SOME HYPERCOMPLEX NUMBER SYSTEMS

by

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ABSTRACT

Thesis Title : SOME HYPERCOMPLEX NUMBER SYSTEMS

The aim of this thesis is to study the properties and applications of some hypercomplex number systems. A hypercomplex number system is a vector space over a field of real or rational number. Let x_i ($i = o, 1, 2, \ldots\ldots n$) belong to a field of real of rational numbers. Then a hypercomplex number of n dimensional space can be written $H = \sum_{o}^{n} e_i x_i$ where e_i ($i = o, 1, \ldots\ldots\ldots n$) are basis elements. If $n = 1$, H is a complex number; if $n = 3$, H is a quaternion; and if $n = 7$, H is a Cayley number. Equality, addition and multiplication are defined as follows.

Let $H_1 = \sum_{o}^{n} e_i x_i$ and $H_2 = \sum_{o}^{n} e_i y_i$, where $e_0 = 1$.

1. $H_1 = H_2$ iff corresponding x_i are equal to y_i

2. $H_1 + H_2 = \sum_{o}^{n} e_i x_i + \sum_{o}^{n} e_i y_i = \sum_{o}^{n} e_i (x_i + y_i)$

3. Products of the basis elements are defined in the table below,

$H_1 \backslash H_2$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	e_2	$-e_3$	-1	e_1	$-e_6$	e_7	e_4	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	e_4	$-e_5$	e_6	$-e_7$	-1	e_1	$-e_2$	e_3
e_5	e_5	e_4	$-e_7$	$-e_6$	$-e_1$	-1	e_3	e_2
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Two of the hypercomplex number systems can be represented by matrices as follows:

A complex number $z = x_o + e_1 e_1 \leftrightarrow \begin{pmatrix} x_o & x_1 \\ -x_1 & x_o \end{pmatrix}$

A quaternion $Q = x_o + e_1 x_1 + e_2 x_2 + e_3 x_3 \leftrightarrow \begin{pmatrix} x_o & x_1 & x_2 & x_3 \\ -x_1 & x_o & -x_3 & x_2 \\ -x_2 & x_3 & x_o & -x_1 \\ -x_3 & -x_2 & x_1 & x_o \end{pmatrix}$

Cayley Numbers cannot be represented by matrices because in this case multiplication fails to be associative.

The hypercomplex number systems also can be interpreted geometrically. A complex number is interpreted as a point of Cartesian plane. The product of two complex numbers $Z' = AZ$, where $A = a_o + e_1 a_1$, $Z = x_o + e_1 x_1$ and $Z' = x_o' + e_1 x_1'$, represents a rotation of the point Z about the origin. The product of two quaternions $Q' = AQ$ where $A = a_o + e_1 a_1 + e_2 a_2 + e_3 a_3$, $Q = x_o + e_1 x_1 + e_2 x_2 + e_3 x_3$; represents a rotation of a point in four dimensional Cartesian space about the origin, but not every rotation can be represented in this form. A general rotation of Q can be represented by the product of three quaternions $Q' = AQB$. As a physical application of this Lorentz transformations

$$\begin{pmatrix} x' \\ y' \\ z' \\ \tau' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \frac{\gamma v i}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v i}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \tau \end{pmatrix},$$ can be

written in this form if we put $A = B = \frac{-vi\gamma}{c\sqrt{2(1-\gamma)}} + e_3\sqrt{\frac{1-\gamma}{2}}$.

This is an interesting application of quaternions. Another application of hypercomplex numbers is to prove identities between real numbers containing expressions of the type $(a_1^2 + a_2^2 + \ldots\ldots a_n^2)$. $(b_1^2 + b_2^2 + \ldots\ldots\ldots b_n)$.

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