ทฤษฎีก่อกำเนิดผลเฉลยและสมการโทลแมน-ออพเพนไฮม์เมอร์-โวลคอฟฟ์สำหรับทรงกลมของไหลสมบูรณ์ในพิกัดไอโซทรอปิก

นายอภิสิทธิ์ กินรีวงศ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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SOLUTION GENERATING THEOREMS AND TOLMAN-OPPENHEIMER-VOLKOV EQUATION FOR PERFECT FLUID SPHERES IN ISOTROPIC COORD
INATES

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A Thesis Submitted in Partial Fulfillment of the Requirements
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อัลเบิร์ต ไอน์สไตน์ถือเป็นนักฟิสิกส์ที่มีชื่อเสียงมากที่สุดคนหนึ่งในโลก ได้สร้างสมการที่สามารถอธิบายปฏิกิริยาของแรงโน้มถ่วงขั้นมูลฐานที่สุด เรียกว่า สมการสนามของไอน์สไตน์ เนื่องจากความซับซ้อนของการแก้สมการสนามของไอน์สไตน์ จึงเป็นที่ยอมรับกันว่าต้องตั้งสมมติฐานบางอย่างเพื่อลดความซับซ้อนดังกล่าวลง หนึ่งในสมมติฐานที่นิยมมากที่สุดคือทรงกลมของไหลสมบูรณ์ ซึ่งเป็นสมมติฐานที่ทำให้การจำลองดวงดาวเสมือนจริงง่ายขึ้น ในวิทยานิพนธ์นี้ ได้นำเสนอวิธีที่ตั้งอยู่บนพื้นฐานของการเป็นทรงกลมของไหลสมบูรณ์เพื่อใช้แก้หาผลเฉลยแม่นตรง โดยการใช้หลักการทางคณิตศาสตร์ตรงๆ สร้างวิธีนี้ขึ้นมาจึงเรียกวิธีนี้ว่า ทฤษฎีก่อกำเนิดผลเฉลย ในที่นี้จะได้ศึกษาผลเฉลยในพิกัดไอโซทรอปิก โดยนำเสนอทฤษฎีบทและบทแทรกใหม่สำหรับการแปลงทรงกลมของไหลสมบูรณ์หนึ่งไปเป็นทรงกลมของไหลสมบูรณ์อื่น และทำการวิเคราะห์คุณสมบัติของทรงกลมของไหลสมบูรณ์เหล่านั้น จากนั้นก็ได้ประยุกต์ทฤษฎีบทดังกล่าวกับผลเฉลยบางตัวในโปรแกรมเมเปิ้ล นอกจากนี้ได้นำเสนอเทคนิคใหม่ที่จะช่วยในการสร้างทรงกลมของไหลสมบูรณ์ด้วย สุดท้ายได้สร้างสมการ TOV ดัดแปลง ซึ่งเป็นสมการที่สามารถอธิบายถึงโครงสร้างภายในของดวงดาวเสมือนจริง ได้แก่ ความดัน ความหนาแน่น และมวล

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APISIT KINREEWONG: SOLUTION GENERATING THEOREMS AND TOLMAN-OPPENHEIMER-VOLKOV EQUATION FOR PERFECT FLUID SPHERES IN ISOTROPIC COORDINATES. ADVISOR: ASST. PROF. DR. PETARPA BOONSERM, CO-ADVISOR: DR. TRITOS NGAMPITIPAN, 66 pp.

Albert Einstein is one of the most famous physicists in the world, who formulated an equation to explain the most fundamental interactions of gravitation called the Einstein field equation. Due to the complexity in solving the Einstein field equation, it is accepted that some assumption must be made to reduce the complexity. One of the most popular assumptions is that of a perfect fluid sphere. It is simply performed to simulate the realistic stars. In this thesis, we introduce another method which is based on perfect fluid spheres to solve for the exact solutions. Using pure mathematical principles to construct this method, it is thus called as the solution generating theorems. In currently, we will study these solutions in the isotropic coordinates. We derive a new theorem and a corollary that map a perfect fluid sphere into another perfect fluid sphere, and then we analyze those properties of the perfect fluid spheres. Moreover, we apply this theorem with some example solutions in program Maple. Especially, we also present a new technique for the generation of perfect fluid spheres. Eventually, we obtain a new modified TOV equation, which is an equation to explain the internal structure of realistic stars such as the pressure, density, and mass profiles.

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Introduction

1.1 Motivation

General relativity, also known as the general theory of relativity, was issued by Albert Einstein in 1915. It defines the geometric property of spacetime. As commonly acknowledged, general relativity is a beautiful scheme and a self-explanatory theory in relation to the gravitational field. Actually, general relativity has often been described about the forces between two elementary particles and the geometry of spacetime in the universe. Moreover, Einstein has proposed an important equation called the "*Einstein field equation*". The equation describes the fundamental interactions of gravitation as a result of spacetime being curved by matter and energy. The Einstein field equation is a difficult-to-solve equation due to its mathematical complexity. It is a fully non-linear partial differential equation which, in general, cannot directly be solved. Therefore, some assumptions must be made in order to reduce the complexity of the equation. One of the popular assumptions is that of a perfect fluid sphere [1-5].

Perfect fluid spheres are simply developed as idealized models of stars. Using the perfect fluid constraints that are composed of features like no-viscosity, no conduction of heat and isotropy, has led us to the ordinary differential equations. These differential equations have to be solved to find the exact solutions. Normally, the solutions can be described in term of physically realistic stars. Therefore, perfect fluid spheres are well known as an assumption to represent idealized stars. In this thesis, we consider these solutions in isotropic coordinates [6-9].

Furthermore, we can also derive the solutions in another form. According to astrophysics, the Tolman-Oppenheimer-Volkov (TOV) equation can be used to describe the internal structure of general relativistic static perfect fluid spheres [1, 10, 11]. The significance of the TOV equation constrains is to study the interior structure of perfect fluid spheres, including the density and pressure profiles.

Despite the complexity of finding the exact solutions to the Einstein field equations, there is another way to obtain the new exact solutions without directly solving the Einstein field equations. This method is the so called "*solution generating theorems*" [6-9, 11].

For the investigation of the solution generating theorems and the TOV equation in the isotropic coordinates, the study considers using pure mathematical principles as a way of finding several new solutions.

In our research, we develop the relative solution generating theorems that map perfect fluid spheres into perfect fluid spheres in isotropic coordinates [6-8, 12, 13]. In this framework, we derive a new corollary and a new theorem by combining two linking theorems that is also a perfect fluid sphere, and investigate those properties. Moreover, we apply this theorem in the Maple program to generate new perfect fluid spheres. In addition, we also present the new theorem using a new technique to expand some exact solutions [14]. Finally, we study and develop new solutions for the TOV equation, thereby also directly giving information about the pressure and density profiles of general relativistic static perfect fluid spheres.

1.2 Objectives

The goal of this research is to analyze the properties of perfect fluid spheres and derive the new theorem for the generation of the exact solution in isotropic coordinates. In addition, we also modify the Tolman-Oppenheimer-Volkov equations and convert it into the form of pressure and density profiles.

1.3 Structure of the thesis

This thesis looks for two problems in general relativity.

- We shall present the technical ways in finding the exact solutions in the isotropic coordinates using pure mathematical principles. These solutions can be described using physically realistic stars.
- We shall convert the exact solutions into another form. In terms of the TOV equation, which can be explained using the internal structure of realistic stars including the pressure and the density profiles.

The thesis is divided as follows

In chapter 2, we explain the basic knowledge behind both special and general relativity. Moreover, we also introduce the concept of perfect fluid spheres, which are simply the assumption used in building and developing idealized models of stars.

In chapter 3, we introduce the solution generating theorems in the isotropic coordinates using pure mathematical principles as a way of finding several new solutions. We present the new theorem by linking two theorems and analyzing the property of perfect fluid spheres. Moreover, we also generate a new theorem using a new technique. Towards the end of chapter 3, for convenience we introduce the Maple program to apply with some example solutions to help generating perfect fluid spheres.

In chapter 4, we introduce the TOV equation in isotropic coordinates. Furthermore, we discuss the TOV equation in isotropy with other coordinates in order to construct a modified TOV equation based on the main principles of the TOV equation.

In chapter 5, conclusions are drawn and a discussion is provided on all aspects of the thesis. Moreover, interesting issues are further suggested in last section.

Chapter 2
General Introduction

Relativity is an extremely well acknowledged theory, and is perhaps one of the most famous theories in physics. It was formulated by Albert Einstein in 1905. The theory of relativity is very useful in predicting everything, including the existence of black holes. Moreover, we can use this theory to study some phenomena such as the bending of light due to gravity, the behavior of the moon in its orbit, and various other occurrences in the universe. Einstein formulated these concepts with respect to special as well as general situations [1].

In this chapter, we introduce all the fundamental knowledge required in the comprehension of this thesis. We shall also brief the basic ideas behind both special and general relativity. In the third section, we introduce an assumption called "perfect fluid spheres" to provide an ordinary differential equation.

2.1 Special relativity

The primary purpose of this section is to offer an essential idea for the understanding of the completely unfamiliar special relativity. The preliminary concept is first provided, which is immediately followed by the inertial frame, Newtonian physics, and the postulate of special relativity.

2.1.1 Inertial frame

In order to discuss space and time without being ambiguous, it is more helpful to introduce the notation of a *reference body*, which we usually call as an "*inertial frame of reference*". This is one in which Newton's law of motion holds. By considering many freely moving objects in different space and time, one may deduce that all parts of an inertial frame move along together. An inertial frame has a constant velocity and a zero acceleration that is related to other frames [4, 15].

2.1.2 Newtonian physics

Classical physics is concerned with the behavior of physical objects, as developed by Newton, Galileo, and others. These ideas obey the laws of both space and time. Consider two inertial frames S and S' having all their axes aligned. Let S' move along the X-directional axes relative to S at speed v as in figure 2.1 [1, 4, 15].

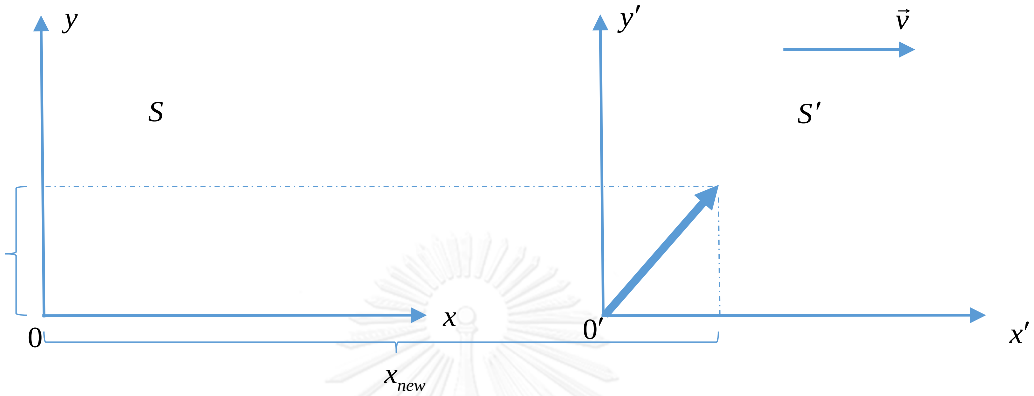


Figure 2.1: Galilean transformation.

As determined in two inertial frames in standard configuration, according to *Newtonian physics*,

$$\begin{aligned} t' &= t, \\ x' &= x - vt, \\ y' &= y, \\ z' &= z. \end{aligned} \tag{2.1}$$

The above equations represent what is called as the *Galilean transformation*. It can also be written in a matrix form as

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$$

Due to we considered in high velocity, the Galilean transformation was deformed by the *Lorentz transformation*. It can be represented in this transformation

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ x' &= \gamma\left(x - vt\right), \\ y' &= y, \\ z' &= z, \end{aligned} \tag{2.2}$$

where $\gamma = \dfrac{1}{\sqrt{1-\left(v^2/c^2\right)}} \geq 1,$ and c is the speed of light. Note that the limit of $v/c \to 0,$ v remains finite, the Lorentz transformations approach to the Galilean transformation.

This transformation can be obeyed with two postulates as following below.

2.1.3 Two postulates of special relativity

Now, we shall introduce special relativity. We find that the principle of relativity is still obeyed, but the Galilean transformation is broken. The main postulates of special relativity are

(1) First postulate (principle of relativity)

The laws of physics must be the same for all inertial reference frame.

(2) Second postulate (speed of light postulate)

The speed of light in vacuum, determined in any inertial reference frame, always has the same value of *c* (*c* is a universal physical constant). The speed of light is approximately 3×10^8 m/s, no matter how fast the source of light and the observer are moving relative to one another [10, 16-18].

We now need to complete the theory of gravitation, particularly looking at general relativity.

2.2 General relativity

General relativity, also known as the general theory of relativity, was developed by Albert Einstein in 1915. This theory is best known as an essence in modern astrophysics. According to general relativity, the observed gravitational attraction between matters is a direct result of spacetime being curved by matters and energy. This spacetime curvature is the fundamental idea behind general relativity. Eventually, Einstein presented an equation that explains the fundamental interaction of gravitation called the “Einstein field equation” [2, 3, 17]. In the previous section, we presented the idea of special relativity, but the idea of gravitation has its limits with Newton’s theory of gravitation. Einstein began by modifying Newton’s theory of gravity to match with the ideas of special relativity. However, it failed to explain particular circumstances. Consequently, Einstein came to offer the ideas of general relativity.

General relativity is the main basis of this thesis. In particular, we have divided the body of the thesis into three sections. First, we describe the physical meaning of the values used in this thesis. In the second section, we focus on the Einstein field equation. Finally, we consider the algorithm for calculating the Einstein field equation.

2.2.1 Physical meaning of values

1) Einstein summation

There are essentially three rules to Einstein summation notation, namely: repeated indices are implicitly summed over, each index can appear at most twice in any term, and each term must contain identical non-repeated indices. Therefore, Einstein summation is a notational convention for simplifying expressions. For example, the indices can range over the set $\{1,2,3\}$,

$$y=\sum_{i=1}^{3} c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3.$$

From the above expression, it happens that for a sum involved, we can simplify by the usage of $y=c_i x^i$. In this case, we call "i" as *a dummy index*.

Note that the typical coordinates $\{x^1, x^2, x^3\}$ would be used instead of the traditional coordinates $\{x, y, z\}$, respectively. In general relativity, we determine the index in a commonly recognized form, that is [19]

- The ***Greek alphabets*** are used for space-time components, where indices often start from values 0,1,2, or 3 (normally used for letters $\mu, \nu, \ldots$)
- The classical ***Latin alphabet*** is used for the spatial component only, where indices accept values 1,2, or 3 (normally used for letters $i, j, \ldots$)

2) Tensor

Tensor is an important quantity, which is useful in representing two different coordinate systems. When objects have to change from one coordinate to another, a set of components can be changed in a given way [4, 20, 21]. We consider such relation as the definition of tensor. For each $p,q=0,1,2,\ldots$, all tensors of rank (p,q) form a vector space,

$$T=T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}\otimes\partial_{\mu_1}\otimes\partial_{\mu_2}\otimes\ldots\otimes\partial_{\mu_p}\otimes dx^{\nu_1}\otimes dx^{\nu_2}\otimes\ldots\otimes dx^{\nu_q}, \tag{2.3}$$

where $\otimes$ is the tensor product. Notice that ∂_{μ_1} means $\dfrac{\partial}{\partial x^{\mu_1}}$.

In the previous relation, tensor is the operation between a set of components and the basic component. This relationship can be regarded in the transformation of $T\to T'$ at point a:

$$\begin{aligned}T(a)&=T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}(X)\otimes\partial_{\mu_1}\otimes\partial_{\mu_2}\otimes\ldots\otimes\partial_{\mu_p}\otimes dx^{\nu_1}\otimes dx^{\nu_2}\otimes\ldots\otimes dx^{\nu_q}\Big|_a\\&=T'^{\rho_1\rho_2\ldots\rho_p}{}_{\sigma_1\sigma_2\ldots\sigma_q}(X')\otimes\partial'_{\rho_1}\otimes\partial'_{\rho_2}\otimes\ldots\otimes\partial'_{\rho_p}\otimes dx^{\nu'_1}\otimes dx^{\nu'_2}\otimes\ldots\otimes dx^{\nu'_q}\Big|_a.\end{aligned}$$

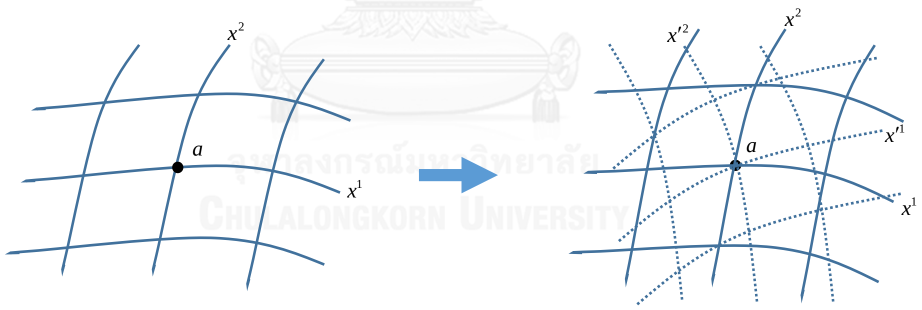


Figure 2.2: The transformation maps coordinate $X=(x^1,x^2)$ onto coordinate $X'=(x'^1,x'^2)$ containing a point a.

The transformation to coordinates T' can be written as

$$T'^{\rho_1\rho_2\ldots\rho_p}{}_{\sigma_1\sigma_2\ldots\sigma_q}(X')=\left(\frac{\partial x^{\rho_1'}}{\partial x^{\mu_1}}\frac{\partial x^{\rho_2'}}{\partial x^{\mu_2}}\cdots\frac{\partial x^{\rho_p'}}{\partial x^{\mu_p}}\right)\left(\frac{\partial x^{\nu_1}}{\partial x^{\sigma_1'}}\frac{\partial x^{\nu_2}}{\partial x^{\sigma_2'}}\cdots\frac{\partial x^{\nu_q}}{\partial x^{\sigma_q'}}\right)T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}(X).$$

From the above expression, there is an exact sequence of transformation, which is one by one (i.e. ρ_1 pairs with $\mu_1, \ldots, \nu_q$ pairs with σ_q). It does not jump over indices. Hence, we will prescribe the tensors as following:

- Tensor (0,0): scalar

$$S(X') = S(X)$$

- Tensor (1,0): vector

$$V^{\rho_1}(X') = \frac{\partial x^{\rho_1'}}{\partial x^{\mu_1}} V^{\mu_1}(X)$$

- Tensor (0,1): dual vector, gradient

$$V_{\sigma_1}(X') = \frac{\partial x^{\nu_1}}{\partial x^{\sigma_1'}} V_{\nu_1}(X)$$

- Tensor (2,0):

$$T^{\rho_1\rho_2}(X') = \frac{\partial x^{\rho_1'}}{\partial x^{\mu_1}} \frac{\partial x^{\rho_2'}}{\partial x^{\mu_2}} T^{\mu_1\mu_2}(X)$$

- Tensor (0,2):

$$T_{\sigma_1\sigma_2}(X') = \frac{\partial x^{\nu_1}}{\partial x^{\sigma_1'}} \frac{\partial x^{\nu_2}}{\partial x^{\sigma_2'}} T_{\nu_1\nu_2}(X)$$

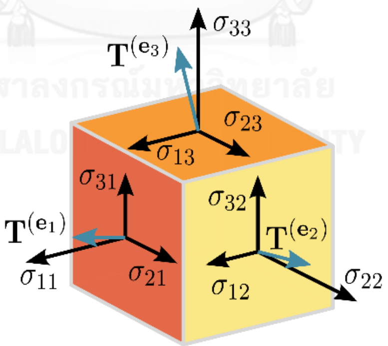


Figure 2.3: The Cauchy stress tensor, with tensor of rank (0, 2) for which the tensor's component is in a 3-dimensional Cartesian coordinate system [21].

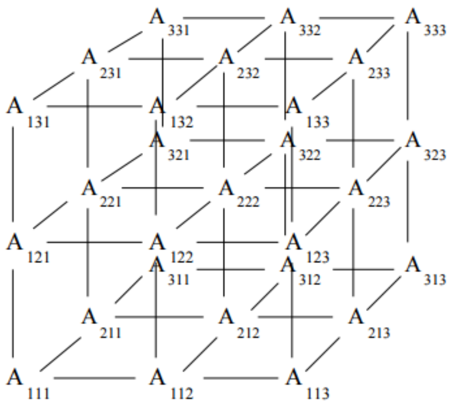


Figure 2.4: A tensor of rank (0, 3) [20].

Rule for tensor algebra

(1) The *sum* of two tensors; if T and S are tensors of type (p,q), then $T+S$ is also a tensor defined as

$$U^{ab\ldots}_{cd\ldots} = T^{ab\ldots}_{cd\ldots} + S^{ab\ldots}_{cd\ldots}.$$

(2) The *outer product* of two tensors can be explained as

$$T^a S^b = U^{ab} \text{ or } T^a_b S^c_{de} = U^{ac}_{bde}.$$

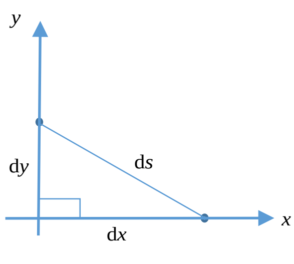
(3) *Contraction* is often used to reduce the rank of the tensor. In general, one could have combination such as [4]

$$U^{ab} = A^{a\lambda} B_\lambda{}^b \text{ and } T^{ab}_c = S^{a\lambda b}_{\lambda c} \ .$$

3) Metric tensor

In particular, let us introduce the metric tensor, which is significant in the beginning of the calculation. We have a metric (also known as $\mathrm{d}s^2$) which represents the shortest distance between two points in space. As for the basics of $\mathrm{d}s^2$, for simplicity, we first consider it in a two-dimensional coordinate.

- Cartesian coordinate (x, y):



Using the Pythagorean theorem, we attain $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$.

On the other hand, we can also generalize $\mathrm{d}s^2$ from

$$\mathrm{d}s^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$= g_{11}\mathrm{d}x^1\mathrm{d}x^1 + g_{12}\mathrm{d}x^1\mathrm{d}x^2 + g_{21}\mathrm{d}x^2\mathrm{d}x^1 + g_{22}\mathrm{d}x^2\mathrm{d}x^2,$$

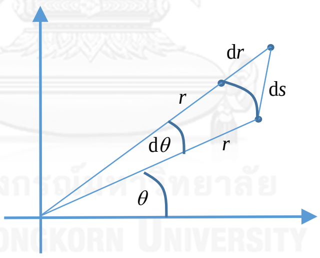
where $x^1 = x,\ x^2 = y$ and $g_{11} = 1,\ g_{12} = 0,\ g_{21} = 0,\ g_{22} = 1.$ Then we can bring g_{ij} into the matrix form, where g_{ij} can be defined as

$$\left[g_{ij}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{for } i, j = 1, 2.$$

Similarly, in three-dimensional coordinate, we obtain

$$\left[g_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{for } i, j = 1, 2, 3.$$

- Polar coordinate (r, θ):



Since $\mathrm{d}\theta$ is spread over an angle of a small value, then the region of the curve is equal to $r\mathrm{d}\theta$. We will further look into the line. With the same logic as the Cartesian coordinate, we derive that $\mathrm{d}s^2$ is equal to $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$. Therefore, g_{ij} can also be derived as

$$\begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}, \qquad \text{for } i, j = 1, 2.$$

We can see that $g_{ij} = 0,$ where $i \neq j$. This can definitely be used in creating a diagonal matrix g_{ij} by forming a trace of matrix from the coefficients in each set of coordinates, such as the coefficient in the r-coordinate being equal to 1, while the θ-coordinate being equal to r^2.

In general relativity, we can neglect these summation signs by using the Einstein summation convention. Then the metric can be generalized to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \tag{2.4}$$

Further discussing about spacetime, it is a four-dimensional space with indices μ,ν often starting from values 0, 1, 2, and 3, respectively. Consequently, we call $\left[g_{\mu\nu}\right]$ as "*metric tensor*", with a tensor rank of (0, 2). In addition, we can also define the metric tensor as taking the form

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}. \tag{2.5}$$

Furthermore, we can also define the "*inverse metric tensor*" (representing $g^{\mu\nu}$) as $\left[g^{\mu\nu}\right] = \left[g_{\mu\nu}\right]^{-1}$.

4) Einstein tensor

In this section, we shall refer to the Einstein tensor. This is an essential quantity because we have to use it for calculating the left hand side of the Einstein field equation, while the right hand side will be introduced later in section 2.2. Currently, we shall discuss the relative values that satisfy the Einstein tensor. Moreover, we also offer an overview of the Christoffel symbol, Riemann curvature tensor, Ricci tensor, and Ricci scalar, respectively.

- **Christoffel symbol**

We present the Christoffel symbol in general relativity, which is a connector between vector space. That tells us how the basis of vectors changes as we move from one point to another. In order to understand it in more detail, we must consider the shifting of vector $\vec{A}$ and $\vec{B}$ using the rule for vector addition. On flat space, vectors perform as figure 2.5;

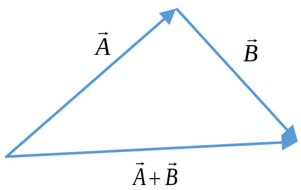


Figure 2.5: Vector addition on flat space.

However, if we conduct an investigation on curved space, vector $\vec{B}$ may be changed. Therefore, $\tilde{\vec{B}}$ is equivalent to $\Gamma\vec{B}$, where Γ is a connector for linking vector spaces. The main assumption here is that the metric tensor is faultless. Generally, we can now obtain the connector as

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left[\frac{\partial g_{\sigma\nu}}{\partial x^{\rho}} + \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\sigma}} \right], \tag{2.6}$$

where $g_{\mu\nu}$ is a metric tensor, and $g^{\mu\nu}$ is the inverse metric tensor. Then the affine connection coefficients that are built-up by metric tensor, we call as the *Christoffel symbol*, which is denoted as $\begin{Bmatrix} \mu \\ \nu\rho \end{Bmatrix}$. Furthermore, in this basis, the connection coefficients are symmetric, i.e., $\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$ [22, 23].

- **Riemann curvature tensor**

Now, we consider the parallel transportation of vector v along curve C by moving along two separate paths as follows (figure 2.6);

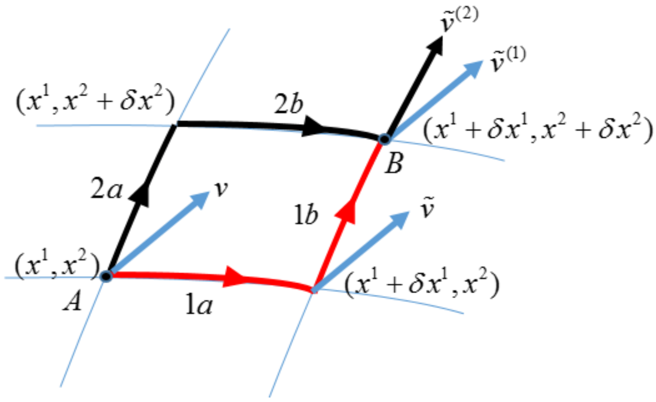


Figure 2.6: The parallel transportation of vector v on a curved surface.

We will move vector v from point A to B. Let δx^1 be a slight shift along x^1 - axis, and δx^2 be a slight shift along x^2 - axis, respectively.

Initially, we look at path *1a* since vector v^μ has a covariant derivative (i.e. $\nabla_\nu v^\mu = 0$). Note that $\nabla_1 v^\mu = \partial_1 v^\mu + \Gamma^\mu_{\nu 1} v^\nu$, then we get

$$\tilde{v}^\mu(x^1+\delta x^1, x^2) = v^\mu(x^1, x^2) - \Gamma^\mu_{\nu 1}(x^1, x^2) v^\nu(x^1, x^2)\delta x^1.$$

Therefore, in path *1b*:

$$\tilde{v}^\mu_{(1)} = \tilde{v}^\mu(x^1+\delta x^1, x^2) - \Gamma^\mu_{\nu 2}(x^1+\delta x^1, x^2)\tilde{v}^\nu(x^1+\delta x^1, x^2)\delta x^2,$$

where $\Gamma^\mu_{\nu 2}(x^1+\delta x^1, x^2) = \Gamma^\mu_{\nu 2}(x^1, x^2) + \partial_1 \Gamma^\mu_{\nu 2}(x^1, x^2)\delta x^1 + O\left(\left(\delta x^1\right)^2\right)$.

Similarly, we can consider a continuation toward path 2, while using the results from path 1, and interchanging the indices $1 \leftrightarrow 2$. Actually, we can also derive it as

$$\tilde{v}^\mu_{(1)} - \tilde{v}^\mu_{(2)} = -\left\{\partial_1 \Gamma^\mu_{\nu 2} - \partial_2 \Gamma^\mu_{\nu 1} + \Gamma^\mu_{\rho 1}\Gamma^\rho_{\nu 2} - \Gamma^\mu_{\rho 2}\Gamma^\rho_{\nu 1}\right\} v^\nu \delta x^1 \delta x^2 + \cdots,$$

So, we determine the *Riemann curvature tensor* as

$$R^\mu_{\nu 12} = \left\{\partial_1 \Gamma^\mu_{\nu 2} - \partial_2 \Gamma^\mu_{\nu 1} + \Gamma^\mu_{\rho 1}\Gamma^\rho_{\nu 2} - \Gamma^\mu_{\rho 2}\Gamma^\rho_{\nu 1}\right\}.$$

Generally, the Riemann curvature tensor can also be written in the form [22, 24]

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.7}$$

Consequently, the Riemann curvature tensor is an essential quantity, which we shall study in a small area. The Riemann curvature tensor refers to the curvature of spacetime.

Moreover, we also determine the *Ricci tensor* $(R_{\mu\nu})$ and the *Ricci scalar*, which are related to the curvature of spacetime. It is easy to find the value of the Ricci tensor because it occurs from $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$, which is a contraction over two indices. In contrast, the value of the Ricci scalar is defined as $R = g^{\mu\nu}R_{\mu\nu}$.

Finally, we derive the Einstein tensor (representing $G_{\mu\nu}$) as following,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.8}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor, and R is the Ricci scalar.

2.2.2 Einstein field equation

The acceleration due to gravity has an effect on the curvature of spacetime. As the theory proposes, the principles of general relativity relates to the effect of spacetime being curved by matter, which consequently affects the path of other moving matter within that curvature of spacetime. Therefore, Albert Einstein published an equation that explains the fundamental interactions of gravitation as a result of this curvature of spacetime by matter and energy. The equation is called the "Einstein field equation" or "Einstein's equation", which can be explained by

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.9}$$

where $G_{\mu\nu}$ is the "*Einstein tensor*" describing the geometry of spacetime and

$T_{\mu\nu}$ is the "*stress energy tensor*" describing the distribution of matter and energy.

2.2.3 The procedure in the calculation of the Einstein field equation

Now, we shall consider the algorithm for the calculation of the Einstein field equation. In this section, we will focus on the left hand side of the Einstein field equation, that is, we will present the procedure in the calculation of the Einstein tensor. For the part of stress energy tensor, the calculations are more gradual, details of which will be provided more in section 2.2.

Let us introduce the left hand side of the Einstein field equation. We begin with the metric ds^2 for spacetime, which has four-dimensional coordinates t, r, θ, ϕ. Then we will consider the metric tensor $g_{\mu\nu}$ from

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.10}$$

In this thesis, we study matters with symmetric property. That is, the off-diagonal components of the metric tensor that are equal to zero, while the components of the diagonal metric tensor $g_{\mu\nu}$ have determinable values. So, $g_{\mu\nu}$ and $g^{\mu\nu}$ take to the forms

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix}, \tag{2.11}$$

and

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{g_{00}} & 0 & 0 & 0 \\ 0 & \frac{1}{g_{11}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{22}} & 0 \\ 0 & 0 & 0 & \frac{1}{g_{33}} \end{bmatrix}. \tag{2.12}$$

The Einstein tensor is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.13}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. The Ricci tensor and Ricci scalar are

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu}, \tag{2.14}$$

and

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.15}$$

Using the above information, we can derive the Christoffel symbol $\Gamma^{\mu}_{\nu\rho}$. Moreover, we must also derive the other Christoffel symbols. These symbols are used in general relativity for which there is a connection with each of the coordinates,

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left[\frac{\partial g_{\sigma\nu}}{\partial x^{\rho}} + \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\nu\rho}}{\partial x^{\sigma}} \right]. \tag{2.16}$$

And then we write the Riemann curvature tensor as

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{2.17}$$

We start off with replacing the Christoffel symbols and the Riemann curvature tensors in the equation with the Ricci tensor. Then we find the value of the Ricci scalar. Finally, we can derive the Einstein tensor from the above process.

As for the stress energy tensor, we shall present the details later in section 2.3, which is related to our beginning assumption. Theories of relativity are continuously being developed. The additional knowledge has corresponded with previous theories. These theories can be associated with Newtonian physics for the explanation of natural events related to gravitation in the universe, such as studying the expansion of the universe or calculating the orbit of planet Mercury. For this reason, general relativity is a necessary inclusion in the theory of gravitation of spacetime.

2.3 Introduction to perfect fluid spheres

We shall now provide an introduction to perfect fluid spheres. Perfect fluid spheres are simply an assumption developed as idealized models of stars. In this section, we present perfect fluid spheres in various coordinates. Furthermore, we also introduce the different properties of perfect fluid spheres. Finally, we will show how perfect fluid constraint can be used to build several new exact solutions for any relativistic static perfect fluid spheres.

2.3.1 Introduction

In the elementary step of the approximation of stars, we use perfect fluid spheres to develop idealized models of stars. We also make use of the properties of perfect fluid spheres; i.e., non-viscosity, no conduction of heat, and isotropy, which implies that in orthonormal components, the stress energy tensor takes the form

$$T_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix}, \tag{2.18}$$

where ρ is the density, and p_r, p_t are the radial pressure and the transverse pressure, respectively.

Using the condition of isotropy, we obtain $p_r = p_t$ which can be implied as $T_{\hat{r}\hat{r}} = T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}}$. Equating the appropriate orthonormal component of the Einstein tensor [25], we obtain

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}. \tag{2.19}$$

This equation is a "perfect fluid constraint", which supplies us with the ordinary differential equation [ODE] for any relativistic static perfect fluid sphere.

2.3.2 Coordinate systems for perfect fluid spheres

First we introduce the coordinate systems for perfect fluid spheres in accordance with expectations. There are exact solutions of the Einstein field equation, which can be written in a closed form ($\mathrm{d}s^2$). The perfect fluid constraint of the static perfect fluid spheres is now considered in some coordinate systems. To place the use of the overall coordinate system, observations made through by Finch and Skea [9] project shows that about 55% of all work relates to fluid spheres is used in the Schwarzschild curvature coordinates, about 35% of related research is used in the isotropic coordinates, while and the remaining 10% is expanded into the specialized coordinate systems [12].

Coordinates	Metric
Schwarzschild curvature	$\mathrm{d}s^2 = -\zeta^2(r)\mathrm{d}t^2 + \frac{1}{B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$
Isotropic	$\mathrm{d}s^2 = -\zeta^2(r)\mathrm{d}t^2 + \frac{1}{\zeta^2(r)B^2(r)}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right\}.$
Others coordinates • Gaussian polar • Sygnge isothermal	• $\mathrm{d}s^2 = -\zeta^2(r)\mathrm{d}t^2 + \mathrm{d}r^2 + R^2(r)\mathrm{d}\Omega^2.$ • $\mathrm{d}s^2 = -\zeta^{-2}(r)\left\{\mathrm{d}t^2 - \mathrm{d}r^2\right\} + \left\{\zeta^{-2}(r)R^2(r)\mathrm{d}\Omega^2\right\}.$

• Buchdahl	• $ds^2 = -\zeta^2(r)dt^2 + \zeta^{-2}(r)\{dr^2 + R^2(r)d\Omega^2\}.$

Note that the arbitrary function $\zeta(r)$, $B(r)$, and $R(r)$ are scaling factors, which have an effect on the curvature of spacetime.

2.3.3 Properties of perfect fluid sphere

As perfect fluid spheres are the natural assumption generated as idealized models of stars, the properties of perfect fluid spheres can be used to formulate perfect fluid constraint. This leads to an ordinary differential equation.

We begin with the Einstein field equation,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \quad (2.20)$$

Turning now to stress energy tensor (on the right hand side), we shall introduce the term of the stress energy tensor $T_{\mu\nu}$. Figure 2.7 summarizes the physical interpretation of the elements of the stress energy tensor matrix as follow [4];

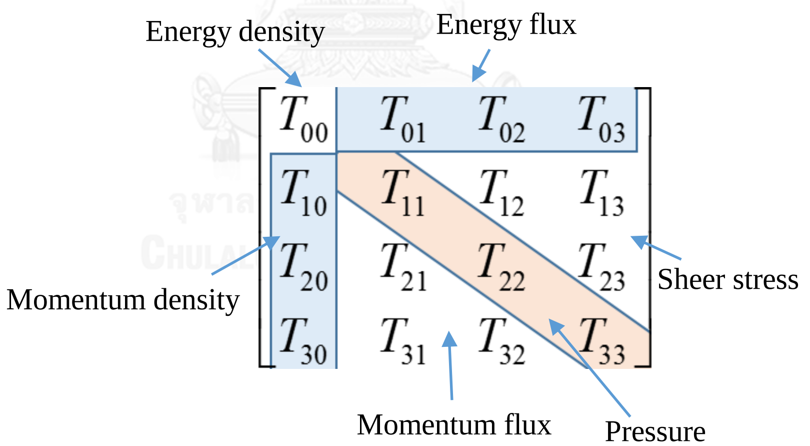


Figure 2.7: The matrix form of stress energy tensor.

From figure 2.7, we can attend to the physical interpretation, in detail, of the various quantities. For the physical meaning of each component, we shall display that as expressions.

Firstly, we shall start by describing *a surface of constant* x^{ν}. Let us consider a three dimensional coordinate system xyz as shown below,

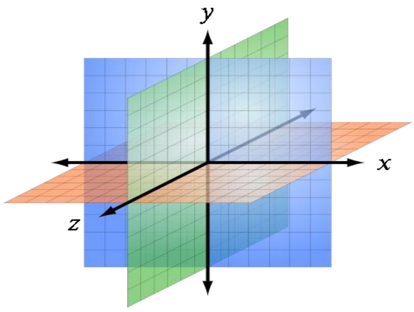


Figure 2.8: Three dimensional coordinates xyz [26].

"*A surface of constant* x" means plane yz at $x = c$, where c is a constant. Similarly, a surface of constant y and z are the planes xz and xy at $y = k$ and $z = l$, respectively. For a four dimensional coordinate $txyz$, we use the same logic as in a three dimensional coordinate.

Next, we write *a four-momentum* as following;

Since spacetime has four dimensions, we represent spacetime into a matrix form as

$$\begin{bmatrix} ct & x & y & z \end{bmatrix}^T.$$

Dividing spacetime (4-vectors) by t, we obtain *a four-velocity* as follows;

$$\begin{bmatrix} c & v_x & v_y & v_z \end{bmatrix}^T.$$

Then we multiply a four-velocity by m to derive a four-momentum as

$$\begin{bmatrix} mc = \dfrac{mc^2}{c} = \dfrac{E}{c} \\ mv_x = p_x \\ mv_y = p_y \\ mv_z = p_z \end{bmatrix}.$$

Let us introduce the physical interpretation of the stress energy tensor (definition 1).

Definition 1. *The stress energy tensor* $T_{\mu\nu}$ *is the flux of the* μ*-th component of four-momentum across a surface of constant* x^ν.

By the definition, we can write out each component of the stress energy tensor. In particular, we shall display some significant components of perfect fluid spheres as

$$T_{00} := \frac{E}{xyz} = \rho,$$

$$T_{11} := \frac{mv_x}{tyz} = \frac{ma_x}{yz} = \frac{F_x}{A} = p,$$

$$T_{22} := \frac{mv_y}{txz} = \frac{ma_y}{xz} = \frac{F_y}{A} = p,$$

$$T_{33} := \frac{mv_z}{txy} = \frac{ma_z}{xy} = \frac{F_z}{A} = p.$$

That explains the physical meaning in each element of the stress energy tensor as in figure 2.7.

As mentioned earlier, there are three general properties of perfect fluid spheres; namely, no-viscosity, no conduction of heat, and isotropy. The isotropic property means that the pressure remains the same regardless of the direction of measure. Using the properties of perfect fluid spheres, we derive the stress energy tensor as

$$T_{\hat{\mu}\hat{\nu}} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix},$$

where ρ is the density, p_r and p_t are the radial pressure and the transverse pressure, respectively.

Consequently, we obtain the perfect fluid constraint

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}. \tag{2.20}$$

This condition leads us to the ordinary differential equation [ODE] for any relativistic static perfect fluid sphere.

In this section, we introduce perfect fluid spheres in various coordinates, which is, as mentioned earlier, the assumption necessary in the development of idealized models of stars. Then we derive a perfect fluid constraint such that it can also be implied as ordinary differential equation.

In the next chapter, we consider isotropic coordinates. A perfect fluid constraint can lead to solution generating theorems. Furthermore, we also present the solution generating theorems in isotropic coordinates, which is an important tool in mapping a perfect fluid sphere to a perfect fluid sphere, in the isotropic coordinates.

Chapter 3
Solution generating theorems

Although the determination of exact solutions can be possible for the Einstein field equations; however, there is another way to obtain new exact solutions without having to directly solve the Einstein field equations. This method is the so called "*solution generating theorems*". In the descriptive approximation of stars, we will apply these solutions to analyze the realistic stars, investigate several well-known spacetime, and maybe generate a new solution as an unexpected solution. At present, we use the "solution generating theorems" to solve for new exact solutions, which can comfortably generate the class of several new perfect fluid spheres. As for the concept behind of this work, we have studied the solution generating theorems by mapping a perfect fluid sphere into a perfect fluid sphere. If we have a perfect fluid sphere and we apply the solution generating theorems with some coordinates of the sphere, we obtain a new solution. These solution generating theorems have been derived using perfect fluid constrain. We can see a brief illustration of this concept, as shown below;

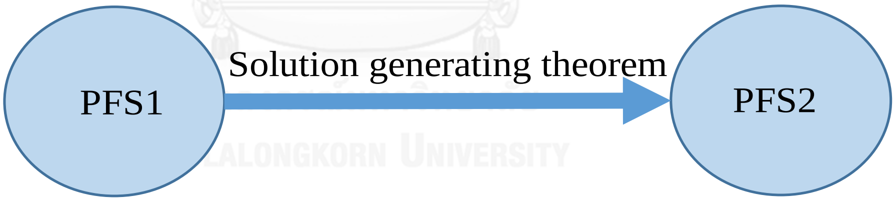


Figure 3.1: Using solution generating theorems for the mapping of a perfect fluid sphere to a perfect fluid sphere.

In this chapter, we regard the solution generating theorems in isotropic coordinates using pure mathematical principles as a way of finding several new solutions. In the current chapter, we study the isotropic coordinates. Let us first refer to the characteristics of isotropic coordinates, which are provided in the section below.

3.1 Isotropic coordinates

In the previous chapter, we have learnt about the coordinate systems of perfect fluid spheres. There are now several coordinate systems of perfect fluid spheres in use. One of the well-known coordinates is the isotropic coordinates. A significant characteristic of the isotropic coordinates is that the coefficients of radial and angular coordinates are equal. The metric takes the form

$$\mathrm{d}s^2 = -\zeta(r)^2 \mathrm{d}t^2 + \frac{1}{\zeta(r)^2 B(r)^2}\left\{\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right\}, \tag{3.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$. It is simply constructed to define $g_{\mu\nu}$, which is equal to

$$\begin{bmatrix} -\zeta(r)^2 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\zeta(r)^2 B(r)^2} & 0 & 0 \\ 0 & 0 & \dfrac{r^2}{\zeta(r)^2 B(r)^2} & 0 \\ 0 & 0 & 0 & \dfrac{r^2 \sin^2\theta}{\zeta(r)^2 B(r)^2} \end{bmatrix}.$$

Then we calculate the Einstein tensor as follows

$$G_{\hat{r}\hat{r}} = -2B'B\zeta^2/r + (B')^2\zeta^2 - (\zeta')^2 B^2, \tag{3.2}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -B'B\zeta^2/r + (B')^2\zeta^2 - BB''\zeta^2 + (\zeta')^2 B^2. \tag{3.3}$$

Note that rr and $\hat{r}\hat{r}$ have the same meaning when the stress energy tensor is a diagonal matrix. Otherwise, rr is not equivalent to $\hat{r}\hat{r}$.

3.2 Ordinary differential equation

In the elementary step of the approximation of stars, we use perfect fluid spheres to develop idealized models of stars. We also make use of the properties of perfect fluid spheres. So we obtain

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}, \tag{3.4}$$

which provides us with an ordinary differential equation [ODE]. Then we derive the ordinary differential equation for isotropic coordinates when we make use of perfect fluid constraint as [6-8, 11-13]:

$$\left(\frac{\zeta'}{\zeta}\right)^2 = \frac{B'' - B'/r}{2B}. \tag{3.5}$$

Let us define $g(r)=\dfrac{\zeta'(r)}{\zeta(r)}$. We can find $\zeta(r)$ as $\exp\left(\int g(r)\mathrm{d}r\right)$ and equation (3.5) becomes

$$g(r)^2=\frac{B''-B'/r}{2B}. \tag{3.6}$$

Equation (3.6) can be rewritten in terms of $B(r)$ as:

$$B''-\frac{B'}{r}-2g^2B=0. \tag{3.7}$$

We can then employ the ordinary differential equation for isotropic coordinates as the equation (3.5) to develop several new solution generating theorems. Accordingly, section 3.3 consists of the analysis and expansions of perfect fluid spheres in isotropic coordinates.

3.3 Solution generating theorems for isotropic coordinates

As derived in [13], we have developed several “algorithmic” techniques that permit one to generate large classes of perfect fluid spheres. Let us now develop two solution generating theorems appropriate for isotropic coordinates [6-8].

Theorem 1. *(7^{th} BVW or Buchdahl transformation (T_7)) Suppose that* $\{\zeta_0(r),B_0(r)\}$ *represents an initial perfect fluid sphere as*

$$\mathrm{d}s^2=-\zeta_0(r)^2\mathrm{d}t^2+\frac{1}{\zeta_0(r)^2B_0(r)^2}\left\{\mathrm{d}r^2+r^2\mathrm{d}\Omega^2\right\}.$$

Let $B_0(r)$ *be fixed, then we obtain*

$$\mathrm{d}s^2=-\frac{1}{\zeta_0(r)^2}\mathrm{d}t^2+\frac{\zeta_0(r)^2}{B_0(r)^2}\left\{\mathrm{d}r^2+r^2\mathrm{d}\Omega^2\right\}$$

is also a perfect fluid sphere. That is, the mapping

$$T_7:\{\zeta_0(r),B_0(r)\}\mapsto\left\{\frac{1}{\zeta_0(r)},B_0(r)\right\}. \tag{3.8}$$

takes perfect fluid spheres into perfect fluid spheres.

Proof Suppose that $\{\zeta_0(r),B_0(r)\}$ solves equation (3.5).

Let $B_0(r)$ be fixed, we have changed $\zeta_0(r)$ to $\zeta_1(r)$. We write $\zeta_1(r)=\dfrac{1}{\zeta_0(r)}$.

Then

$$\left(\frac{\zeta_1'}{\zeta_1}\right)^2 = \left(\frac{\left(\frac{1}{\zeta_0}\right)'}{\frac{1}{\zeta_0}}\right)^2 = \left(\frac{-\frac{1}{\zeta_0^{\,2}}\zeta_0'}{\frac{1}{\zeta_0}}\right)^2 = \left(\frac{-\zeta_0'}{\zeta_0}\right)^2 = \left(\frac{\zeta_0'}{\zeta_0}\right)^2.$$

That $\{\zeta_1(r), B_0(r)\}$ also satisfies equation (3.5).

Therefore $\{\zeta_1(r), B_0(r)\} = \left\{\frac{1}{\zeta_0(r)}, B_0(r)\right\}$ is also a perfect fluid sphere. #

In addition, solution generating theorem, as defined in theorem 1, is a "square root of unity" in the sense that $T_7 \circ T_7 = I,$ where I is an identity. That is, when we apply theorem 1 twice, we obtain the initial perfect fluid spheres as the following [6-8]

$$T_7 \circ T_7 : \{\zeta_0(r), B_0(r)\} \mapsto \left\{\frac{1}{\zeta_0(r)}, B_0(r)\right\} \mapsto \{\zeta_0(r), B_0(r)\}. \tag{3.9}$$

After applying theorem 1 n-times, we can represent $\zeta(r)$ as:

$$\zeta(r) = \begin{cases} \dfrac{1}{\zeta_0(r)} & \text{if } n \text{ is odd number,} \\ \zeta_0(r) & \text{if } n \text{ is even number.} \end{cases}$$

Theorem 2. *(8th BVW transformation (T8)) Let $\zeta_0(r)$ be fixed and extend $B_0(r)$ to $B_0(r)Z_0(r)$. Define*

$$Z_0(r) = \left\{\sigma + \varepsilon \int \frac{r\mathrm{d}r}{B_0(r)^2}\right\},$$

for arbitrary σ and ε. Then

$$\mathrm{d}s^2 = -\zeta_0(r)^2 \mathrm{d}t^2 + \frac{1}{\zeta_0(r)^2 B_0(r)^2 Z_0(r)^2}\left\{\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right\}$$

is also a perfect fluid sphere. That is, the mapping of

$$T_8 : \{\zeta_0(r), B_0(r)\} \mapsto \{\zeta_0(r), B_0(r) Z_0(B_0(r); r)\} \tag{3.10}$$

takes perfect fluid spheres into perfect fluid spheres.

Proof Assume that $\{\zeta_0(r), B_0(r)\}$ satisfies equation (3.7).

We require that $\{\zeta_0(r), B_0(r)Z_0(r)\}$ is also a perfect fluid sphere. Therefore $\{\zeta_0(r), B_0(r)Z_0(r)\}$ must satisfies equation (3.7), i.e.,

$$\left(B_0Z_0\right)'' - \frac{\left(B_0Z_0\right)'}{r} - 2g_0^2 B_0 Z_0 = 0,$$

where $g_0 = \frac{\zeta_0'}{\zeta_0}$. Expanding the above differential equation to

$$\left(B_0''Z_0 + 2B_0'Z_0' + B_0Z_0''\right) - \left(B_0'Z_0\right)/r - \left(B_0Z_0'\right)/r - 2g_0^2\left(B_0Z_0\right) = 0,$$

we can then rearrange terms to get

$$\left(B_0'' - \frac{B_0'}{r} - 2g_0^2 B_0\right)Z_0 + 2B_0'Z_0' + B_0Z_0'' - \left(B_0Z_0'\right)/r = 0. \tag{3.11}$$

By the above assumption, the first term in equation (3.11) vanishes and we have a linear homogeneous 2nd order ODE in terms of Z_0,

$$B_0Z_0'' + \left(2B_0' - \frac{B_0}{r}\right)Z_0' = 0. \tag{3.12}$$

Equation (3.12) can be solved in two steps. First, we use method of separation of variables to get

$$\frac{Z_0''}{Z_0'} = -2\frac{B_0'}{B_0} + \frac{1}{r}, \tag{3.13}$$

followed by integration both sides of the equation (3.13) to obtain

$$\int \frac{\left(Z_0'\right)'}{Z_0'}\mathrm{d}r = \int\left(-2\frac{B_0'}{B_0} + \frac{1}{r}\right)\mathrm{d}r$$

$$\int \frac{1}{Z_0'}\mathrm{d}Z_0' = -2\int \frac{1}{B_0}\mathrm{d}B_0 + \int \frac{1}{r}\mathrm{d}r$$

$$\ln\left|Z_0'\right| = -2\ln\left|B_0\right| + \ln\left|r\right| + \ln\left|\varepsilon\right|$$

$$Z_0' = \frac{\varepsilon r}{B_0^2}.$$

Finally, integrating the above equation leads to

$$Z_0(r) = \sigma + \varepsilon\int \frac{r}{B_0^2(r)}\mathrm{d}r, \tag{3.14}$$

where σ and ε are the arbitrary integration constants. #

Definition 2. *A transformation T is called "idempotent" if $T \circ T \triangleq T$, where the symbol $\triangleq$ represents equality up to the relabeling of the parameters. In this sense, any theorem is idempotent if we apply the transformation more than once, with no further solutions being obtained.*

To see this, consider the mapping of theorem 2

$$T_8 \circ T_8 : \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 Z_0(B_0;r)\} \mapsto \{\zeta_0, B_0 Z_0(B_0;r) Z_1(B_0 Z_0(B_0;r))\}, \tag{3.15}$$

where $Z_1 = \sigma + \varepsilon \int \frac{r}{B_0^2(r) Z_0^2(r)} \mathrm{d}r.$

When we apply the transformation of theorem 2 the second time, it does not lead to a new yield (no additional information). Therefore, theorem 2 is idempotent [6-8].

Having now found two solution generating theorems and the ordinary differential equation, this leads to a new corollary.

Corollary 1. *Let $\{\zeta_0(r), B_a(r)\}$ and $\{\zeta_0(r), B_b(r)\}$ be perfect fluid spheres and let $\zeta_0(r)$ be fixed. Then for real arbitrary constant α, β,*

$$\{\zeta_0, \alpha B_a + \beta B_b\} \tag{3.16}$$

is also a perfect fluid sphere.

Proof Suppose that $\{\zeta_0(r), B_a(r)\}$ and $\{\zeta_0(r), B_b(r)\}$ are perfect fluid spheres.

Recall a linear homogeneous 2nd order ODE in terms of $B(r)$,

written as

$$B'' - \frac{B'}{r} - 2g^2 B = 0. \tag{3.17}$$

Then we obtain

$$B_a'' - \frac{B_a'}{r} - 2g^2 B_a = 0, \tag{3.18}$$

and

$$B_b'' - \frac{B_b'}{r} - 2g^2 B_b = 0. \tag{3.19}$$

We want to show that $\{\zeta_0(r), B_1(r)\}$, where $B_1(r) = \alpha B_a + \beta B_b$ solves equation (3.5). Since

$$B_1 = \alpha B_a + \beta B_b,$$
$$B_1' = \alpha B_a' + \beta B_b',$$
$$B_1'' = \alpha B_a'' + \beta B_b'',$$

we have

$$B_1'' - \frac{B_1'}{r} - 2g^2 B_1 = \alpha B_a'' + \beta B_b'' - \frac{\alpha B_a'}{r} - \frac{\beta B_b'}{r} - 2g^2 \alpha B_a - 2g^2 \beta B_b,$$

$$= \alpha(B_a'' - \frac{B_a'}{r} - 2g^2 B_a) + \beta(B_b'' - \frac{B_b'}{r} - 2g^2 B_b),$$

$$= 0. \qquad \text{(by assumption)}$$

Hence, $\{\zeta_0, \alpha B_a + \beta B_b\}$ is also a perfect fluid sphere. #

3.4 New theorems

We shall now present new theorems which can conveniently generate a large class of perfect fluid spheres. The new theorems have the same technique as [6-8, 12, 13]. In section 3.4.1, we offer the new theorems by composing T_7 and T_8, then we analyze the property of these perfect fluid spheres. In section 3.4.2, we introduce a new technique of finding a new solution that is also in the form of a perfect fluid sphere.

3.4.1 The new theorem of two linking theorems

The solution generating theorem we shall present is slightly different from those developed so far. We can also simultaneously apply T_7 and T_8. So, the transformation can be represented as:

Theorem 3. *Suppose that* $\{\zeta_0(r), B_0(r)\}$ *is an initial perfect fluid sphere. Let* $\zeta_0(r)$ *and* $B_0(r)$ *be changed to* $\frac{1}{\zeta_0(r)}$ *and* $B_0(r)Z_0(r)$, *respectively. Define*

$$Z_0(r) = \left\{\sigma + \varepsilon \int \frac{r}{B_0(r)^2}\,\mathrm{d}r\right\}, \qquad (3.20)$$

for arbitrary σ *and* ε. *The transformation*

$$T_9 : \{\zeta_0(r), B_0(r)\} \mapsto \left\{ \frac{1}{\zeta_0(r)}, B_0(r) Z_0(B_0(r); r) \right\} \tag{3.21}$$

maps perfect fluid spheres into perfect fluid spheres.

Proof. We shall transform $\{\zeta_0(r), B_0(r)\}$ to $\{\zeta_1(r), B_1(r)\}$.

Let $\zeta_1(r) = \dfrac{1}{\zeta_0(r)}$ and $B_1(r) = B_0(r) Z_0(r)$.

Then $\zeta_1' = -\dfrac{1}{\zeta_0^2} \cdot \zeta_0'$, which implies that $\dfrac{\zeta_1'}{\zeta_1} = -\dfrac{\zeta_0'}{\zeta_0^2} \cdot \zeta_0 = -\dfrac{\zeta_0'}{\zeta_0}$.

Also we have

$$B_1' = B_0 Z_0' + Z_0 B_0', \text{ and } B_1'' = B_0 Z_0'' + 2B_0' Z_0' + B_0'' Z_0.$$

If $\{\zeta_1(r), B_1(r)\}$ would be a perfect fluid sphere, it has to satisfy perfect fluid constraint in equation (3.5), i.e.,

$$\left(\frac{\zeta_1'}{\zeta_1}\right)^2 = \frac{B_1'' - B_1'/r}{2B_1}.$$

So we have

$$\left(-\frac{\zeta_0'}{\zeta_0}\right)^2 = \frac{B_0 Z_0'' + 2B_0' Z_0' + B_0'' Z_0}{2B_0 Z_0} - \frac{B_0 Z_0' + Z_0 B_0'}{2rB_0 Z_0},$$

then

$$\frac{B_0'' - B_0'/r}{2B_0} = \frac{Z_0''}{2Z_0} + \frac{B_0' Z_0'}{B_0 Z_0} + \frac{B_0''}{2B_0} - \frac{Z_0'}{2rZ_0} - \frac{B_0'}{2rB_0},$$

and therefore

$$\frac{Z_0''}{2Z_0} + \frac{B_0' Z_0'}{B_0 Z_0} - \frac{Z_0'}{2rZ_0} = 0.$$

Multiplying the above equation by $2Z_0$, we get

$$Z_0'' = \frac{Z_0'}{r} - \frac{2B_0' Z_0'}{B_0},$$

and

$$\frac{Z_0''}{Z_0'} = \frac{1}{r} - \frac{2B_0'}{B_0}.$$

Integrating twice on both sides of the above equation, we eventually have

$$Z_0(r) = \left\{ \sigma + \varepsilon \int \frac{r}{B_0(r)^2} \mathrm{d}r \right\}. \qquad \#$$

We can see that the term $Z_0(r)$ has the same as theorem 2. Now, we verify

$$T_9 \circ T_9 : \{\zeta_0, B_0\} \mapsto \left\{ \frac{1}{\zeta_0}, B_0 Z_0(B_0; r) \right\} \mapsto \{\zeta_0, B_0 Z_0(B_0; r) Z_1(B_0 Z_0(B_0; r))\}, \tag{3.22}$$

where $Z_1 = \sigma + \varepsilon \int \frac{r}{B_0^2(r) Z_0^2(r)} \mathrm{d}r.$

When we apply T_9 once, $\zeta(r)$ perpetually changes. Therefore, after applying T_9 n-times, the function $\zeta(r)$ has the same result as T_7. Therefore, T_9 is not idempotent.

3.4.2 A new technique for generating perfect fluid spheres

There are several techniques to solve for the solutions, which can be applied in the classification of a large class of perfect fluid spheres. In this framework, we focus on the isotropic coordinates. We present a technique to expand a new solution generating theorem based on using the assumption of perfect fluid spheres, which implies an ordinary differential equation as following [6-8, 14]:

$$\left(\frac{\zeta'}{\zeta} \right)^2 = \frac{B'' - B'/r}{2B}. \tag{3.23}$$

Let $g(r) = \frac{\zeta'(r)}{\zeta(r)}$, we can find that

$$\zeta(r) = \exp\left(\int g(r) \mathrm{d}r \right),$$

equation (3.23) becomes

$$g^2(r) = \frac{B'' - B'/r}{2B} \tag{3.24}$$

or

$$g(r) = \pm \sqrt{\frac{B'' - B'/r}{2B}}. \tag{3.25}$$

Therefore,

$$\zeta(r) = \exp\left(\pm \int \sqrt{\frac{B'' - B'/r}{2B}} \mathrm{d}r \right). \tag{3.26}$$

From equation (3.25), we obtain

$$B'' - B'/r - 2g^2(r)B = 0. \tag{3.27}$$

Now, let $h(r) = \dfrac{B'(r)}{2B(r)}$. We can find that

$$B(r) = \exp\left(\int 2h(r)\mathrm{d}r \right). \tag{3.28}$$

Differentiate $h(r)$ we get

$$\begin{aligned} h'(r) &= \frac{1}{2}\left[\frac{B(r)B''(r) - B'(r)B'(r)}{B(r)^2} \right] \\ &= \frac{B''(r)}{2B(r)} - \frac{1}{2}\left(\frac{B'(r)}{B(r)} \right)^2 \\ &= \frac{B''(r)}{2B(r)} - 2h^2(r). \end{aligned}$$

Therefore, we get

$$\frac{B''(r)}{2B(r)} = h'(r) + 2h^2(r). \tag{3.29}$$

From equation (3.24) the above steps have led to a new equation

$$g^2(r) = 2h^2(r) + h'(r) - h(r)/r. \tag{3.30}$$

Definition 3. *The general Riccati equation can be written in the form*

$$y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x),$$

for arbitrary function $q_0(x)$, $q_1(x)$, *and* $q_2(x)$. *Without knowing a solution, we shall need a particular solution to solve the Riccati equation. If we are given a particular solution* $y_0(x)$, *then the general solution is obtained as*

$$y(x) = y_0(x) + \frac{k\exp\left\{ \int \left[2q_2(x)y_0(x) + q_1(x) \right] \mathrm{d}r \right\}}{1 - k\int q_2(x)\exp\left\{ \int \left[2q_2(x)y_0(x) + q_1(x) \right] \mathrm{d}r \right\} \mathrm{d}r},$$

for a real arbitrary constant [6, 11, 27, 28].

Using equation (3.30), we can rearrange it into the form of the Riccati equation

$$h'(r) = g^2(r) + \frac{1}{r}h(r) - 2h^2(r), \tag{3.31}$$

where $q_0(r) = g_0^2(r)$, $q_1(r) = \frac{1}{r}$, and $q_2(r) = -2$.

Through the algorithmic solving of the Riccati equation, if we know an initial solution (i.e. one particular solution) then we can derive its general solution [27].

Let $\{g_0(r), h_0(r)\}$ be a known solution of equation (3.31).

Then

$$h_0'(r) = g_0^2(r) + \frac{1}{r}h_0(r) - 2h_0^2(r), \tag{3.32}$$

- Assume the general solution

$$h(r) = h_0(r) + \frac{1}{z(r)}, \tag{3.33}$$

where $h_0(r)$ is a particular solution of the Riccati differential equation, which satisfies the equation

$$h'(r) = q_0(r) + q_1(r)h(r) + q_2(r)h^2(r). \tag{3.34}$$

Substituting $h(r) = h_0(r) + \frac{1}{z(r)}$ in equation (3.34)

$$\left(h_0(r) + \frac{1}{z(r)}\right)' = q_0(r) + q_1(r)\left[h_0(r) + \frac{1}{z(r)}\right] - 2\left[h_0(r) + \frac{1}{z(r)}\right]^2$$

$$h_0'(r) + \left(-\frac{1}{z^2(r)}\right)z'(r) = q_0(r) + q_1(r)h_0(r) + \frac{q_1(r)}{z(r)} - 2\left[h_0^2(r) + 2\frac{h_0(r)}{z(r)} + \frac{1}{z^2(r)}\right]$$

$$-\frac{1}{z^2(r)}z'(r) - \frac{q_1(r)}{z(r)} + 4\frac{h_0(r)}{z(r)} + \frac{2}{z^2(r)} = q_0(r) + q_1(r)h_0(r) - 2h_0^2(r) - h_0'(r) \tag{3.35}$$

From the first assumption, we get the right hand side of the equation equals to zero and equation (3.35) becomes

$$-\frac{1}{z^2(r)}z'(r) - \frac{[q_1(r) - 4h_0(r)]}{z(r)} = -\frac{2}{z^2(r)}. \tag{3.36}$$

Multiplying equation (3.36) by $-z^2(r)$, we obtain it to be

$$z'(r)+\left[q_1(r)-4h_0(r)\right]z(r)=2, \quad (3.37)$$

where $z(r)$ is the general solution of the first order linear differential equation.

- On the other hand, we regroup the equation in terms of $g_0(r),$ where

$$g_0^2(r)=h'(r)-\frac{1}{r}h(r)+2h^2(r). \quad (3.39)$$

Assuming that $g(r)$ is a solution of the above equation,

$$g^2(r)=h'(r)-\frac{1}{r}h(r)+2h^2(r). \quad (3.40)$$

Let us extend $g(r)$ to $g_0(r)+g^*(r)$ such that $g_0(r)+g^*(r)$ is a solution of equation (3.40). Then we substitute $g_0(r)+g^*(r)$ into equation (3.40), taking us to

$$\left(g_0(r)+g^*(r)\right)^2=h'(r)-\frac{1}{r}h(r)+2h^2(r)$$

$$g_0^2(r)+2g_0(r)g^*(r)+g^{*2}(r)=h'(r)-\frac{1}{r}h(r)+2h^2(r)$$

Based on the first assumption of equation (3.39), we can derive it as

$$2g_0(r)g^*(r)+g^{*2}(r)=0.$$

For $g^*(r)\neq 0,$ we obtain

$$g^*(r)=-2g_0(r)$$

$$\therefore \quad g_0(r)+g^*(r)=-g_0(r) \quad (3.41)$$

Therefore, the above expression suggests that, if we have $\{g_0(r),h_0(r)\}$ as a perfect fluid sphere, then $\left\{-g_0(r),h_0(r)+\frac{1}{z(r)}\right\}$ is also a perfect fluid sphere, where $\frac{1}{z(r)}$ can be solved from equation (3.37).

Next, let us start with the initial solution $\{g_0(r),h_0(r)\}=\left\{\sqrt{2-\frac{1}{r}},1\right\}.$

Substituting $q_1(r)=\frac{1}{r}$ and $h_0(r)=1$ into equation (3.37) as

$$z'(r)+\left[\frac{1}{r}-4\right]z(r)=2, \quad (3.42)$$

which is a first order linear inhomogeneous ODE. Then we can solve it using the integrating factor method,

$$z(r)=e^{-(\ln r-4r)}\int 2e^{(\ln r-4r)}\mathrm{d}r.$$

Therefore, we get

$$z(r)=-\frac{1}{2}-\frac{1}{8r}+c. \tag{3.43}$$

We choose $c=0$ as an example.

Consequently, the function $h(r)$ is equal to

$$h_0(r)+\frac{1}{z(r)}=1-\frac{1}{\frac{1}{2}+\frac{1}{8r}}.$$

Hence,

$$h(r)=\frac{1-4r}{1+4r}. \tag{3.44}$$

Consider equation (3.41),

$$g^*(r)=-2g_0(r).$$

That is the function $g(r)$ is equal to

$$g_0(r)+g^*(r)=-g_0(r)=-\sqrt{2-\frac{1}{r}}. \tag{3.45}$$

From $\zeta(r)=\exp\left(\int g(r)\mathrm{d}r\right),$ we obtain

$$\zeta(r)=\exp\left(-\int\sqrt{2-\frac{1}{r}}\mathrm{d}r\right). \tag{3.46}$$

Next, we consider in term $\int\sqrt{2-\frac{1}{r}}\mathrm{d}r.$ Integrating by part,

let $u=\sqrt{2-\frac{1}{r}},$ then $\mathrm{d}u=\dfrac{1}{2r^2\sqrt{2-\frac{1}{r}}},$

and $\mathrm{d}v=\mathrm{d}r,$ then $v=r.$ Therefore,

$$\int \sqrt{2-\frac{1}{r}}\mathrm{d}r = r\sqrt{2-\frac{1}{r}} - \int r\left(\frac{1}{2r^2\sqrt{2-\frac{1}{r}}}\right)\mathrm{d}r + c_1$$

$$= r\sqrt{2-\frac{1}{r}} - \int \frac{1}{2\sqrt{2r^2-r}}\,\mathrm{d}r + c_1$$

$$= r\sqrt{2-\frac{1}{r}} - \int \frac{1}{2\sqrt{2\left[\left(r-\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2\right]}}\,\mathrm{d}r + c_1$$

$$= r\sqrt{2-\frac{1}{r}} - \frac{1}{2\sqrt{2}}\int \frac{1}{\sqrt{\left(r-\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2}}\,\mathrm{d}\left(r-\frac{1}{4}\right) + c_1$$

$$= r\sqrt{2-\frac{1}{r}} - \frac{1}{2\sqrt{2}}\ln\left|\left(r-\frac{1}{4}\right)+\sqrt{\left(r-\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2}\right| + c_2$$

$$= r\sqrt{2-\frac{1}{r}} - \frac{1}{2\sqrt{2}}\ln\left|\left(r-\frac{1}{4}\right)+\sqrt{r^2-\frac{r}{2}}\right| + c_2$$

$$= r\sqrt{2-\frac{1}{r}} - \frac{1}{2\sqrt{2}}\ln\left|(4r-1)+2r\sqrt{4-\frac{2}{r}}\right| + c_3$$

$$= r\sqrt{2-\frac{1}{r}} - \frac{\ln\left|\left(4+2\sqrt{4-\frac{2}{r}}\right)r-1\right|}{2\sqrt{2}} + c_3.$$

Finally, we have

$$\zeta(r) = \exp\left\{-\left(r\sqrt{2-\frac{1}{r}} - \frac{\ln\left(\left(-4-2\sqrt{4-\frac{2}{r}}\right)r+1\right)}{2\sqrt{2}} + \gamma_1\right)\right\}, \tag{3.47}$$

and

$$B(r) = \exp\left(\int 2\frac{1-4r}{1+4r}\,\mathrm{d}r\right)$$

$$= \exp\left(\int 2\left(\frac{2}{1+4r}-1\right)\mathrm{d}r\right)$$

$$= \exp\left(\ln\left(4r+1\right)-2r+\gamma_2\right). \quad (3.48)$$

as required, where γ_1, and γ_2 are arbitrary constants.

3.5 Maple program: GRTensor program

In this section, we introduce some Maple programs to explain the basics of general relativity. The feature of GRTensor is a package specially built to deal with GR. In addition, we also use package of GRTensor with an example and verify a perfect fluid. Particularly, we built the solution of the Einstein field equation in isotropic coordinates. Furthermore, we also describe how one can use Maple and GRTensor for general relativity [6-8, 29].

3.5.1 Using maple and GRTensor

GRTensor is built as a package in the Maple platform as a special set of libraries. It deal with tensors and other geometric objects in general relativity. In this section, we present some of the main features offered by GRTensor.

3.5.2 Examples and the application of theorem 3

For the application with general relativity, we have based it on the Einstein field equation. Currently, there is an exact solution to the equation. We shall show the examples when we apply theorem 3 to some perfect fluid spheres, based on isotropic coordinates. We have derived theorem 3 by linking the two related theorems, T_7 and T_8. Furthermore, we have investigated these perfect fluid spheres, which can transform into a new solution or the initial solution. For example, we bring the solution of Minkowski and apply it with theorem 3 using the program Maple. In addition, for the other metric of S1 and K-O III, we can see the codes of the program in appendix A. Parts of the program code can be represented as follows;

- **Minkowski Metric**

It takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.49}$$

There is a maple code for applying the Minkowski metric based on the following

```
> #----------------------------------
# The new theorem of two linking theorems
  (applied to Minkowski metric)
#----------------------------------
restart;
  with(tensor);[Christoffel1, Christoffel2, Einstein,
Jacobian,   Killing_eqns, Levi_Civita, Lie_diff, Ricci,
Ricciscalar, Riemann, RiemannF, Weyl, act,
antisymmetrize, change_basis, commutator, compare, conj,
connexF, contract, convertNP, cov_diff, create, d1metric,
d2metric, directional_diff, displayGR, display_allGR,
dual, entermetric, exterior_diff, exterior_prod, frame,
geodesic_eqns, get_char, get_compts, get_rank, init,
invars, invert, lin_com, lower, npcurve, npspin,
partial_diff, permute_indices, petrov, prod, raise,
symmetrize, tensorsGR, transform];
 coords:=[t,r,theta,phi];
```

[*Christoffel1, Christoffel2, Einstein, Jacobian, Killing_eqns, Levi_Civita, Lie_diff, Ricci, Ricciscalar, Riemann, RiemannF, Weyl, act, antisymmetrize, change_basis, commutator, compare, conj, connexF, contract, convertNP, cov_diff, create, d1metric, d2metric, directional_diff, displayGR, display_allGR, dual, entermetric, exterior_diff, exterior_prod, frame, geodesic_eqns, get_char, get_compts, get_rank, init, invars, invert, lin_com, lower, npcurve, npspin, partial_diff, permute_indices, petrov, prod, raise, symmetrize, tensorsGR, transform*]

$$coords := [t, r, \theta, \phi]$$

```
> #---------------------------
# Seed metric (initial solution)
#---------------------------
 zeta0(r):=1;
```

$$\zeta 0(r) := 1$$

```
> B0(r):=1;
```

$$B0(r) := 1$$

```
> # Minkowski metric
#------------------------------------
# Theorem 0: checking the metric is a perfect fluid
#------------------------------------
 g:=array (symmetric, sparse, 1..4, 1..4):
 g[1,1]:=-zeta0(r)^2:
 g[2,2]:=1/B0(r):
 g[3,3]:=r^2:
```

```
g[4,4]:=(r^2)*sin(theta)^2:
metric:=create([-1,-1], eval(g));
```

$$metric := table\left(\left[compts = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{bmatrix}, index\_char = [-1, -1]\right]\right)$$

```
> tensorsGR(coords,metric,contra_metric,'det_met', C1,
C2, Rm, Rc, Rs, G, C);
 Grr:=simplify(G[compts][2,2]/metric[compts][2,2]):
 Gthth:=simplify(G[compts][3,3]/metric[compts][3,3]):
 Gphph:=simplify(G[compts][4,4]/metric[compts][4,4]):
 simplify(Gthth-Gphph);
# consistency check --- To verify a perfect fluid sphere,
this should be zero!
```

$$0$$

```
> dp0:=numer(simplify(Grr-Gthth));
# checking a perfect fluid
```

$$dp0 := 0$$

```
> #-----------------------------------
  # Theorem 3: B0->B1=B0*Z0;
  #-----------------------------------
 Z0(r):=sigma + epsilon*int(r/(B0(r)^2),r);
```

$$Z0(r) := \sigma + \frac{1}{2}\varepsilon r^2$$

```
> B1(r):=B0(r)*Z0(r);
```

$$B1(r) := \sigma + \frac{1}{2}\varepsilon r^2$$

```
> zeta1(r):=1/zeta0(r);
```

$$\zeta 1 := r \rightarrow \frac{1}{\zeta 0(r)}$$

```
> g:=array (symmetric, sparse, 1..4, 1..4):
 g[1,1]:=-1/(zeta0(r)^2):
 g[2,2]:=zeta0(r)^2/(B0(r)^2):
 g[3,3]:=r^2*zeta0(r)^2/(B0(r)^2):
 g[4,4]:=r^2*(sin(theta))^2*zeta0(r)^2/(B0(r)^2):
 metric:=create([-1,-1], eval(g));
```

$$metric := table\left(\left[compts = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{bmatrix}, index\_char = [-1, -1]\right]\right)$$

```
> tensorsGR(coords,metric,contra_metric,'det_met', C1,
C2, Rm, Rc, Rs, G, C);
 Grr:=simplify(G[compts][2,2]/metric[compts][2,2]):
```

```
 Gthth:=simplify(G[compts][3,3]/metric[compts][3,3]):
 Gphph:=simplify(G[compts][4,4]/metric[compts][4,4]):
 simplify(Gthth-Gphph);
# consistency check --- For statisfying the perfect fluid
constraint, this must be equal to zero!
```

$$0$$

```
> dp1:=numer(simplify(Grr-Gthth));
```

$$dp1 := 0$$

```
>   #------------------------------------------
    #CONCLUSION: Theorem 3 with a Minkowski seed gives
                 the Minkowski.
    #------------------------------------------
>   # END OF PROOFS.
 #-------------------------------------------------------
 # END WORKSHEET
 #-------------------------------------------------------
>
```

When we apply theorem 3 with the Minkowski metric, the output solution is the same as the initial solution. In addition, when we apply this theorem with the S1 and K-O III metrics, we derive a new solution, which is a perfect fluid sphere (See appendix A).

This section describes a way in which some simple computer programs in Maple and GRTensor can be used in general relativity. It shows that the speed of learning the main concepts in general relativity can be increased by avoiding large hand computation steps and a lot of errors or typos.

3.6 Conclusion

In this chapter, we focused on isotropic coordinates for extending a class of several new solutions. We analyzed the relationship between the solution generating theorems that map perfect fluid spheres into perfect fluid spheres. These theorems have led us to a new corollary and some additional properties. In addition, we have also presented two new solution generating theorems. The first theorem, we have derived by applying T_7 and T_8. Furthermore, we also investigated the idempotent property of this theorem. In addition, we obtained the new theorem using a new technique. In the last section, we applied theorem 3 with some example solutions such as Minkowski, S1, and K-O III in the GRTensor package of Maple. Consequently, when we apply this theorem on a known perfect fluid sphere, the solutions are often given as the same initial solution or an obtained new solution.

Chapter 4

The TOV equation in isotropic coordinates

Following the previous chapter, we recommend an applied technique, which is a simple method in finding solutions, thus called "*the solution generating theorems*". We have also presented new theorems for the generation of perfect fluid spheres in isotropic coordinates. These theorems are even more helpful in classifying a large class of perfect fluid spheres.

In the present chapter, we shall explain the use of the TOV equation in isotropic coordinates. We shall first describe in general the TOV equation.

Earlier, the Einstein field equation was denoted by the terms stress energy tensor, such that the equation was a variable of the functions of pressure and density. Karl Schwarzschild was the physicist who found and published the first exact solution to the Einstein field equation in 1916. This solution is called as the "Schwarzschild solution", which describes the gravitational field outside a static spherically symmetric object. Therefore, the Schwarzschild solution is a useful approximation for describing slowly rotating astronomical objects such as many stars and planets. The Tolman-Oppenheimer-Volkov equation attempts to find the solution inside a static spherically symmetric object. In order for the solution to have continuity within the surface, the TOV equation explains the interior structure of the static perfect fluid spheres.

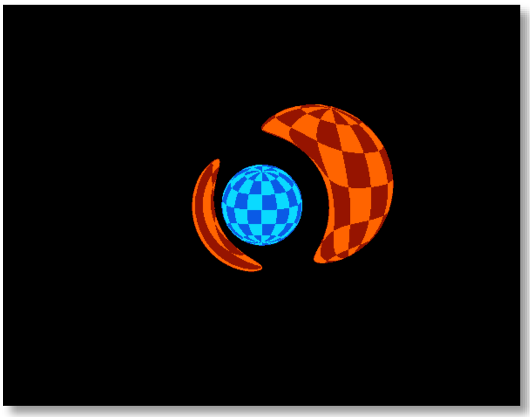


Figure 4.1: The TOV equation describes the deflected spectacle of a neutron star, including the density and pressure profile [30].

Chapter 4 is composed of the TOV equation in isotropic coordinates, the main theorems of the TOV equation, and the modified TOV equation. We will express each topic in more detail.

4.1 The characteristic of the TOV equation in isotropic coordinates

In chapter 3, the solution can be written in the form of spacetime ($\mathrm{d}s^2$). However, in this current study, our main focus is the isotropic coordinates. The solution can thereby also be expressed using the TOV equation. The TOV equation for isotropic coordinates can be represented by [6, 11]:

$$\mathrm{d}s^2 = -\zeta(r)^2 \mathrm{d}t^2 + \frac{1}{\zeta(r)^2 B(r)^2}\left\{\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right\}. \tag{4.1}$$

The metric tensor and inverse metric tensor can be written as

$$g_{\mu\nu} = \begin{bmatrix} -\zeta(r)^2 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\zeta(r)^2 B(r)^2} & 0 & 0 \\ 0 & 0 & \dfrac{r^2}{\zeta(r)^2 B(r)^2} & 0 \\ 0 & 0 & 0 & \dfrac{r^2 \sin^2\theta}{\zeta(r)^2 B(r)^2} \end{bmatrix},$$

$$g^{\mu\nu} = \begin{bmatrix} -\dfrac{1}{\zeta(r)^2} & & & \\ & \zeta(r)^2 B(r)^2 & & \\ & & \dfrac{\zeta(r)^2 B(r)^2}{r^2} & \\ & & & \dfrac{\zeta(r)^2 B(r)^2}{r^2 \sin^2\theta} \end{bmatrix}.$$

Considering the stress energy tensor,

$$T_{\mu\nu} = \begin{bmatrix} \zeta(r)^2 \rho & 0 & 0 & 0 \\ 0 & \dfrac{p}{\zeta(r)^2 B(r)^2} & 0 & 0 \\ 0 & 0 & \dfrac{pr^2}{\zeta(r)^2 B(r)^2} & 0 \\ 0 & 0 & 0 & \dfrac{pr^2 \sin^2\theta}{\zeta(r)^2 B(r)^2} \end{bmatrix}.$$

Then we get

$$T^{tt} = g^{tt} g^{tt} T_{tt} = \left(-\zeta(r)^{-2}\right)\left(-\zeta(r)^{-2}\right)\zeta(r)^2 \rho = \zeta(r)^{-2}\rho,$$

$$T^{rr} = g^{rr} g^{rr} T_{rr} = \zeta(r)^2 B(r)^2 p,$$

$$T^{\theta\theta} = g^{\theta\theta} g^{\theta\theta} T_{\theta\theta} = \frac{\zeta(r)^2 B(r)^2}{r^2} p,$$

$$T^{\phi\phi} = g^{\phi\phi} g^{\phi\phi} T_{\phi\phi} = \frac{\zeta(r)^2 B(r)^2}{r^2 \sin^2\theta} p.$$

Definition 4. *(Conservation of energy and momentum). A rule in physics, the total energy momentum 4-vector of a system of particles not acted upon by external forces is constant in measure and direction irrespective of any reactions among the parts of the system. That is* [4],

$$\nabla_\mu T^{\mu\nu} = 0.$$

Consider the conservation of momentum [1, 5, 10]; $\nabla_\mu T^{\mu\nu} = 0.$

Since the metric is nontrivial, $\nu = r.$

Then

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu r} + \Gamma^\mu_{\mu\nu} T^{\nu r} + \Gamma^r_{\mu\nu} T^{\mu\nu},$$

$$0 = \partial_r T^{rr} + \Gamma^\mu_{\mu r} T^{rr} + \left[\Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\theta\theta} T^{\theta\theta} + \Gamma^r_{\phi\phi} T^{\phi\phi}\right],$$

$$= \partial_r T^{rr} + \left(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}\right) T^{rr} + \left[\Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\theta\theta} T^{\theta\theta} + \Gamma^r_{\phi\phi} T^{\phi\phi}\right].$$

The Christoffel symbols can be defined by

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\rho}\left[\frac{\partial g_{\rho\alpha}}{\partial x^\beta} + \frac{\partial g_{\rho\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\rho}\right].$$

We derive the Christoffel symbols as shown below;

- $\Gamma^{t}_{tr} = \dfrac{\zeta'(r)}{\zeta(r)}$
- $\Gamma^{r}_{rr} = -\dfrac{B'(r)}{B(r)} - \dfrac{\zeta'(r)}{\zeta(r)}$
- $\Gamma^{\theta}_{\theta r} = \dfrac{1}{r} - \dfrac{B'(r)}{B(r)} - \dfrac{\zeta'(r)}{\zeta(r)}$
- $\Gamma^{\phi}_{\phi r} = \dfrac{1}{r} - \dfrac{B'(r)}{B(r)} - \dfrac{\zeta'(r)}{\zeta(r)}$
- $\Gamma^{r}_{tt} = \zeta^{3}(r)B^{2}(r)\zeta'(r)$
- $\Gamma^{r}_{\theta\theta} = -r + \dfrac{r^{2}B'(r)}{B(r)} + \dfrac{r^{2}\zeta'(r)}{\zeta(r)}$
- $\Gamma^{r}_{\phi\phi} = -r\sin^{2}\theta + \dfrac{r^{2}\sin^{2}\theta B'(r)}{B(r)} + \dfrac{r^{2}\sin^{2}\theta\zeta'(r)}{\zeta(r)}$
- $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$
- $\Gamma^{\phi}_{\theta\phi} = \sin\theta\cos\theta$

So that

$$0 = \partial_r\left(\zeta(r)^2 B(r)^2 p\right) + \left(\frac{\zeta'(r)}{\zeta(r)} - \frac{B'(r)}{B(r)} - \frac{\zeta'(r)}{\zeta(r)} + \frac{2}{r} - \frac{2B'(r)}{B(r)} - \frac{2\zeta'(r)}{\zeta(r)}\right)\left(\zeta(r)^2 B(r)^2 p\right)$$
$$+\left(\zeta(r)^2 B(r)^2 \zeta'(r)\right)\left(\frac{\rho}{\zeta(r)^2}\right) + \left(-\frac{B'(r)}{B(r)} - \frac{\zeta'(r)}{\zeta(r)}\right)\left(\zeta(r)^2 B(r)^2 p\right)$$
$$+\left(-r + \frac{r^2 B'(r)}{B(r)} + \frac{r^2\zeta'(r)}{\zeta(r)}\right)\left(\frac{\zeta(r)^2 B(r)^2}{r^2} p\right)$$
$$+\left(-r + \frac{r^2 B'(r)}{B(r)} + \frac{r^2\zeta'(r)}{\zeta(r)}\right)\sin^2\theta\left(\frac{\zeta(r)^2 B(r)^2}{r^2\sin^2\theta} p\right).$$

$$0 = \left[\zeta(r)^2 B(r)^2 \frac{\partial p}{\partial r} + p\left(\zeta(r)^2 2B(r)B'(r) + B(r)^2 2\zeta(r)\zeta'(r)\right)\right]$$
$$+\left[\frac{2\zeta(r)^2 B(r)^2}{r} p - 3\zeta(r)^2 B'(r)B(r)p - 2B(r)^2\zeta'(r)\zeta(r)p\right]$$
$$+\zeta(r)\zeta'(r)B(r)^2\rho + \left(-\zeta(r)^2 B'(r)B(r)p - B(r)^2\zeta(r)\zeta'(r)p\right)$$

$$+2\left[-\frac{\zeta(r)^2 B(r)^2}{r}p + B'(r)B(r)\zeta(r)^2 p + \zeta'(r)\zeta(r)B(r)^2 p\right].$$

Therefore,

$$\zeta^2(r)B^2(r)\frac{\partial p}{\partial r} + \zeta(r)\zeta'(r)B^2(r)\rho + \zeta(r)\zeta'(r)B^2(r)p = 0.$$

Finally, we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\zeta'(r)}{\zeta(r)}(\rho + p). \tag{4.2}$$

Let us set $\zeta(r) = \exp(\int g(r)\mathrm{d}r).$

Define an auxiliary function $g(r)$ is defined to be [11]

$$g(r) = \frac{m(r) + 4\pi p(r)r^3}{r^2\left[1 - 2m(r)/r\right]}.$$

Hence, directly

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{\left[\rho(r) + p(r)\right]\left[m(r) + 4\pi p(r)r^3\right]}{r^2\left[1 - 2m(r)/r\right]}; \tag{4.3}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi\rho(r)r^2. \tag{4.4}$$

The equations (4.3) and (4.4), called the “TOV equation” in isotropic coordinates, describes the internal structure of general relativistic static perfect fluid spheres. The significance of the TOV equation is associated with the constraints in studying the interior structures.

We can see that it gives the same results as the TOV equations using the Schwarzschild metric as in [31]. Because the TOV equation describes the interior of stars in terms of pressure and density, they are considered as the real measures. Regardless of the changing coordinates, the form of the TOV equation is not different. Therefore, we can apply the theorems of the TOV equation with the Schwarzschild coordinates and the isotropic coordinates together.

4.2 The main theorems of the TOV equation

We now study the main theorems of the TOV equation. Having already found the solution generating theorems in isotropic coordinates, we can then derive these theorems directly in terms of the pressure and density profiles, $p(r)$ and $\rho(r)$, which are useful in generating the interior solutions for perfect fluid spheres [6, 11].

Theorem 4. *(P1) Defines an auxiliary function* $g_0(r)$ *by*

$$g_0(r)=\frac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]}. \tag{4.5}$$

Then the general solution to the TOV equation is given by $p(r)=p_0(r)+\delta p(r),$ *where*

$$\delta p(r)=\frac{\delta p_c\sqrt{1-2m_0/r}\,\exp\left\{-2\int_0^r g_0\mathrm{d}r\right\}}{1+4\pi\delta p_c\int_0^r \frac{r}{\sqrt{1-2m_0/r}}\exp\left\{-2\int_0^r g_0 dr\right\}\mathrm{d}r}, \tag{4.6}$$

and δp_c *is the shift in the central pressure.*

Proof. From the TOV equation (4.3), we can see that

$$\begin{aligned}\frac{\mathrm{d}p(r)}{\mathrm{d}r}&=-\frac{\left[\rho(r)+p(r)\right]\left[m(r)+4\pi p(r)r^3\right]}{r^2\left[1-2m(r)/r\right]}\\&=-\frac{\rho(r)m(r)+\rho(r)\left[4\pi p(r)r^3\right]+p(r)m(r)+4\pi r^3p^2(r)}{r^2\left[1-2m(r)/r\right]}.\end{aligned}$$

We can see that the above equation to be the Riccati equation.

The general solution $p(r)$ of the TOV equation is $p(r)=p_0(r)+\delta p(r),$ where $p_0(r)$ is a particular solution.

Therefore, the function $\delta p(r)$ is equal to

$$\begin{aligned}&\frac{k\exp\left\{\int\left[2\left(-\frac{4\pi r^3}{r^2\left[1-2m_0/r\right]}\right)p_0-\frac{4\pi r^3\rho_0+m_0}{r^2\left[1-2m_0/r\right]}\right]\mathrm{d}r\right\}}{1-k\int-\frac{4\pi r^3}{r^2\left[1-2m_0/r\right]}\exp\left\{\int\left[2\left(-\frac{4\pi r^3}{r^2\left[1-2m_0/r\right]}\right)p_0-\frac{4\pi r^3\rho_0+m_0}{r^2\left[1-2m_0/r\right]}\right]\mathrm{d}r\right\}\mathrm{d}r}\\&=\frac{k\exp\left\{-\int\frac{m_0+4\pi r^3\rho_0+8\pi r^3p_0}{r^2\left[1-2m_0/r\right]}\mathrm{d}r\right\}}{1+4\pi k\int\frac{r}{1-2m_0/r}\exp\left\{-\int\frac{m_0+4\pi r^3\rho_0+8\pi r^3p_0}{r^2\left[1-2m_0/r\right]}\mathrm{d}r\right\}\mathrm{d}r}\end{aligned} \tag{4.7}$$

Next, we will simplify the term of

$$\exp\left\{-\int \frac{m_0 + 4\pi r^3 \rho_0 + 8\pi r^3 p_0}{r^2\left[1-2m_0/r\right]}\mathrm{d}r\right\}.$$

Consider $\int \frac{4\pi r^3 \rho_0}{r^2\left[1-2m_0/r\right]}\mathrm{d}r = \int \frac{4\pi r \rho_0}{1-2m_0/r}\mathrm{d}r.$

Solving the above equation by substitution; $u = 1-2m_0/r,\quad \mathrm{d}u = \left(\frac{-2rm_0' + 2m_0}{r^2}\right)\mathrm{d}r.$

Since $\frac{\mathrm{d}m_0}{\mathrm{d}r} = 4\pi r^2 \rho_0,$ then we obtain

$$\mathrm{d}u = \left(-\frac{8\pi r^2 \rho_0}{r} + \frac{2m_0}{r^2}\right)\mathrm{d}r.$$

This leads us to the following equation

$$4\pi\rho_0 r\mathrm{d}r = \frac{m_0}{r^2}\mathrm{d}r - \frac{1}{2}\mathrm{d}u. \tag{4.8}$$

Therefore,

$$\int \frac{4\pi r \rho_0}{1-2m_0/r}\mathrm{d}r = \int \frac{m_0}{r^2\left(1-2m_0/r\right)}\mathrm{d}r - \frac{1}{2}\int \frac{\mathrm{d}u}{u}$$

$$= \int \frac{m_0}{r^2\left(1-2m_0/r\right)}\mathrm{d}r - \ln\left[1-2m_0/r\right]^{1/2}.$$

So that

$$\exp\left\{-\int \frac{m_0 + 4\pi r^3 \rho_0 + 8\pi r^3 p_0}{r^2\left[1-2m_0/r\right]}\mathrm{d}r\right\}$$

$$= \exp\left\{-\int\left(\frac{8\pi r p_0}{1-2m_0/r} + \frac{m_0 + m_0}{r^2\left[1-2m_0/r\right]}\right)\mathrm{d}r + \ln\left[1-2m_0/r\right]^{1/2}\right\}$$

$$= \sqrt{1-2m_0/r}\,\exp\left[-2\int g_0 \mathrm{d}r\right]. \tag{4.9}$$

Hence, from equation (4.7)

$$\delta p(r) = \frac{k\sqrt{1-2m_0/r}\,\exp\left[-2\int g_0 \mathrm{d}r\right]}{1 + 4\pi k \int \frac{r}{\sqrt{1-2m_0/r}}\exp\left[-2\int g_0 \mathrm{d}r\right]\mathrm{d}r}. \tag{4.10}$$

To find the constant k, we must apply the boundary condition $p(0) = p_c$ at $r = 0.$

We can consider the limits of integration from 0 to r as

$$\delta p_c = \lim_{r\to 0} \frac{k\sqrt{1-2m_0(r)/r}\,\exp\left[-2\int_0^r g_0 \mathrm{d}r\right]}{1+4\pi k\int_0^r \frac{r}{\sqrt{1-2m_0(r)/r}}\exp\left[-2\int_0^r g_0\mathrm{d}r\right]\mathrm{d}r}, \tag{4.11}$$

where $\lim_{r\to 0}\frac{2m_0(r)}{r}$ approach to zero.

Using the fundamentals theorem of calculus, the integrals go to zero. We can directly use mathematical methods to solve the equation (4.11), to obtain

$$k = \delta p_c.$$

Consequently,

$$\delta p(r) = \frac{\delta p_c \sqrt{1-2m_0/r}\,\exp\left\{-2\int_0^r g_0\mathrm{d}r\right\}}{1+4\pi\delta p_c\int_0^r \frac{r}{\sqrt{1-2m_0/r}}\exp\left\{-2\int_0^r g_0 dr\right\}\mathrm{d}r}, \tag{4.12}$$

as required.

Theorem 5. *(P2) Defines an auxiliary function* $g_0(r)$ *by*

$$g_0(r) = \frac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]} = \frac{m(r)+4\pi p(r)r^3}{r^2\left[1-2m(r)/r\right]}. \tag{4.13}$$

Then the general solution to the TOV equation is given by $p(r) = p_0(r)+\delta p(r)$ *and* $m(r) = m_0(r)+\delta m(r)$, *where*

$$\delta m(r) = \frac{4\pi r^3 \delta\rho_c}{3\left[1+rg_0\right]^2}\exp\left\{2\int_0^r g_0\frac{1-rg_0}{1+rg_0}\mathrm{d}r\right\}, \tag{4.14}$$

and

$$\delta p(r) = -\frac{\delta m}{4\pi r^3}\frac{1+8\pi p_0 r^2}{1-2m_0/r}. \tag{4.15}$$

Here, $\delta\rho_c$ *is the shift in the central density. Explicitly combining these formulae, we have*

$$\delta p(r) = \frac{\delta p_c}{\left[1+rg_0\right]^2}\frac{1+8\pi p_0 r^2}{1-2m_0/r}\exp\left\{2\int_0^r g_0\frac{1-\tilde{r}g_0}{1+\tilde{r}g_0}\mathrm{d}\tilde{r}\right\}, \tag{4.16}$$

where $\delta p_c = -\frac{\delta\rho_c}{3}$.

Proof. From equation (4.13), we can see that

$$m(r) = g_0(r)r^2\left[1-2m(r)/r\right]-4\pi p(r)r^3.$$

Solving for $m(r)$ we get

$$m(r) = \frac{g_0(r)r^2-4\pi p(r)r^3}{1+2rg_0(r)}.$$

Considering

$$m(r)+\delta m(r) = \frac{g_0(r)r^2-4\pi\left[p(r)+\delta p(r)\right]r^3}{1+2rg_0(r)}$$

$$= \frac{g_0(r)r^2-4\pi p(r)r^3-4\pi\delta p(r)r^3}{1+2rg_0(r)}$$

$$= \frac{g_0(r)r^2-4\pi p(r)r^3}{1+2rg_0(r)}-\frac{4\pi\delta p(r)r^3}{1+2rg_0(r)}$$

Thus,

$$\delta m(r) = -\frac{4\pi\delta p(r)r^3}{1+2rg_0(r)}. \tag{4.17}$$

We substitute $g_0(r) = \dfrac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]}$ into equation (4.17), then

$$\delta m(r) = -4\pi\delta p(r)r^3\left\{\frac{1-2m_0/r}{1+8\pi p_0 r^2}\right\}.$$

Given $m = m_0+\delta m$ and $\rho = \rho_0+\delta\rho$. From equation (4.4) becomes

$$\delta\rho(r) = \frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\delta m(r)\right). \tag{4.18}$$

Substituting $\delta m(r)$ from equation (4.17) into equation (4.18),

$$\delta\rho(r) = -\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\delta p(r)r^3}{1+2rg_0(r)}\right]. \tag{4.19}$$

Then we consider the TOV equation (4.3) by adjusting the terms of pressure and density, so it can be written as

$$\frac{\mathrm{d}\delta p(r)}{\mathrm{d}r} = -\left[\delta\rho(r)+\delta p(r)\right]g_0(r). \tag{4.20}$$

Replacing $\delta\rho(r)$ from equation (4.19) into equation (4.20), we have

$$\frac{\mathrm{d}\delta p(r)}{\mathrm{d}r} = \left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\delta p(r) r^3}{1+2rg_0(r)}\right] - \delta p(r)\right] g_0(r), \tag{4.21}$$

which leads to

$$\frac{1}{\delta p(r)}\mathrm{d}\delta p(r) = \left(\frac{2g_0(r)\left[1 - r^2 g_0'(r) - 2r^2 g_0^2(r)\right]}{\left(1+2rg_0(r)\right)\left(1+rg_0(r)\right)}\right)\mathrm{d}r. \tag{4.22}$$

Given the boundary condition as $\delta p(0) = \delta p_c$ and taking integration on both sides from $\tilde{r} = 0$ to $\tilde{r} = r$, we get

$$\int_{\delta p_c}^{\delta p(r)} \frac{1}{\delta p(\tilde{r})}\mathrm{d}\delta p(\tilde{r}) = \int_0^r \left(\frac{2g_0(\tilde{r})\left[1 - \tilde{r}^2 g_0'(\tilde{r}) - 2\tilde{r}^2 g_0^2(\tilde{r})\right]}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r}.$$

$$\ln\frac{\delta p(r)}{\delta p_c} = \int_0^r \left(\frac{2g_0(\tilde{r})\left[1 - \tilde{r}^2 g_0'(\tilde{r}) - 2\tilde{r}^2 g_0^2(\tilde{r})\right]}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r}$$

$$= \int_0^r \frac{2g_0(\tilde{r})\left[1 - 2\tilde{r}^2 g_0^2(\tilde{r})\right]}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\mathrm{d}\tilde{r} - \int_0^r \left(\frac{2g_0(\tilde{r})\left[\tilde{r}^2 g_0'(\tilde{r})\right]}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r}.$$

From the above equation, let $u = 1 + \tilde{r}g_0(\tilde{r})$. Then $\mathrm{d}u = \left(\tilde{r}g_0' + g_0\right)\mathrm{d}\tilde{r}$,

$$\therefore \quad \tilde{r}g_0'\mathrm{d}\tilde{r} = \mathrm{d}u - g_0\mathrm{d}\tilde{r}.$$

Therefore,

$$\ln\frac{\delta p(r)}{\delta p_c} = \int_0^r \frac{2g_0(\tilde{r})\left[1 - 2\tilde{r}^2 g_0^2(\tilde{r})\right]}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\mathrm{d}\tilde{r} - \int_0^u \frac{2\left(u-1\right)}{\left(2u-1\right)u}\mathrm{d}u$$

$$+ \int_0^r \left(\frac{2\tilde{r}g_0(\tilde{r})^2}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r}.$$

$$= \ln\left|\frac{2u-1}{u^2}\right| + 2\int_0^r \left(\frac{g_0(\tilde{r})\left[1 + \tilde{r}g_0(\tilde{r})^2 - 2\tilde{r}^2 g_0(\tilde{r})^2\right]}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r} + c_1$$

$$= \ln\left|\frac{2u-1}{u^2}\right| + 2\int_0^r \left(\frac{g_0(\tilde{r})\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1-\tilde{r}g_0(\tilde{r})\right)}{\left(1+2\tilde{r}g_0(\tilde{r})\right)\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r} + c_1.$$

For simplicity, we choose $c_1 = 0,$ gives

$$\ln\frac{\delta p(r)}{\delta p_c} = \ln\left|\frac{2u-1}{u^2}\right| + 2\int_0^r \left(\frac{g_0(\tilde{r})\left(1-\tilde{r}g_0(\tilde{r})\right)}{\left(1+\tilde{r}g_0(\tilde{r})\right)}\right)\mathrm{d}\tilde{r}.$$

Then

$$\delta p(r)=\delta p_c\cdot\frac{1+2rg_0(r)}{\left(1+rg_0(r)\right)^2}\exp\left\{2\int_0^r g_0(\tilde{r})\frac{1-\tilde{r}g_0(\tilde{r})}{1+\tilde{r}g_0(\tilde{r})}\mathrm{d}\tilde{r}\right\} \tag{4.23}$$

Finally, we get

$$\delta p(r)=\frac{\delta p_c}{\left[1+rg_0\right]^2}\frac{1+8\pi p_0 r^2}{1-2m_0/r}\exp\left\{2\int_0^r g_0\frac{1-\tilde{r}g_0}{1+\tilde{r}g_0}\mathrm{d}\tilde{r}\right\}. \qquad \#$$

4.3 The modified TOV equation

In this section, we shall introduce a new modified theorem for the TOV equation. We study the extension of the values of pressure, density, and mass of objects in order to analyze the arbitrary functions in terms of r, and how the form should be represented.

Theorem 6. *Let* $p_0(r)$ *and* $\rho_0(r)$ *solve the TOV equation, and hold* $g_0(r)$ *fixed, in the sense that*

$$g_0(r)=\frac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]}=\frac{m(r)+4\pi p(r)r^3}{r^2\left[1-2m(r)/r\right]}. \tag{4.24}$$

Then the general solution to the TOV equation is given by $p(r)=Z_0(r)p_0(r)$, $\rho(r)=A_0(r)\rho_0(r)$, *and* $m(r)=B_0(r)m_0(r)$. *We define functions* $Z_0(r)$ *and* $A_0(r)$ *by*

$$Z_0(r)=-e^{-\nu(r)}\int e^{\nu(r)}\frac{A_0(r)\rho_0(r)g_0(r)}{p_0(r)}\mathrm{d}r, \tag{4.25}$$

where $\nu(r)=-\int\frac{\rho_0(r)g_0(r)}{p_0(r)}\mathrm{d}r$, *and*

$$A_0(r)=B_0(r)+\frac{B_0'(r)m_0(r)}{4\pi\rho_0(r)r^2}, \tag{4.26}$$

for all arbitrary $B_0(r)$.

Proof. From the TOV equation

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r}=-\frac{\left[\rho(r)+p(r)\right]\left[m(r)+4\pi p(r)r^3\right]}{r^2\left[1-2m(r)/r\right]} \tag{4.27}$$

Then we substitute $p(r)=Z_0(r)p_0(r)$, $\rho(r)=A_0(r)\rho_0(r)$, and $m(r) = B_0(r)m_0(r)$ into equation (4.28) to give

$$Z_0(r)\frac{\mathrm{d}p_0(r)}{\mathrm{d}r} + \frac{\mathrm{d}Z_0(r)}{\mathrm{d}r}p_0(r)$$

$$= -\frac{\left[A_0(r)\rho_0(r) + Z_0(r)p_0(r)\right]\left[B_0(r)m_0(r) + 4\pi Z_0(r)p_0(r)r^3\right]}{r^2\left[1 - 2B_0(r)m_0(r)/r\right]}.$$

Expanding the above equation as: the left hand side is equal to

$$-Z_0(r)\frac{\left[\rho_0(r) + p_0(r)\right]\left[m_0(r) + 4\pi p_0(r)r^3\right]}{r^2\left[1 - 2m_0(r)/r\right]} + \frac{\mathrm{d}Z_0(r)}{\mathrm{d}r}p_0(r).$$

While the right hand side is

$$-\frac{\left[A_0(r)\rho_0(r) + Z_0(r)p_0(r)\right]\left[B_0(r)m_0(r) + 4\pi Z_0(r)p_0(r)r^3\right]}{r^2\left[1 - 2B_0(r)m_0(r)/r\right]}.$$

So, we get

$$-Z_0(r)\left[\rho_0(r) + p_0(r)\right]g_0(r) + \frac{\mathrm{d}Z_0(r)}{\mathrm{d}r}p_0(r) = -\left[A_0(r)\rho_0(r) + Z_0(r)p_0(r)\right]g_0(r).$$

Accordingly, we can write the equation in the form

$$Z_0'(r) - \frac{\rho_0(r)g_0(r)}{p_0(r)}Z_0(r) = -\frac{A_0(r)\rho_0(r)g_0(r)}{p_0(r)}, \qquad (4.28)$$

which is the first order linear non-homogeneous equation in terms of $Z_0(r)$.

We can solve for $Z_0(r)$ using the integrating factor method.

Finally,

$$Z_0(r) = -e^{-\nu(r)}\int e^{\nu(r)}\frac{A_0(r)\rho_0(r)g_0(r)}{p_0(r)}\mathrm{d}r, \qquad (4.29)$$

where $\nu(r) = -\int\frac{\rho_0(r)g_0(r)}{p_0(r)}\mathrm{d}r.$

Next, we consider

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi\rho(r)r^2. \qquad (4.30)$$

The extension of $m(r)$ and $\rho(r)$, in a more suggestive form, gives;

$$A_0(r) = B_0(r) + \frac{B_0'(r)m_0(r)}{4\pi\rho_0(r)r^2}. \qquad \#$$

4.4 Conclusion

In this chapter, the Tolman-Oppenheimer-Volkov equation attempts to find the solution inside a static spherically symmetric object. In order for the solution to have continuity within the surface, the TOV equation explains the interior structure of static perfect fluid spheres. Furthermore, we can see that the TOV equation of isotropic coordinates gives the same results as the TOV equations of the Schwarzschild metric.

We deformed these solutions into the TOV equation in terms of pressure and density, and developed a new modified theorem for the TOV equation.

Chapter 5
Conclusion

5.1 Conclusion and discussion

This thesis has been written with the aim of making several physical concepts more comprehensible to people with a basic background in general relativity; especially concepts related to isotropic coordinates, perfect fluid spheres in general relativity, and the Tolman-Oppenheimer-Volkov equation.

In chapter 2, we introduced the basic concepts of relativity, which consist of special and general relativity. In addition, we also presented the procedure for the calculation of the Einstein field equation. General relativity, as specified, defines the geometric property of spacetime. Einstein presented an equation that explains the fundamental interaction of gravitation called the "*Einstein field equation*". Moreover, we also referred to a perfect fluid sphere in general relativity, in which it is used as an assumption to idealize models of stars. Then we derived a perfect fluid constraint such that imply us to ordinary differential equation. Perfect fluid spheres are simply developed as idealized models of stars.

In chapter 3, we used the perfect fluid constraint to build several new exact solutions for any relativistic static perfect fluid sphere. In the descriptive approximation of stars, we made use of these solutions to analyze the realistic stars. Due to the coordinate systems of perfect fluid spheres in perspective, perfect fluid spheres in the isotropic coordinates forms about 35% of this research. A significant characteristic of isotropic coordinates is that the coefficients of radial and angular coordinates are equal. Therefore, we focus on these solutions in isotropic coordinates for extending several classes of new solutions.

In fact, we have found a new relationship between the solution generating theorems that map perfect fluid spheres into perfect fluid spheres. According to theorem 3, what we have presented is slightly different from the previous theorems. What we have done is simultaneously apply T_7 and T_8. So, the transformation can be represented as

$$T_9 : \{\zeta_0(r), B_0(r)\} \mapsto \left\{\frac{1}{\zeta_0(r)}, B_0(r)Z_0(r)\right\}.$$

Furthermore, these theorems have also led us to some additional properties, which are idempotent property and the square root of unity. For the convenience in the investigation of perfect fluid spheres, we applied theorem 3 to some example solutions such as Minkowski, S1, and K-O III in the GRTensor package of Maple.

Working with the previous theorems, we obtain a new corollary, which can be verified with all perfect fluid spheres in isotropic coordinates. Furthermore, $\{\zeta_0, \alpha B_a + \beta B_b\}$ is also a perfect fluid sphere, for real arbitrary constant α, β.

We also specially constructed a new technique for the generation of perfect fluid spheres. The methodology used has additionally been considered in isotropic coordinates using the perfect fluid constraint. We have analyzed an ordinary differential equation, and have also developed a technical solution with the Riccati equation. Finally, we obtained the new functions of $\zeta(r)$ and $B(r)$ with respect to r.

In chapter 4, we introduced the Tolman-Oppenheimer-Volkov equation. Previously, we established the solution inside a static spherically symmetric object. In order for the solution to have continuity within the surface, the TOV equation can be explained through the interior structure of static perfect fluid spheres, including the pressure and the density profiles. In this chapter, we found that the TOV equation in isotropic coordinates gives the same results as the TOV equations of the Schwarzschild metric. We deformed these solutions into the TOV equation in terms of pressure and density. Moreover, we developed a new modified theorem for the TOV equation to study the extension of the values of pressure, density, and mass of objects, which have all been expressed to be a part of this theorem.

To summarize, we have found the relative transformation theorems that map perfect fluid spheres into perfect fluid spheres in isotropic coordinates. We analyzed the properties of perfect fluid spheres and derived a new theorem for the generation of exact solutions in isotropic coordinates. In addition, we also modified new solutions for the TOV equation, which directly provides information about the pressure and density profiles of general relativistic static perfect fluid spheres.

This thesis developed and provided a platform for a better understanding of isotropic coordinates and perfect fluid spheres in general relativity. Furthermore, we generated two new theorems for the construction of perfect fluid spheres in isotropic coordinates.

5.2 Further interesting issues

For a supplementary study of this thesis, we would like to

- Extend the TOV equation in other coordinates. This will be done to generate several new theorems, which can be useful in learning physical meanings, including the pressure and density profiles of stars.
- Study the physical meaning in terms of the temperature of realistic stars. It will be based on the TOV equation.

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APPENDIX

In chapter 3, we have already presented the application of theorem three with the Minkowski metric in the GRTensor package of Maple. Now, we shall apply this theorem with S1 and K-O III, respectively. This idea is obtained from [6-8].

- **S1**

The form of metric S1 write to be

$$\mathrm{d}s^2 = -r^4\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2.$$

```
> #-----------------------------------
  # The new theorem of two linking theorems
   (applied to S1 metric)
  #-----------------------------------
restart;
with(tensor);
[Christoffel1, Christoffel2, Einstein, Jacobian,
Killing_eqns, Levi_Civita, Lie_diff, Ricci, Ricciscalar,
Riemann, RiemannF, Weyl, act, antisymmetrize,
change_basis, commutator, compare, conj, connexF,
contract, convertNP, cov_diff, create, d1metric,
d2metric, directional_diff, displayGR, display_allGR,
dual, entermetric, exterior_diff, exterior_prod, frame,
geodesic_eqns, get_char, get_compts, get_rank, init,
invars, invert, lin_com, lower, npcurve, npspin,
partial_diff, permute_indices, petrov, prod, raise,
symmetrize, tensorsGR, transform];
coords:=[t,r,theta,phi];
```

[*Christoffel1*, *Christoffel2*, *Einstein*, *Jacobian*, *Killing_eqns*, *Levi_Civita*, *Lie_diff*, *Ricci*, *Ricciscalar*, *Riemann*, *RiemannF*, *Weyl*, *act*, *antisymmetrize*, *change_basis*, *commutator*, *compare*, *conj*, *connexF*, *contract*, *convertNP*, *cov_diff*, *create*, *d1metric*, *d2metric*, *directional_diff*, *displayGR*, *display_allGR*, *dual*, *entermetric*, *exterior_diff*, *exterior_prod*, *frame*, *geodesic_eqns*, *get_char*, *get_compts*, *get_rank*, *init*, *invars*, *invert*, *lin_com*, *lower*, *npcurve*, *npspin*, *partial_diff*, *permute_indices*, *petrov*, *prod*, *raise*, *symmetrize*, *tensorsGR*, *transform*]

$$coords := [t, r, \theta, \phi]$$

```
> #---------------------------
  # SEED metric (initial metric)
  #---------------------------
  zeta0(r):=1/r^2;
```

$$\zeta 0(r) := \frac{1}{r^2}$$

```
> B0(r):=1/r^2;
```

$$B0(r) := \frac{1}{r^2}$$

```
> # S1 seed
  #-----------------------------------
  # Theorem 0: check the seed is a perfect fluid
  #------------------------------------
 g:=array (symmetric, sparse, 1..4, 1..4):
 g[1,1]:=-1/zeta0(r)^2:
 g[2,2]:=zeta0(r)^2/B0(r)^2:
 g[3,3]:=r^2*zeta0(r)^2/B0(r)^2:
 g[4,4]:=(r^2)*sin(theta)^2*zeta0(r)^2/B0(r)^2:
 metric:=create([-1,-1], eval(g));
```

$$metric := table\left(\left[compts = \begin{bmatrix} -r^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin(\theta)^2 \end{bmatrix}, index\_char = [-1, -1]\right]\right)$$

```
> tensorsGR(coords,metric,contra_metric,'det_met', C1,
C2, Rm, Rc, Rs, G, C);
 Grr:=simplify(G[compts][2,2]/metric[compts][2,2]):
 Gthth:=simplify(G[compts][3,3]/metric[compts][3,3]):
 Gphph:=simplify(G[compts][4,4]/metric[compts][4,4]):
 simplify(Gthth-Gphph);
# consistency check --- To verify a perfect fluid sphere,
this should be zero!
```

$$0$$

```
> dp0:=numer(simplify(Grr-Gthth));
# checking a perfect fluid
```

$$dp0 := 0$$

```
> #-----------------------------------
  # Theorem 3: B0->B1=B0*Z0;
  #-----------------------------------
 Z0(r):=sigma + epsilon*int(r/(B0(r)^2),r);
```

$$Z0(r) := \sigma + \frac{1}{6}\varepsilon r^6$$

```
> B1(r):=B0(r)*Z0(r);
```

$$B1(r) := \frac{\sigma + \frac{1}{6}\varepsilon r^6}{r^2}$$

```
> zeta1(r):=1/zeta0(r);
```

$$\zeta 1 := r^2$$

```
> g:=array (symmetric, sparse, 1..4, 1..4):
 g[1,1]:=-1/(zeta1(r)^2):
 g[2,2]:=zeta1(r)^2/(B1(r)^2):
```

```
 g[3,3]:=r^2*zeta1(r)^2/(B1(r)^2):
 g[4,4]:=r^2*(sin(theta))^2*zeta1(r)^2/(B1(r)^2):
metric:=create([-1,-1], eval(g));
>
```

$$metric := table\left(\left[compts = \begin{bmatrix} -\frac{1}{r(r)^4} & 0 & 0 & 0 \\ 0 & \frac{r(r)^4 r^4}{\left(\sigma + \frac{1}{6}\varepsilon r^6\right)^2} & 0 & 0 \\ 0 & 0 & \frac{r^6 r(r)^4}{\left(\sigma + \frac{1}{6}\varepsilon r^6\right)^2} & 0 \\ 0 & 0 & 0 & \frac{r^6 \sin(\theta)^2 r(r)^4}{\left(\sigma + \frac{1}{6}\varepsilon r^6\right)^2} \end{bmatrix}, index\_char = [-1, -1]\right]\right)$$

```
tensorsGR(coords,metric,contra_metric,'det_met', C1, C2,
Rm, Rc, Rs, G, C);
 Grr:=simplify(G[compts][2,2]/metric[compts][2,2]):
 Gthth:=simplify(G[compts][3,3]/metric[compts][3,3]):
 Gphph:=simplify(G[compts][4,4]/metric[compts][4,4]):
simplify(Gthth-Gphph);
# consistency check --- For satisfying the perfect fluid
constraint, this must be equal to zero!
```

$$0$$

```
> dp1:=numer(simplify(Grr-Gthth));
```

$$dp1 := 0$$

```
> #-------------------------------------------------------
  #CONCLUSION: Theorem 3 with an S1 seed gives
               the new solution.
  #-------------------------------------------------------
> # END OF PROOFS.
  #-------------------------------------------------------
  # END WORKSHEET
  #-------------------------------------------------------
>
```

- **K-O III**

The metric K-O III has the form

$$\mathrm{d}s^2 = -A\left(1+ar^2\right)^2 \mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2,$$

where A is a real arbitrary constant.

```
> #-------------------------------------
  # The new theorem of two linking theorems
    (applied to K-O-III metric)
  #-------------------------------------
restart;
with(tensor);
[Christoffel1, Christoffel2, Einstein, Jacobian,
Killing_eqns, Levi_Civita, Lie_diff, Ricci, Ricciscalar,
Riemann, RiemannF, Weyl, act, antisymmetrize,
change_basis, commutator, compare, conj, connexF,
contract, convertNP, cov_diff, create, d1metric,
d2metric, directional_diff, displayGR, display_allGR,
dual, entermetric, exterior_diff, exterior_prod, frame,
geodesic_eqns, get_char, get_compts, get_rank, init,
invars, invert, lin_com, lower, npcurve, npspin,
partial_diff, permute_indices, petrov, prod, raise,
symmetrize, tensorsGR, transform];
coords:=[t,r,theta,phi];
```

[*Christoffel1*, *Christoffel2*, *Einstein*, *Jacobian*, *Killing_eqns*, *Levi_Civita*, *Lie_diff*, *Ricci*, *Ricciscalar*, *Riemann*, *RiemannF*, *Weyl*, *act*, *antisymmetrize*, *change_basis*, *commutator*, *compare*, *conj*, *connexF*, *contract*, *convertNP*, *cov_diff*, *create*, *d1metric*, *d2metric*, *directional_diff*, *displayGR*, *display_allGR*, *dual*, *entermetric*, *exterior_diff*, *exterior_prod*, *frame*, *geodesic_eqns*, *get_char*, *get_compts*, *get_rank*, *init*, *invars*, *invert*, *lin_com*, *lower*, *npcurve*, *npspin*, *partial_diff*, *permute_indices*, *petrov*, *prod*, *raise*, *symmetrize*, *tensorsGR*, *transform*]

$$coords := [t, r, \theta, \phi]$$

```
> #---------------------------
  # SEED metric (initial metric)
  #---------------------------
 zeta0(r):=1/(1 + r^2/2);
```

$$\zeta 0(r) := \frac{1}{1+\frac{1}{2}r^2}$$

```
> B0(r):=1/(1 + r^2/2);
```

$$B0(r) := \frac{1}{1+\frac{1}{2}r^2}$$

```
> # K-O-III seed
  #-------------------------------------
  # Theorem 0: check the seed is a perfect fluid
  #-------------------------------------
 g:=array (symmetric, sparse, 1..4, 1..4):
 g[1,1]:=-1/zeta0(r)^2:
 g[2,2]:=zeta0(r)^2/B0(r)^2:
 g[3,3]:=r^2*zeta0(r)^2/B0(r)^2:
 g[4,4]:=(r^2)*sin(theta)^2*zeta0(r)^2/B0(r)^2:
 metric:=create([-1,-1], eval(g));
```

$$metric := table\left(\left[compts = \begin{bmatrix} -\left(1+\frac{1}{2}r^2\right)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin(\theta)^2 \end{bmatrix}, index\_char = [-1, -1]\right]\right)$$

```
> tensorsGR(coords,metric,contra_metric,'det_met', C1,
C2, Rm, Rc, Rs, G, C);
 Grr:=simplify(G[compts][2,2]/metric[compts][2,2]):
 Gthth:=simplify(G[compts][3,3]/metric[compts][3,3]):
 Gphph:=simplify(G[compts][4,4]/metric[compts][4,4]):
 simplify(Gthth-Gphph);
# consistency check --- To verify a perfect fluid sphere,
this should be zero!
```

$$0$$

```
> dp0:=numer(simplify(Grr-Gthth));
# checking a perfect fluid
```

$$dp0 := 0$$

```
> #-----------------------------------
  # Theorem 3: B0->B1=B0*Z0;
  #-----------------------------------
 Z0(r):=sigma + epsilon*int(r/(B0(r)^2),r);
```

$$Z0(r) := \sigma + \varepsilon\left(\frac{1}{24}r^6 + \frac{1}{4}r^4 + \frac{1}{2}r^2\right)$$

```
> B1(r):=B0(r)*Z0(r);
```

$$B1(r) := \frac{\sigma + \varepsilon\left(\frac{1}{24}r^6 + \frac{1}{4}r^4 + \frac{1}{2}r^2\right)}{1 + \frac{1}{2}r^2}$$

```
> zeta1(r):=1/zeta0(r);
```

$$\zeta 1 := 1 + \frac{1}{2}r^2$$

```
> g:=array (symmetric, sparse, 1..4, 1..4):
 g[1,1]:=-1/(zeta1(r)^2):
 g[2,2]:=zeta1(r)^2/(B1(r)^2):
 g[3,3]:=r^2*zeta1(r)^2/(B1(r)^2):
 g[4,4]:=r^2*(sin(theta))^2*zeta1(r)^2/(B1(r)^2):
 metric:=create([-1,-1], eval(g));
```

$$metric := table\left(\left[compts = \left[\left[-\frac{1}{\left(1+\frac{1}{2}r(r)^2\right)^2}, 0, 0, 0\right],\right.\right.\right.$$

$$\left[0, \frac{\left(1+\frac{1}{2}r(r)^2\right)^2\left(1+\frac{1}{2}r^2\right)^2}{\left(\sigma+\varepsilon\left(\frac{1}{24}r^6+\frac{1}{4}r^4+\frac{1}{2}r^2\right)\right)^2}, 0, 0\right],$$

$$\left[0, 0, \frac{r^2\left(1+\frac{1}{2}r(r)^2\right)^2\left(1+\frac{1}{2}r^2\right)^2}{\left(\sigma+\varepsilon\left(\frac{1}{24}r^6+\frac{1}{4}r^4+\frac{1}{2}r^2\right)\right)^2}, 0\right],$$

$$\left.\left.\left.\left[0, 0, 0, \frac{r^2\sin(\theta)^2\left(1+\frac{1}{2}r(r)^2\right)^2\left(1+\frac{1}{2}r^2\right)^2}{\left(\sigma+\varepsilon\left(\frac{1}{24}r^6+\frac{1}{4}r^4+\frac{1}{2}r^2\right)\right)^2}\right]\right], index\_char = [-1, -1]\right]\right)$$

```
> tensorsGR(coords,metric,contra_metric,'det_met', C1,
C2, Rm, Rc, Rs, G, C);
 Grr:=simplify(G[compts][2,2]/metric[compts][2,2]):
```

```
 Gthth:=simplify(G[compts][3,3]/metric[compts][3,3]):
 Gphph:=simplify(G[compts][4,4]/metric[compts][4,4]):
 simplify(Gthth-Gphph);
# consistency check --- For satisfying the perfect fluid
constraint, this must be equal to zero!
                              0
> dp1:=numer(simplify(Grr-Gthth));
                          dp1 := 0
> #-------------------------------------------------------
  #CONCLUSION: Theorem 3 with an K-O-III seed gives the
new solution.
  #-------------------------------------------------------
> # END OF PROOFS.
  #-------------------------------------------------------
  # END WORKSHEET
  #-------------------------------------------------------
>
```

VITA

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