{i=1}^{3}\Lambda_{rri}^{m0}A_{0i}^{m}+\Lambda_{rrb}^{m0}B_{00}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}f_{c0ni}^{m2}\Lambda_{rrni}^{m2}F_{ni}^{m}+\Lambda_{rrI}^{m3}A_{0}^{m\prime}+\Lambda_{rrK}^{m3}B_{0}^{m\prime}=\frac{p_{0}^{rrm}}{2} \quad (3.36a)$$

$$\sum_{i=1}^{3}\Lambda_{Irrji}^{m1}A_{ji}^{m}+\sum_{i=1}^{3}\Lambda_{Krrji}^{m1}B_{ji}^{m}+\sum_{n=1}^{\infty}\sum_{i=1}^{3}\Lambda_{nji}^{\sigma 2\_rrmb}F_{ni}^{m}+\Lambda_{rrI}^{m3}A_{n}^{m\prime}+\Lambda_{rrK}^{m3}B_{n}^{m\prime}=p_{j}^{rrm} \quad (3.36b)$$

Where Λ_{rri}^{m0}, Λ_{rrb}^{m0}, Λ_{Irrji}^{m1}, Λ_{Krrji}^{m1}, Λ_{rrni}^{m2}, Λ_{rrI}^{m3}, Λ_{rrK}^{m3}, C_{rr}^{m1}, C_{rr}^{m2}, C_{rr}^{m3}, C_{rrb}^{m} and C_{rrb}^{m-} see appendix

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{rr}^{m}=\bar{P}(z,s)$ is $1+Q$.

The boundary condition at outer surface (3.13), $\bar{\sigma}_{rz}^{m}=\bar{V}(z,s)$ can be expressed as

$$\sum_{i=1}^{3}\Lambda_{Irzji}^{m1}A_{ji}^{m}+\sum_{i=1}^{3}\Lambda_{Krzji}^{m1}B_{ji}^{m}-\Lambda_{Irzj}^{m3}A_{j}^{m\prime}+\Lambda_{Krzj}^{m3}B_{j}^{m\prime}=v_{j}^{rrm} \quad (3.37)$$

Where Λ_{Irzji}^{m1}, Λ_{Krzji}^{m1}, Λ_{Irzj}^{m3} and Λ_{Krzj}^{m3} see appendix

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{rz}^{m}=\bar{V}(z,s)$ is Q.

The boundary condition at outer surface (3.14), $\bar{D}_{r}^{m}=\bar{D}(z,s)$ can be expressed as

$$\sum_{i=1}^{3}\Lambda_{ni}^{D1\_rmb\_I}A_{ni}^{m}+\sum_{i=1}^{3}\Lambda_{ni}^{D1\_rmb\_K}B_{ni}^{m}+\Lambda_{Irn}^{m3}A_{n}^{m\prime}+\Lambda_{Krn}^{m3}B_{n}^{m\prime}=D_{n}^{rm} \quad (3.38)$$

Where Λ_{Irji}^{m1}, Λ_{Krji}^{m1}, Λ_{Irn}^{m3} and Λ_{Krn}^{m3} see appendix.

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{D}_{r}^{m}=\bar{D}(z,s)$ is Q.

Then, the linear equations due to the boundary condition at the interface (3.15) $\bar{\sigma}_{rr}^{m} = \bar{\sigma}_{rr}^{f}$ can be expressed as

$$\sum_{i=1}^{3} \Lambda_{rri}^{m0} A_{0i}^{m} + \Lambda_{rra}^{m0} B_{00}^{m} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} f_{c0ni}^{m2} \Lambda_{rrni}^{m2} F_{ni}^{m} + \Lambda_{rrI}^{m3} A_{0}^{m\prime} + \Lambda_{rrK}^{m3} B_{0}^{m\prime}$$

$$= \Lambda_{rrI}^{f3} A_{0}^{f\prime} + \sum_{i=1}^{3} \Lambda_{rri}^{f0} A_{0i}^{f} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} f_{c0ni}^{f2} \Lambda_{rrni}^{f2} E_{ni}^{f} \tag{3.39a}$$

$$\sum_{i=1}^{3} \Lambda_{Irrji}^{m1} A_{ji}^{m} + \sum_{i=1}^{3} \Lambda_{Krrji}^{m1} B_{ji}^{m} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} \Lambda_{nji}^{\sigma 2\_rrma} F_{ni}^{m} + \Lambda_{rrI}^{m3} A_{n}^{m\prime} + \Lambda_{rrK}^{m3} B_{n}^{m\prime}$$

$$= \sum_{i=1}^{3} \Lambda_{Irrji}^{f1} A_{ji}^{f} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} \Lambda_{nji}^{\sigma 2\_rrfa} F_{ni}^{m} + \Lambda_{rrI}^{f3} A_{n}^{m\prime} \tag{3.39b}$$

Where Λ_{rri}^{m0}, Λ_{rra}^{m0}, Λ_{Irrji}^{m1}, Λ_{Krrji}^{m1}, Λ_{rrni}^{m2}, Λ_{rrI}^{m3}, Λ_{rrK}^{m3}, C_{rra}^{m}, C_{rra}^{m-}, Λ_{rri}^{f0}, Λ_{rra}^{f0}, Λ_{Irrji}^{f1},

Λ_{rrni}^{m2}, Λ_{rrI}^{f3}, C_{rr}^{f1}, C_{rr}^{f2}, C_{rr}^{f3} and C_{rra}^{f-} see appendix.

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{rr}^{m} = \bar{\sigma}_{rr}^{f}$ is $1+Q$.

$$\sum_{i=1}^{3} \Lambda_{Irzni}^{m1} A_{in}^{m} + \sum_{i=1}^{3} \Lambda_{Krzni}^{m1} B_{ni}^{m} + \Lambda_{Irzn}^{m3} A_{n}^{m\prime} + \Lambda_{Krzn}^{m3} B_{n}^{m\prime} = \sum_{i=1}^{3} \Lambda_{Irzni}^{f1} A_{in}^{f} + \Lambda_{Irzn}^{f3} A_{n}^{f\prime} \tag{3.40}$$

Where Λ_{Irzni}^{m1}, Λ_{Krzni}^{m1}, Λ_{Krzni}^{m1}, Λ_{Irzn}^{m3}, Λ_{Krzn}^{m3}, Λ_{Irzni}^{f1} and Λ_{Irzn}^{f3} see appendix.

The boundary condition at the interface (3.15) $\bar{\sigma}_{rz}^{m} = \bar{\sigma}_{rz}^{f}$ can be expressed as

$$2a\sum_{i=1}^{3} A_{0i}^{m} + \frac{1}{a} B_{00}^{m} + \Lambda_{0}^{Cu3\_rm\_I} A_{0}^{m\prime} - \Lambda_{0}^{Cu3\_rm\_K} B_{0}^{m\prime} = 2a\sum_{i=1}^{3} A_{0i}^{f} + \Lambda_{0}^{Cu3\_rf\_I} A_{0}^{f\prime} \tag{3.41a}$$

$$\sum_{i=1}^{3} \Lambda_{Iur}^{m1} A_{ni}^{m} + \sum_{i=1}^{3} \Lambda_{Kur}^{m1} B_{ni}^{m} + \Lambda_{Iurn}^{m3} A_{n}^{m\prime} + \Lambda_{Kurn}^{m3} B_{n}^{m\prime} = \sum_{i=1}^{3} \Lambda_{Iur}^{f1} A_{ni}^{f} + \Lambda_{Iurn}^{f3} A_{n}^{f\prime} \tag{3.41b}$$

Where Λ_{Iur}^{m3}, Λ_{Iur}^{f3}, Λ_{Iur}^{f3}, Λ_{Iur}^{m1}, Λ_{Kur}^{m1}, Λ_{Iur}^{f1}, Λ_{Iurn}^{m3}, Λ_{Kurn}^{m3} and Λ_{Iurn}^{f3} see appendix.

To limit range of summation, we truncate the upper limit to Q. Therefore, the number of equation for $\bar{\sigma}_{rz}^{m} = \bar{\sigma}_{rz}^{f}$ is $1+Q$.

$$\left\{\begin{array}{l} \sum_{i=1}^{3} \frac{f_{s0i}^{m0} l_{1i}^{m}}{\left(\gamma_i^m\right)^2} A_{0i}^{m} + \sum_{i=1}^{3} \frac{f_{s0i}^{f0} l_{1i}^{f}}{\left(\gamma_i^f\right)^2} A_{0i}^{f} + \sum_{i=1}^{3} \Lambda_{Iuz}^{m1} A_{ji}^{m} + \sum_{i=1}^{3} \Lambda_{Kuz}^{m1} B_{ji}^{m} \\ + \sum_{i=1}^{3} \Lambda_{Ijz}^{f1} A_{ij}^{f} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} f_{s0ni}^{m2} \Lambda_{uzji}^{m2} F_{ni}^{m} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} f_{s0ni}^{f2} \Lambda_{uzji}^{f2} E_{ni}^{f} \end{array}\right\} = \left\{\begin{array}{l} -f_{s0j}^{m3} \Lambda_{Iuzj}^{m3} A_j^{m\prime} \\ -f_{s0j}^{m3} \Lambda_{Kuzj}^{m3} B_j^{m\prime} \\ +f_{s0j}^{f3} \Lambda_{Iuzj}^{f3} A_j^{f\prime} \end{array}\right\} \qquad (3.42)$$

Where Λ_{Iuz}^{m1}, Λ_{Kuz}^{m1}, Λ_{Iuz}^{f3}, Λ_{uzji}^{m2}, Λ_{uzji}^{f2}, Λ_{Iuzj}^{m3}, Λ_{Kuzj}^{m3} and Λ_{Iuzj}^{f3} see appendix.

$$\left\{\begin{array}{l} \sum_{i=1}^{3} \frac{f_{s0i}^{m0} l_{2i}^{m}}{\left(\gamma_i^m\right)^2} A_{0i}^{m} + \sum_{i=1}^{3} \frac{f_{s0i}^{f0} l_{2i}^{f}}{\left(\gamma_i^f\right)^2} A_{0i}^{f} + \sum_{i=1}^{3} \Lambda_{I\varphi}^{m1} A_{ji}^{m} + \sum_{i=1}^{3} \Lambda_{K\varphi}^{m1} B_{ji}^{m} \\ + \sum_{i=1}^{3} \Lambda_{I\varphi ji}^{f1} A_{ij}^{f} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} f_{s0ni}^{m2} \Lambda_{\varphi ji}^{m2} F_{ni}^{m} + \sum_{n=1}^{\infty}\sum_{i=1}^{3} f_{s0ni}^{f2} \Lambda_{\varphi ji}^{f2} E_{ni}^{f} \end{array}\right\} = \left\{\begin{array}{l} -f_{s0j}^{m3} \Lambda_{I\varphi j}^{m3} A_j^{m\prime} \\ -f_{s0j}^{m3} \Lambda_{K\varphi j}^{m3} B_j^{m\prime} \\ +f_{s0j}^{f3} \Lambda_{I\varphi j}^{f3} A_j^{f\prime} \end{array}\right\} \qquad (3.43)$$

Where $\Lambda_{I\varphi}^{m1}$, $\Lambda_{K\varphi}^{m1}$, $\Lambda_{I\varphi}^{f3}$, $\Lambda_{\varphi ji}^{m2}$, $\Lambda_{\varphi ji}^{f2}$, $\Lambda_{I\varphi j}^{m3}$, $\Lambda_{K\varphi j}^{m3}$ and $\Lambda_{I\varphi j}^{f3}$ see appendix.

$$\left\{\begin{array}{l} \sum_{i=1}^{3} \Lambda_{ni}^{CD1\_rm\_I} A_{ni}^{m} + \sum_{i=1}^{3} \Lambda_{ni}^{CD1\_rm\_K} B_{ni}^{m} \\ + \sum_{i=1}^{3} \Lambda_{ni}^{CD1\_rf\_I} A_{in}^{f} \end{array}\right\} = \left\{\begin{array}{l} \Lambda_{n}^{CD3\_rm\_I} A_{4n}^{m} - \Lambda_{n}^{CD3\_rm\_K} B_{4n}^{m} \\ -\Lambda_{n}^{CD3\_rm\_I} A_{4n}^{f} \end{array}\right\} \qquad (3.44)$$

Where Λ_{IDrnia}^{m1}, Λ_{KDrnia}^{m1}, Λ_{IDrnia}^{f1}, Λ_{IDrnia}^{m3}, Λ_{KDrnia}^{m3} and Λ_{IDrnia}^{f3} see appendix.

The equation $(3.26),(3.30),(3.31)-(3.33),(3.36)-(3.44)$ are used to generate a system of linear algebraic equations of order $(7+15J)$ with arbitrary constant A_{0i}^{m}, B_{0i}^{m}, A_{0i}^{f}, A_{ni}^{m}, B_{ni}^{m}, F_{ni}^{m}, A_{ni}^{f} and E_{ni}^{m} . This system of linear algebraic equations can be solved numerically.

CHAPTER IV

NUMERICAL RESULTS

In this chapter, the result for special case of transient response of thermopiezoelectric cylinder which homogenous material is presented. A computer code has been developed to obtain the numerical results from the boundary value problems that formulated in the previous chapter. Since the problem is formulated in the Laplace domain, the transient solutions are obtained by employing a numerical Laplace inversion scheme. In this thesis, Laplace inversion scheme proposed by Stehfest (1970) is employed. The formula due to Stehfest is given by

$$f(t) \approx \frac{\log 2}{t} \sum_{n=1}^{N} c_n \tilde{f}\left(n \frac{\log 2}{t}\right) \tag{4.1a}$$

Where $c_n = (-1)^{n+N/2} \sum_{k=[(n+1)/2]}^{\min(n,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(n-k)!(2k-n)!}$ (4.2b)

and N is an even number.

4.1 Piezoelectric Cylinder under Uniform Temperature

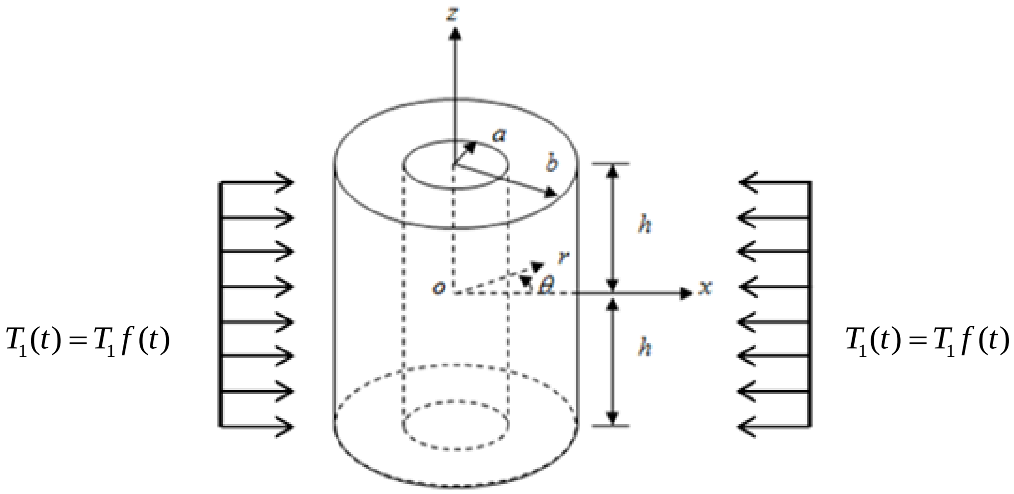

Fig 4.1 Piezoelectric cylinder subjected to zero heat flux at both ends with uniform temperature at its curve surface.

Consider a cylinder subjected to uniform temperature $T_1(t) = T_1 f(t)$ at its curve surface is shown in Fig 3.1. The inner and outer diameter of the cylinder are $a = 0.1h$, and $b = 0.4h$ respectively, with same material properties $k_1^{m,f} / k_3^{m,f} = 1$ and the thermal expansion coefficient $\alpha = 4.4x10^{-6} K^{-1}$. Let the constant positive reference temperature be $T_0 = 300K$. The complete solutions in this case are given by

$$\bar{\theta}^m = A_0^{m\prime} I_0\left(\frac{\chi_0^m r}{\delta^m}\right) + B_0^{m\prime} K_0\left(\frac{\chi_0^m r}{\delta^m}\right) \qquad \text{for} \qquad a \le r \le b \qquad (4.2a)$$

$$\bar{\theta}^f = A_0^{f\prime} I_0\left(\frac{\chi_0^f r}{\delta^f}\right) \qquad \text{for} \qquad 0 \le r \le a \qquad (4.2b)$$

where

$$A_0^{m\prime} = T_1 \bar{f}(s) \frac{\Gamma K_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^f a}{\delta^f}\right) + K_0\left(\frac{\chi_0^m a}{\delta^m}\right) I_1\left(\frac{\chi_0^f a}{\delta^f}\right)}{\Pi_1 + \Pi_2} \qquad (4.2c)$$

$$B_0^{m\prime} = T_1 \bar{f}(s) \frac{\Gamma I_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^f a}{\delta^f}\right) - I_0\left(\frac{\chi_0^m a}{\delta^m}\right) I_1\left(\frac{\chi_0^f a}{\delta^f}\right)}{\Pi_1 + \Pi_2} \qquad (4.2d)$$

$$A_0^{f\prime} = T_1 \bar{f}(s) \frac{\Gamma\left[I_1\left(\frac{\chi_0^m a}{\delta^m} \hat{a}\right) K_0\left(\frac{\chi_0^m a}{\delta^m}\right) - K_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^m a}{\delta^m}\right)\right]}{\Pi_1 + \Pi_2} \qquad (4.2e)$$

$$\Pi_1 = K_0\left(\frac{\chi_0^m b}{\delta^m}\right)\left[\Gamma I_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^f a}{\delta^f}\right) - I_0\left(\frac{\chi_0^m a}{\delta^m}\right) I_1\left(\frac{\chi_0^f a}{\delta^f}\right)\right] \qquad (4.2f)$$

$$\Pi_2 = I_0^m\left(\frac{\chi_0^m b}{\delta^m}\right)\left[\Gamma K_1\left(\frac{\chi_0^m a}{\delta^m}\right) I_0\left(\frac{\chi_0^f a}{\delta^f}\right) + K_0\left(\frac{\chi_0^m a}{\delta^m}\right) I_1\left(\frac{\chi_0^f a}{\delta^f}\right)\right] \qquad (4.2g)$$

There are two cases of the time-dependency $f(t)$ for the thermal loading considered in the numerical study as shown in Fig. 3.2.

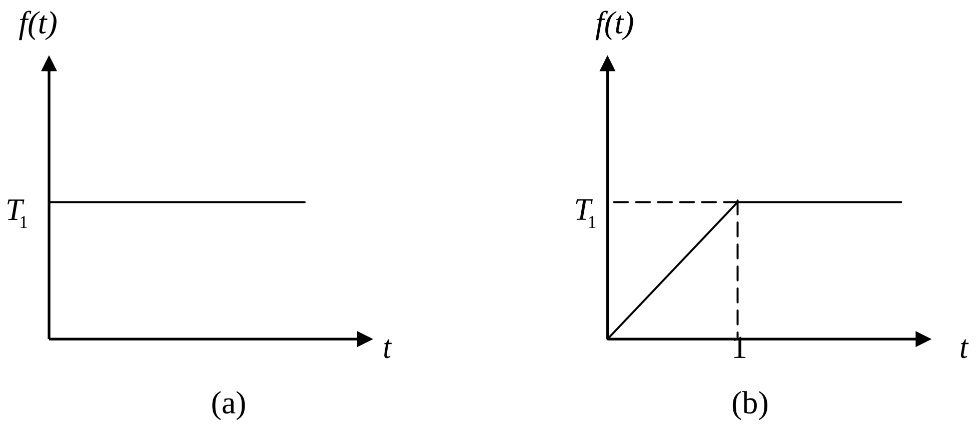

Fig 4.2 Time-dependency of thermal loading considered in the numerical study.

The Laplace transforms for the loading cases (a) and (b) are given by Eqs. (4.3) and (4.4) respectively.

$$\bar{f}_a(s) = \frac{T_1}{s} \qquad \text{for case (a)} \qquad (4.3)$$

$$\bar{f}_b(s) = \frac{\left(1-e^{-s}\right)T_1}{s^2} \qquad \text{for case (b)} \qquad (4.4)$$

Transient response of thermopiezoelectric cylinders under thermal loading is presented next. It is noted that the temperature distributions in the fiber and matrix parts given by Eqs. (4.2) are independent along the length of the cylinder (the z-direction). Therefore, only radial variation is shown in the numerical results. First, it is important to examine the convergence of the numerical Laplace inversion scheme given by Eq. (3.1). Radial profiles of nondimensional temperature change in a thermopiezocomposite cylinder for different values of N at $t = 0.1$ under thermal loading case (a) are shown in Fig. 4.3. Numerical results presented in Fig. 4.3 indicate that the converged time domain solution for this problem is obtained when N is greater than 12.

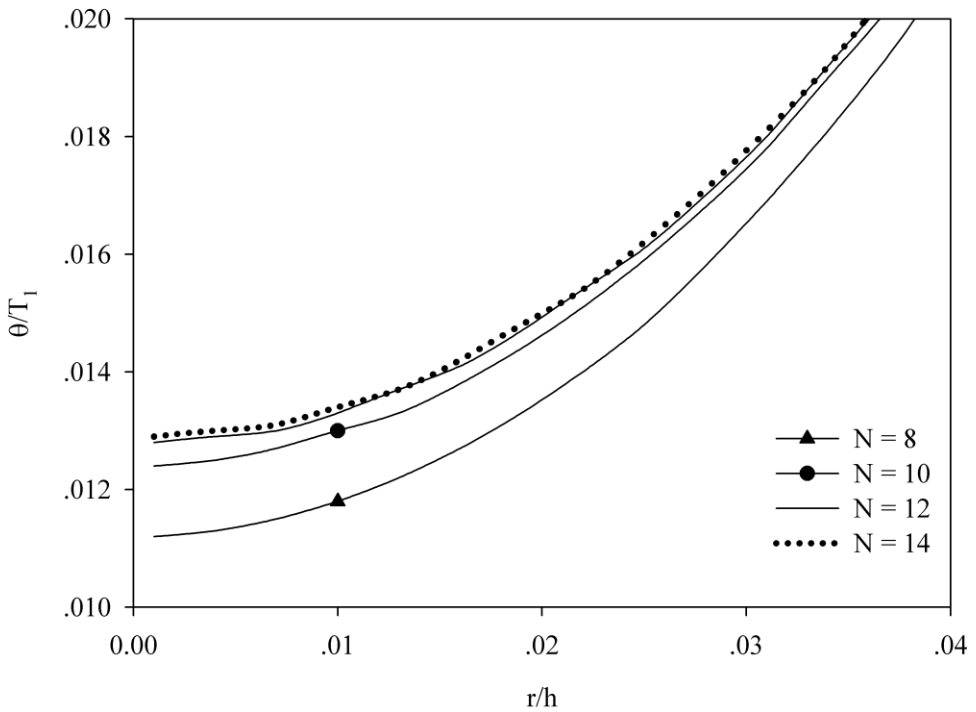

Fig 4.3 Convergence of solutions with respect to the number of term in Laplace inversion scheme (N) at $t = 0.1$.

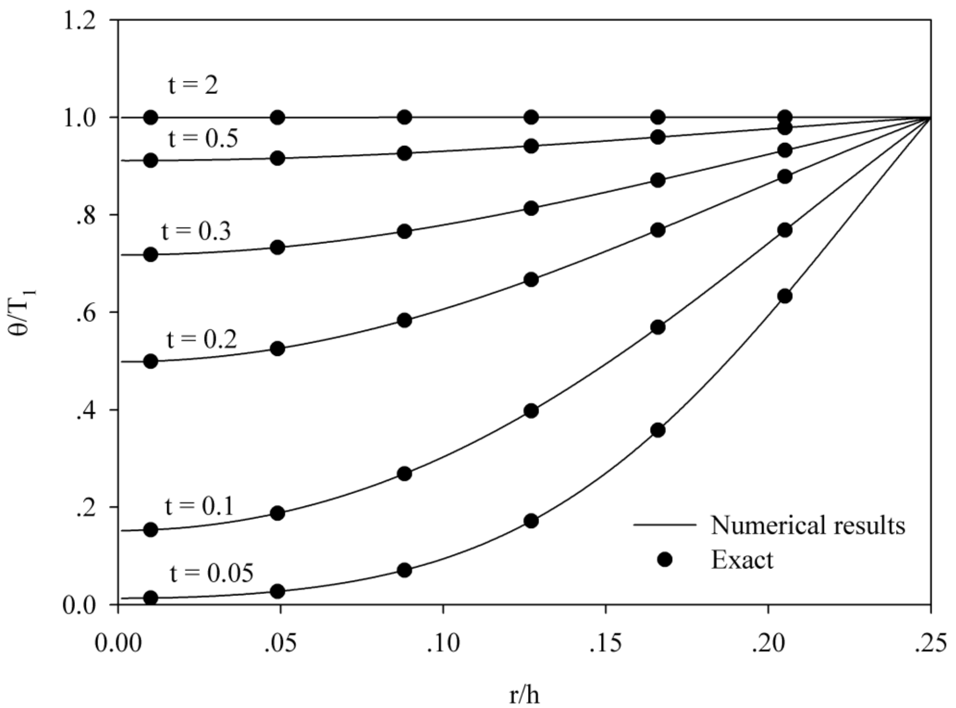

Fig 4.4 Radial profiles of transient temperature in a thermopiezocomposite solid cylinder under loading case (a).

Radial profiles of transient temperature in a thermopiezocomposite solid cylinder for different values of time are shown in Figs. 4.4. The numerical solutions for those figures are compared with the exact solution (conduction of heat in solids - carslaw and jaeger). The temperature change throughout the cylinder is nearly zero except in the vicinity of the applied temperature. Thereafter, the temperature in the cylinder gradually increases with time before reaching the steady-state temperature, which is $\theta = T_1$ throughout the cylinder.

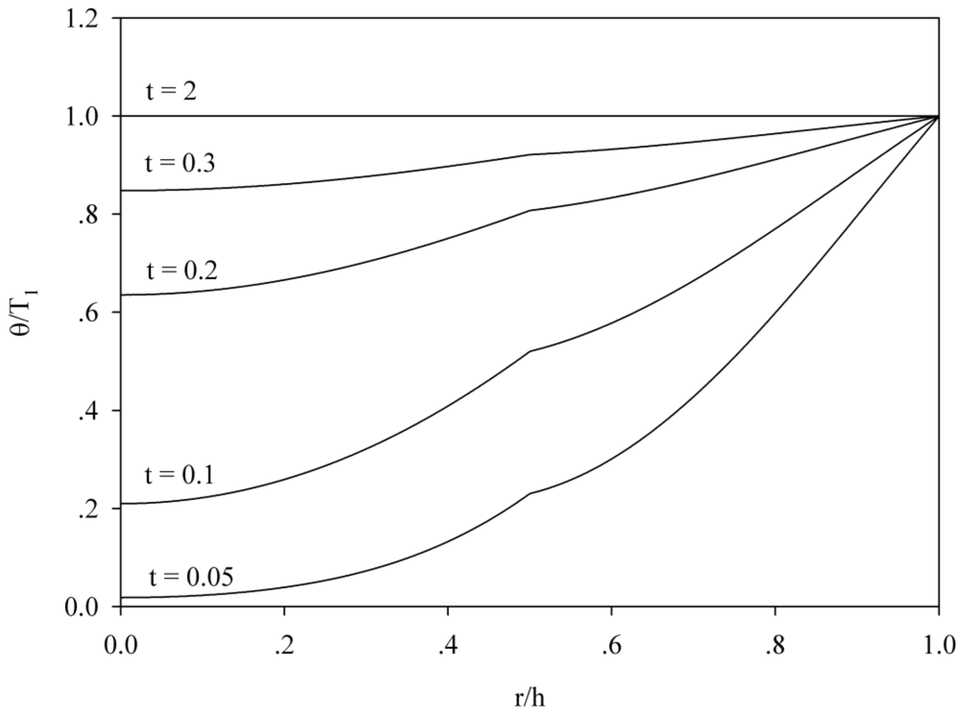

Fig 4.5 Radial profiles of transient temperature in a thermopiezocomposite composite cylinder under loading case (a).

Radial profiles of transient temperature in a thermopiezocomposite composite cylinder for different values of time are shown in Figs. 4.5. The ration of heat conduction coefficients properties between matrix and fiber is $K^m / K^f = 3$. The temperature distribution of matrix is faster than fiber because the value of heat conduction coefficient of matrix is greater than fiber. Thereafter, the temperature in the cylinder gradually increases with time before reaching the steady-state temperature, which is $\theta = T_1$ throughout the cylinder. The temperature at any value of time from thermal cylinder problem is used to input data to thermopiezoelectric problem.

4.2 Comparison Elastic Finite Solid Cylinder under Mechanical loading.

In this section, Firstly, let compare numerical results with Meleshko and Yu (2012) in case of mechanical band loading. The cylinder is elastic with $h = a$ and Poisson's ratio $\nu = 1/3$. And boundary condition at outer surface are defined by

$$P(z) = \begin{cases} -p, & |z| \le h/2 \\ 0, & h/2 < |z| \le h \end{cases} \tag{4.5}$$

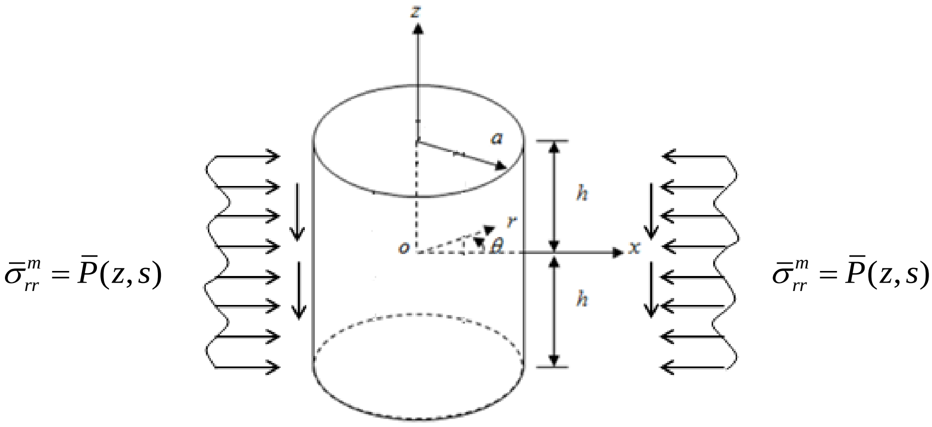

Fig 4.6 Thermopiezoelectric cylinder subjected to mechanical loading at its curve surface.

In Figs. (4.5) – (4.7) are shown the comparison of radial, axial and shear stresses along z-direction at various radius r. The number of term for converge solution at inside solid cylinder is greater than 30. But for satisfied boundary condition at outer surface, the number of them is greater than 200.

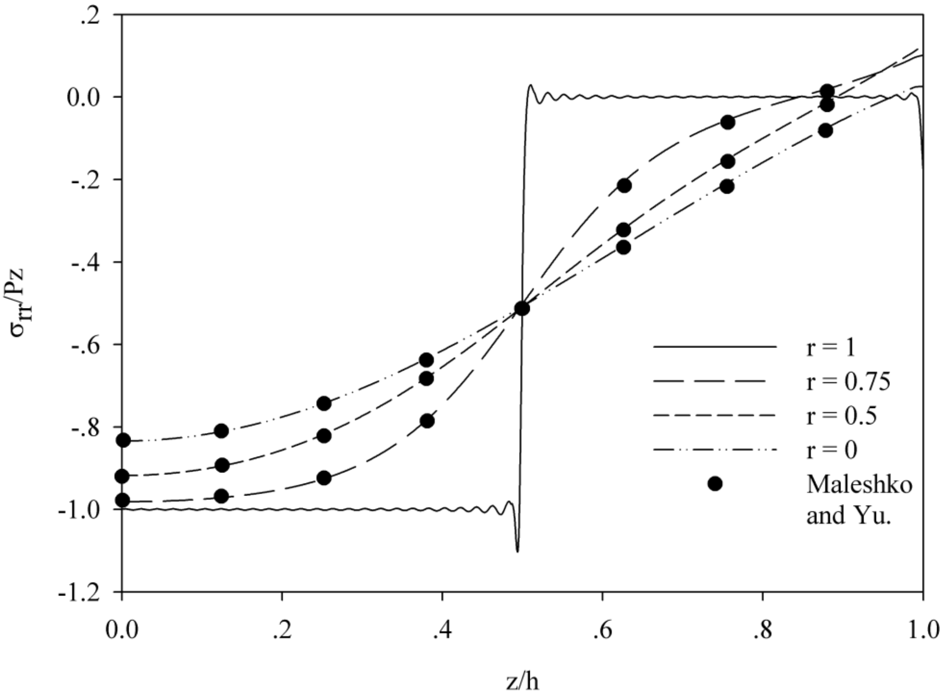

Fig 4.7 Radial stress along z-direction of elastic finite cylinder compare with Meleshko and Yu (2012).

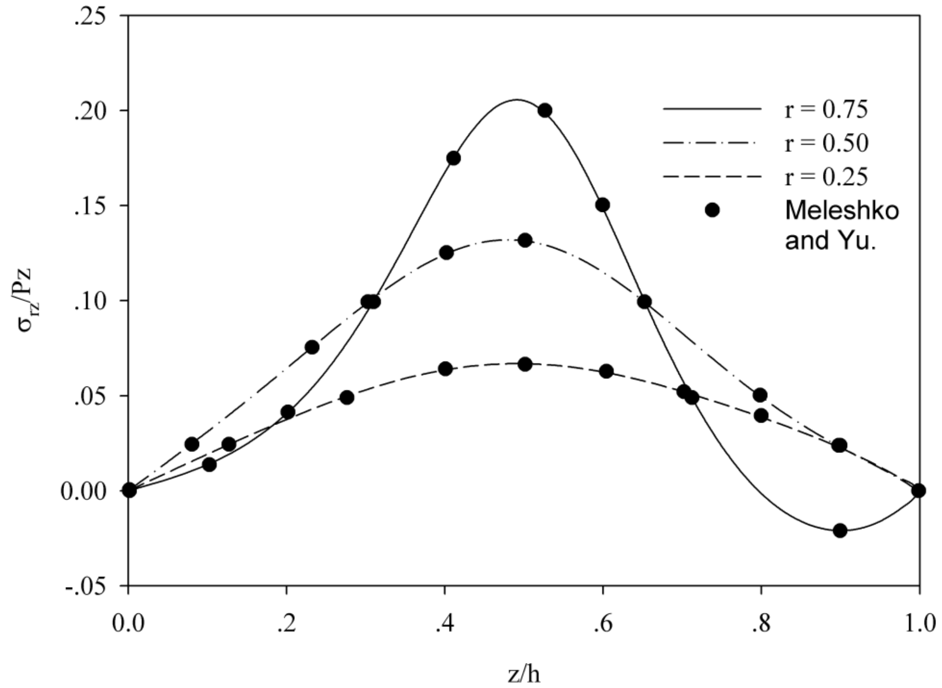

Fig 4.8 Shear stress along z-direction of elastic finite cylinder compare with Meleshko and Yu (2012).

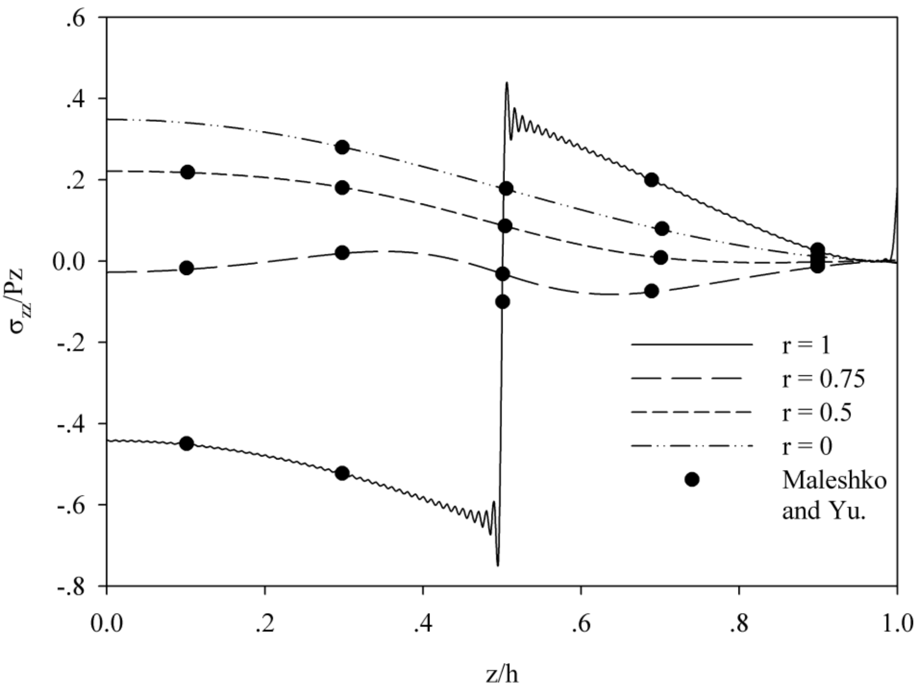

Fig 4.9 Axial stress along z-direction of elastic finite cylinder compare with Meleshko and Yu (2012).

4.3 Thermopiezoelectric Finite Solid Cylinder under Mechanical loading.

The solution of thermopiezoelectric solid cylinder is presented. This solution is used for a benchmark solution for compare with thermopiezoelectric composite cylinder. The numerical results for thermopiezoelectric finite cylinder subjected to constant mechanic and temperature loading at curve surface are presented. The thermopiezoelectric solid cylinder is composed of radius a and height $2h$. Material properties of thermopiezoelectric for this thesis are defined by

$$c_{11} = 74.1,\ c_{12} = 45.2,\ c_{13} = 39.3,\ c_{33} = 83.6,\ c_{44} = 13.2 \quad \left(10^{9}\right) N/m^{2} \tag{4.6}$$

$$e_{15} = -0.138,\ e_{31} = -0.160,\ e_{33} = 0.347 \quad C/m^{2} \tag{4.7}$$

$$\lambda_{11} = 0.621,\ \lambda_{33} = 0.551 \quad \left(10^{6}\right) N/Km^{2} \tag{4.8}$$

$$\varepsilon_{11} = 82.6,\ \varepsilon_{33} = 90.3 \quad \left(10^{-12}\right) C^2/Nm^2 \tag{4.9}$$

$$p_3 = -2.94 \quad \left(10^{-6}\right) C/N \tag{4.10}$$

In Figs. 4.8, the number of series solution (Q) for complete solution of radial stress of thermopiezoelectric solid cylinder due to temperature loading in transient state at $z = 0$ are varied. For convergence of solution, the number of series solution (Q) have to greater than 15. Therefore, for boundary of constant temperature and mechanical loading, the number of series solution (Q) is selected 15 terms.

Figs. 4.9 – 4.10 is shown the radial and axial displacement due to constant temperature. The radial and axial displacement is increasing when time is increasing in transient state. It is corresponding that the cylinder is extend when its subjected to temperature. When time is come to steady state $t = 2$, the maximum radial and axial displacement is equal to the thermal linear expansion of thermopiezoeletric material $(u_r = 0.00000425)$ and $(u_z = 0.000002749)$ respectively.

Figs 4.11 is shown the radial stress due to constant temperature. In transient state $t < 2$, the expansion of cylinder is not uniform, strain components are exist. Therefore, the stresses are exist in cylinder. When time is increasing to steady state $t = 2$, the expansion of cylinder is uniform, strain component are not exist. Therefore, the stresses in cylinder are reduced to zero.

Figs. 4.12 – 4.14 is shown the radial displacement, axial displacement and radial stress in r-direction due to constant temperature and constant mechanical loading. The effect of temperature is more decreasing when time is increasing in transient state. Because of applied temperature when its come to steady state is seem to applied permanent displacement to cylinder.

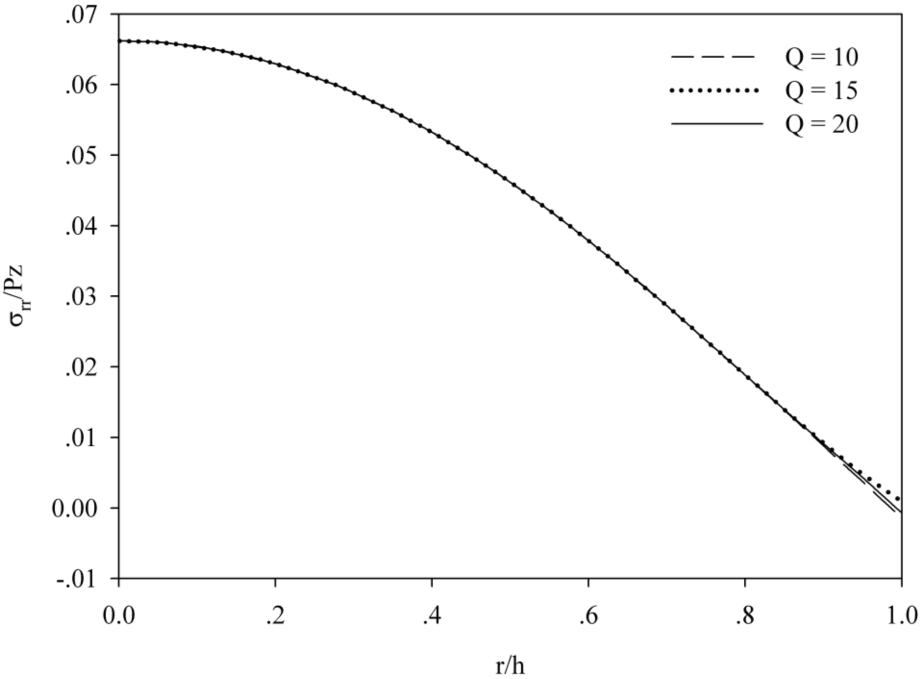

Fig 4.10 Convergence of solutions of radial stress due to constant temperature at $z=0$ with respect to the number of term in series complete solution (Q) at $t=0.1$

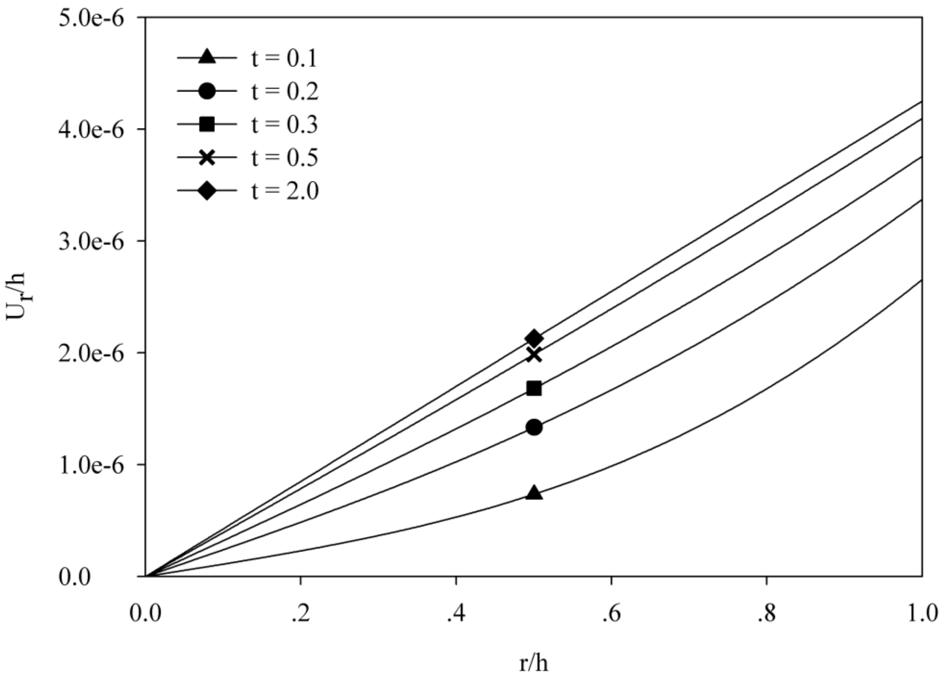

Fig 4.11 Radial displacement due to constant temperature at $z=0$ with respect to various normalize time.

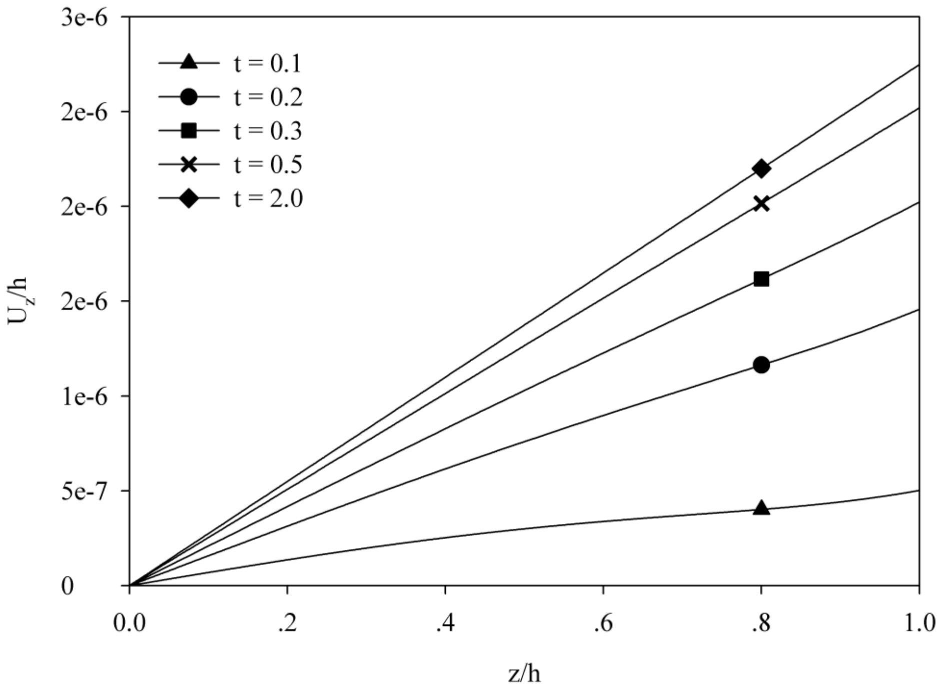

Fig 4.12 Axial displacement due to constant temperature at $r = 0$ with respect to various normalize time.

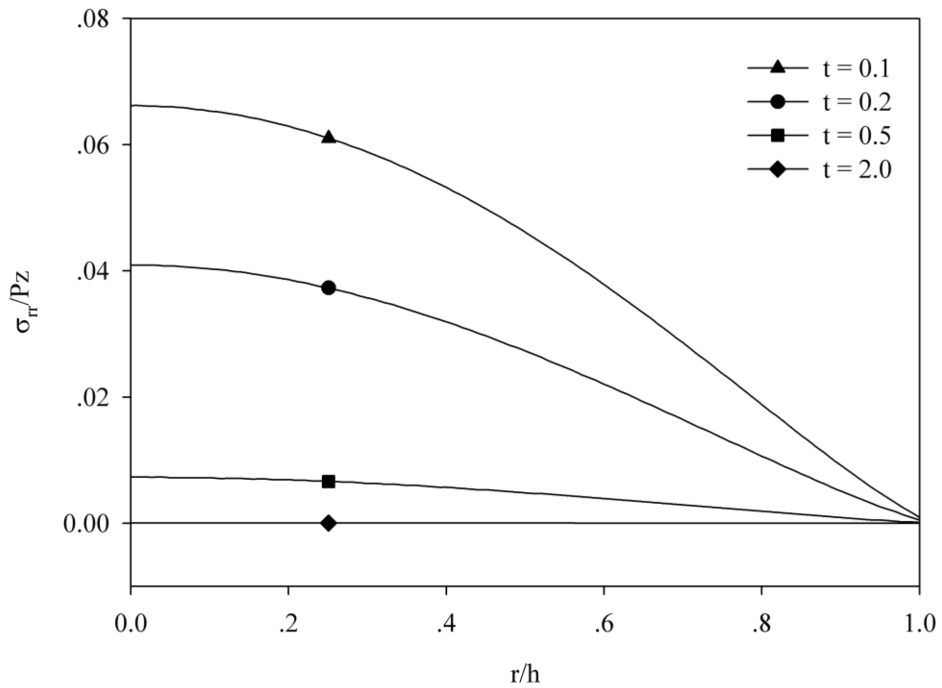

Fig 4.13 Radial stress due to constant temperature at $z = 0$ with respect to various normalize time.

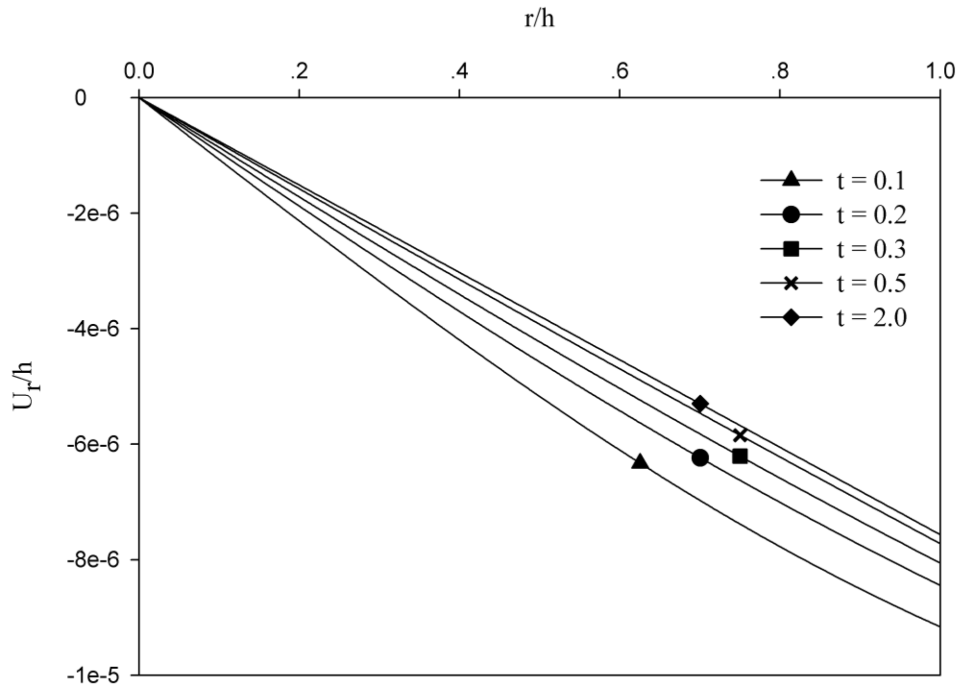

Fig 4.14 Radial displacement due to constant temperature and mechanical loading at $r = 0$ with respect to various normalize time.

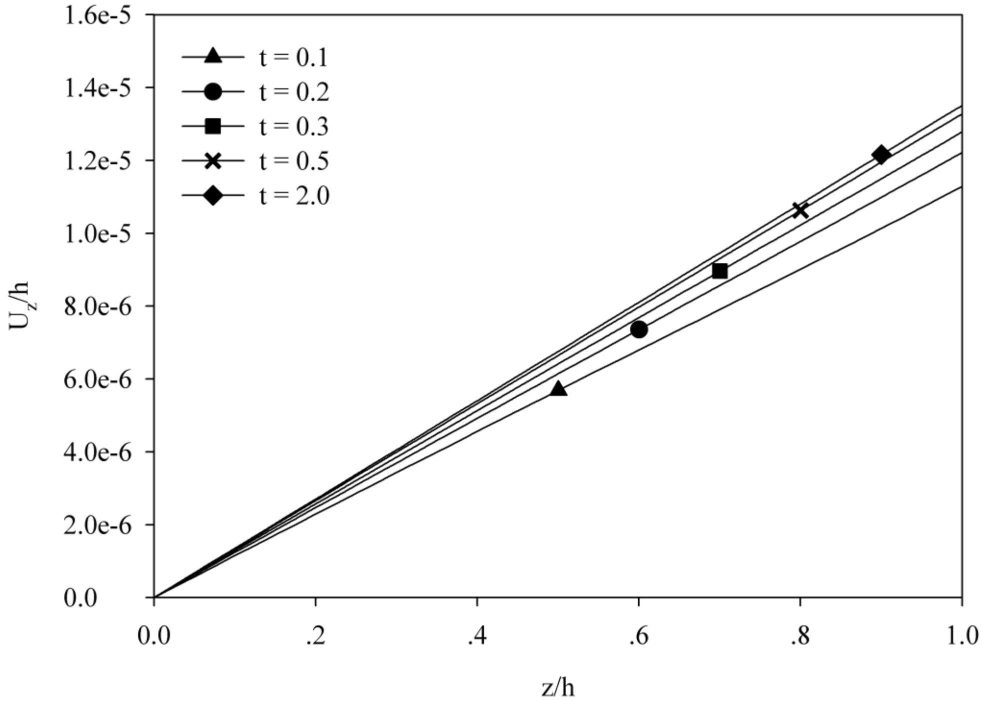

Fig 4.15 Axial displacement due to constant temperature and mechanical loading at $r = 0$ with respect to various normalize time.

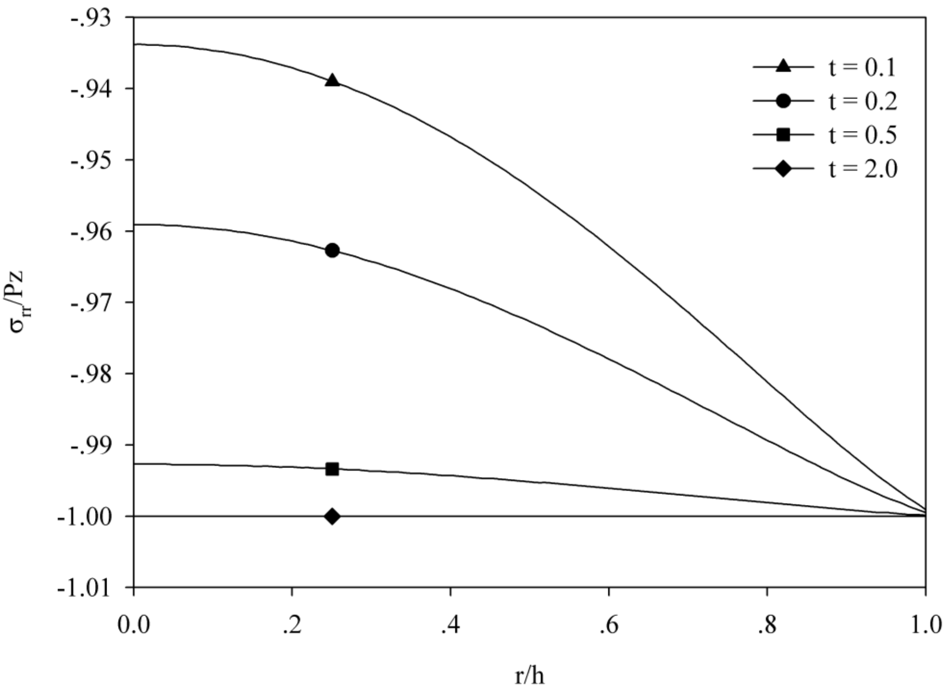

Fig 4.16 Axial displacement due to constant temperature and mechanical loading at $r = 0$ with respect to various normalize time.

4.4 Thermopiezoelectric Finite Solid Cylinder under Mechanical loading in transient state.

In this section, the numerical results for thermopiezoelectric finite solid cylinder subjected to constant temperature and band mechanical loading with same material properties as previous section are presented. The boundary condition at outer surface are defined by

$$P(z,s) = \begin{cases} -p(s), & |z| \le h_0 \\ 0, & h_0 < |z| \le h \end{cases} \tag{4.11}$$

The number of series solution (Q) in this case is greater than 50 for converge solution. Because in transient state is very sensitive, therefore the number of series solution is more required than regular case.

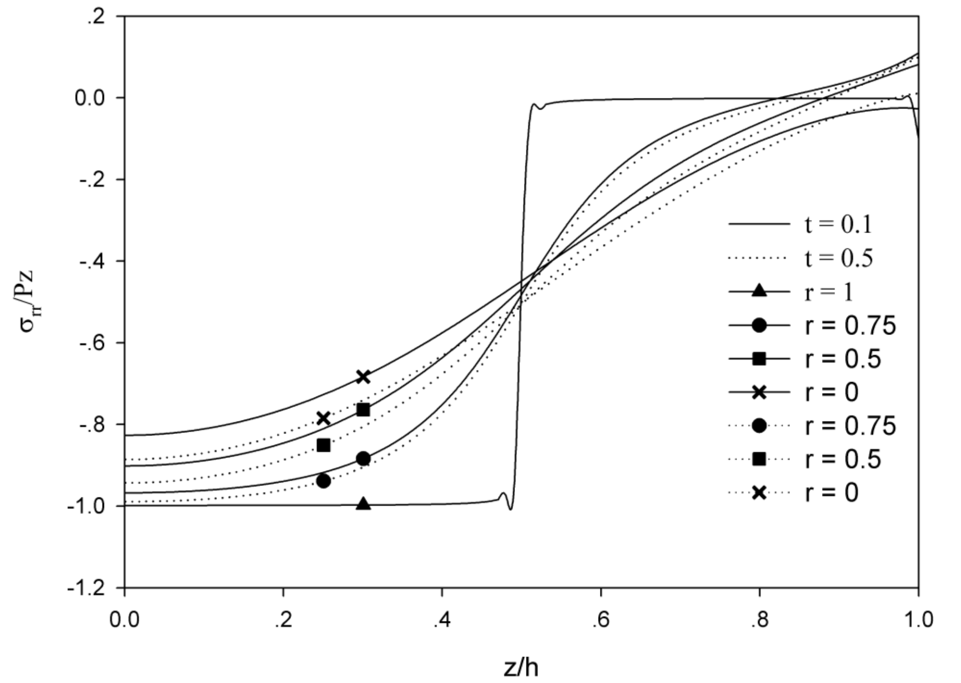

Fig 4.17 Radial stress along z-direction due to constant temperature and band mechanical loading $h_0 = h/2$ with respect to various normalize radius.

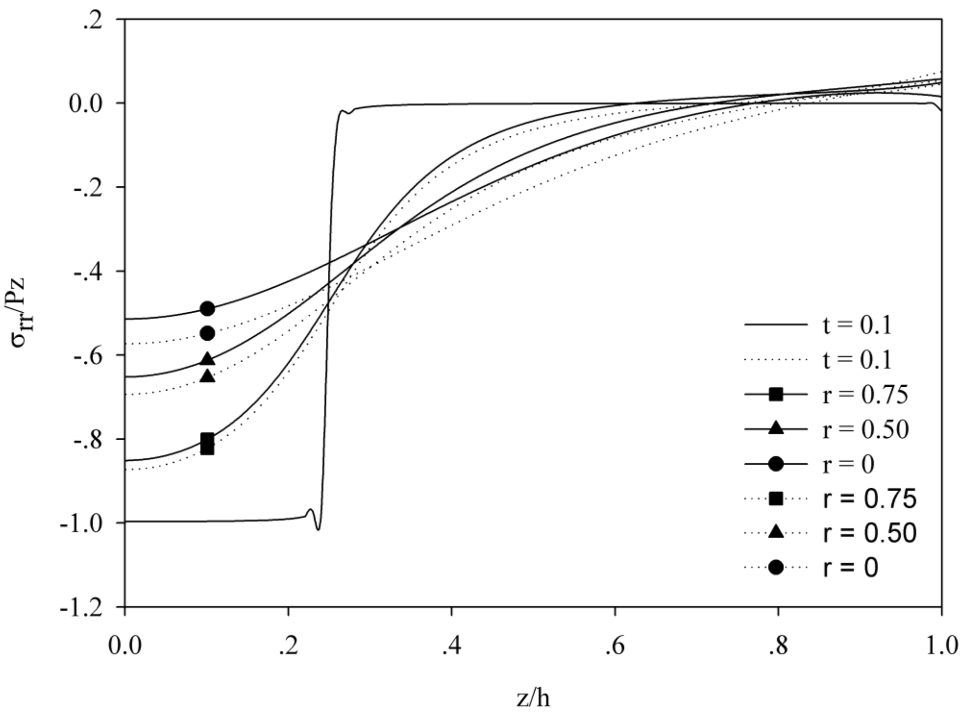

Fig 4.18 Radial stress along z-direction due to constant temperature and band mechanical loading $h_0 = h/4$ with respect to various normalize radius.

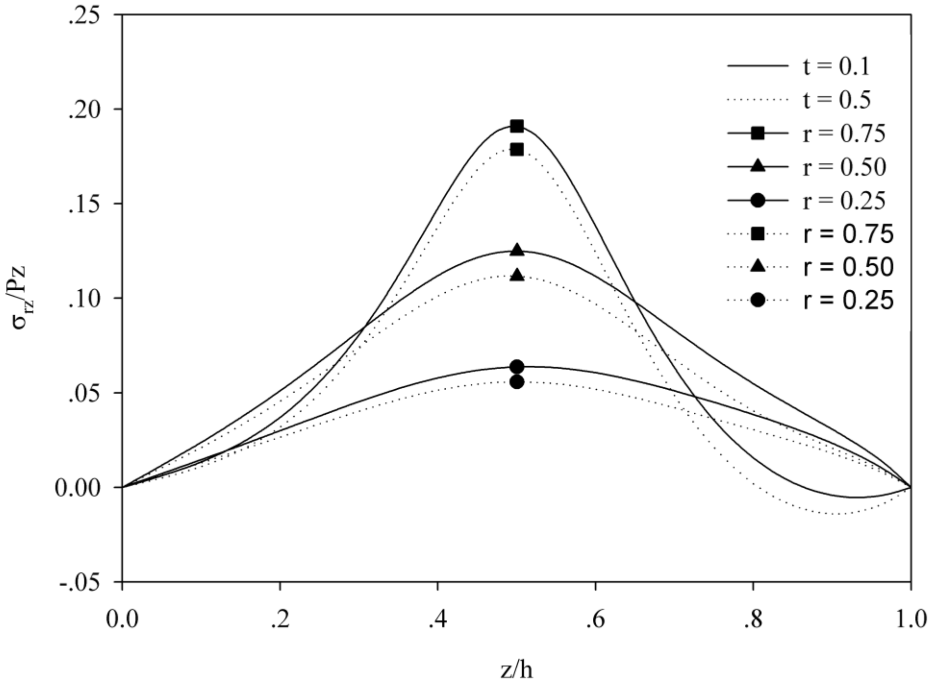

Fig 4.19 Shear stress along z-direction due to constant temperature and band mechanical loading $h_0 = h/2$ with respect to various normalize radius.

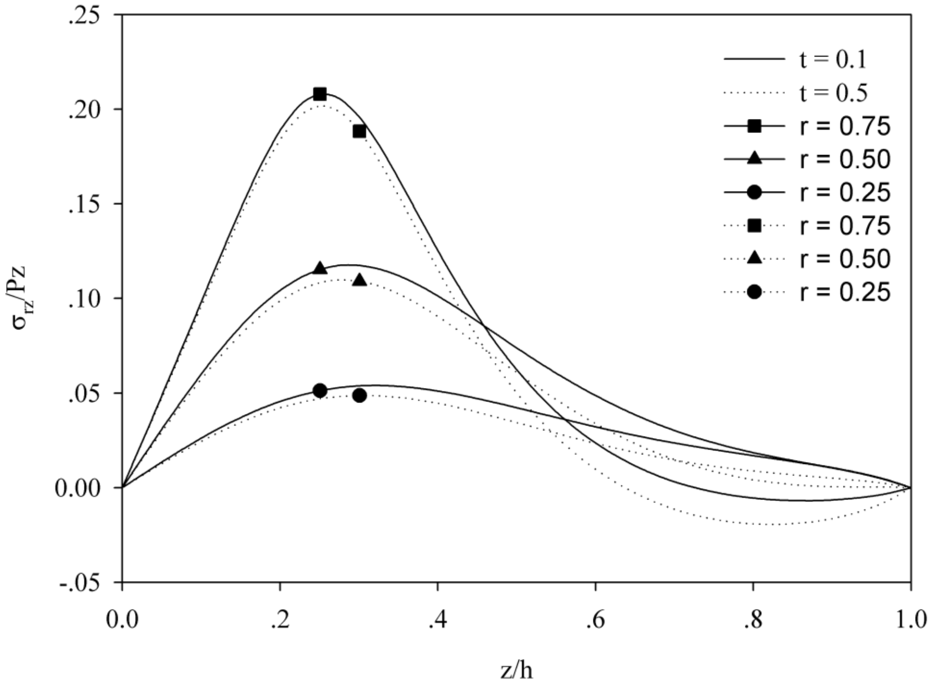

Fig 4.20 Shear stress along z-direction due to constant temperature and band mechanical loading $h_0 = h/4$ with respect to various normalize radius.

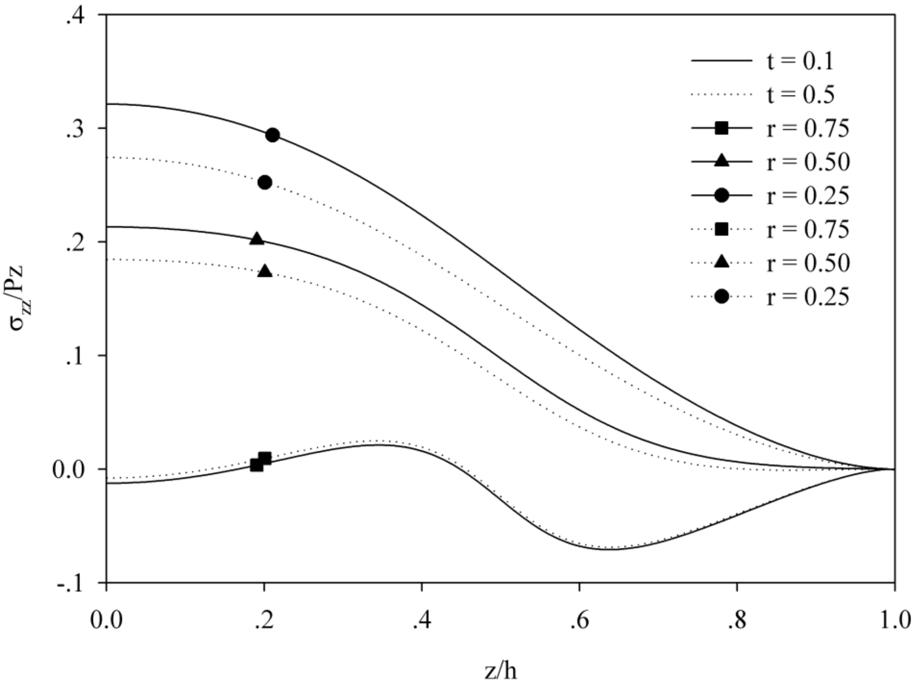

Fig 4.21 Axial stress along z-direction due to constant temperature and band mechanical loading $h_0 = h/2$ with respect to various normalize radius.

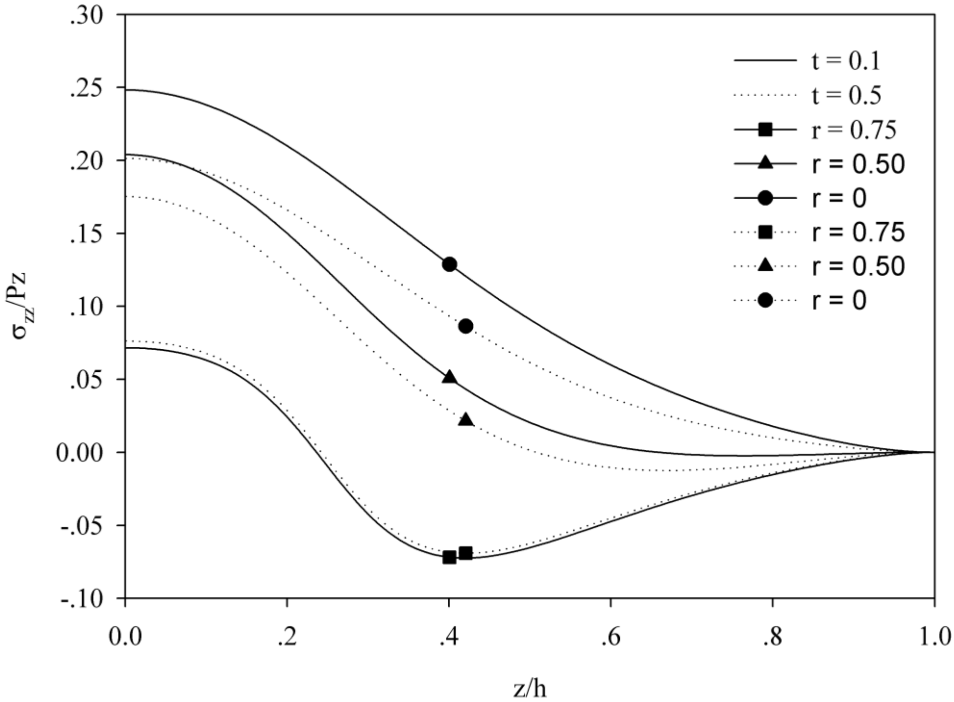

Fig 4.22 Axial stress along z-direction due to constant temperature and band mechanical loading $h_0 = h/4$ with respect to various normalize radius.

4.5 Thermopiezoelectric Finite Composite Cylinder under Mechanical loading in transient state.

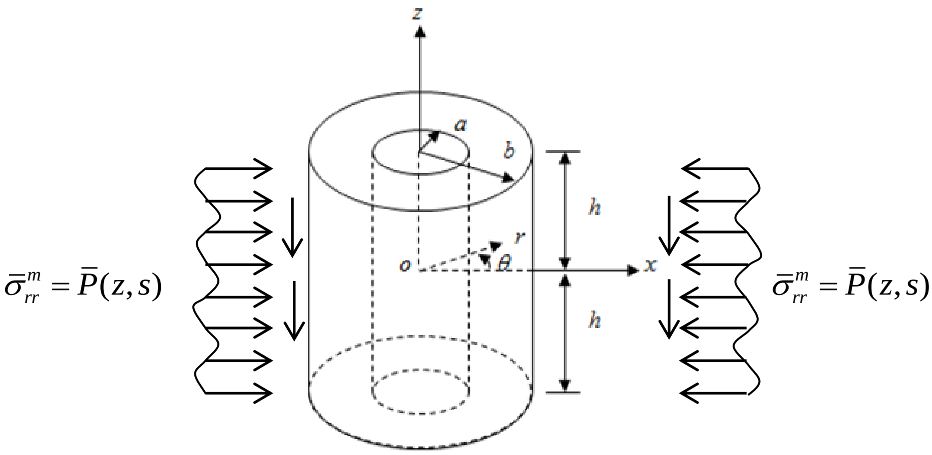

Fig 4.23 Thermopiezoelectric Finite Composite Cylinder under Mechanical loading at curve surface.

In this section, the numerical results for thermopiezoelectric finite composite cylinder subjected to constant temperature and constant mechanical loading at outer surface are presented. Material properties for both fiber and matrix are same as previous section. Therefore, the results of solid cylinder in section 4.2 are used to compare the results in this section.

The problem statement are defined by radius of fiber is $a = 0.1h$, the radius of matrix is $b = h$ and height of cylinder is $2h$. The reason for choose size of fiber equal to $a = 0.1h$ is generated domain of composite closer to single domain. The mechanic boundary conditions are assumed to be a perfectly bonded and the electric boundary conditions are assumed to be an permeable. The composite problem is more complicated than solid problem because it has an interface between two materials. Therefore, the number of series solution (Q) is more required than solid problem. For this problem, the used number of series solution is 150 terms.

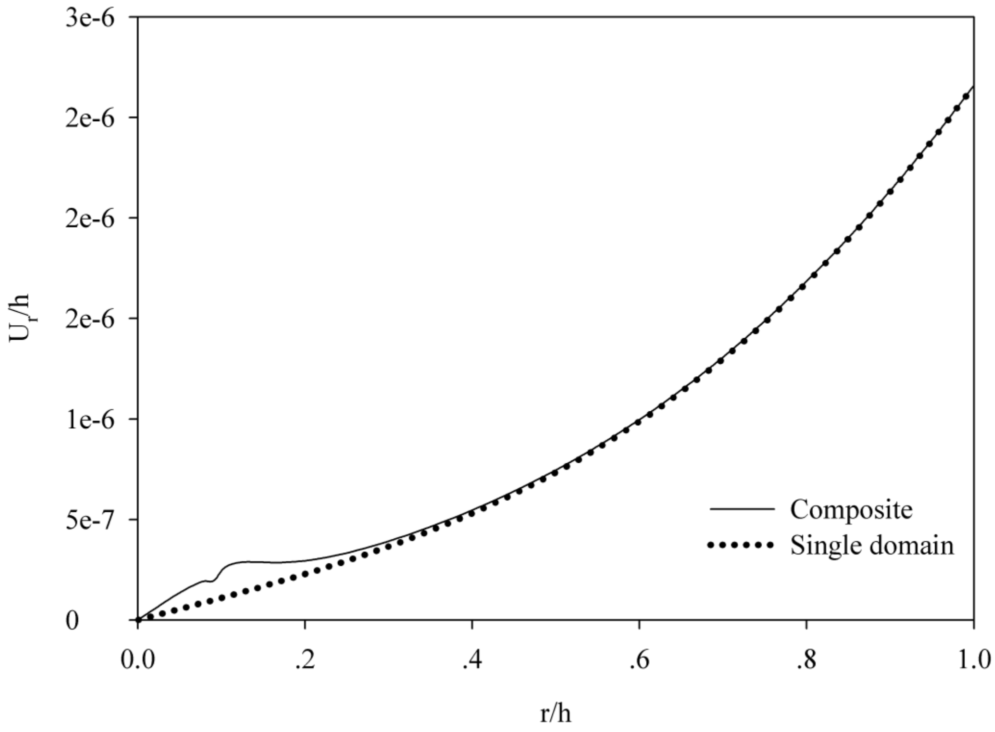

Fig 4.24 Comparison of radial displacement along r-direction between single domain and composite domain due to constant temperature at $t = 0.1$

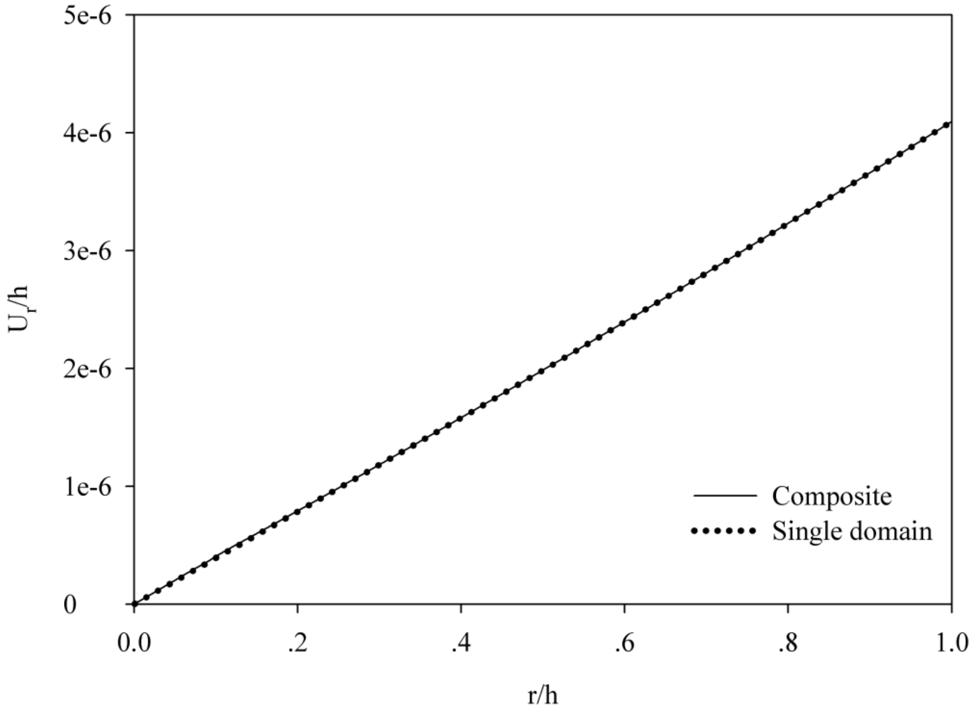

Fig 4.25 Comparison of radial displacement along r-direction between single domain and composite domain due to constant temperature at $t = 0.5$

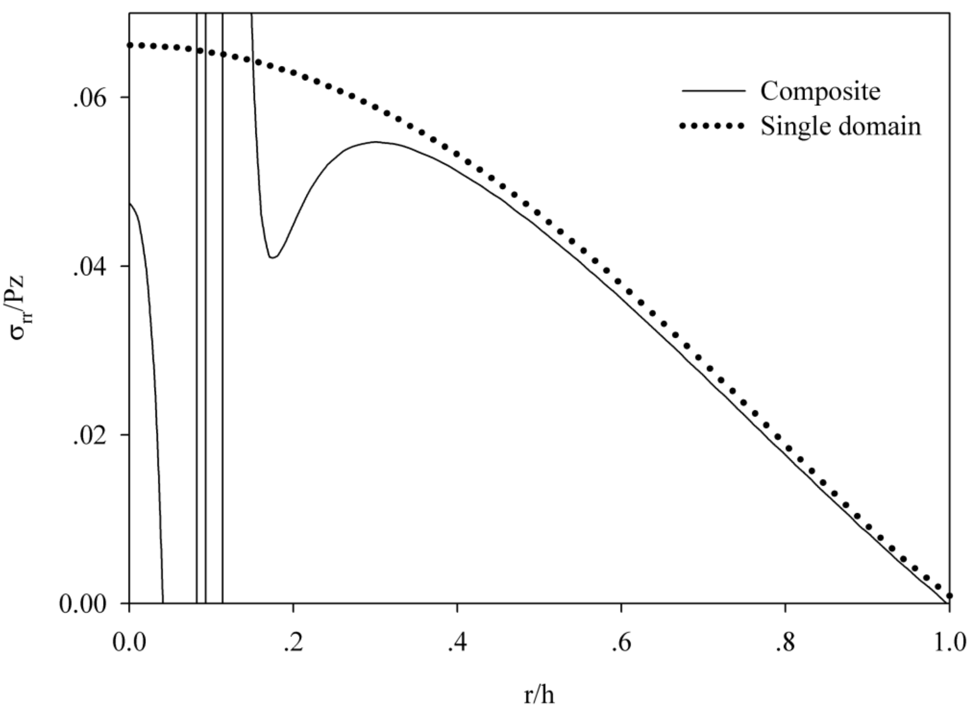

Fig 4.26 Comparison of radial displacement along r-direction between single domain and composite domain due to constant temperature at $t = 0.1$

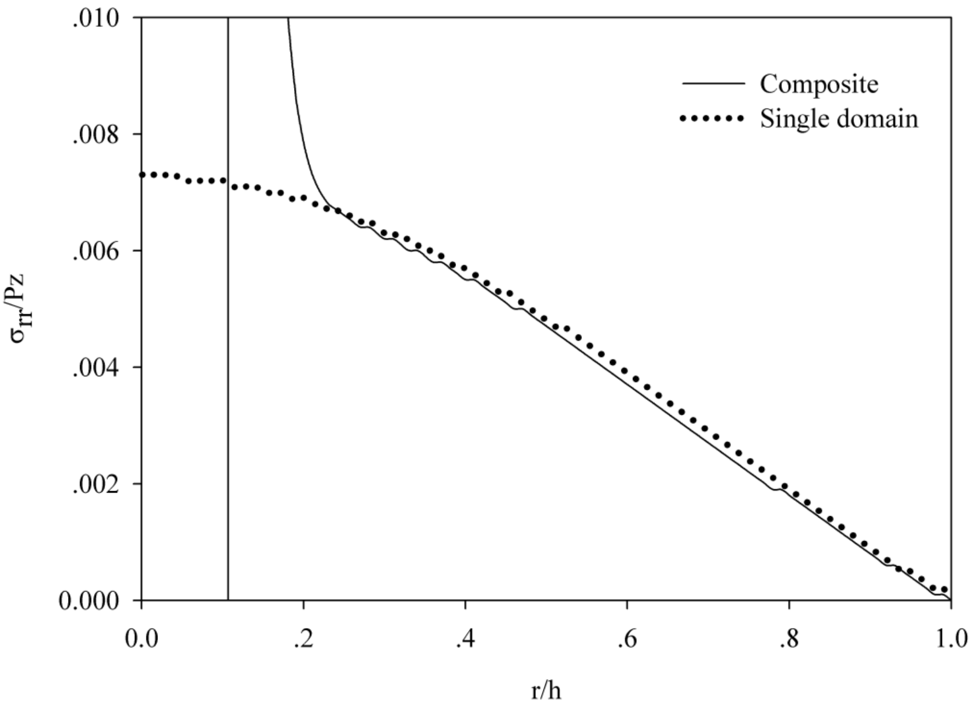

Fig 4.27 Comparison of radial displacement along r-direction between single domain and composite domain due to constant temperature at $t = 0.5$

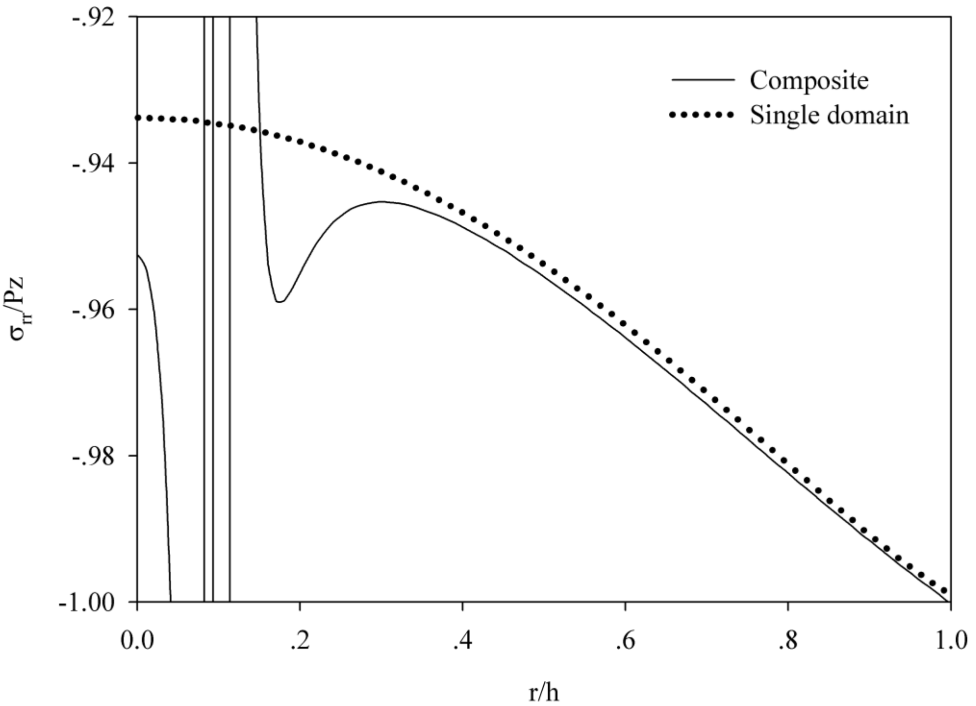

Fig 4.28 Comparison of radial stress along r-direction between single domain and composite domain due to constant temperature and constant loading at $t = 0.1$

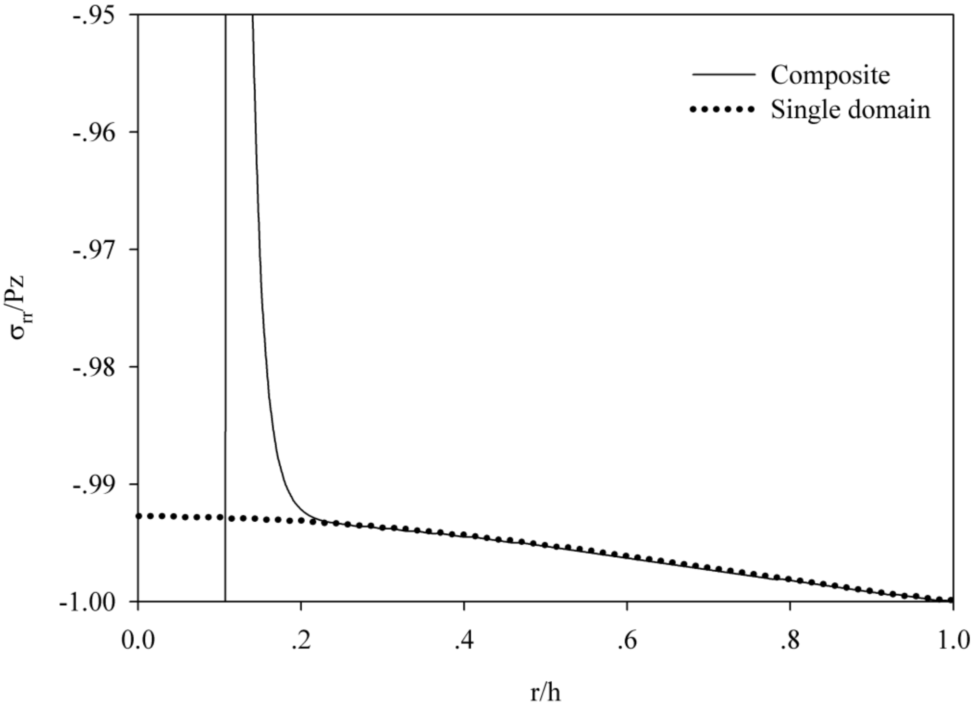

Fig 4.29 Comparison of radial displacement along r-direction between single domain and composite domain due to constant temperature and constant loading at $t = 0.5$

In Figs. 4.20 – 4.25, the results from composite domain in transient state are close or exactly the same with the results from single domain at any radius except near the interface. But at the interface, all quantities look jumping and make curve not smooth. Because the interface of fiber, it seems to be an outer surface of the cylinder in section 4.2 and the interface of matrix it seems to be and inner surface of hollow. In section 4.2, to satisfied boundary condition at outer surface, number of series solution (Q) have to greater than 200 terms. Because, at the interface, the functions that explain the behavior of thermopiezoelectric is very complicated. Therefore, number of series solutions (Q) in case of composite materials are more needed than 150 terms. However, to increase more terms, the computer program cannot get the value of coefficient because the numerical error of operation the large value in the system of linear equations. Therefore, one of the solutions to solve this problem is try to manipulate or scale the large value. When time is closer to steady state, the jumping behavior at the interface is decreasing. And the results of composite domain are same as the results of single domain.

CHAPTER V

CONCLUSIONS

5.1 Summary and Major Findings

The complete solutions for transinet response of thermopiezoelectric finite composite cylinder subjected to axisymmetric loading are presented. The temperature field are solved separately by using separation of variable method in Laplace domain. Then solved mechanic and electric filed by potential function method in Laplace domain. After matching general solution wiht boudary conditions in Laplace domain, the numerical inversion of Laplace scheme is needed. In this thesis, Gaver-Stehfest scheme is used to transform solution on Laplace domain to time domain. The number of Gaver-Stehfest term for converge is greater than 12.

For complete solution of thermopiezoelectric finite solid cylinder, the number of series solution (Q) for stress field, displacement field and are required at least 15 terms. The stress field due to temperature in transient state are less significant when the time is incresing and not producing when the time is in steady state. But for displacement field, the effect of temperature is more significant when the time incresing in transient state and constant when the time is steady state.

For complete solution of thermopiezoelectric finite composite cylinder, the number of series solution (Q) for stress field, displacement field and are required more than 150 terms for capture the bahavior at the interface. The limitation of our computer programming is not maniputale the very large value of number. Therfore, the solution at the interface still have an error in all quantitles in transient state. The solution of composite domain are almost or exactly the same with the solution of single domain when the radius is not close to the interface. For steady state, the solution of composite domain and single domain are exaclty the same at any point of cylinder. The stress field due to temperature in transient state are less significant when the time is incresing and not producing when the time is in steady state. But for displacement field, the effect of temperature is more significant when the time incresing in transient state and constant when the time is steady state.

5.2 Suggestions for Future Work

The boundary value problem focused on in the current study is restricted only to the axisymmetric boundary conditions and temperature boundary condition is not depends on z-direction. Therefore, too many choice for improve the solution for general case. For instance,

(i) To manipulate the operation of the very large number to get a correct solution at the interface.

(ii) The boundary condition of temperature field can be depends on z-direction.

(iii) The mechanic and electric boundary condition can be arbitary.

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APPENDICES

APPENDIX A

$$\Lambda_{zzi}^{m0} = \left(c_{13}^m - \frac{c_{33}^m l_{1i}^m}{\left(\gamma_i^m\right)^2} - \frac{e_{33}^m l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \tag{A1}$$

$$\Lambda_{zzmi}^{m1} = \left(m\pi\gamma_i^m\right)^2 \left(c_{13}^m - \frac{c_{33}^m l_{1i}^m}{\left(\gamma_i^m\right)^2} - \frac{e_{33}^m l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \tag{A2}$$

$$\Lambda_{zzni}^{m2} = \left(\xi_n^m\right)^2 \left(-c_{13}^m + \frac{c_{33}^m l_{1i}^m}{\left(\gamma_i^m\right)^2} + \frac{e_{33}^m l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \tag{A3}$$

$$\Lambda_{zzp}^{mT} = c_{13}^m \left(\frac{\chi_p^{m\prime}}{\delta} \right)^2 - c_{33}^m l_{14}^m \left(p\pi\right)^2 - e_{33}^m l_{24}^m \left(p\pi\right)^2 \tag{A4}$$

$$\Lambda_{zzi}^{f0} = \left(c_{13}^f - \frac{c_{33}^f l_{1i}^f}{\left(\gamma_i^f\right)^2} - \frac{e_{33}^f l_{2i}^f}{\left(\gamma_i^f\right)^2} \right) \tag{A5}$$

$$\Lambda_{zzmi}^{f1} = \left(m\pi\gamma_i^f\right)^2 \left(c_{13}^f - \frac{c_{33}^f l_{1i}^f}{\left(\gamma_i^f\right)^2} - \frac{e_{33}^f l_{2i}^f}{\left(\gamma_i^f\right)^2} \right) \tag{A6}$$

$$\Lambda_{zzni}^{f2} = \left(\xi_n^f\right)^2 \left(-c_{13}^f + \frac{c_{33}^f l_{1i}^f}{\left(\gamma_i^f\right)^2} + \frac{e_{33}^f l_{2i}^f}{\left(\gamma_i^f\right)^2} \right) \tag{A7}$$

$$\Lambda_{zzp}^{fT} = c_{13}^f \left(\frac{\chi_p^{f\prime}}{\delta} \right)^2 - c_{33}^f l_{14}^f \left(p\pi\right)^2 - e_{33}^f l_{24}^f \left(p\pi\right)^2 \tag{A8}$$

$$\Lambda_{zrni}^{m2} = -\frac{\left(\xi_n^m\right)^2 \left(c_{44}^m + c_{44}^m l_{1i}^m + e_{15}^m l_{2i}^m\right)}{\gamma_i^m} \sinh\left(\frac{\xi_n^m}{\gamma_i^m} \right) \tag{A9}$$

$$\Lambda_{zrni}^{f2} = \frac{\left(\xi_n^f\right)^2 \left(\hat{c}_{44}^f + \hat{c}_{44}^f l_{1i}^f + \hat{e}_{15}^f l_{2i}^f\right)}{\gamma_i^f} \sinh\left(\frac{\xi_n^f}{\gamma_i^f} \right) \tag{A10}$$

$$\Lambda_{zi}^{m0} = 4\left(e_{31}^m - e_{33}^m \frac{l_{1i}^m}{\left(\gamma_i^m\right)^2} + \in_{33}^m \frac{l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \tag{A11}$$

$$\Lambda_{zni}^{m1} = (-1)^n \left(n\pi\gamma_i^m\right)^2 \left(e_{31}^m - e_{33}^m \frac{l_{1i}^m}{\left(\gamma_i^m\right)^2} + \in_{33}^m \frac{l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \tag{A12}$$

$$\Lambda_{zrni}^{m2} = -\frac{\left(\xi_n^m\right)^2 \left(c_{44}^m + c_{44}^m l_{1i}^m + e_{15}^m l_{2i}^m\right)}{\gamma_i^m} \sinh\left(\frac{\xi_n^m}{\gamma_i^m}\right) \tag{A13}$$

$$\Lambda_{zi}^{f0} = 4\left(e_{31}^f - e_{33}^f \frac{l_{1i}^f}{\left(\gamma_i^f\right)^2} + \in_{33}^f \frac{l_{2i}^f}{\left(\gamma_i^f\right)^2} \right) \tag{A14}$$

$$\Lambda_{zni}^{f1} = (-1)^n \left(n\pi\gamma_i^f\right)^2 \left(e_{31}^f - e_{33}^f \frac{l_{1i}^f}{\left(\gamma_i^f\right)^2} + \in_{33}^f \frac{l_{2i}^f}{\left(\gamma_i^f\right)^2} \right) \tag{A15}$$

$$\Lambda_{zji}^{f2} = \left(\xi_j^f\right)^2 \left(-e_{31}^f + e_{33}^f \frac{l_{1i}^f}{\left(\gamma_i^f\right)^2} - \in_{33}^f \frac{l_{2i}^f}{\left(\gamma_i^f\right)^2} \right) \cosh\left(\frac{\xi_j^f}{\gamma_i^f}\right) \tag{A16}$$

$$\Lambda_{zn}^{f3} = (-1)^n \left[\left\{ e_{31}^f \left(\frac{\chi_n^{f\prime}}{\delta^f}\right) - e_{33}^f l_{14}^f (n\pi)^2 + \in_{33}^f l_{24}^f (n\pi)^2 \right\} \varpi^f + p_3^f \right] \tag{A17}$$

$$\Lambda_{zji}^{m3} = \left(\xi_j^m\right)^2 \left(-e_{31}^m + e_{33}^m \frac{l_{1i}^m}{\left(\gamma_i^m\right)^2} - \in_{33}^m \frac{l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \cosh\left(\frac{\xi_j^m}{\gamma_i^m}\right) \tag{A18}$$

$$\Lambda_{rri}^{m0} = \left(2c_{11}^m + 2c_{12}^m - 4c_{13}^m \frac{l_{1i}^m}{\left(\gamma_i^m\right)^2} - 4e_{31}^m \frac{l_{2i}^m}{\left(\gamma_i^m\right)^2} \right) \tag{A19}$$

$$\Lambda_{rrb}^{m0} = \left(-\frac{c_{11}^m}{b^2} - \frac{c_{12}^m}{b^2} \right) \tag{A20}$$

$$\Lambda_{Irrji}^{m1} = j\pi\gamma_i^m \left[j\pi\gamma_i^m C_{rr}^{m1} I_0\left(j\pi\gamma_i^m b\right) + C_{rrb}^{m-} I_1\left(j\pi\gamma_i^m b\right) \right] \tag{A21}$$

$$\Lambda_{Krrji}^{m1} = j\pi\gamma_i^m \left[j\pi\gamma_i^m C_{rr}^{m1} K_0\left(j\pi\gamma_i^m b\right) + C_{rrb}^{m} K_1\left(j\pi\gamma_i^m b\right) \right] \tag{A22}$$

$$\Lambda_{rrni}^{m2} = \frac{\xi_n^m}{2} \left\{ \left[\xi_n^m C_{rrb}^{m2} J_0\left(\xi_n^m b\right) + C_{rrb}^{m-} J_1\left(\xi_n^m b\right) \right] \left[-\frac{Y_1\left(\xi_n^m a\right)}{J_1\left(\xi_n^m a\right)} \right] + \left[\xi_n^m C_{rrb}^{m2} Y_0\left(\xi_n^m b\right) + C_{rrb}^{m-} Y_1\left(\xi_n^m b\right) \right] \right\} \tag{A23}$$

$$\Lambda_{rrI}^{m3}=\left[C_{rrb}^{m3}I_0\left(\frac{\chi_0^{m\prime}b}{\delta^m}\right)+C_{rrb}^{m-}I_1\left(\frac{\chi_0^{m\prime}b}{\delta^m}\right)\right]\varpi^m-\lambda_{11}^m \tag{A24}$$

$$\Lambda_{rrK}^{m3}=\left[C_{rrb}^{m3}K_0\left(\frac{\chi_0^{m\prime}b}{\delta^m}\right)+C_{rrb}^{m}K_1\left(\frac{\chi_0^{m\prime}b}{\delta^m}\right)\right]\varpi^m-\lambda_{11}^m \tag{A25}$$

$$C_{rr}^{m1}=\left(c_{11}^m-c_{13}^m\frac{l_{1i}^m}{\left(\gamma_i^m\right)^2}-e_{31}^m\frac{l_{2i}^m}{\left(\gamma_i^m\right)^2}\right) \tag{A26}$$

$$C_{rr}^{m2}=\left(-c_{11}^m+c_{13}^m\frac{l_{1i}^m}{\left(\gamma_i^m\right)^2}+e_{31}^m\frac{l_{2i}^m}{\left(\gamma_i^m\right)^2}\right) \tag{A27}$$

$$C_{rr}^{m3}=\left[c_{11}^m\left(\frac{\chi_p^{m\prime}}{\delta^m}\right)^2-c_{13}^m l_{14}^m\left(p\pi\right)^2-e_{31}^m l_{24}^m\left(p\pi\right)^2\right] \tag{A28}$$

$$C_{rrb}^{m}=\left(\frac{c_{11}^m}{b}-\frac{c_{12}^m}{b}\right) \tag{A29}$$

$$C_{rrb}^{m-}=\left(-\frac{c_{11}^m}{b}+\frac{c_{12}^m}{b}\right) \tag{A30}$$

$$\Lambda_{Irzji}^{m1}=-\left(n\pi\right)^2\gamma_i^m\left(c_{44}^m+c_{44}^m l_{1i}^m+e_{15}^m l_{2i}^m\right)I_1\left(n\pi\gamma_i^m b\right) \tag{A31}$$

$$\Lambda_{Krzji}^{m1}=\left(n\pi\right)^2\gamma_i^m\left(c_{44}^m+c_{44}^m l_{1i}^m+e_{15}^m l_{2i}^m\right)K_1\left(n\pi\gamma_i^m b\right) \tag{A32}$$

$$\Lambda_{Irzj}^{m3}=\frac{\chi_j^{m\prime}j\pi\varpi^m}{\delta^m}\left(c_{44}^m+c_{44}^m l_{14}^m+e_{15}^m l_{24}^m\right)I_1\left(\frac{\chi_j^{m\prime}b}{\delta^m}\right) \tag{A33}$$

$$\Lambda_{Irji}^{m1}=-\left(n\pi\right)^2\gamma_i^m\left(e_{15}^m+e_{15}^m l_{1i}^m-\in_{11}^m l_{2i}^m\right)I_1\left(n\pi\gamma_i^m b\right) \tag{A34}$$

$$\Lambda_{Krji}^{m1}=\left(n\pi\right)^2\gamma_i^m\left(e_{15}^m+e_{15}^m l_{1i}^m-\in_{11}^m l_{2i}^m\right)K_1\left(n\pi\gamma_i^m b\right) \tag{A35}$$

$$\Lambda_{Irn}^{m3}=\frac{\chi_n^{m\prime}n\pi\varpi^m}{\delta^m}\left(e_{15}^m+e_{15}^m l_{14}^m-\in_{11}^m l_{24}^m\right)I_1\left(\frac{\chi_n^{m\prime}b}{\delta^m}\right) \tag{A36}$$

$$\Lambda_{Krn}^{m3}=\frac{\chi_n^{m\prime}n\pi\varpi^m}{\delta^m}\left(e_{15}^m+e_{15}^m l_{14}^m-\in_{11}^m l_{24}^m\right)K_1\left(\frac{\chi_n^{m\prime}b}{\delta^m}\right) \tag{A37}$$

$$\Lambda_{Krn}^{m3} = \frac{\chi_n^{m\prime} n\pi\varpi^m}{\delta^m}\left(e_{15}^m + e_{15}^m l_{14}^m - \in_{11}^m l_{24}^m\right)K_1\left(\frac{\chi_n^{m\prime} b}{\delta^m}\right) \tag{A38}$$

$$\Lambda_{Krzj}^{m3} = \frac{\chi_j^{m\prime} j\pi\varpi^m}{\delta^m}\left(c_{44}^m + c_{44}^m l_{14}^m + e_{15}^m l_{24}^m\right)K_1\left(\frac{\chi_j^{m\prime} b}{\delta^m}\right) \tag{A39}$$

$$\Lambda_{rri}^{m0} = \left(2c_{11}^m + 2c_{12}^m - 4c_{13}^m \frac{l_{1i}^m}{\left(\gamma_i^m\right)^2} - 4e_{31}^m \frac{l_{2i}^m}{\left(\gamma_i^m\right)^2}\right) \tag{A40}$$

$$\Lambda_{rra}^{m0} = \left(-\frac{c_{11}^m}{a^2} - \frac{c_{12}^m}{a^2}\right) \tag{A41}$$

$$\Lambda_{Irrji}^{m1} = j\pi\gamma_i^m\left[j\pi\gamma_i^m C_{rr}^{m1} I_0\left(j\pi\gamma_i^m a\right) + C_{rrb}^{m-} I_1\left(j\pi\gamma_i^m a\right)\right] \tag{A42}$$

$$\Lambda_{Krrji}^{m1} = j\pi\gamma_i^m\left[j\pi\gamma_i^m C_{rr}^{m1} K_0\left(j\pi\gamma_i^m a\right) + C_{rrb}^{m} K_1\left(j\pi\gamma_i^m a\right)\right] \tag{A43}$$

$$\Lambda_{rrni}^{m2} = \frac{\xi_n^m}{2}\left\{\left[\xi_n^m C_{rrb}^{m2} J_0\left(\xi_n^m a\right) + C_{rrb}^{m-} J_1\left(\xi_n^m a\right)\right]\left[-\frac{Y_1\left(\xi_n^m a\right)}{J_1\left(\xi_n^m a\right)}\right] + \left[\xi_n^m C_{rrb}^{m2} Y_0\left(\xi_n^m a\right) + C_{rrb}^{m-} Y_1\left(\xi_n^m a\right)\right]\right\} \tag{A44}$$

$$\Lambda_{rrI}^{m3} = \left[C_{rrb}^{m3} I_0\left(\frac{\chi_0^{m\prime} a}{\delta^m}\right) + C_{rrb}^{m-} I_1\left(\frac{\chi_0^{m\prime} a}{\delta^m}\right)\right]\varpi^m - \lambda_{11}^m \tag{A45}$$

$$\Lambda_{rrK}^{m3} = \left[C_{rrb}^{m3} K_0\left(\frac{\chi_0^{m\prime} a}{\delta^m}\right) + C_{rrb}^{m} K_1\left(\frac{\chi_0^{m\prime} a}{\delta^m}\right)\right]\varpi^m - \lambda_{11}^m \tag{A46}$$

$$C_{rra}^{m} = \left(\frac{c_{11}^m}{a} - \frac{c_{12}^m}{a}\right) \tag{A47}$$

$$C_{rra}^{m-} = \left(-\frac{c_{11}^m}{a} + \frac{c_{12}^m}{a}\right) \tag{A48}$$

$$\Lambda_{rri}^{f0} = \left(2c_{11}^f + 2c_{12}^f - 4c_{13}^f \frac{l_{1i}^f}{\left(\gamma_i^f\right)^2} - 4e_{31}^f \frac{l_{2i}^f}{\left(\gamma_i^f\right)^2}\right) \tag{A49}$$

$$\Lambda_{rra}^{f0} = \left(-\frac{c_{11}^f}{a^2} - \frac{c_{12}^f}{a^2}\right) \tag{A50}$$

$$\Lambda_{Irrji}^{f1} = j\pi\gamma_i^f\left[j\pi\gamma_i^f C_{rr}^{f1} I_0\left(j\pi\gamma_i^f a\right) + C_{rrb}^{f-} I_1\left(j\pi\gamma_i^f a\right)\right] \tag{A51}$$

$$\Lambda_{rrni}^{m2} = \frac{\xi_n^m}{2}\left[\xi_n^m C_{rrb}^{m2} J_0\left(\xi_n^m b\right) + C_{rrb}^{m-} J_1\left(\xi_n^m b\right)\right] \tag{A52}$$

$$\Lambda_{rrI}^{f3} = \left[C_{rrb}^{f3} I_0\left(\frac{\chi_0^{f\prime} a}{\delta^f}\right) + C_{rrb}^{f-} I_1\left(\frac{\chi_0^{f\prime} a}{\delta^f}\right)\right]\varpi^f - \lambda_{11}^f \tag{A53}$$

$$C_{rr}^{f1} = \left(c_{11}^f - c_{13}^f \frac{l_{1i}^f}{\left(\gamma_i^f\right)^2} - e_{31}^f \frac{l_{2i}^f}{\left(\gamma_i^f\right)^2}\right) \tag{A54}$$

$$C_{rr}^{f2} = \left(-c_{11}^f + c_{13}^f \frac{l_{1i}^f}{\left(\gamma_i^f\right)^2} + e_{31}^f \frac{l_{2i}^f}{\left(\gamma_i^f\right)^2}\right) \tag{A55}$$

$$C_{rr}^{f3} = \left(c_{11}^f \left(\frac{\chi_p^{f\prime}}{\delta^f}\right)^2 - c_{13}^f l_{14}^f \left(p\pi\right)^2 - e_{31}^f l_{24}^f \left(p\pi\right)^2\right) \tag{A56}$$

$$C_{rra}^{f-} = \left(-\frac{c_{11}^f}{a} + \frac{c_{12}^f}{a}\right) \tag{A57}$$

$$\Lambda_{Irzni}^{m1} = -\left(n\pi\right)^2 \gamma_i^m \left(c_{44}^m + c_{44}^m l_{1i}^m + e_{15}^m l_{2i}^m\right) I_1\left(n\pi\gamma_i^m a\right) \tag{A58}$$

$$\Lambda_{Krzni}^{m1} = \left(n\pi\right)^2 \gamma_i^m \left(c_{44}^m + c_{44}^m l_{1i}^m + e_{15}^m l_{2i}^m\right) K_1\left(n\pi\gamma_i^m a\right) \tag{A59}$$

$$\Lambda_{Irzn}^{m3} = -\frac{\chi_n^{m\prime} n\pi\varpi^m}{\delta^m}\left(c_{44}^m + c_{44}^m l_{14}^m + e_{15}^m l_{24}^m\right) I_1\left(\frac{\chi_n^{m\prime} a}{\delta^m}\right) \tag{A60}$$

$$\Lambda_{Krzn}^{m3} = \frac{\chi_n^{m\prime} n\pi\varpi^m}{\delta^m}\left(c_{44}^m + c_{44}^m l_{14}^m + e_{15}^m l_{24}^m\right) K_1\left(\frac{\chi_n^{m\prime} a}{\delta^m}\right) \tag{A61}$$

$$\Lambda_{Irzni}^{f1} = \left(n\pi\right)^2 \gamma_i^f \left(c_{44}^f + c_{44}^f l_{1i}^f + e_{15}^f l_{2i}^f\right) I_1\left(n\pi\gamma_i^f a\right) \tag{A62}$$

$$\Lambda_{Iur}^{m3} = \frac{\chi_0^{m\prime}\varpi^m}{\delta^m} I_1\left(\frac{\chi_0^{m\prime} a}{\delta^m}\right) \tag{A63}$$

$$\Lambda_{Kur}^{m3} = \frac{\chi_0^{m\prime}\varpi^m}{\delta^m} K_1\left(\frac{\chi_0^{m\prime} a}{\delta^m}\right) \tag{A64}$$

$$\Lambda_{Iur}^{f3} = \frac{\chi_0^{f\prime}\varpi^f}{\delta^f} I_1\left(\frac{\chi_0^{f\prime} a}{\delta^f}\right) \tag{A65}$$

$$\Lambda_{Iur}^{m1} = n\pi\gamma_i^m I_1\left(n\pi\gamma_i^m a\right) \tag{A66}$$

$$\Lambda_{Kur}^{m1} = -n\pi\gamma_i^m K_1\left(n\pi\gamma_i^m a\right) \tag{A67}$$

$$\Lambda_{Iur}^{f1} = n\pi\gamma_i^f I_1\left(n\pi\gamma_i^f a\right) \tag{A68}$$

$$\Lambda_{Iurn}^{m3} = \frac{\chi_n^{m\prime}\varpi^m}{\delta^m} I_1\left(\frac{\chi_n^{m\prime} a}{\delta^m}\right) \tag{A69}$$

$$\Lambda_{Kurn}^{m3} = \frac{\chi_n^{m\prime}\varpi^m}{\delta^m} K_1\left(\frac{\chi_n^{m\prime} a}{\delta^m}\right) \tag{A70}$$

$$\Lambda_{Iurn}^{f3} = \frac{\chi_n^{f\prime}\varpi^m}{\delta^m} I_1\left(\frac{\chi_n^{f\prime} a}{\delta^m}\right) \tag{A71}$$

$$\Lambda_{Irzn}^{f3} = -\frac{\chi_n^{f\prime} n\pi\varpi^f}{\delta^f}\left(c_{44}^f + c_{44}^f l_{14}^f + e_{15}^f l_{24}^f\right) I_1\left(\frac{\chi_n^{f\prime} a}{\delta^f}\right) \tag{A72}$$

$$\Lambda_{Iuz}^{m1} = -n\pi l_{1i}^m I_0\left(n\pi\gamma_i^m a\right) \tag{A73}$$

$$\Lambda_{Kuz}^{m1} = -n\pi l_{1i}^m K_0\left(n\pi\gamma_i^m a\right) \tag{A74}$$

$$\Lambda_{Iuz}^{f3} = -n\pi l_{1i}^f I_0\left(n\pi\gamma_i^f a\right) \tag{A75}$$

$$\Lambda_{uzji}^{m2} = \frac{C_{uzji}^{m2}\xi_j^m l_{1i}^m}{\gamma_i^m} H_0\left(\xi_j^m a\right) \tag{A76}$$

$$\Lambda_{uzji}^{f2} = \frac{-\xi_j^f C_{uzji}^{f2} l_{1i}^f}{\gamma_i^f} J_0\left(\xi_j^f a\right) \tag{A77}$$

$$\Lambda_{Iuzj}^{m3} = l_{14}^m\varpi^m\left(j\pi\right) I_0\left(\frac{\chi_j^{m\prime} a}{\delta^m}\right) \tag{A78}$$

$$\Lambda_{Kuzj}^{m3} = l_{14}^m\varpi^m\left(j\pi\right) K_0\left(\frac{\chi_j^{m\prime} a}{\delta^m}\right) \tag{A79}$$

$$\Lambda_{I\varphi}^{m1} = -n\pi l_{2i}^m I_0\left(n\pi\gamma_i^m a\right) \tag{A80}$$

$$\Lambda_{K\varphi}^{m1} = -n\pi l_{2i}^m K_0\left(n\pi\gamma_i^m a\right) \tag{A81}$$

$$\Lambda_{I\varphi}^{f3} = -n\pi l_{2i}^f I_0\left(n\pi\gamma_i^f a\right) \tag{A82}$$

$$\Lambda_{\varphi ji}^{m2} = \frac{C_{\varphi ji}^{m2}\xi_j^m l_{2i}^m}{\gamma_i^m} H_0\left(\xi_j^m a\right) \tag{A83}$$

$$\Lambda_{\varphi ji}^{f2} = \frac{-\xi_j^f C_{\varphi ji}^{f2} l_{2i}^f}{\gamma_i^f} J_0\left(\xi_j^f a\right) \tag{A84}$$

$$\Lambda_{I\varphi j}^{m3} = l_{24}^m \varpi^m \left(j\pi\right) I_0\left(\frac{\chi_j^{m\prime} a}{\delta^m}\right) \tag{A85}$$

$$\Lambda_{K\varphi j}^{m3} = l_{24}^m \varpi^m \left(j\pi\right) K_0\left(\frac{\chi_j^{m\prime} a}{\delta^m}\right) \tag{A86}$$

$$\Lambda_{I\varphi j}^{f3} = l_{24}^f \varpi^f \left(j\pi\right) I_0\left(\frac{\chi_j^{f\prime} a}{\delta^f}\right) \tag{A87}$$

$$\Lambda_{Iuzj}^{f3} = l_{14}^f \varpi^f \left(j\pi\right) I_0\left(\frac{\chi_j^{f\prime} a}{\delta^f}\right) \tag{A88}$$

$$\Lambda_{IDrnia}^{m1} = -\left(n\pi\right)^2 \gamma_i^m \left(e_{15}^m + e_{15}^m l_{1i}^m - \in_{11}^m l_{2i}^m\right) I_1\left(n\pi\gamma_i^m a\right) \tag{A89}$$

$$\Lambda_{KDrnia}^{m1} = \left(n\pi\right)^2 \gamma_i^m \left(e_{15}^m + e_{15}^m l_{1i}^m - \in_{11}^m l_{2i}^m\right) K_1\left(n\pi\gamma_i^m a\right) \tag{A90}$$

$$\Lambda_{IDrnia}^{f1} = \left(n\pi\right)^2 \gamma_i^f \left(e_{15}^f + e_{15}^f l_{1i}^f - \in_{11}^f l_{2i}^f\right) I_1\left(n\pi\gamma_i^f a\right) \tag{A91}$$

$$\Lambda_{IDrnia}^{m3} = \frac{\chi_n^{m\prime} n\pi}{\delta^m}\left(e_{15}^m + e_{15}^m l_{1i}^m - \in_{11}^m l_{2i}^m\right) I_1\left(\frac{\chi_n^{m\prime} a}{\delta^m}\right) \tag{A92}$$

$$\Lambda_{KDrnia}^{m3} = \frac{\chi_n^{m\prime} n\pi}{\delta^m}\left(e_{15}^m + e_{15}^m l_{1i}^m - \in_{11}^m l_{2i}^m\right) K_1\left(\frac{\chi_n^{m\prime} a}{\delta^m}\right) \tag{A93}$$

$$\Lambda_{IDrnia}^{f3} = \frac{\chi_n^{f\prime} n\pi}{\delta^f}\left(e_{15}^f + e_{15}^f l_{1i}^f - \in_{11}^f l_{2i}^f\right) I_1\left(\frac{\chi_n^{f\prime} a}{\delta^f}\right) \tag{A94}$$

$$c_{I0mi}^m = \frac{2\int_a^b \hat{r} I_n\left(m\pi\gamma_i^m r\right) dr}{b^2 - a^2} \tag{A95}$$

$$c_{K0ni}^{m} = \frac{2\int_{a}^{b} rK_n\left(m\pi\gamma_i^m r\right)dr}{b^2 - a^2} \tag{A96}$$

$$c_{Ijmi}^{m} = \frac{\int_{a}^{b} I_n\left(m\pi\gamma_i^m r\right) rH_0(\xi_j^m r)dr}{\int_{a}^{b} r\left[H_0(\xi_j^m r)\right]^2 dr} \tag{A97}$$

$$c_{Kjni}^{m} = \frac{\int_{a}^{b} K_n\left(m\pi\gamma_i^m r\right) rH_0(\xi_j^m r)dr}{\int_{a}^{b} r\left[H_0(\xi_j^m r)\right]^2 dr} \tag{A98}$$

$$c_{I0mi}^{f} = \frac{2}{b^2}\int_{0}^{b} rI_n\left(m\pi\gamma_i^f r\right)dr \tag{A99}$$

$$c_{Ijmi}^{f} = \frac{2}{b^2\left[J_0\left(\xi_j^f b\right)\right]^2}\int_{0}^{b} rJ_0\left(\xi_j^f r\right) I_n\left(m\pi\gamma_i^f r\right)dr \tag{A100}$$

APPENDIX B

From Eq. (2.43) the coefficients $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ are given by

$$\bar{A} = e_{15}^2 + c_{44} \in_{11}$$

$$\bar{B} = \frac{\left[2e_{15}^2 c_{13} - c_{44} e_{31}^2 + 2e_{15} e_{31} c_{13} - 2e_{15} c_{11} e_{33} + \in_{11} c_{13}^2 + 2c_{31} c_{44} \in_{11} - c_{33} c_{11} \in_{11} - c_{44} c_{11} \in_{33}\right]}{c_{11}}$$

$$\bar{C} = \frac{\left[e_{33}\left(c_{11} e_{33} + 2e_{15} c_{44}\right) - 2e_{33}\left(e_{15} + e_{31}\right)\left(c_{13} + c_{44}\right) - \in_{33}\left(c_{13}^2 + 2c_{13} c_{44} - c_{33} c_{11}\right) + c_{33}\left(e_{15} + e_{31}\right)^2\right]}{c_{11}}$$

$$\bar{D} = \frac{-\left(e_{33}^2 + \in_{33} c_{33}\right) c_{44}}{c_{11}}$$

From Eq. (2.48), the value of l_{14} and l_{24} for $\bar{\theta}$ can be determined in term of material properties by solving 2 equations as shown below.

$$\frac{\lambda_{11}}{\left(\frac{\chi}{\delta}\right)^2 c_{11} - \vartheta^2 M_4} = \frac{\lambda_{33}}{\left(\frac{\chi}{\delta}\right)^2 N_4 - \vartheta^2 P_4} \quad \text{(B.1)}$$

$$\frac{\lambda_{33}}{\left(\frac{\chi}{\delta}\right)^2 N_4 - \vartheta^2 P_4} = \frac{-p_3}{\left(\frac{\chi}{\delta}\right)^2 F_4 - \vartheta^2 G_4} \quad \text{(B.2)}$$

From Eq. (2.48), the value of l_{14} and l_{24} for $\bar{\theta}$ can be determined in term of material properties by solving 2 equations as shown below

$$\frac{\lambda_{11}}{\left(\frac{\mu}{\delta}\right)^2 \hat{c}_{11} - \eta^2 M_4} = \frac{\lambda_{33}}{\left(\frac{\mu}{\delta}\right)^2 N_4 - \eta^2 P_4} \quad \text{(B.3)}$$

$$\frac{\lambda_{33}}{\left(\frac{\mu}{\delta}\right)^2 N_4-\eta^2 P_4}=\frac{-p_3}{\left(\frac{\mu}{\delta}\right)^2 F_4-\eta^2 G_4} \tag{B.4}$$

where $M_i=c_{44}+\left(c_{13}+c_{44}\right)l_{1i}+\left(e_{15}+e_{31}\right)l_{2i}$

$N_i=c_{13}+c_{44}\left(1+l_{1i}\right)+e_{15}l_{2i}$

$P_i=c_{33}l_{1i}+e_{33}l_{2i}$

$F_i=e_{15}\left(1+l_{1i}\right)+e_{31}-\in_{11} l_{2i}$

$G_i=e_{33}l_{1i}-\in_{33} l_{2i}$

Normalization all field quantities

$$\hat{r}=\frac{r}{h},\ \hat{z}=\frac{z}{h},\ \hat{u}_r=\frac{u_r}{h},\ \hat{u}_z=\frac{u_z}{h}; \tag{B.5}$$

$$\hat{\sigma}_{rr}=\frac{\hat{\sigma}_{rr}}{c_{11}},\ \hat{\sigma}_{\theta\theta}=\frac{\sigma_{\theta\theta}}{c_{11}},\ \hat{\sigma}_{zz}=\frac{\sigma_{zz}}{c_{11}},\ \hat{\sigma}_{zr}=\frac{\sigma_{zr}}{c_{11}},\ \hat{c}_{11}=\frac{c_{11}}{c_{11}},\ \hat{c}_{44}=\frac{c_{44}}{c_{11}},\ \hat{c}_{13}=\frac{c_{13}}{c_{11}}; \tag{B.6}$$

$$\hat{e}_{15}=\frac{e_{15}}{e_{33}},\ \hat{e}_{31}=\frac{e_{31}}{e_{33}},\ \hat{e}_{33}=\frac{e_{33}}{e_{33}},\ \hat{D}_r=\frac{D_r}{e_{33}},\ \hat{D}_z=\frac{D_z}{e_{33}} \tag{B.7}$$

$$\hat{\lambda}_{11}=\frac{\lambda_{11}}{\lambda_{11}},\ \hat{\lambda}_{33}=\frac{\lambda_{33}}{\lambda_{11}} \tag{B.8}$$

$$\hat{\in}_{11}=\frac{\in_{11} c_{11}}{\left(e_{33}\right)^2},\ \hat{\in}_{33}=\frac{\in_{33} c_{11}}{\left(e_{33}\right)^2} \tag{B.9}$$

$$\hat{p}_3=\frac{p_3 c_{11}}{e_{33}\lambda_{11}} \tag{B.10}$$

$$\hat{t}=\frac{\kappa^m t}{h^2} \tag{B.11}$$

$$\hat{\varepsilon}_{rr} = \varepsilon_{rr} \ , \ \hat{\varepsilon}_{zz} = \varepsilon_{zz} \ , \ \hat{\varepsilon}_{rz} = \varepsilon_{rz} \ , \hat{\varepsilon}_{\theta\theta} = \varepsilon_{\theta\theta} \tag{B.12}$$

$$\hat{E}_r = E_r\left(\frac{e_{33}}{c_{11}}\right), \ \hat{E}_z = E_z\left(\frac{e_{33}}{c_{11}}\right) \tag{B.13}$$

$$\hat{e}_r = e_r\left(\frac{\lambda_{11}h}{c_{11}}\right), \ \hat{e}_z = e_z\left(\frac{\lambda_{11}h}{c_{11}}\right) \tag{B.14}$$

$$\hat{h}_r = h_r \frac{\lambda_{11}h}{c_{11}K_{11}} \ , \ \hat{h}_z = h_z \frac{\lambda_{11}h}{c_{11}K_{11}} \tag{B.15}$$

$$\hat{K}_{11} = \frac{K_{11}}{K_{11}} \ , \ \hat{K}_{33} = \frac{K_{33}}{K_{11}} \tag{B.16}$$

$$\hat{\varphi} = \frac{\varphi e_{33}}{c_{11}h} \tag{B.17}$$

$$\hat{\theta} = \frac{\theta\lambda_{11}}{c_{11}} \tag{B.18}$$

BIOGRAPHY

The author, Mr. Panupong Boonphennimit graduated his Bachelor of Engineering degree in Civil Engineering from Chulalongkorn University in 2009. As he would like to obtain the advanced knowledge of structural engineering, he continued his Master's degree in structural civil engineering at Chulalongkorn University in the same year under the supervision of Assistant Professor Dr. Jaroon Rungamornrat and Professor Dr. Teerapong Senjuntichai. For three years of studying in Master's degree, he had studied new interesting knowledge of advanced solid mechanics solved by the powerful numerical techniques and decided to do his research on this kind of field. He successfully fulfilled the requirements for the Master of Engineering degree in 2012.