CHAPTER III

THE REPRESENTATION OF HYPERCOMPLEX NUMBER SYSTEMS

I. MATRIX REPRESENTATION

I. The complex numbers can be represented by 2 × 2 matrices whose elements are real numbers.

Let $z = x + iy$ be a complex number and consider the representation:

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \ldots\ldots\ldots\ldots (1)$$

$$i \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \ldots\ldots\ldots\ldots (2)$$

$$\text{then } x + iy \leftrightarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \quad \ldots\ldots\ldots\ldots (3)$$

These representations (1), (2) and (3) preserve the properties of equatity, addition, and multiplication defined for complex numbers in Chapter II.

$$\text{Let } z_1 = x_1 + iy_1 \leftrightarrow \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \quad \text{and}$$

$$z_2 = x_2 + iy_2 \leftrightarrow \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}$$

Then (1) $\begin{pmatrix} x_1 & y_1 \\ -y & x_1 \end{pmatrix} = \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}$ iff $x_1 = x_2$ and $y_1 = y_2$

that is $\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} = \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}$ iff $z_1 = z_2$

(2) $\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} + \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{pmatrix}$

that is $\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} + \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} \leftrightarrow z_1 + z_2$

$$(3)\quad \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & x_1y_2 + y_1x_2 \\ -(y_1x_2 + x_1y_2) & -y_1y_2 + x_1x_2 \end{pmatrix}$$

that is $\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} \leftrightarrow z_1z_2$

All the properties of complex numbers in Chapter II which follow from the definitions can therefore equally well be obtained by manipulating the above matrices.

Note:

The modulus of z is related determinant of the corresponding matrix as follows:

$$|z|^2 = x^2 + y^2 = \begin{vmatrix} x & y \\ -y & x \end{vmatrix}$$

A quaternion may be represented by a 4×4 matrix of real numbers or a 2×2 matrix of complex numbers as follows:

$$Q \leftrightarrow \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \leftrightarrow \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

where z and w are complex numbers, a, b, c, d are real numbers.

To show that the above representations preserve the properties of quaternions;

Let $z_1 = a_1 + ib_1$, $w_1 = c_1 + id_1$

$z_2 = a_2 + ib_2$, $w_2 = c_2 + id_2$

and consider the correspondence

$$Q_1 \leftrightarrow \begin{pmatrix} z_1 & w_1 \\ -\bar{w}_1 & \bar{z}_1 \end{pmatrix}$$

$$Q_2 \leftrightarrow \begin{pmatrix} z_2 & w_2 \\ -\bar{w}_2 & \bar{z}_2 \end{pmatrix}$$

1. The matrices representing Q_1 and Q_2 are equal iff $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$

2. The sum of the matrices representing Q_1 and Q_2 is the matrix

$$\begin{pmatrix} z_1 + z_2 & w_1 + w_2 \\ -\bar{w}_1 - \bar{w}_2 & \bar{z}_1 + \bar{z}_2 \end{pmatrix} = \begin{pmatrix} z_1 + z_2 & w_1 + w_2 \\ -\overline{(w_1 + w_2)} & \overline{(z_1 + z_2)} \end{pmatrix}$$

This represents the quaternion $(a_1 + a_2) + i\,(b_1 + b_2) + j\,(c_1 + c_2) + k\,(d_1 + d_2)$, which is the sum of Q_1 and Q_2.

3. The product of the matrices representing Q_1 and Q_2 is the matrix

$$\begin{pmatrix} z_1 z_2 - w_1 \bar{w}_2 & z_1 w_2 + w_1 \bar{z}_2 \\ -z_2 \bar{w}_1 - \bar{z}_1 \bar{w}_2 & -\bar{w}_1 w_2 + \bar{z}_1 \bar{z}_2 \end{pmatrix}$$

This represents the quaternion

$A + iB + jC + kD$ where

$$A = a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2$$

$$B = a_1 b_2 + a_2 b_1 + c_1 d_2 - c_2 d_1$$

$$C = a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2$$

$$D = b_1 c_2 + a_1 d_2 + d_1 a_2 - c_1 b_2 ,$$

which is the product of Q_1 and Q_2.

All properties proved for quaternions from the definitions in Chapter II can therefore equally well be obtained by manipulating the matrices above.

Note:

The modulus of Q is related to the determinant of the matrices for quaternion Q, as follows.

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2$$

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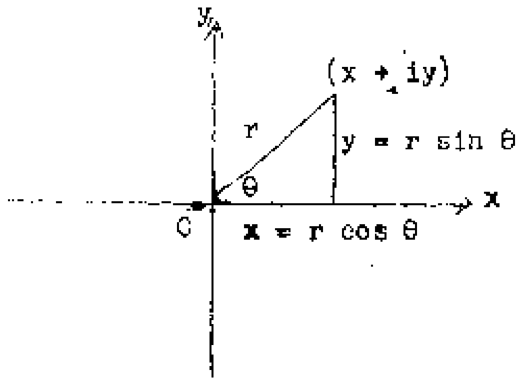
3. Cayley numbers can not be represented by matrices because the associative law of multiplication does not hold in the Cayley number system, but matrices must obey associative law.

II GEOMETRIC INTERPRETATION

The complex numbers system is a two - dimensional vector space, so that complex numbers may be represented by the points of a cartesian plane. (2, p. 104). The point in the cartesian plane can be specified by rectangular or polar co-ordinates as follows.



Here, $r = x + iy = \sqrt{x^2 + y^2}$,

and, $\theta = \arg |x + iy| = \arctan \frac{y}{x}$,

where

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \text{when} \quad x > 0$$

$$\frac{\pi}{2} < \theta < \frac{3\pi}{2} \quad \text{when} \quad x < 0$$

$$\theta = \frac{\pi}{2} \quad \text{when} \quad x = 0 \text{ and } y > 0$$

$$\theta = -\frac{\pi}{2} \quad \text{when} \quad x = 0 \text{ and } y < 0$$

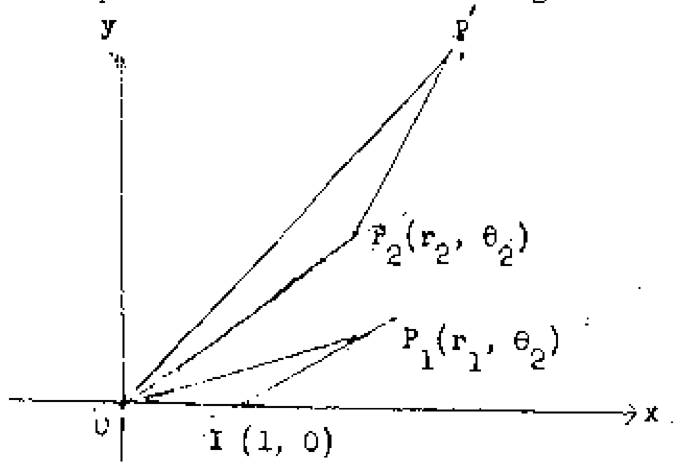
The product of the complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ can be written in polar form,

$$(x_1 + iy_1)(x_2 + iy_2) = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$$

where $r_1 = |x_1 + iy_1|$; $r_2 = |x_2 + iy_2|$,

and $\theta_1 = \arg(x_1 + iy_1)$; $\theta_2 = \arg(x_2 + iy_2)$.

This product can be constructed geometrically as follows.



On the segment OP_2 in the diagram construct a triangle OP_2P' similar to the triangle OIP_1, where $P'\hat{O}P_2 = P_1\hat{O}I$, $O\hat{P_2}P' = O\hat{I}P_1$ then P' is the point $(r_1r_2, \theta_1 + \theta_2)$

Proof.

since $\Delta\ OIP_1$ and $\Delta\ OP_2P'$ are similar,

we have $\dfrac{OP'}{OP_1} = \dfrac{OP_2}{OI}$

therefore $OP' = OP_1 \cdot OP_2 = r_1r_2$

Also since $I\hat{O}P_2 = \theta_2$ and $P_2\hat{O}P' = \theta_1$,

we have $I\hat{O}P' = \theta_1 + \theta_2$

Hence the point P' is $(r_1r_2\ , \theta_1 + \theta_2)$.

It is clear from the above discussion that the multiplication of a complex number z by another complex number A has the effect of rotation the vector z through an angle arg A and multiplying its length by $|A|$ to produce a new vector z ; where $z' = Az$

This can be written $x' + iy' = (a + ib)\ (x + iy)$

$= (ax - by) + i\ (bx + ay)$ .... (1)

Also $(x' + iy')(x' - iy') = (a + ib)(x + iy)(a - ib)(x - iy)$

$= (a + ib)(a - ib)(x + iy)(x - iy)$

Therefore $(x'^2 + y'^2) = (a^2 + b^2)(x^2 + y^2)$

or $|z'| = |A| \, |z|$ .... (2)

The above equations (1) and (2) show that the linear transformation $z' = Az$ is orthogonal when $|A| \neq 0$. (3, p. 67)

This linear transformation represents a rotation about the origin and a magnification of the vector representing z.

If $|A| = 1$, the linear transformation $z' = Az$ represents a pure rotation about the origin.

The multiplication of a quaternion Q by another quaternion A has the effect of rotation and expansion.

Let $Q' = AQ$ and $\bar{Q}' = \bar{Q}\bar{A}$, this can be written

$$x' + iy' + jz' + kt' = (a + ib + jc + kd)(x + iy + jz + kt)$$

$$= (ax - by - cz - dt) + i\,(ay + bx - dz + ct)$$

$$+ j\,(cx + dy + az - bt) + k\,(dx - cy + bz + at) \quad \ldots (1)$$

Also $Q'\bar{Q}' = (A\,Q)(\bar{Q}\,\bar{A})$

$= A\,(Q\,\bar{Q})\,\bar{A}$, by the associative law,

$= (A\bar{A})\,(Q\bar{Q})$, since $Q\bar{Q}$ is real and commentes with any quaternion.

Therefore $|Q'|^2 = |A|^2 \, |Q|^2$, or $|Q'| = |A| \, |Q|$ .... (2)

The above equations (1) and (2) show that the linear transformation $Q' = AQ$ is orthogonal when $|A| \neq 0$. This linear transformation represents a rotation about the origin with an expansion by the factor A about origin. (3, p. 67)

If $|A| = 1$, the linear transformation

$Q' = AQ$, represents a pure rotation about the origin.

Note : Modern mathematicians differ from Klein (ref. 3) in using " orthogonal " to a pure rotation without magnifigation.

The multiplication of a Cayley number C with other Cayley number A will have an effect of rotation the vector C.

Consider the equation $C' = AC$,

where

$$C' = \sum_{0}^{7} e_i x_i'$$

$$A = \sum_{0}^{7} e_i a_i$$

$$C = \sum_{0}^{7} e_i x_i$$

Therefore

$$\sum_{0}^{7} e_i x_i' = \left(\sum_{0}^{7} e_i a_i\right)\left(\sum_{0}^{7} e_i x_i\right)$$

$$\begin{pmatrix} x_0' \\ x_1' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_7' \end{pmatrix} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & -a_7 & a_6 \\ a_2 & a_3 & a_0 & -a_1 & a_6 & -a_7 & -a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & -a_6 & a_5 & a_4 \\ a_4 & a_5 & -a_6 & a_7 & a_0 & -a_1 & a_2 & -a_3 \\ a_5 & -a_4 & a_7 & a_6 & a_1 & a_0 & -a_3 & -a_2 \\ a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & a_0 & -a_1 \\ a_7 & -a_6 & -a_5 & -a_4 & a_3 & a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_7 \end{pmatrix}$$

This may be written

$$\xi' = BT .$$

Since

$$BB^t = (a_0^2 + a_1^2 + a_2^2 + \ldots\ldots + a_7^2)^4 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

the transformation above is orthogonal, matrix B will rotate the vector $(x_0, x_1, \ldots\ldots\ldots\ldots x_7)^t$. If $|D| = 1$, this transformation is a pure rotation.