CHAPTER II

THEORETICAL CONSIDERATIONS

2.1 Basic Equations and General Solutions

The constitutive relations of a homogeneous poroelastic material with compressible constituents can be expressed with respect to a conventional cylindrical polar coordinate system (r,θ,z) as shown in Fig. 2.1 by using the standard indicial notation as (Rice and Cleary,1976)

$$\sigma_{ij} = 2\mu\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon\right] - \frac{3(\nu_u-\nu)}{B(1-2\nu)(1+\nu_u)}\delta_{ij}p \ , \ i,j=r,\theta,z \quad (2.1)$$

$$p = -\frac{2\mu B(1+\nu_u)}{3(1-2\nu_u)}\varepsilon + \frac{2\mu B^2(1-2\nu)(1+\nu_u)^2}{9(1-2\nu_u)(\nu_u-\nu)}\zeta \quad (2.2)$$

In the above equations , σ_{ij} is the total stress component of bulk material ; ε_{ij} and ε are the strain component and the dilatation of solid matrix, respectively ; p is defined as the excess pore fluid pressure (suction is considered negative) ; μ , ν and ν_u denote the shear modulus, drained and undrained Poisson's ratios, respectively ; B is Skempton's pore pressure coefficient (Skempton, 1954) ; δ_{ij} is the Kronecker delta. In addition , ζ is defined as the variation of pore fluid volume per unit reference volume. It is noted that $0 \leq B \leq 1$ and $\nu \leq \nu_u \leq 0.5$ for all poroelastic materials (Rajapakse and Senjuntichai,1993). The limiting case of a poroelastic solid with incompressible constituents and a dry elastic material are obtained when $\nu_u = 0.5$ and $B = 1$, and $B \to 0$, respectively.

The quasi-static governing equations of a poroelastic medium with compressible constituents , expressed in terms of stresses and pore pressure as basic variables, can be transformed into Navier equations with coupling terms and a diffusion equation by treating the displacement and the variation of fluid volume as basic unknowns. The governing equations can be expressed as (Rajapakse and Senjuntichai ,1993)

$$\nabla^2 u_r + \frac{1}{1-2\nu_u}\frac{\partial\varepsilon}{\partial r} - \frac{1}{r}\left[\frac{2}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right] - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial\zeta}{\partial r} = 0 \qquad (2.3)$$

$$\nabla^2 u_\theta + \frac{1}{1-2\nu_u}\frac{\partial\varepsilon}{r\partial\theta} - \frac{1}{r}\left[\frac{u_\theta}{r} - \frac{2}{r}\frac{\partial u_r}{\partial\theta}\right] - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{1}{r}\frac{\partial\zeta}{\partial\theta} = 0 \qquad (2.4)$$

$$\nabla^2 u_z + \frac{1}{1-2\nu_u}\frac{\partial\varepsilon}{\partial z} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial\zeta}{\partial z} = 0 \qquad (2.5)$$

$$\nabla^2 \zeta = \frac{\partial\zeta}{c\partial t} \qquad (2.6)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \qquad (2.7)$$

$$\varepsilon = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} \qquad (2.8)$$

$$c = \frac{2\mu\kappa B^2(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u-\nu)} \qquad (2.9)$$

In the above equations, u_i and κ denote the average displacement of the solid matrix in the *i*-direction $(i = r, \theta, z)$ and the permeability of the medium respectively. κ is greater than zero and equal to k/γ_w where k is the coefficient of permeability and γ_w is the unit weight of pore fluid. It can be shown that general solutions for eqns(2.3)-(2.6) can be derived by applying Fourier expansion , Laplace and Hankel transforms with respect to circumferential coordinate, time and radial coordinate, respectively (Rajapakse and Senjuntichai, 1993).

The application of Fourier expansion with respect to the circumferential coordinate θ for displacement u_i and pore pressure p results in

$$u_i(r,\theta,z,t) = \sum_{m=0}^{\infty} u_{im}(r,z,t) f(\theta) - \sum_{m=0}^{\infty} \widetilde{u}_{im}(r,z,t) f'(\theta) \qquad (2.10)$$

$$p(r,\theta,z,t) = \sum_{m=0}^{\infty} p_m(r,z,t) \cos m\theta + \sum_{m=0}^{\infty} \widetilde{p}_m(r,z,t) \sin m\theta \qquad (2.11)$$

where

$$f(\theta) = \begin{cases} \cos m\theta, & i \neq \theta \\ \sin m\theta, & i = \theta \end{cases} \qquad (2.12)$$

In eqns (2.10) and (2.11) , u_{im} and p_m are symmetric components and $\widetilde{u}_{im}$ and $\widetilde{p}_{im}$ are anti-symmetric components corresponding to the *m*th harmonic , respectively . In addition, $f'(\theta)$ denotes the derivative of $f(\theta)$ with respect to the circumferential coordinate θ.

The Laplace-Hankel transform (*m*th order) of function $\phi(r,z,t)$ with respect to the variables t and r , respectively , is defined as (Sneddon, 1951)

$$\overline{\chi}_m\{\phi(r,z,t)\} = \int_0^{\infty}\int_0^{\infty} \phi(r,z,t) e^{-st} J_m(\xi r) r\,dr\,dt \qquad (2.13)$$

where s and ξ denote the Laplace and Hankel tranform parameters respectively, and J_m denotes the Bessel function of the first kind of order m.The inverse relationship is given by

$$\phi(r,z,t)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\int_{0}^{\infty}\bar{\chi}_m[\phi(r,z,t)]e^{st}J_m(\xi r)\xi d\xi ds \quad (2.14)$$

in which β is greater than the real part of all singularities of $\bar{\chi}_m[\phi(r,z,t)]$ and $i=\sqrt{-1}$

The general solutions for the mth harmonic of fluid and solid displacement , pore pressure and stresses in the Laplace-Hankel transform space can be expressed in the following matrix form (Senjuntichai and Rajapakse, 1995).

$$\mathbf{v}(\xi,z,s)=\mathbf{R}(\xi,z,s)\mathbf{C}(\xi,s) \quad (2.15)$$

$$\mathbf{f}(\xi,z,s)=\mathbf{S}(\xi,z,s)\mathbf{C}(\xi,s) \quad (2.16)$$

where

$$\mathbf{v}(\xi,z,s)=\langle \mathrm{v}_i(\xi,z,s)\rangle^T \quad , \; i=1\,,2\,,3\,,4 \quad (2.17)$$

$$\mathbf{f}(\xi,z,s)=\langle f_i(\xi,z,s)\rangle^T \quad , \; i=1\,,2\,,3\,,4 \quad (2.18)$$

$$\mathrm{v}_1(\xi,z,s)=\frac{1}{2}[\bar{\chi}_{m+1}(u_{rm}+u_{\theta m})-\bar{\chi}_{m-1}(u_{rm}-u_{\theta m})] \quad (2.19)$$

$$\mathrm{v}_2(\xi,z,s)=\frac{1}{2}[\bar{\chi}_{m+1}(u_{rm}+u_{\theta m})+\bar{\chi}_{m-1}(u_{rm}-u_{\theta m})] \quad (2.20)$$

$$\mathrm{v}_3(\xi,z,s)=\bar{\chi}_m(u_{zm}) \quad (2.21)$$

$$\mathrm{v}_4(\xi,z,s)=\bar{\chi}_m(p_m) \quad (2.22)$$

$$f_1(\xi,z,s) = \frac{1}{2}[\overline{\chi}_{m+1}(\sigma_{zrm} + \sigma_{z\theta m}) - \overline{\chi}_{m-1}(\sigma_{zrm} - \sigma_{z\theta m})] \tag{2.23}$$

$$f_2(\xi,z,s) = \frac{1}{2}[\overline{\chi}_{m+1}(\sigma_{zrm} + \sigma_{z\theta m}) + \overline{\chi}_{m-1}(\sigma_{zrm} - \sigma_{z\theta m})] \tag{2.24}$$

$$f_3(\xi,z,s) = \overline{\chi}_m(\sigma_{zzm}) \tag{2.25}$$

$$f_4(\xi,z,s) = \overline{\chi}_m(w_{zm}) \tag{2.26}$$

$$\mathbf{C}(\xi,s) = < A_m \ B_m \ C_m \ D_m \ E_m \ F_m \ G_m \ H_m >^T \tag{2.27}$$

In eqn(2.26) , w_{zm} denotes the symmetric components corresponding to the mth Fourier harmonic of the fluid displacement relative to the solid matrix in the z-direction. The matrices $\mathbf{R}(\xi,z,s)$ and $\mathbf{S}(\xi,z,s)$ in eqns (2.15) and (2.16) are given by eqns (A1) and (A2) in appendix A. The arbitrary functions $A_m(s,\xi)$,$B_m(s,\xi)$,..., $G_m(s,\xi)$, $H_m(s,\xi)$ in eqn(2.27) are to be determined by employing appropriate boundary and/or continuity conditions.

2.2 Stiffness Matrices and Influence Functions

The analysis of the problem in Fig 1.1 by using the discretization technique proposed by Rajapakse(1988) requires the computation of a set of displacement influence functions of a multi-layered half-space due to unit vertical pressure applied over an annular region on the surface of the half-space as shown in Fig 2.2. An exact stiffness matrix method is used to calculate the required influence functions. A brief outline of the exact stiffness matrix scheme proposed by Senjuntichai and Rajapakse (1995) is presented in this section.

Consider a multi-layered poroelastic half-space as shown in Fig 2.2, for an nth layer , the following relationships can be established by using eqns (2.15) and (2.16);

$$\mathbf{U}^{(n)} = \begin{bmatrix} \mathbf{R}^{(n)}(\xi, z_n, s) \\ ---------- \\ \mathbf{R}^{(n)}(\xi, z_{n+1}, s) \end{bmatrix} \mathbf{C}^{(n)}(\xi, s) \tag{2.28}$$

$$\mathbf{F}^{(n)} = \begin{bmatrix} -\mathbf{S}^{(n)}(\xi, z_n, s) \\ ---------- \\ \mathbf{S}^{(n)}(\xi, z_{n+1}, s) \end{bmatrix} \mathbf{C}^{(n)}(\xi, s) \tag{2.29}$$

where

$$\mathbf{U}^{(n)} = < \mathbf{v}^{(n)}(\xi, z_n, s) \quad \mathbf{v}^{(n)}(\xi, z_{n+1}, s) >^T \tag{2.30}$$

$$\mathbf{F}^{(n)} = < -\mathbf{f}^{(n)}(\xi, z_n, s) \quad \mathbf{f}^{(n)}(\xi, z_{n+1}, s) >^T \tag{2.31}$$

In the above equations , $\mathbf{U}^{(n)}$ denotes a vector of generalized coordinates for the nth layer whose elements are related to the Laplace-Hankel transforms of the mth Fourier harmonic of displacements and pore pressure of the top and bottom surfaces of the nth layer. Similarly, $\mathbf{F}^{(n)}$ denotes a generalized force vector whose elements are related to the Laplace-Hankel transform of the mth harmonic of tractions and fluid displacement of the top and bottom surfaces of the nth layer.

The vectors $\mathbf{v}^{(n)}$ and $\mathbf{f}^{(n)}$ in eqns (2.30) and (2.31) are identical to $\mathbf{v}$ and $\mathbf{f}$ defined in eqns (2.15) and (2.16) except that the material properties of the nth layer are employed in the definition and $z = z_n$ or z_{n+1}. The eqn(2.28) can be inverted to express $\mathbf{C}^{(n)}$ in terms of $\mathbf{U}^{(n)}$ and the substitution into eqn(2.29) yields.

$$\mathbf{F}^{(n)} = \mathbf{K}^{(n)}\mathbf{U}^{(n)} \quad , \quad n = 1,2,\ldots,N \tag{2.32}$$

where $\mathbf{K}^{(n)}$ can be considered as an exact stiffness matrix in the Laplace-Hankel transform space describing the relationship between the generalized coordinate vector $\mathbf{U}^{(n)}$ and the generalized force vector $\mathbf{F}^{(n)}$ for the nth layer. Similarly , the stiffness matrix for an underlying half-space $\mathbf{K}^{(N+1)}$ can be obtained by establishing a relationship between the generalized coordinate vector $\mathbf{U}^{(N+1)}$ and the generalized force vector $\mathbf{F}^{(N+1)}$ for the underlying half-space. The stiffness matrix for the half-space can be expressed as

$$\mathbf{F}^{(N+1)} = \mathbf{K}^{(N+1)}\mathbf{U}^{(N+1)} \tag{2.33}$$

where

$$\mathbf{U}^{(N+1)} = < \mathbf{v}^{(N+1)}(\xi, z_{N+1}, s) >^T \tag{2.34}$$

$$\mathbf{F}^{(N+1)} = < -\mathbf{f}^{(N+1)}(\xi, z_{N+1}, s) >^T \tag{2.35}$$

The global stiffness matrix of a multi-layered half-space is assembled by using the layer and half-space stiffness matrices together with the continuity of tractions and fluid flow at the layer interfaces. For example , the continuity conditions at the nth interface can be expressed as

$$\mathbf{f}^{(n-1)}(\xi, z_n, s) - \mathbf{f}^{(n)}(\xi, z_n, s) = \mathbf{T}^{(n)} \tag{2.36}$$

where $\mathbf{f}^{(n-1)}$ and $\mathbf{f}^{(n)}$ are as defined in eqn(2.18) . In addition, $\mathbf{T}^{(n)}$ denotes the tractions and fluid source applied at the nth interface and

$$\mathbf{T}^{(n)} = < T_1^{(n)} \quad T_2^{(n)} \quad T_3^{(n)} \quad \frac{Q^{(n)}}{s} >^T \tag{2.37}$$

in which

$$T_1^{(n)} = \frac{1}{2}\left[\bar{\chi}_{m+1}(T_{rm}^{(n)} + T_{\theta m}^{(n)}) - \bar{\chi}_{m-1}(T_{rm}^{(n)} - T_{\theta m}^{(n)})\right] \tag{2.38}$$

$$T_2^{(n)} = \frac{1}{2}\left[\bar{\chi}_{m+1}(T_{rm}^{(n)} + T_{\theta m}^{(n)}) + \bar{\chi}_{m-1}(T_{rm}^{(n)} - T_{\theta m}^{(n)})\right] \tag{2.39}$$

$$T_3^{(n)} = \bar{\chi}_m(T_{zm}^{(n)}) \tag{2.40}$$

$$Q^{(n)} = \bar{\chi}_m(Q_m^{(n)}) \tag{2.41}$$

where $T_{im}^{(n)}(i = r,\theta,z)$ and $Q_m^{(n)}$ denote the *m*th Fourier harmonic of the tractions and fluid source applied at the *n*th interface , respectively .

The consideration of eqn(2.36) at each layered interface together with the layers and the bottom half-space stiffness matrices defined in eqns(2.32) and (2.33) results in the following global stiffness equation system.

$$\begin{bmatrix} K^{(1)} & & & \\ & K^{(2)} & & \\ & & \ddots & \\ & & & K^{(N)} \\ & & & & K^{(N+1)} \end{bmatrix} \begin{Bmatrix} U^{(1)} \\ U^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ U^{(N)} \\ U^{(N+1)} \end{Bmatrix} = \begin{Bmatrix} T^{(1)} \\ T^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ T^{(N)} \\ T^{(N+1)} \end{Bmatrix} \tag{2.42}$$

The solutions of the above equation are the influence functions required to establish the flexibility equation for the deriviation of the strain energy of a multi-layered poroelastic half-space in the section 2.3.2 .

2.3 Variational Formulation for Interaction Problem

2.3.1 Strain Energy of Circular Elastic Plate

By adopting the displacement function given by Rajapakse (1988) , the vertical displacement of a circular elstic plate resting on a surface of a multi-layered poroelastic half-space as shown in Fig 1.1 can be represented by the following kinematically admissible form :

$$w(r,t) = a_0(t) r^2 \ln r + \sum_{n=0}^{N} \alpha_n(t) r^{2n} \quad , \qquad 0 \le r \le 1 \qquad (2.43)$$

where

$$a_0(t) = P_o(t)/8\pi D \qquad (2.44)$$

$$D = E_p t_p^3 / \left[12(1 - \nu_p^2)\right] \qquad (2.45)$$

From eqns(2.43) to (2.45) , $\alpha_n(t)$ where $n = 0,1,2,...,N$ denotes a set of generalized coordinates, $P_0(t)$ is the central vertical force acting on the plate, t_p is the thickness of the plate and E_p and ν_p are defined as the Young's modulus and Poisson's ratio of the plate material , respectively. The term $r^2 \ln r$ in eqn(2.43) simulates the singular stress resultants at the plate origin due to the presence of concentrated force $P_0(t)$. In addition , only the even terms in the power of r have been used in the second term of eqn(2.43) because the term r is inadmissible to represent the plate displacement and the proposed form is compact and convenient for manipulation.

Bending moment and shear force per unit length of a circular elastic plate subjected to axisymmetric loadings are given by (Timoshenko and Woinowsky-Krieger,1959)

$$M_r = -D\left(\frac{d^2 w}{dr^2} + \frac{\nu_p}{r}\frac{dw}{dr}\right) \qquad (2.46)$$

$$Q = -D\left(\frac{d^3 w}{dr^3} + \frac{1}{r}\frac{d^2 w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \qquad (2.47)$$

where M_r and Q denote the bending moment and shearing force per unit length acting along circumferential sections of the plate , respectively.

By substituting of eqn (2.43) into the above equations the bending moment and shear force per unit length of the plate can be expressed as

$$M_r(r,t) = -D\left\{a_0(3+2\ln r+\nu_p+2\nu_p\ln r)+\sum_{n=1}^{N}\alpha_n\left[2n(2n-1)+2n\nu_p\right]r^{2n-2}\right\}$$

(2.48)

$$Q(r,t) = -D\left[\frac{4a_0}{r}+\sum_{n=2}^{N}4n^2(2n-2)\alpha_n r^{2n-3}\right] \tag{2.49}$$

By using the Laplace transformation , the eqn (2.43) can be transformed to the Laplace domain and can be rewritten as

$$\overline{w}(r,s) = \overline{a}_0(s)r^2\ln r+\sum_{n=0}^{N}\overline{\alpha}_n(s)r^{2n} \quad , \qquad 0\le r\le 1 \tag{2.50}$$

where

$$\overline{a}_0(s) \;= \overline{P}_o(s)/8\pi D \tag{2.51}$$

and $\overline{\alpha}_n(s)$ is defined as the Laplace transform of $\alpha_n(t)$. In addition , $\overline{P}_0(s)$ is the Laplace transform of $P_0(t)$ where s is the Laplace transform parameter.

The strain energy of a thin circular elastic plate undergoing axisymmetric flexural deformations is given by (Timoshenko and Woinowsky-Krieger,1959)

$$U_p = \int_0^1 \pi D\left[\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)^2-\frac{2(1-\nu_p)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right]r\,dr \tag{2.52}$$

The substitution of eqn (2.50) into the above equation results in an expression for U_p in terms of the generalized coordinates. The strain energy of the plate corresponding to the assumed displacement function can be expressed in the Laplace domain as

$$U_p = \pi D(3+\nu_p)\overline{a}_0^2 + < Q^p > \{\overline{\alpha}\} + \{\overline{\alpha}\}^T \left[K^p\right]\{\overline{\alpha}\} \tag{2.53}$$

where

$$Q_1^p = 0 \tag{2.54}$$

$$Q_i^p = \pi D\left[4\overline{a}_0\left(4(i-1)-(1-\nu_p)i\right) - 4\overline{a}_0(1+\nu_p)\right] \quad ; \quad 2 \le i \le (N+1) \tag{2.55}$$

$$K_{1j}^p = K_{j1}^p = 0 \tag{2.56}$$

$$K_{ij}^p = \frac{4(i-1)(j-1)\pi D}{(2i+2j-6)}\left[4(i-1)(j-1) - 2(1-\nu_p)(2i-3)\right] \quad ; \quad 2 \le i,j \le (N+1) \tag{2.57}$$

2.3.2 Boundary Conditions for Circular Elastic Plate

For a circular elastic plate resting on the surface of a multi-layered poroelastic half-space, the plate deflection is such that it satisfies the following Kirchhoff boundary conditions along the plate edge in the Laplace domain as follows :

$$\overline{M}_r(1,s) = 0 \tag{2.58}$$

$$\overline{Q}(1,s) = 0 \tag{2.59}$$

The above expressions can be expressed in the following matrix form

$$[B]\{\overline{\alpha}\} = \{R\} \tag{2.60}$$

where

$$\{\overline{\alpha}\} = < \overline{\alpha}_0(s) \quad \overline{\alpha}_1(s) \ldots\ldots\ldots \overline{\alpha}_N(s) >^T \tag{2.61}$$

$$B_{11} = B_{21} = B_{22} = 0 \tag{2.62}$$

$$B_{12} = 2(1+\nu_p) \tag{2.63}$$

$$B_{1j} = \left[4(j-1)^2 - 2(1-\nu_p)(j-1)\right] \ , \ 3 \le j \le (N+1) \tag{2.64}$$

$$B_{2j} = 4(j-1)^2(2j-4) \ , \ 3 \le j \le (N+1) \tag{2.65}$$

$$R_1 = -\overline{a}_0(3+\nu_p) \tag{2.66}$$

$$R_2 = -4\overline{a}_0 \tag{2.67}$$

2.3.3 Strain Energy of Multi-Layered Poroelastic Half-Space

The resultant of the traction in the vertical direction acting on the plate surface in Fig 2.3(a) may be represented by a traction field $\overline{T}_z(r,s)$ acting on a circular surface with $0 \le r \le 1$ as shown in Fig 2.3(b). The strain energy of the half-space can be expressed as follows :

$$U_h = \frac{1}{2}\int_0^1 \left[2\pi r \overline{T}_z(r,s).\overline{w}(r,s)\right] dr \tag{2.68}$$

The traction $\overline{T}_z(r,s)$ can be expressed in terms of the generalized coordinates $\overline{\alpha}_n$ as

$$\overline{T}_z(r,s) = \sum_{n=0}^{N} \overline{\alpha}_n(s)\breve{T}_{nz}(r) + \overline{a}_0(s).T_z^*(r) \tag{2.69}$$

where $\breve{T}_{nz}(r)$ and $T_z^*(r)$ denote the tractions in the vertical direction applied over circular surface such that the vertical displacement within circular surface is equal to r^{2n} and $r^2 \ln r$, respectively.

A convenient way to obtain these tractions is to discretize the circular surface into M concentric ring elements , and to evaluate $\breve{T}_{nz}(r)$, $T_z^*(r)$ acting on each ring element by solving a flexibility equation based on the influence functions that can be computed from eqn (2.42). The flexibility equation is given by

$$[F^{zz}][\overline{T}_{zn}] = [\overline{w}_n] \quad , \quad n = 0,1,2,.........N \qquad (2.70)$$

In the above equation , the element F_{ij}^{zz} where i , j = 1, 2,....., M , in $[F^{zz}]$ denotes the influence function which is the displacement at the center of the i^{th} ring element caused by a unit vertical load at the j^{th} ring element. These influence functions are obtained from the global stiffness equation, eqn(2.42) .

In view of eqn(2.50) and eqns(2.68) to (2.70) the strain energy of the half-space U_h can be expressed as

$$U_h = \{\overline{\alpha}\}^T[K^h]\{\overline{\alpha}\} + \langle Q^h \rangle \{\overline{\alpha}\} + \sum_{j=1}^{M} \pi r_j T_{zj}^* \overline{a}_0^2 r_j^2 \ln r_j \Delta r_j \qquad (2.71)$$

where

$$K_{ij}^h = \sum_{l=1}^{M} \breve{T}_{z(i-1)}(r_l).\pi r_l^{2j-1} \Delta r_l \quad , \quad 1 \le i,j \le (N+1) \qquad (2.72)$$

$$Q_i^h = \sum_{j=1}^{M} \left[\pi r^{2i-1} T_{zj}^* \Delta r_j + \pi r_j^3 \ln r_j . \breve{T}_{z(i-1)}(r_j).\Delta r_j\right]\overline{a}_0 \quad , \quad 1 \le i \le (N+1)$$

$$(2.73)$$

2.3.4 Total Potential Energy of Plate - Layered Half-Space System

The total potential energy functional of the system shown in Fig 1.1 consists of the strain energy of the plate and the multi-layered poroelastic half-space given by eqns (2.53) and (2.71) respectively , together with the potential energy of applied loadings. The loadings consist of a centrally applied force P_0 and a uniformly distributed load of intensity q_0 as shown in Fig 1.1. The total potential energy can be expressed in terms of the generalized coordinates as

$$U = \{\overline{\alpha}\}^T\{[K^p]+[K^h]\}\{\overline{\alpha}\}+[< Q^p > + < Q^h >]\{\overline{\alpha}\}+\pi D(3+\nu_p)\overline{\alpha}_0^2$$
$$+\sum_{j=1}^{M}\pi r_j^3\overline{\alpha}_0^2 \ln r_j T_{zj}^* \Delta r_j - \overline{P}_0\overline{\alpha}_0 - 2\pi\overline{q}_0\sum_{n=0}^{N}\overline{\alpha}_n/(2n+2)$$

(2.74)

The energy functional defined by eqn(2.74) does not satisfy the boundary conditions along the plate edge. In order to satisfy these conditions, given by eqn(2.60) , it is necessary to define a constraint energy function $\overline{U}$ of the form

$$\overline{U} = U + \frac{1}{2}\lambda[[B]\{\alpha\}-\{R\}]^T[[B]\{\alpha\}-\{R\}] \qquad (2.75)$$

where λ is a penalty number associated with the constraint term and $\lambda >> \max(K_{ii}^p + K_{ii}^h)$.

The generalized coordinates $\overline{\alpha}_n$ are determined by using the principle of minimum potential energy , which requires that

$$\frac{\partial\overline{U}}{\partial\overline{\alpha}_n} = 0 \qquad , \qquad n = 0,1,\ldots\ldots N \qquad (2.76)$$

Eqn. (2.76) yields the linear simultaneous equation system that can be expressed as

$$[K^s]\{\overline{\alpha}\} = \{\overline{F}\} \tag{2.77}$$

where

$$[K^s] = [K^p] + [K^p]^T + [K^h] + [K^h]^T + \lambda[B]^T[B] \tag{2.78}$$

$$\{\overline{F}\} = <\pi q_0/i>^T + <\overline{P}_0\delta_{1i}>^T + \lambda[B]^T\{R\} - <Q^p>^T - <Q^h>^T \tag{2.79}$$

and K_{ij}^p , K_{ij}^h , Q_{ij}^p and Q_{ij}^h are given in eqns (2.54)-(2.57) and eqns (2.72)-(2.73) . In addition , B_{ij} and R_i are defined as in eqn(2.60).

The solution of the system of simultaneous linear equations given by eqn(2.77) yields the numerical values of the generalized coordinates $\overline{\alpha}_n$. By using the inverse Laplace transformation , $\overline{\alpha}_n$ can be transformed into time domain. Finally , the substitution of α_n into eqn(2.43) and eqns (2.48) and (2.49) results in the quasi-static behaviour of the circular elastic plate.

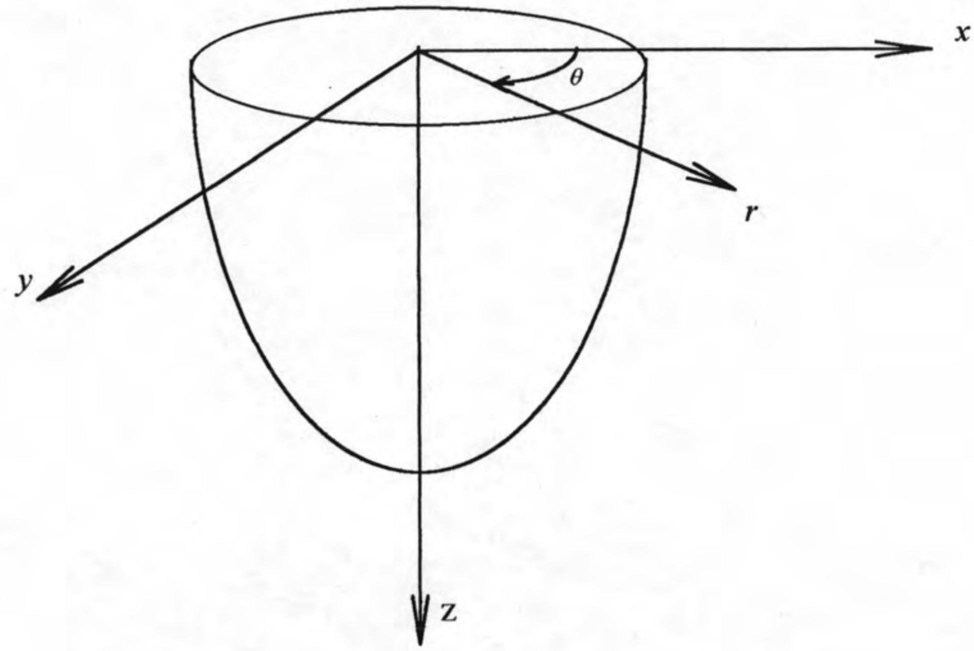


Figure 2.1 : A homogeneous poroelastic half-space

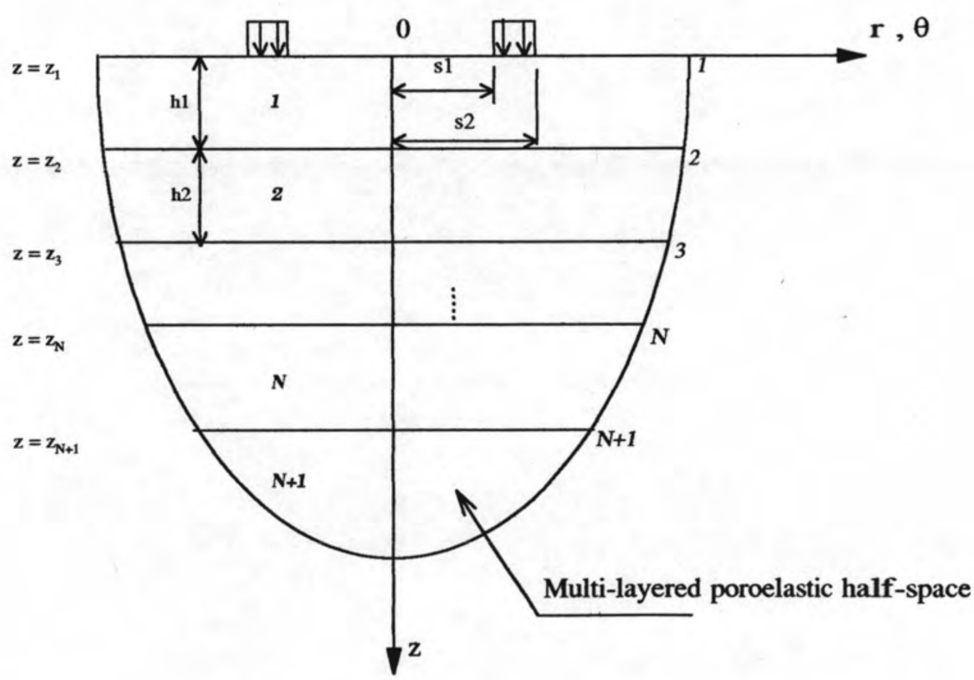
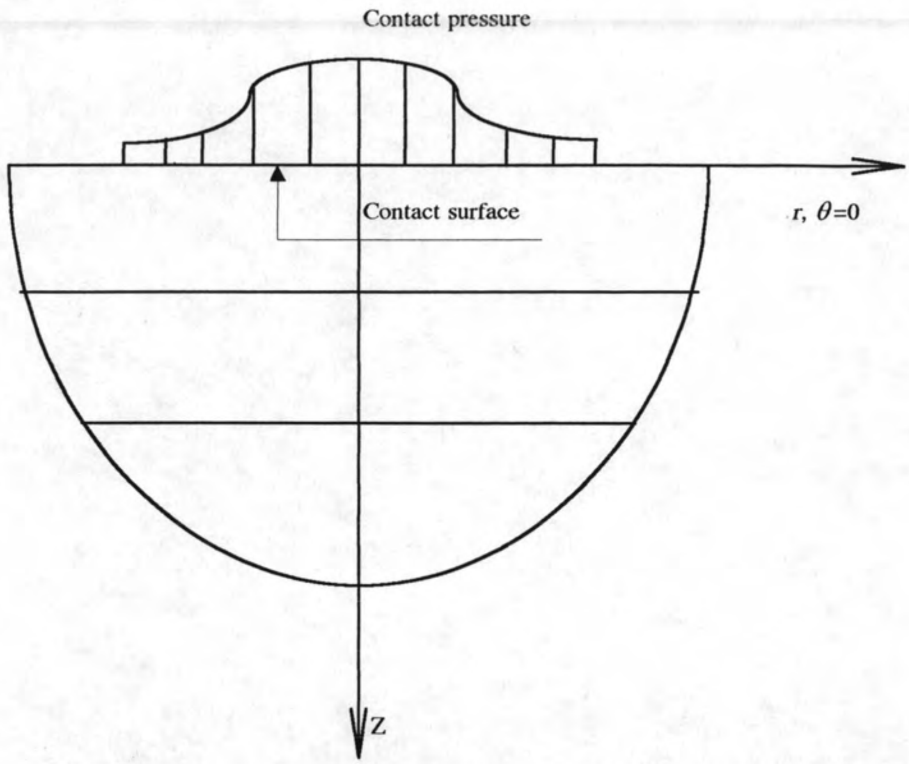
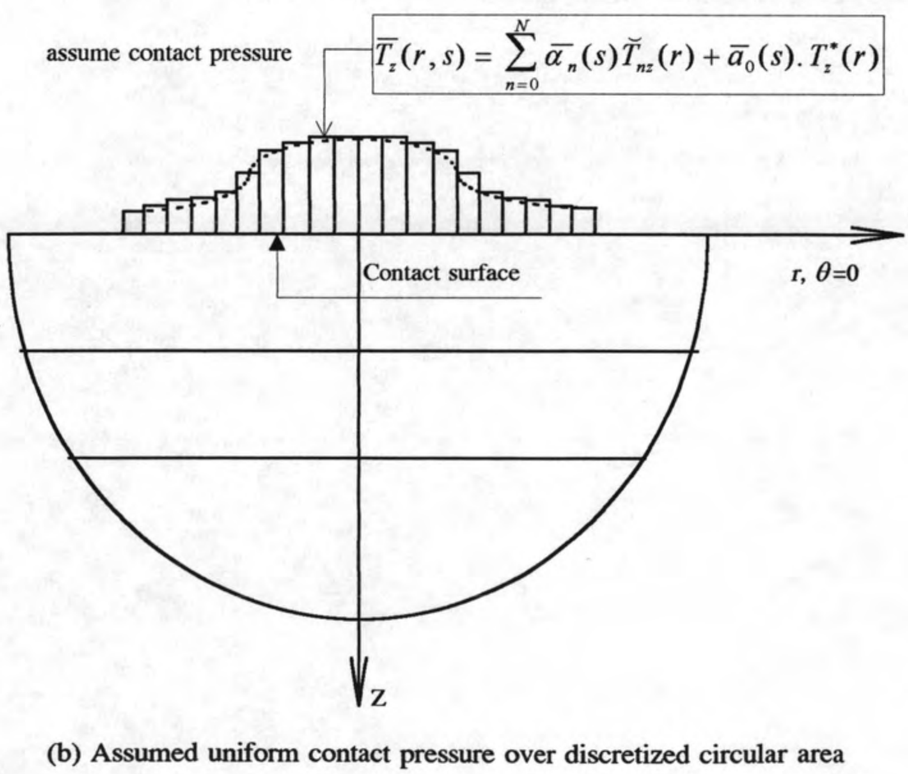


Figure 2.2 : Unit vertical pressure acting over an annular region on the surface of a multi-layered poroelastic half-space



(a) Contact pressure on the circular area of multi-layered half-space.



(b) Assumed uniform contact pressure over discretized circular area

Figure 2.3 : Model used in computation of strain energy of multi-layered poroelastic half-space.