CHAPTER V

SELF AND MUTUAL IMPEDANCES

5-1. Introduction

The impedance presented by an antenna to a transmission line is called the terminal or driving - point impedance. If the antenna is isolated, that is, remote from the ground or other objects, and is lossless, its terminal impedance is the same as the self - impedance of the antenna. This impedance has a real part called the self - resistance (radiation resistance) and an imaginary part called the self-reactance.

In case there are nearby subjects, say several other antennas, the terminal impedance is determined not only by the self - impedance of the antenna but also by the mutual impedances between it and the other antennas and the currents flowing in them.

5-2. Self - impedance of a Thin Linear Antenna.[20]

In this section an induced emf method as used by Carter[21] is applied to the determination of the self - impedance of a thin linear antenna. The antenna is center - fed with the lower end located at the origin of the coordinates as shown in Fig. 5-1. The antenna is situated in air or vacuum and is remote from other subjects. Since the antenna is thin, a sinusoidal current distribution will be assumed with the maximum current I_1 at the terminals. Only lengths L which are odd multiple of $\frac{1}{2}$ - wavelength will be considered so that the current distribution is symmetrical, with a current maximum at the terminals. The current distribution shown in Fig. 5-1 is for the case where $L = \lambda/2$. The terminal impedance Z_{11} of the antenna is given by the ratio of applied emf V_{11} to the total terminal current I_1. Thus,

$$Z_{11} = \frac{V_{11}}{I_1} \qquad (5\text{-}1)$$

Applying the reciprocity theorem,[22] to Fig. 5-1, we have

$$V_{11} = -\frac{1}{I_1}\int_0^L I_z E_z \, dz \qquad (5\text{-}2)$$

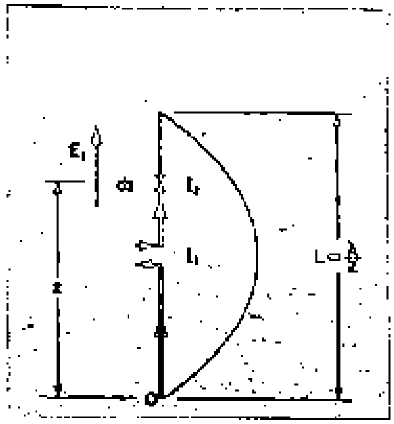

Fig. 5-1. Center - fed linear $\frac{1}{2}$ - wavelength antenna.

Then,

$$Z_{11} = \frac{V_{11}}{I_1} = -\frac{1}{I_1^2}\int_0^L I_z E_z \, dz \qquad (5\text{-}3)$$

Since the antenna is isolated, this impedance is called the self-impedance. In (5-3) E_z is the z component of the electric field at the antenna caused by its own current.

And according to Maxwell's equation, the Z component of the electric field everywhere is

$$E_z = -j30I_1\left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2}\right) \qquad (5\text{-}4)$$

where r_1 and r_2 are the distances from both ends of the antenna to the point considered.

In case of the Z component of electric field at the antenna due to its own current in Fig. 5-1, we have

$$r_1 = Z \qquad (5-5)$$

$$r_2 = L - Z \qquad (5-6)$$

Substituting (5-4), (5-5), (5-6) into (5-3), we obtain the self - impedance of a thin linear antenna an odd number of $\frac{1}{2}$ - wavelengths long to be

$$Z_{11} = -15 \int_0^L \left[\frac{e^{-j2\beta z} - 1}{Z} - \frac{e^{-j\beta L}\,(e^{j2\beta z} - 1)}{L - Z} \right] dz \qquad (5-7)$$

For $L = n\lambda/2$ where $n = 1, 3, 5, \ldots,\ e^{-j\beta L} = e^{-j\pi n} = -1$, so that upon integration of (5-7) by Carter[23] becomes

$$Z_{11} = R_{11} + jX_{11} = 30 \left[\mathrm{Cin}\,(2\pi n) + j\,\mathrm{Si}\,(2\pi n) \right] \qquad (5-8)$$

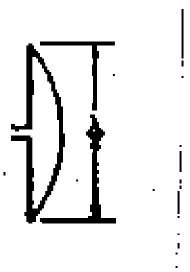

Fig. 5-2. One - half wavelength antennas.

In the case of a $\frac{1}{2}$ - wavelength antenna as shown in Fig. 5-2, n = 1, so we have for the self - resistance and self - reactance

$$R_{11} = 30\ \mathrm{Cin}\,(2\pi) \qquad (5-9)$$

and

$$X_{11} = 30\ \mathrm{Si}\,(2\pi) \qquad (5-10)$$

The value of (5-9) is identical with that given for the radiation resistance of a $\frac{1}{2}$ - wavelength antenna, in Sec. 4-5, Eq. (4-25). Evaluating (5-9), (5-10), we obtain for the self - impedance

$$Z_{11} = R_{11} + jX_{11} = 73 + j42.5 \text{ ohms} \qquad (5-11)$$

Since X_{11} is not zero, an antenna of an exact $\frac{1}{2}$ - wavelength long is not resonant. To obtain a resonant antenna, it is common practice to shorten the antenna a few percent to make $X_{11} = 0$. In this case the self resistance is somewhat less than 73 ohms.

It is interesting that the self - reactance of center - fed antennas, an exact odd number of $\frac{1}{2}$ - wavelength long, is always positive since the sine integral Si ($2\pi n$) is always positive. It should be noted that for antenna lengths not an exact odd number of $\frac{1}{2}$ - wavelengths the reactance may be positive or negative as illustrated for example by Fig. 5-3. in Sec. 5-3.

For large n, the self - resistance expression approaches the value

$$R_{11} = 30 \left[0.577 + \ln (2\pi n) \right] \qquad (5\text{-}12)$$

Thus, the self - resistance continues to increase indefinitely with the increasing n but at a logarithmic rate.

5-3. Self Impedance of Thin Linear Antenna not an Exact Number of $\frac{1}{2}$ - wavelength Long.

It should be noted that for antenna length L not an exact odd number of $\frac{1}{2}$ - wavelength the reactance may be positive or negative according to the length. The self - resistance for this case is [24]

$$R_{11} = 30 \Big[(1 - \cot^2 \frac{\beta L}{2}) \operatorname{Cin} 2\beta L + 4 \cot^2 \frac{\beta L}{2} \operatorname{Cin} \beta L + 2 \cot \frac{\beta L}{2} (\operatorname{Si} 2\beta L - 2 \operatorname{Si} \beta L) \Big] \text{ ohms} \qquad (5\text{-}13)$$

When the length L is small, (5-13) reduces very nearly to

$$R_{11} = 5 (\beta L)^2 \quad \text{ohms} \qquad (5\text{-}14)$$

Another approach from Hallén's[25] which expresses the input impedance Z_T of a center - fed cylindrical antenna to be

$$Z_T = -j\,60\Omega \left[\frac{\cos\beta l + (d_1/\Omega)}{\sin\beta l + (b_1/\Omega)} \right] \qquad (5\text{-}15)$$

This is a first - order approximation for the input impedance. If the second - order terms are included, Hallén input - impedance expression has the form

$$Z_T = -j60\Omega \left[\frac{\cos\beta l + (d_1/\Omega) + (d_2/\Omega^2)}{\sin\beta l + (b_1/\Omega) + (b_2/\Omega^2)} \right] \qquad (5\text{-}16)$$

where $2l$ = total length

$2a$ = diameter

Ω = length - thickness parameter $= 2 \ln \frac{2l}{a}$

This relation has been evaluated by Hallén who has also presented the results in chart form[26]. Impedance spirals based on Hallén's data are presented in Fig. 5-3. for center - fed cylindrical antennas with ratios of total length to diameter (l/a) of 60 and 2,000. The half-length l of the antenna is given along the spirals in free - space wavelengths. The impedance variation is that which would be obtained as a function of frequency for an antenna of fixed physical dimensions. The difference in the impedance behavior of the thinner antenna (l/a = 2,000) and of the thicker antenna (l/a = 60) is striking, the variation in impedance with frequency of the thicker antenna being

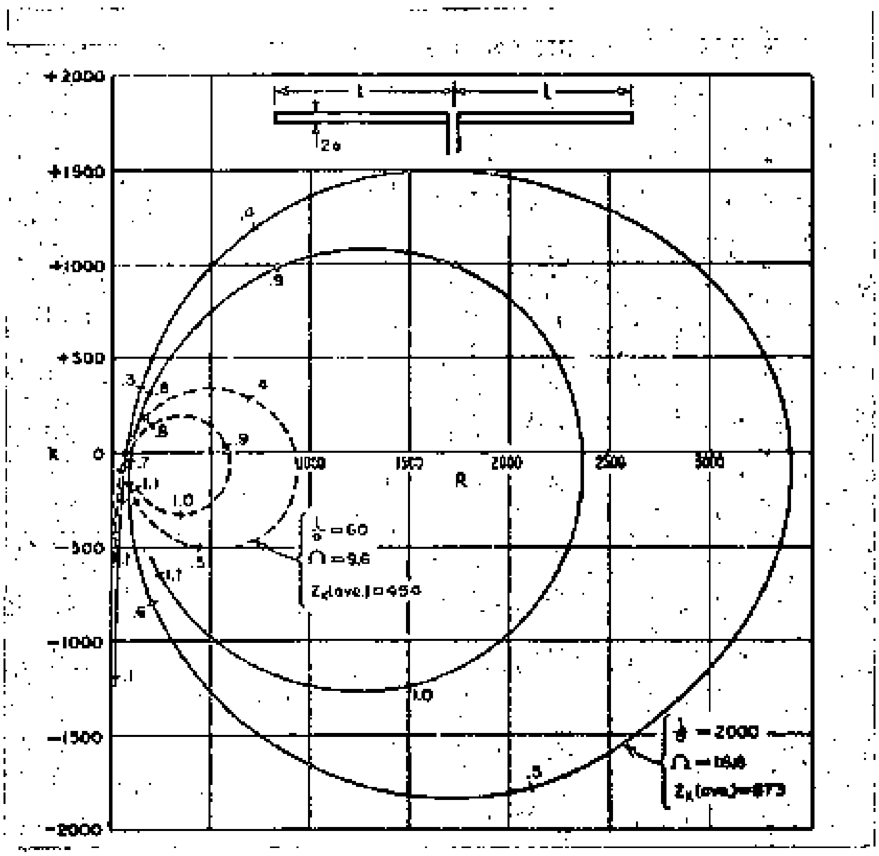

Fig. 5-3. Calculated input impedance (R + jX) in ohms for cylindrical center - fed antennas with ratios of total length to diameter (2l/2a) of 60 and 2,000 (after Hallén).

much less than that of the thinner antenna.

The impedance diagrams, showing antenna resistance and reactance seperately are produced in Fig. 5-4 and 5-5. Also according to King-Middleton[27] expansion are reproduced in Fig. 5-6 and 5-7. For the case of thin linear antenna, the curve of $\frac{H}{a} \doteqdot 20,000$ is suitable to indicate the input impedance.

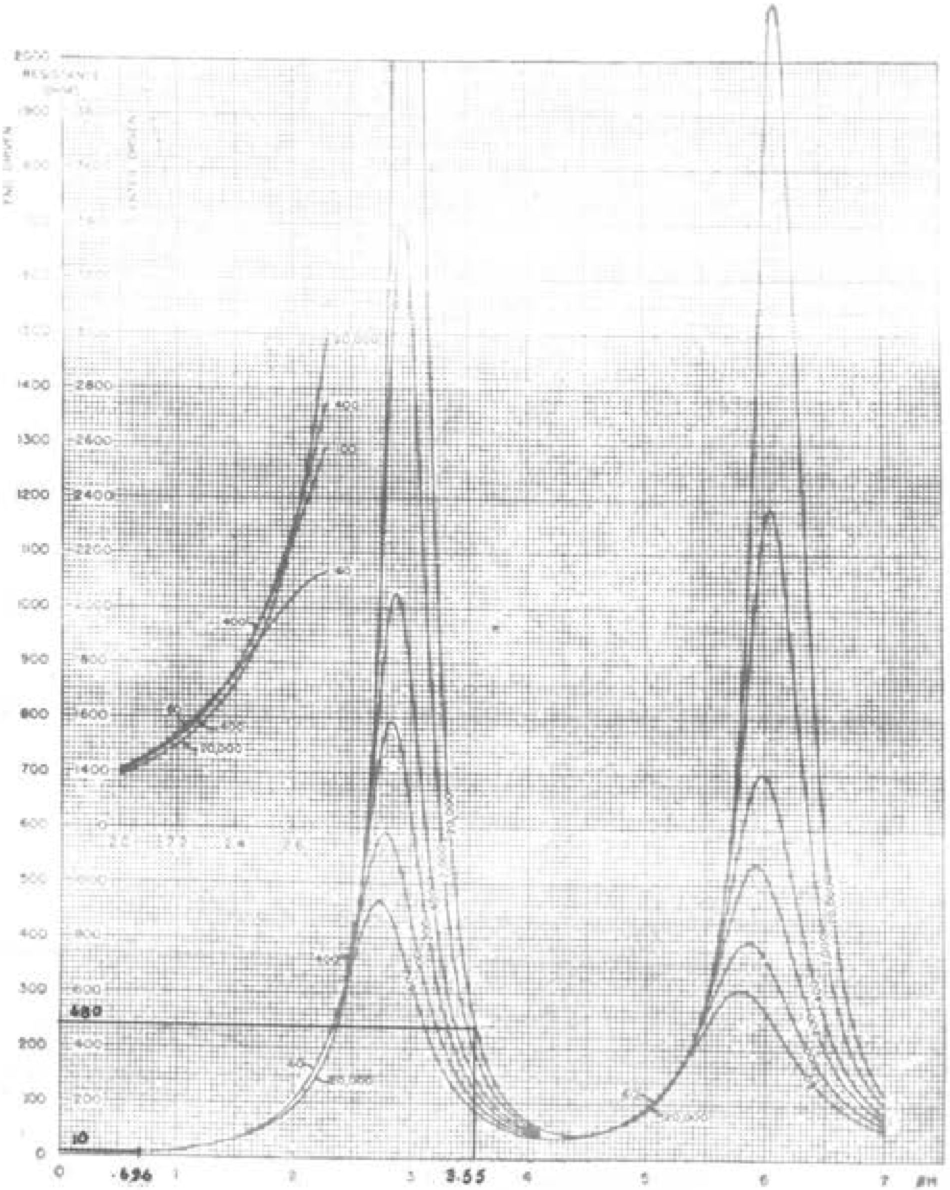

FIG. 5-4. Antenna resistance according to Hallén. The resistance of center-fed dipoles is plotted as a function of $2\pi H/\lambda$, the antenna half-length in radians, for various ratios of H/a, half-length to radius. **For monopoles of length H, the ordinates should be divided by two.**

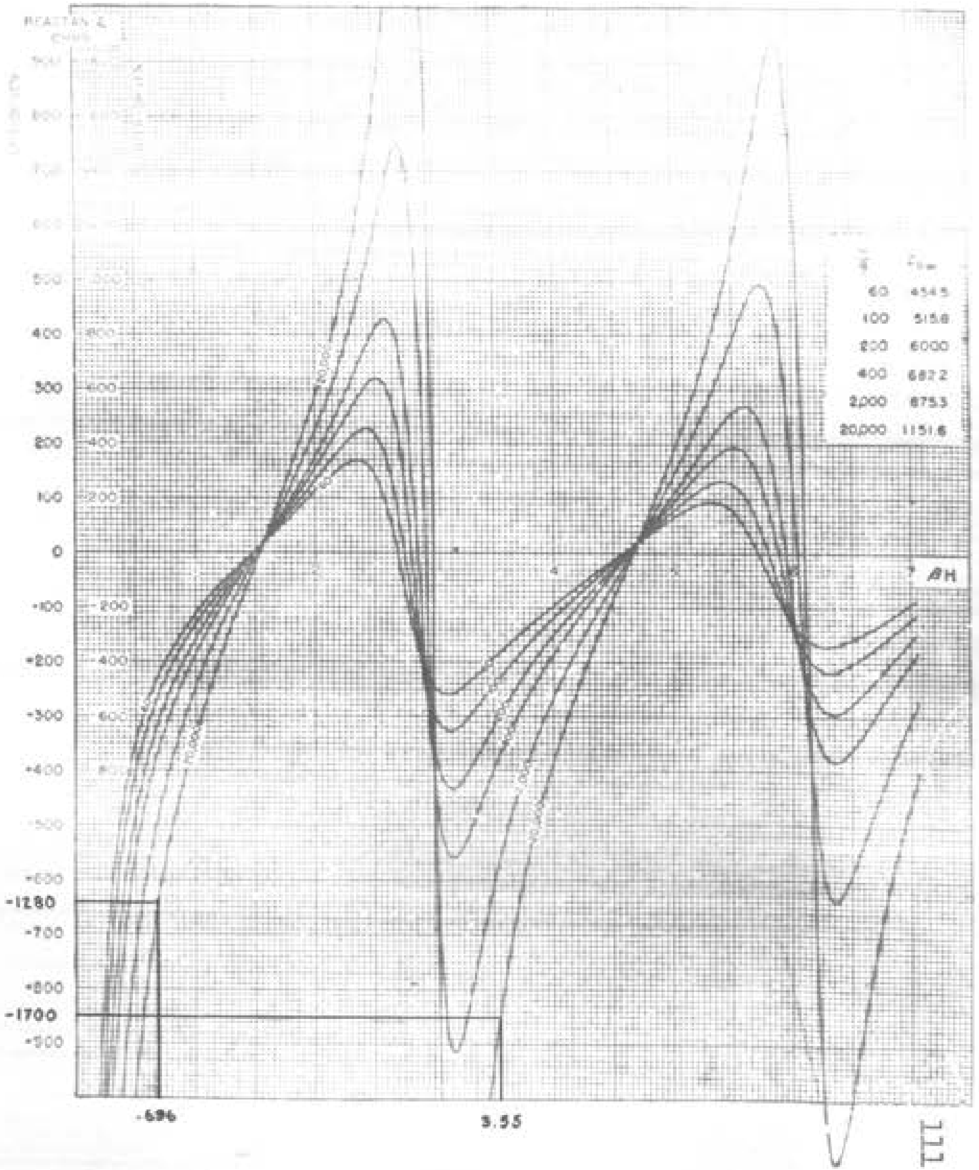

FIG. 5-5. Antenna reactance according to Hallén (see legend for Fig. 5-4).

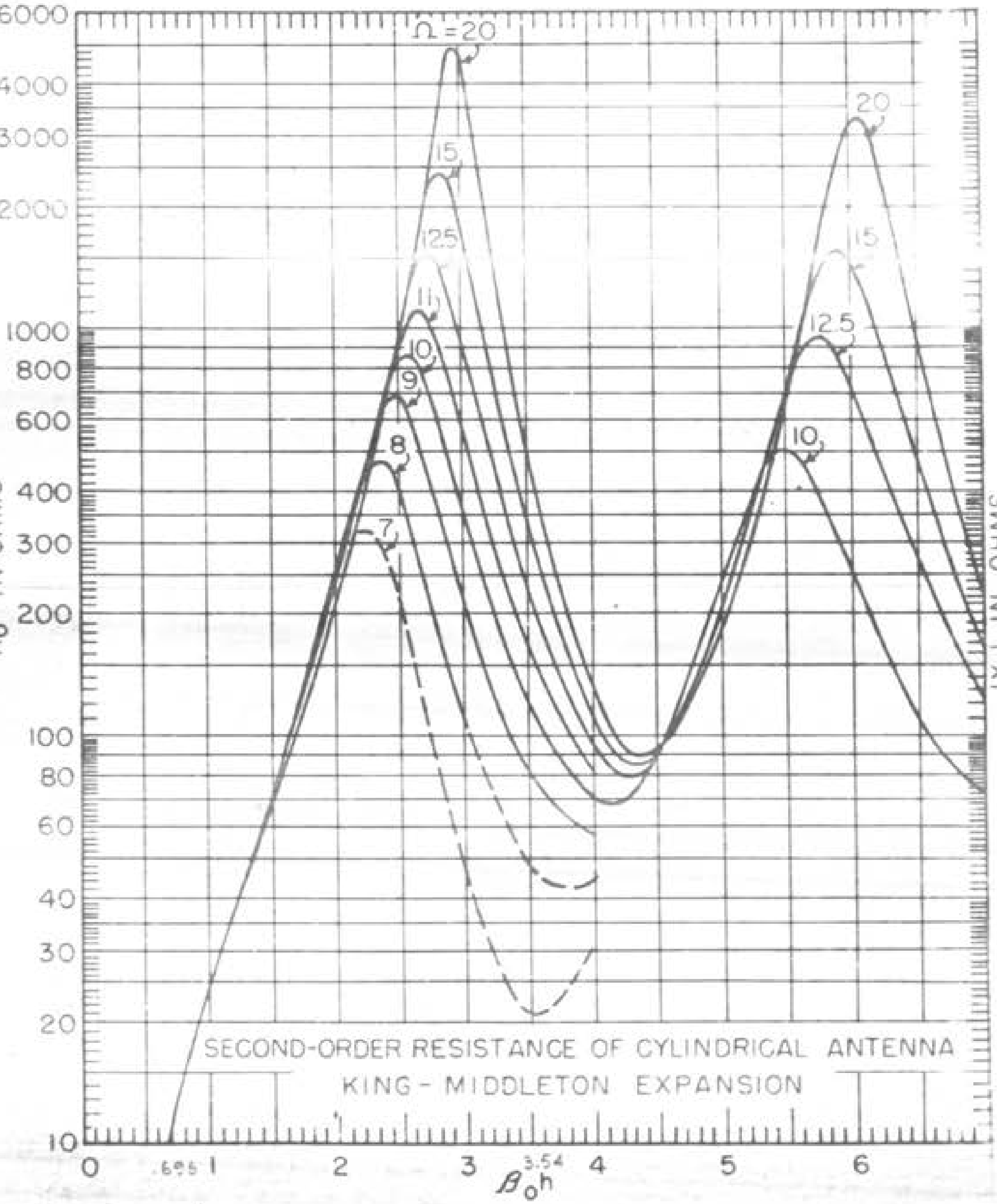

Fig. 5-6. Resistance R_0 of cylindrical antenna. King-Middleton second-order expansion.

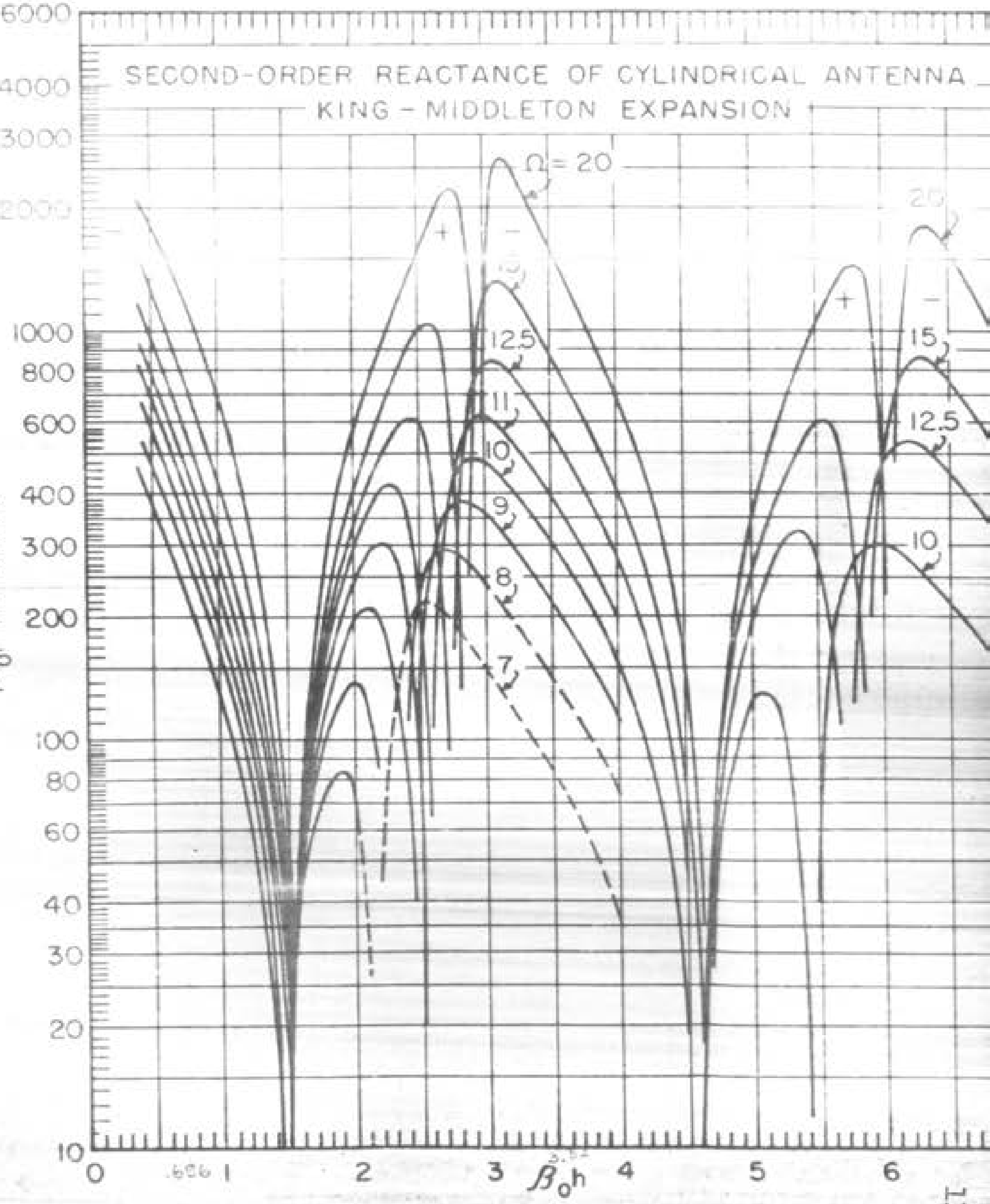

Fig. 5-7. Reactance X_0 of cylindrical antenna, King-Middleton second-order expansion.

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5-4. Mutual Impedance of Two Parallel Linear Antennas

The mutual impedance of two coupled circuits is defined in circuit - theory as the negative of the ratio of the emf V_{21} induced in circuit 2 to the current I_1 flowing in circuit 1 with circuit 2 open.

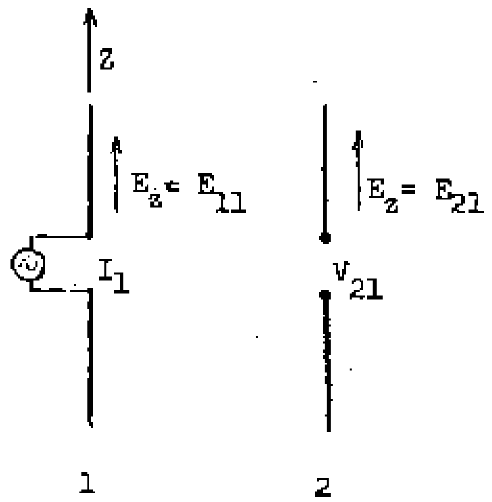

Fig. 5-8. Parallel coupled antennas.

Consider the case of two coupled antennas 1 and 2 as shown in Fig. 5-8. Suppose a current I_1 in antenna 1 induces an emf V_{21} at the open terminals of antenna 2. Then the ratio of $-V_{21}$ to I_1 is the mutual impedance Z_{21}. Thus,

$$Z_{21} = -\frac{V_{21}}{I_1} \qquad (5\text{-}17)$$

If the generator is moved to the terminals of antenna 2, then by reciprocity the mutual impedance Z_{12} or ratio of $-V_{12}$ to I_2 is the same as before, where V_{12} is the emf induced at the open terminals of antenna 1 by the current I_2 in antenna 2. Thus,

$$-\frac{V_{21}}{I_1} = Z_{21} = Z_{12} = -\frac{V_{12}}{I_2} \qquad (5\text{-}18)$$

To calculate Z_{21}, we need to know V_{21} and I_1. Let the antennas be in the Z direction as shown in Fig. 5-8. The emf $-V_{11}$ induced by its own current is indicated by (5-2), i.e.,

$$V_{11} = -\frac{1}{I_1}\int_0^L I_z E_z \, dz$$

where V_{11} is the emf that must be applied to produce I_1 at the terminals. To obtain the emf V_{21} induced at the open terminals of antenna 2 by the current in antenna 1, we set $E_z = E_{21}$, $V_{11} = -V_{21}$, $I_1 = I_2$ into (5-2). Then,

$$V_{21} = \frac{1}{I_2}\int_0^L I_z E_{21} \, dz \qquad (5\text{-}19)$$

where I_2 is the maximum current and I_z the value at a distance Z from the lower end of antenna 2 with its terminals closed, and where E_{21} is the electric field along antenna 2 produced by the current in antenna 1. Assuming that this current distribution is sinusoidal as given by

$$I_z = I_2 \sin\beta Z \qquad (5\text{-}20)$$

so that (5-19) becomes

$$V_{21} = \int_0^L E_{21} \sin\beta Z \, dZ \qquad (5\text{-}21)$$

then

$$Z_{21} = -\frac{V_{21}}{I_1} = -\frac{1}{I_1}\int_0^L E_{21} \sin\beta Z \, dZ \qquad (5\text{-}22)$$

This is the general expression for the mutual impedance of two thin linear, parallel, center - fed antennas with sinusoidal current distribution.

We will consider the situation where both antennas are the same length L, where L is an odd number of $\frac{1}{2}$ - wavelengths long ($L = n\lambda/2$; $n = 1, 3, 5, \ldots$). A case of particular interest is where both antennas are $\frac{1}{2}$ - wavelength long ($n = 1$) situated side by side as given in the following section.

5-5. Mutual Impedance of Parallel Antennas Side by Side.[29]

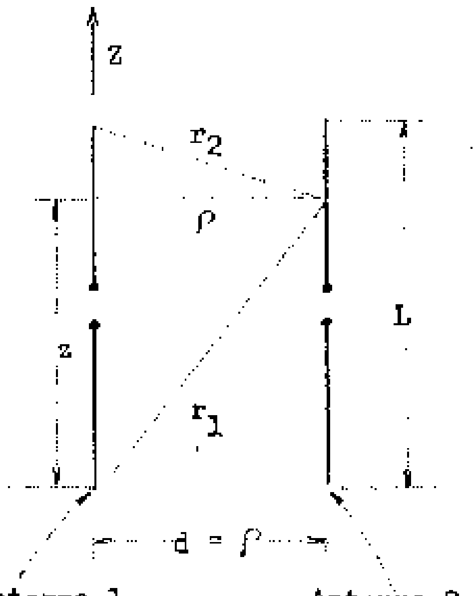

Fig. 5-9. Parallel coupled antennas with dimensions.

Referring to the arrangement of Fig. 5-9, with E_{21} given by (5-4) where

$$r_1 = \sqrt{d^2 + z^2} \qquad (5\text{-}23)$$

and

$$r_2 = \sqrt{d^2 + (L - z)^2} \qquad (5\text{-}24)$$

Substituting this into (5-22), the mutual impedance becomes

$$Z_{21} = j30 \int_0^L \left[\frac{e^{-j\beta\sqrt{d^2 + z^2}}}{\sqrt{d^2 + z^2}} + \frac{e^{-j\beta\sqrt{d^2 + (L - z)^2}}}{\sqrt{d^2 + (L - z)^2}} \right] \sin \beta Z dZ \qquad (5\text{-}25)$$

Carter has shown that upon integration of (5-25)

$$R_{21} + jX_{21} = Z_{21} = Z_{12} = R_{12} + jX_{12} \qquad (5\text{-}26)$$

$$R_{21} = 30\left\{2\,\mathrm{Ci}\,(\beta d) - \mathrm{Ci}\left[\beta\left(\sqrt{d^2 + L^2} + L\right)\right] - \mathrm{Ci}\left[\beta\left(\sqrt{d^2 + L^2} - L\right)\right]\right\} \qquad (5\text{-}27)$$

$$X_{21} = -30\left\{2\,\mathrm{Si}\,(\beta d) - \mathrm{Si}\left[\beta\left(\sqrt{d^2 + L^2} + L\right)\right] - \mathrm{Si}\left[\beta\left(\sqrt{d^2 + L^2} - L\right)\right]\right\} \qquad (5\text{-}28)$$

The mutual resistance and reactance calculated by (5-27) and (5-28) for the case of $\frac{1}{2}$ - wavelength antenna ($L = \lambda/2$) are presented by the solid curve in Fig. 5-10 as a function of the spacing d.

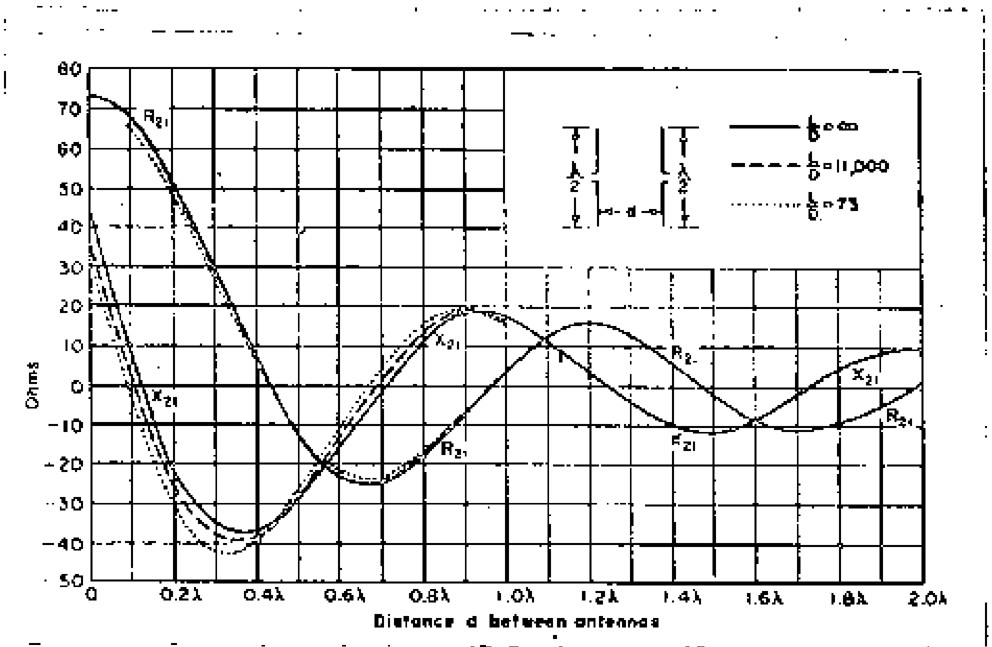

Fig. 5-10. Curves of mutual resistance (R_{21}) and reactance (X_{21}) of two parallel side - by - side linear $\frac{1}{2}$ - wavelength antennas as a function of distance between them.

5-6. Mutual Impedance of Parallel Antennas Side by Side but Not of the Same Length.[30]

The problem on hand is illustrated in Fig. 5-11 where h_1 and h_2 are the half - lengths of dipoles 1 and 2, d is their seperation, Z is the coordinate of a typical element dZ, and r_o, r_1 and r_2 are distances from fixed points on one dipole to a typical element

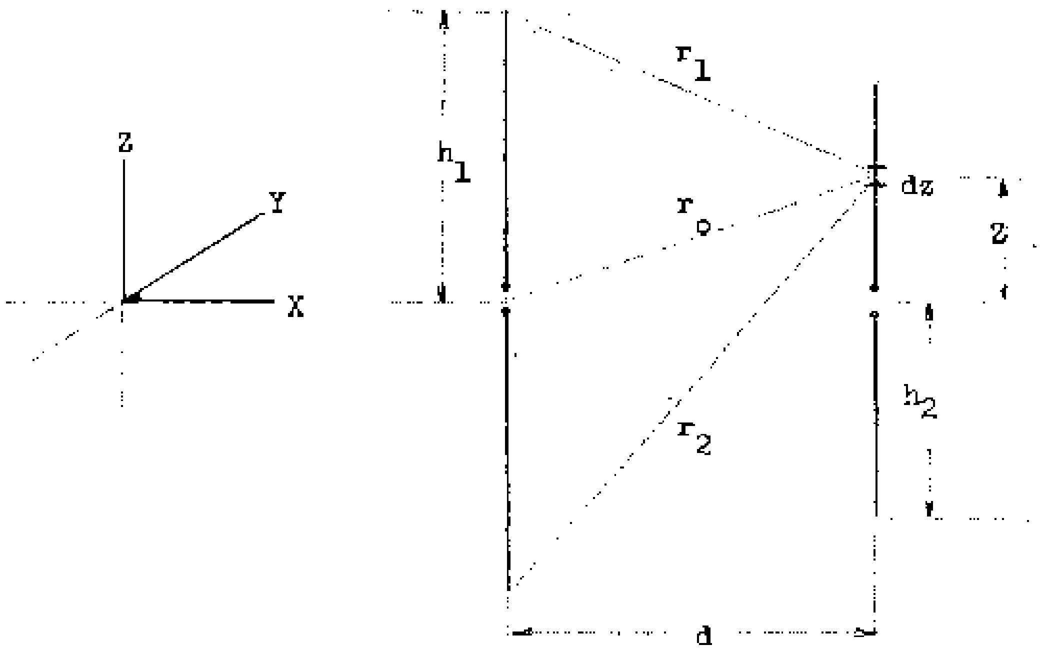

Fig. 5-11. Geometry and notation used in calculation of mutual impedances.

on the other. The mutual impedance between the two antennas of Fig. 5-11 is designed by

$$Z_{21} = \frac{V_{21}}{I_1(0)} \qquad (5\text{-}29)$$

where V_{21} is the open circuit voltage at the terminals of antenna 2 due to a base current $I_1(0)$ at antenna 1. The induced emf at the open terminals of antenna 2 may be found by the application of the reciprocity theorem

$$\text{emf} = -V_{21} = \frac{1}{I_2(0)} \int_{-h_2}^{h_2} E_{z1}\, I_2(Z)\, dZ \qquad (5\text{-}30)$$

where E_{z1} is the Z component of electric field intensity at the location of antenna 2 due to the current on antenna 1, specified by $I_1(0)$, when 2 is removed. The current distribution on antenna 2 is assumed to be sinusoidal and is given by

$$I_2(Z) = I_{2\,max} \sin\beta\,(h_2 - |Z|) \qquad (5\text{-}31)$$

The expression for the parallel component of electric field due to a sinusoidal current distribution in antenna 1 is given by

$$E_{z1} = 30\, I_{1\,max} \left[-j\,\frac{e^{-j\beta r_1}}{r_1} - j\,\frac{e^{-j\beta r_2}}{r_2} + \frac{2j\cos\beta h_1 e^{-j\beta r_o}}{r_o} \right] \qquad (5\text{-}32)$$

Inserting (5-31) and (5-32) into (5-30) gives the mutual impedance referred to the base of the antenna

$$Z_{12} = Z_{21} = -30\,\frac{I_{1max}\, I_{2max}}{I_1(0)\, I_2(0)} \int_{-h_2}^{h_2} \sin\beta\,(h_2 - |Z|)$$

$$\left[-j\,\frac{e^{-j\beta r_1}}{r_1} - j\,\frac{e^{-j\beta r_2}}{r_2} + \frac{2j\cos\beta h_1\, e^{-j\beta r_o}}{r_o} \right] dz \qquad (5\text{-}33)$$

From Fig. 3-11.

$$\left.\begin{aligned} r_o &= \sqrt{d^2 + z^2} \\ r_1 &= \sqrt{d^2 + (h_1 - z)^2} \\ r_2 &= \sqrt{d^2 + (h_1 + z)^2} \end{aligned}\right\} \qquad (5\text{-}34)$$

Under the assumption of sinusoidal currents the maximum currents are related to the above currents by

$$\left.\begin{aligned} I_1(0) &= I_{1\,max} \sin\beta h_1 \\ I_2(0) &= I_{2\,max} \sin\beta h_2 \end{aligned}\right\} \qquad (5\text{-}35)$$

Therefore (5-33) can be written as

$$Z_{12} = -30 \operatorname{cosec}\beta h_1 \operatorname{cosec}\beta h_2 \int_{-h_2}^{h_2} \sin\beta (h_2 - |Z|)$$

$$\left[\frac{-je^{-j\beta r_1}}{r_1} - \frac{je^{-j\beta r_2}}{r_2} + \frac{2j \cos\beta h_1 e^{-j\beta r_o}}{r_o} \right] dZ \qquad (5\text{-}36)$$

Integration of (5-36) yields an expression for the mutual impedance in terms of cosine integral and sine integral functions.

$$Z_{12} = \frac{60}{\cos w_2 - \cos w_1} \left\{ e^{jw_1} \left[K (U_o) - K (U_1) - k (U_2) \right. \right.$$

$$+ e^{jw_1} \left[K (V_o) - K (V_1) - K (V_2) + e^{jw_2} \left[K (U_o^1) - K (U_1) - K (V_2) \right] \right.$$

$$+ e^{jw_2}\left\{K(V_o') - K(V_1) - K(U_2)\right\} + 2K(w_o)\left[\cos w_1 + \cos w_2\right]\Big\}^* \quad (5\text{-}37)$$

The * denotes the complex conjugate of the expression in the braces.

Here $$K(X) = Ci(X) + j\,Si(X) \quad (5\text{-}38)$$

where $Ci(X)$ and $Si(X)$ are the cosine integral and sine integral functions of the real argument X;

Also;

$$U_o = \beta\left[\sqrt{d^2 + (h_1 + h_2)^2} - (h_1 + h_2)\right]$$

$$V_o = \beta\left[\sqrt{d^2 + (h_1 + h_2)^2} + (h_1 + h_2)\right]$$

$$U_o' = \beta\left[\sqrt{d^2 + (h_1 - h_2)^2} - (h_1 - h_2)\right]$$

$$V_o' = \beta\left[\sqrt{d^2 + (h_1 - h_2)^2} + (h_1 - h_2)\right]$$

$$U_1 = \beta\left[\sqrt{d^2 + h_1^2} - h_1\right]$$

$$V_1 = \beta\left[\sqrt{d^2 + h_1^2} + h_1\right]$$

$$U_2 = \beta\left[\sqrt{d^2 + h_2^2} - h_2\right]$$

$$V_2 = \beta\left[\sqrt{d^2 + h_2^2} + h_2\right]$$

$$w_1 = \beta(h_1 + h_2)$$

$$w_2 = \beta(h_1 - h_2)$$

$$w_o = \beta d$$

where β is the free space propagation constant, d is the seperation of the two dipoles, and h_1, and h_2 are the half- lengths of dipoles one and two respectively.