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DECOMPOSITION OF COMPLETE MULTIPARTITE GRAPHS INTO DISJOINT UNIONS OF CYCLES

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อุทุมพร จงถาวรวุฒิ : การแยกกราฟหลายส่วนบริบูรณ์ออกเป็นยูเนียนของวัฏจักรที่ไม่มีส่วนร่วมกัน (DECOMPOSITION OF COMPLETE MULTIPARTITE GRAPHS INTO DISJOINT UNIONS OF CYCLES) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.จริยา อุ่ยยะเสถียร, อ. ที่ปรึกษาวิทยานิพธ์ร่วม : Prof. Dr. Saad I. El-Zanati, 76 หน้า.

ให้ G เป็นกราฟที่มี n จุด โดยที่ n เป็นจำนวนคี่ แต่ละจุดมีดีกรี 2 และให้ v เป็นจำนวนเต็มบวก คำถามที่น่าสนใจคือเมื่อไหร่จะสามารถแยกกราฟบริบูรณ์ K_v ออกเป็นกราฟ G ได้ ถ้า $v \equiv 1$ หรือ $n \pmod{2n}$ แล้ว v จะสอดคล้องกับเงื่อนไขจำเป็นของการแยกกราฟบริบูรณ์ K_v ออกเป็นกราฟ G ได้ ถ้ากราฟ G มีกราฟย่อยที่เป็นวัฏจักรที่มีจำนวนจุดเป็นคี่เพียงวงเดียวเท่านั้น แล้วเป็นที่ทราบว่าจะสามารถแยกกราฟบริบูรณ์ K_v ออกเป็นกราฟ G ได้ สำหรับทุก $v \equiv 1 \pmod{2n}$ ในวิทยานิพนธ์ฉบับนี้ เราเน้นการศึกษาการแยกกราฟหลายส่วนบริบูรณ์ออกเป็นกราฟ G สำหรับจำนวนเต็มบวก r และ s ให้ $K_{r\times s}$ แทนกราฟหลายส่วนบริบูรณ์ที่มี r ส่วนแต่ละส่วนมีจำนวนจุดเป็น s เราได้ขยายวิธีการสร้างระบบสามเหลี่ยมสไตน์เนอร์ของโบสเพื่อแสดงการมีอยู่ของการแยกกราฟหลายส่วนบริบูรณ์ $K_{(2k+1)\times n}$ ออกเป็นกราฟ G สำหรับทุกจำนวนเต็มบวก k และการมีอยู่ของการแยกกราฟหลายส่วนบริบูรณ์ $K_{k'\times 2n}$ ออกเป็นกราฟ G สำหรับทุกจำนวนเต็ม $k' \geq 3$ นอกจากนี้ถ้า G ประกอบด้วยกราฟวัฏจักรสองวง แล้วเราสามารถแยกกราฟบริบูรณ์ K_v ออกเป็นกราฟ G สำหรับทุก $v \equiv n \pmod{2n}$ เว้นแต่ $G = C_4 \cup C_5$ และ $v = 9$ ยิ่งไปกว่านั้น ถ้า G ประกอบจากวัฏจักรสามวงที่แต่ละวงมีจำนวนจุดเป็นคี่ แล้วเรายังพบว่าสามารถแยกกราฟหลายส่วนบริบูรณ์ $K_{(2k+1)\times n}$ สำหรับทุกจำนวนเต็มบวก k และ $K_{k'\times 2n}$ สำหรับทุกจำนวนเต็ม $k' \geq 3$ ออกเป็นกราฟ G ได้ และสามารถแยกกราฟบริบูรณ์ K_v ออกเป็นกราฟ G ได้สำหรับทุก $v \equiv 1 \pmod{2n}$ เว้นแต่ $v = 4n+1$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..............................

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..............

ปีการศึกษา 2556 ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม..............

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KEYWORDS : G-DESIGN / G-DECOMPOSITION / STEINER TRIPPLE SYSTEM / THE BOSE CONSTRUCTION / THE OBERWOLFACH PROBLEM

UTHOOMPORN JONGTHAWONWUTH : DECOMPOSITION OF COMPLETE MULTIPARTITE GRAPHS INTO DISJOINT UNIONS OF CYCLES. ADVISOR : ASST. PROF. CHARIYA UIYYASATHIAN, Ph.D., CO-ADVISOR : PROF. SAAD I. EL-ZANATI, Ph.D., 76 pp.

Let G be a 2-regular graph of odd order n and let v be a positive integer. It is of interest to know when there exists a G-decomposition of K_v. If $v \equiv 1$ or $n \pmod{2n}$, then v satisfies the necessary conditions for the existence of a G-decomposition of K_v. If G contains exactly one odd cycle, it is known that there exists a G-decomposition of K_v for all $v \equiv 1 \pmod{2n}$. In this dissertation, we focus on G-decompositions of complete multipartite graphs. For positive integers r and s, let $K_{r\times s}$ denote the complete multipartite graph with r parts of order s each. We use a novel extension of the Bose construction for Steiner triple systems to show that there exists a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k and a G-decomposition of $K_{k'\times 2n}$ for every integer $k' \geq 3$. Furthermore, if G has only two components, we find G-decompositions of K_v for all $v \equiv n \pmod{2n}$ unless $G = C_4 \cup C_5$ and $v = 9$. Additionally, if G consists of three odd cycles, we find G-decompositions of $K_{(2k+1)\times n}$ for every positive integer k, of $K_{k'\times 2n}$ for every integer $k' \geq 3$, and of K_v for all $v \equiv 1 \pmod{2n}$, except $v = 4n + 1$.

Department : Mathematics and Computer Science	Student's Signature : ....
Field of Study : Mathematics	Advisor's Signature : ....
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CONTENTS

LIST OF FIGURES

CHAPTER I
INTRODUCTION

1.1 Prologue

Let G and K be graphs with G a subgraph of K. A *G-decomposition of K*, or a *(K, G)-design*, is a partition of the edge set of K into subgraphs isomorphic to G. A (K_v, G)-design is also known as a *G-design of order v*.

One of the better studied problems in G-designs is the case when G is a cycle. Necessary and sufficient conditions for the existence of C_n-designs of order v were found about a decade ago by Alspach and Gavlas [8] and by Šajna [40]. Necessary and sufficient conditions for the existence of a G-design of order v when G is a 2-regular graph of order at most 10 are found in [5]. For a general 2-regular graph G of order n, the problem of finding necessary and sufficient conditions for the existence of a G-design of order v is far from settled. It is expected however that for such a G, there will exist a G-design of order v for all $v \equiv 1 \pmod{2n}$. This has been confirmed when G is bipartite (see [19] and [10]), when G is almost-bipartite [15], when G is rC_m where m is odd [22], and when G has two components (see [2], [11] and [14]). If in addition n is odd and $(G, v) \notin \{(C_4 \cup C_5, 9), (C_3 \cup C_3 \cup C_5, 11)\}$, then a G-design of order v for all $v \equiv n \pmod{2n}$ is likely to exist.

A well-known problem on decompositions of complete graphs into 2-regular graphs is the Oberwolfach Problem. Let G be a 2-regular graph of odd order n. The problem of determining whether there exists a G-decomposition of K_n is known as the *Oberwolfach Problem.* This problem was settled in 1989 by Alspach, Schellenberg, Stinson, and Wagner [9] in the case when all the components of G are isomorphic to the same cycle. More recently, Traetta [43] settled the case when G consists of two components. The general problem however is far from settled. For example, very little is known when G consists of three components (see [13] for some known results).

It is easy to see that K_{2kn+n} can be decomposed into $K_{(2k+1)\times n}$ and $2k+1$ copies of K_n. Let G of odd order n be a 2-regular graph. Notice that if there is a G-decomposition of K_n and a G-decomposition of $K_{(2k+1)\times n}$, then there is a G-decomposition of K_{2kn+n}. If $G = C_3$, a popular construction for G-decompositions of K_{6k+3} is known as the *the Bose construction* for Steiner triple systems.

This dissertation is organized as follows. The first chapter is the introduction including all definitions and notations of graphs used frequently in this dissertation, and also the definitions of graph decompositions and graph designs.

Chapter 2 is dedicated to a brief survey of the literature. It begins with Steiner triple systems. The Bose construction, a well-known construction for Steiner triple systems of order $3 \pmod 6$, is presented. We then discuss decompositions of complete graphs and of complete multipartite graphs into 2-regular graphs. We also give an overview of the Oberwolfach problem which is concerned with determining whether there exists a G-decomposition K_n, where G is a 2-regular graph of odd order n. Finally, α-labelings of bipartite graphs are discussed.

The next chapter contains our main results. We first show how the Bose construction for Steiner triple systems of order $6k+3$ can be naturally extended to obtain C_n-decompositions of K_{2nk+n} for all odd $n \geq 5$ and all positive integers k. We then show that if G of odd order n is a 2-regular almost-bipartite graph or is the vertex-disjoint union of three odd cycles, then there exists a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k. If G consists of only two components, we combine the G-decomposition the $K_{(2k+1)\times n}$ result with Traetta's result on the Oberwolfach problem to show that there exists a G-decomposition of K_v for all $v \equiv n \pmod{2n}$ unless $G = C_4 \cup C_5$ and $v = 9$. We also show that there exists a G-decomposition of $K_{k\times 2n}$ for all integers $k \geq 3$. Furthermore, when G is the vertex-disjoint union of three odd cycles, we find a G-decomposition of K_{2kn+1} for all positive integers $k \neq 2$. Our research has resulted in three research papers ([17], [31], and [30]). In particular, the results on the decompositions of complete multipartite graphs into the vertex-disjoint union of three odd cycles will appear in the *Australasian Journal of Combinatorics* [31].

Finally, the last chapter contains the summary of our results and several related open problems are presented.

1.2 Definitions and notation

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set of objects called *vertices* and $E(G)$ is a set of 2-element subsets of $V(G)$, called *edges*. We will refer to $V(G)$ as the *vertex set* of G and to $E(G)$ as the *edge set* of G. The *order* and the *size* of G are $|V(G)|$ and $|E(G)|$, respectively.

If $e = \{u, v\}$ is an edge of a graph G, we say that u and v are the *endvertices* of e and that u and v are *adjacent*. In this case, we also say that u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent* edges. It is often convenient to denote an edge by uv or vu rather than by $\{u, v\}$. The *degree* of a vertex v in a graph G is the number of edges in G that are incident with v, which is denoted by $\deg_G v$ or simply by $\deg v$ if G is clear from the context. A vertex of degree 0 is called an *isolated* vertex in G. We write $G - e$ $(G - u)$ for the subgraph of G obtained by deleting an edge e (a vertex u).

It is customary to define or describe a graph G by means of a diagram in which each vertex of G is represented by a point (often drawn as a small circle or some similar object) and each edge $e = \{u, v\}$ of G is represented by a line segment or curve that joins the points corresponding to u and v. We then refer to this diagram as the graph G itself. There are occasions when we are only interested in the structure of a graph defined by a diagram and the vertex set of the graph is irrelevant. In this case, we refer to the graph as an *unlabeled* graph. The two graphs in Figure 3.4 are examples of such unlabeled graphs.

The *union* of graphs $G_1, \ldots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup\limits_{i=1}^{k} V(G_i)$ and edge set $\bigcup\limits_{i=1}^{k} E(G_i)$. The graph obtained by taking the union of graphs G and H with disjoint vertex sets is the *disjoint union*. The vertex-disjoint union of r copies of a graph G will be denoted by rG.

A graph G is a *subgraph* of a graph H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$; in such a case, we also say that H contains G as a subgraph. Whenever a subgraph G of a graph H has the same order as H, then G is called a *spanning subgraph* of H. The *complement* $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ defined by $\{u, v\} \in E(\bar{G})$ if and only if $\{u, v\} \notin E(G)$.

A graph G is *regular of degree* r if $\deg v = r$ for each vertex v of G. Such graphs are called r-*regular*. A graph is *complete* if every two of its vertices are adjacent. A complete graph of order n is therefore $(n-1)$-regular and has size $\binom{n}{2}$. We denote this graph by K_n. The first graph in Figure 1.3 is K_8, the complete graph of order 8.

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$. We say G is *isomorphic to* H, written $G \cong H$, if there is an isomorphism from G to H.

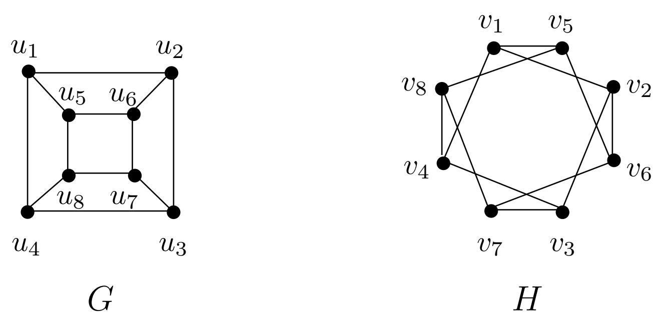


Figure 1.1: Isomorphic graphs

A *path* is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A path is *empty* if it contains only one vertex and thus no edges. Note that a nonempty path starts with a vertex of degree 1 and ends with a vertex of degree 1. These two vertices are called the *endpoints* of the path. All other vertices between the first and the last vertex of a path have degree 2. If the first vertex in a path G is u and the last vertex is v, then G is called a u-v *path* or a *path from* u *to* v. A path with n vertices is often denoted by P_n.

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The number of vertices in a cycle is called its *length*. The cycle with n vertices is denoted by C_n or *n-cycle*. We sometimes denote the cycle with vertex set $\{x_1, x_2, \ldots, x_n\}$ and edge set $\{\{x_i, x_{i+1}\} : 1 \leq i \leq n-1\} \cup \{x_n, x_1\}$ by $\langle x_1, x_2, \ldots, x_n \rangle$. We note that $(x_1, x_2, \ldots, x_n) + (x_n, x_1) = \langle x_1, x_2, \ldots, x_n \rangle$. A cycle is *even* if its length is even; otherwise, it is *odd*. Figure 1.2 shows the path P_5 and the cycle C_6.

A vertex u is said to be *connected* to a vertex v in a graph G if there exists a u-v path in G. A graph G is *connected* if every pair of its vertices is connected. A graph that is not connected is *disconnected*. The relation "is connected to" is an equivalence relation on $V(G)$. The subgraphs of G induced by the resulting equivalence classes are called the *components* of G.

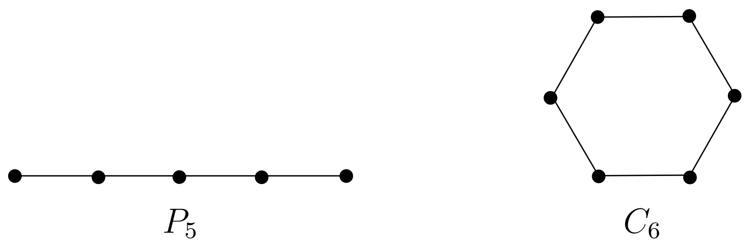


Figure 1.2: A path and a cycle

A spanning subgraph of a graph G is a referred to as a *factor* of G. A k-regular factor is called a *k-factor*. A spanning cycle in a graph G is also called a *Hamiltonian cycle* in G.

A graph G is *k-partite*, $k \geq 1$, if $V(G)$ can be partitioned into into k subsets $V_1, V_2, \ldots, V_k$ (called *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. Note that every graph is k-partite for some k; indeed, if G has order n, then G is n-partite. If G is a 1-partite graph of order n, then $G = \bar{K}_n$. For $k = 2$, such graphs are called *bipartite* graphs, and for $k = 3$ they are are called *tripartite* graphs. A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders G bipartite. For example, cycles of odd length are almost-bipartite.

A *complete k-partite graph* G is a k-partite graph with partite sets $V_1, V_2, \ldots, V_k$ having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $\{u, v\} \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $K(n_1, n_2, \ldots, n_k)$ or $K_{n_1,n_2,\ldots,n_k}$. (The order in which the numbers $n_1, n_2, \ldots, n_k$ are written is not important.) Note that a complete k-partite graph is complete if and only if $n_i = 1$ for all i, in which case it is K_k. A *complete bipartite graph* with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$, is then denoted by $K(r, s)$ or more commonly $K_{r,s}$. We will denote the complete multipartite graph with $r \geq 3$ partite sets of order s each by $K_{r\times s}$. The complete bipartite graph $K_{3,4}$ and the complete tripartite graph $K_{3\times4}$ are shown in Figure 1.3.

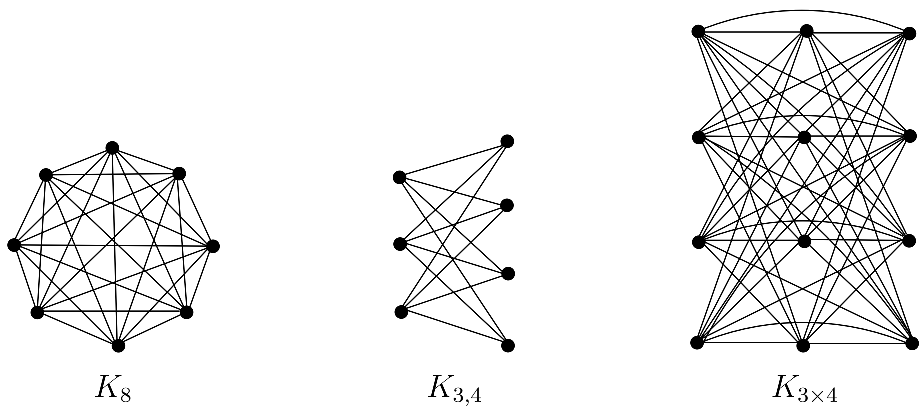


Figure 1.3: A complete graph, a complete bipartite graph and a complete multipartite graph

1.3 Graph decompositions and graph designs

A *decomposition* of a graph K is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that the edge sets of the graphs G_i form a partition of the edge set of K. If G_i is a Hamiltonian cycle, then the decomposition is called *the Hamiltonian decomposition.* If each G_i is isomorphic to a subgraph G of K, such decomposition is called a *G-decomposition of K* or a (K, G)-*design*. A (K_v, G)-design is also known as a *G-design of order v*. The study of graph decompositions is known as the study of graph designs or simply as the study of G-designs. For recent surveys on G-designs, we direct the reader to [3] and [12].

A popular tool for finding (K, G)-designs is through the use of graph labelings. A *labeling* of a graph G is an assignment of integers to the vertices of G subject to certain conditions. Graph labelings were first introduced by Rosa in the late 1960s. Rosa [37] showed that certain basic labelings of a graph G with n edges yielded G-decompositions of K_{2n+1}. Additionally, other stricter labeling yielded G-decomposition of K_{2nk+1} for all positive integers k. A survey of various of Rosa-type labelings that summarize some of the related results can be found in [18]. For a comprehensive look at general graph labelings, we direct the reader to a dynamic survey on the topic by Gallian [21]. We will focus on one of the labelings defined by Rosa [37] for bipartite graphs in Section 2.6.

CHAPTER II

REVIEW OF THE LITERATURE

In this chapter, we give a brief survey of the literature for results related to decompositions of complete graphs and complete multipartite graphs into 2-regular graphs. We begin by looking at Steiner triple systems and one of the popular constructions for them and some of its generalizations. Next, we discuss decompositions of complete graphs and of complete multipartite graphs into 2-regular graphs. We also discuss the Oberwolfach problem and some of the recent progress made on it. Finally, we discuss α-labelings which we will use in obtaining our results.

2.1 Steiner triple systems

A *Steiner triple system of order* v is an ordered pair $(S, \mathcal{T})$, where S is a finite set of v *points* or *symbols*, and $\mathcal{T}$ is a set of 3-element subsets of S called *triples*, such that each pair of distinct elements of S occurs together in exactly one triple of $\mathcal{T}$.

Example 2.1. If $S = \{0, 1, 2, 3, 4, 5, 6\}$ and $\mathcal{T} = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$, then $(S, \mathcal{T})$ is a Steiner triple system of order 7.

Note that a Steiner triple system of order v is equivalent to a C_3-decomposition of K_v.

Steiner triple systems were evidently defined for the first time in 1844 by W.S.B. Woolhouse [44]. In 1847, T.P. Kirkman [32] proved that a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$. In 1939, R.C. Bose published a construction for a Steiner triple of order $v \equiv 3 \pmod 6$ that is much simpler than the one given by Kirkman. In this construction, he made use of idempotent commutative quasigroups. We will refer to this construction as the *Bose construction*. Our work can be viewed as an extension of the Bose construction.

2.2 The Bose construction

Let $\mathbb{N}$ denote the set of nonnegative integers. Let $n \in \mathbb{N}$ and $\mathbb{Z}_n$ the group of integers modulo n. If a and b are integers, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \varnothing$).

A *quasigroup* of *order* q is a pair $(Q, \circ)$ where Q is a set of size q, say $Q = [1, q]$, and $\circ$ is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. The quasigroup is *idempotent* if $i \circ i = i$ for every $i \in Q$ and it is *commutative* if $i \circ j = j \circ i$ for all $i, j \in Q$. Note that in such a quasigroup, if $a \neq b$, then a, b, and $a \circ b$ are distinct. It has long been known that an idempotent commutative quasigroup of order q exists if and only if q is odd (see [34]). The Bose construction is described as follow:

Let $v = 6k + 3$ for some positive integer k, and let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k + 1$, where $Q = [1, 2k + 1]$. Let $S = \mathbb{Z}_3 \times Q$, and define $\mathcal{T}$ to contain the following two types of triples:

Type 1: For $1 \leq i \leq 2k+1$, $\{(0, i), (1, i), (2, i)\} \in \mathcal{T}$.

Type 2: For $1 \leq i < j \leq 2k+1$, $\{(0, i), (0, j), (1, i \circ j)\}, \{(1, i), (1, j), (2, i \circ j)\}, \{(2, i), (2, j), (0, i \circ j)\} \in \mathcal{T}$.

Then $(S, \mathcal{T})$ is a Steiner triple system of order $6k + 3$.

Example 2.2. We will use the Bose construction to produce a Steiner triple system $(S, \mathcal{T})$ of order 15. Let $(Q, \circ)$ be the idempotent commutative quasigroup of order 5 shown in Figure 2.1. Let $S = \mathbb{Z}_3 \times [1, 5]$ and let $\mathcal{T}$ contain the following

35 triples:

Type 1: $\{\{(0,1),(1,1),(2,1)\},\{(0,2),(1,2),(2,2)\},\{(0,3),(1,3),(2,3)\},$
$\{(0,4),(1,4),(2,4)\},\{(0,5),(1,5),(2,5)\}\}$

Type 2:

$i=1, j=2$

$\{(0,1),(0,2),(1,1\circ 2=5)\}$

$\{(1,1),(1,2),(2,1\circ 2=5)\}$

$\{(2,1),(2,2),(0,1\circ 2=5)\}$

$i=1, j=3$

$\{(0,1),(0,3),(1,1\circ 3=2)\}$

$\{(1,1),(1,3),(2,1\circ 3=2)\}$

$\{(2,1),(2,3),(0,1\circ 3=2)\}$

$i=1, j=4$

$\{(0,1),(0,4),(1,1\circ 4=3)\}$

$\{(1,1),(1,4),(2,1\circ 4=3)\}$

$\{(2,1),(2,4),(0,1\circ 4=3)\}$

$i=1, j=5$

$\{(0,1),(0,5),(1,1\circ 5=4)\}$

$\{(1,1),(1,5),(2,1\circ 5=4)\}$

$\{(2,1),(2,5),(0,1\circ 5=4)\}$

$i=2, j=3$

$\{(0,2),(0,3),(1,2\circ 3=4)\}$

$\{(1,2),(1,3),(2,2\circ 3=4)\}$

$\{(2,2),(2,3),(0,2\circ 3=4)\}$

$i=2, j=4$

$\{(0,2),(0,4),(1,2\circ 4=1)\}$

$\{(1,2),(1,4),(2,2\circ 4=1)\}$

$\{(2,2),(2,4),(0,2\circ 4=1)\}$

$i=2, j=5$

$\{(0,2),(0,5),(1,2\circ 5=3)\}$

$\{(1,2),(1,5),(2,2\circ 5=3)\}$

$\{(2,2),(2,5),(0,2\circ 5=3)\}$

$i=3, j=4$

$\{(0,3),(0,4),(1,3\circ 4=5)\}$

$\{(1,3),(1,4),(2,3\circ 4=5)\}$

$\{(2,3),(2,4),(0,3\circ 4=5)\}$

$i=3, j=5$

$\{(0,3),(0,5),(1,3\circ 5=1)\}$

$\{(1,3),(1,5),(2,3\circ 5=1)\}$

$\{(2,3),(2,5),(0,3\circ 5=1)\}$

$i=4, j=5$

$\{(0,4),(0,5),(1,4\circ 5=2)\}$

$\{(1,4),(1,5),(2,4\circ 5=2)\}$

$\{(2,4),(2,5),(0,4\circ 5=2)\}$

$\circ$	1	2	3	4	5
1	1	5	2	3	4
2	5	2	4	1	3
3	2	4	3	5	1
4	3	1	5	4	2
5	4	3	1	2	5

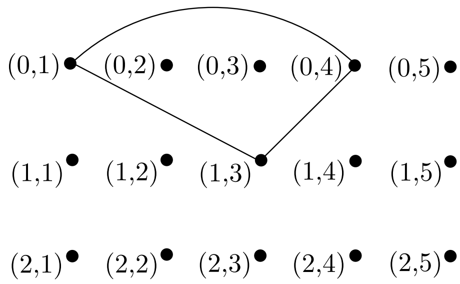


Figure 2.1: An idempotent commutative quasigroup of order 5 and one triple from the Bose construction of a Steiner triple system of order 15.

In terms of graphs, we note that the triples of Type 1 in $\mathcal{T}$ form a C_3-decomposition of $(2k+1)K_3$ and the triples of Type 2 form a C_3-decomposition of $K_{(2k+1)\times 3}$. Since all edges of K_{6k+3} can be separated into edges of $(2k+1)K_3$ and edges of $K_{(2k+1)\times 3}$, we have the desired result.

2.3 The quasigroup with hole construction

A variation on the Bose Construction makes use of quasigroups of even order with holes of size two. For an integer $k \geq 3$, let $Q = [1, 2k]$ and for $i \in [1, k]$, let $h_i = \{2i-1, 2i\}$. Let $H = \{h_1, h_2, \ldots, h_k\}$. In what follows, all elements $h_i \in H$ are called *holes*. A *quasigroup with holes* H is a quasigroup $(Q, \circ)$ of order $2k$ in which for each $h_i \in H$, we have $(h_i, \circ)$ is a subquasigroup of $(Q, \circ)$. It is known that for every $k \geq 3$, there exists a commutative quasigroup $(Q, \circ)$ of order $2k$ with holes H (see [34]). Commutative quasigroups of order $2k$ with holes H are used to construct C_3-decompositions of $K_{k\times 6}$ for every integer $k \geq 3$. This C_3-decompositions of $K_{k\times 6}$ is then combined a C_3-decomposition of K_7 to obtain a Steiner triple system of order $6k+1$.

Let $k \geq 3$ be an integer and for $i \in [1, k]$, let $h_i = \{2i-1, 2i\}$ and $g_i = \mathbb{Z}_3 \times h_i$. Let $Q = [1, 2k]$ and $H = \{h_1, h_2, \ldots, h_k\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $2k$ with holes H. Let $S = \{\infty\} \cup (\mathbb{Z}_3 \times [1, 2k])$. For $1 \leq i \leq k$, let $\mathcal{T}_i$ consist of the triples in a Steiner triple system of order 7 on the symbols $\{\infty\} \cup g_i$.

Consider the following:

(1) let $\mathcal{T}' = \bigcup_{i=1}^{k} \mathcal{T}_i$, and,

(2) for $1 \le i < j \le 2k, \{i,j\} \notin H$, let $\mathcal{T}''$ contain the triples $\{(0,i),(0,j),(1,i \circ j)\}$, $\{(1,i),(1,j),(2,i \circ j)\}, \{(2,i),(2,j),(0,i \circ j)\}$.

Then $(S, \mathcal{T}' \cup \mathcal{T}'')$ is a Steiner triple system of order $6k+1$.

Example 2.3. We will use the quasigroups with hole construction to produce a Steiner triple system of order 19. For $i \in [1,3]$, let $h_i = \{2i-1, 2i\}$ and $g_i = \mathbb{Z}_3 \times h_i$. Let $Q = [1,6]$ and $H = \{h_1, h_2, h_3\}$. Let $(Q, \circ)$ be the commutative quasigroup of order 6 with holes H shown in Figure 2.2. Let $S = \{\infty\} \cup (\mathbb{Z}_3 \times [1,6])$. For $i \in [1,3]$, let $\mathcal{T}_i$ consist of the triples from a Steiner triple system of order 7 on the symbols $\{\infty\} \cup g_i$ and let $\mathcal{T}' = \bigcup_{i=1}^{3} \mathcal{T}_i$. Then each $\mathcal{T}_i$ contains the following triples:

$\{(0,2i-1),(1,2i-1),(0,2i)\}$	$\{\infty,(0,2i-1),(2,2i-1)\}$
$\{(1,2i-1),(2,2i-1),(1,2i)\}$	$\{\infty,(0,2i),(2,2i-1)\}$
$\{(2,2i-1),(0,2i),(2,2i)\}$	$\{\infty,(2,2i),(1,2i-1)\}$
$\{(1,2i),(2,2i),(0,2i-1)\}$	

For $1 \le i < j \le 6$, with $\{i,j\} \notin H$, let $\mathcal{T}''$ contain the following triples:

$i=1, j=3$	$i=1, j=4$
$\{(0,1),(0,3),(1,1 \circ 3 = 5)\}$	$\{(0,1),(0,4),(1,1 \circ 4 = 6)\}$
$\{(1,1),(1,3),(2,1 \circ 3 = 5)\}$	$\{(1,1),(1,4),(2,1 \circ 4 = 6)\}$
$\{(2,1),(2,3),(0,1 \circ 3 = 5)\}$	$\{(2,1),(2,4),(0,1 \circ 4 = 6)\}$

$i=1, j=5$	$i=1, j=6$
$\{(0,1),(0,5),(1,1 \circ 5 = 3)\}$	$\{(0,1),(0,6),(1,1 \circ 6 = 4)\}$
$\{(1,1),(1,5),(2,1 \circ 5 = 3)\}$	$\{(1,1),(1,6),(2,1 \circ 6 = 4)\}$
$\{(2,1),(2,5),(0,1 \circ 5 = 3)\}$	$\{(2,1),(2,6),(0,1 \circ 6 = 4)\}$

$i=2, j=4$	$i=2, j=5$

$$\{(0,2),(0,4),(1,2\circ 4=5)\}$$
$$\{(1,2),(1,4),(2,2\circ 4=5)\}$$
$$\{(2,2),(2,4),(0,2\circ 4=5)\}$$

$$\{(0,2),(0,5),(1,2\circ 5=4)\}$$
$$\{(1,2),(1,5),(2,2\circ 5=4)\}$$
$$\{(2,2),(2,5),(0,2\circ 5=4)\}$$

$$i=2, j=6$$

$$\{(0,2),(0,6),(1,2\circ 6=3)\}$$
$$\{(1,2),(1,6),(2,2\circ 6=3)\}$$
$$\{(2,2),(2,6),(0,2\circ 6=3)\}$$

$$i=3, j=5$$

$$\{(0,3),(0,5),(1,3\circ 5=1)\}$$
$$\{(1,3),(1,5),(2,3\circ 5=1)\}$$
$$\{(2,3),(2,5),(0,3\circ 5=1)\}$$

$$i=3, j=6$$

$$\{(0,3),(0,6),(1,3\circ 6=2)\}$$
$$\{(1,3),(1,6),(2,3\circ 6=2)\}$$
$$\{(2,3),(2,6),(0,3\circ 6=2)\}$$

$$i=4, j=6$$

$$\{(0,4),(0,6),(1,4\circ 6=1)\}$$
$$\{(1,4),(1,6),(2,4\circ 6=1)\}$$
$$\{(2,4),(2,6),(0,4\circ 6=1)\}.$$

Then $(S, \mathcal{T}' \cup \mathcal{T}'')$ is a Steiner triple system of order 19.

$\circ$	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

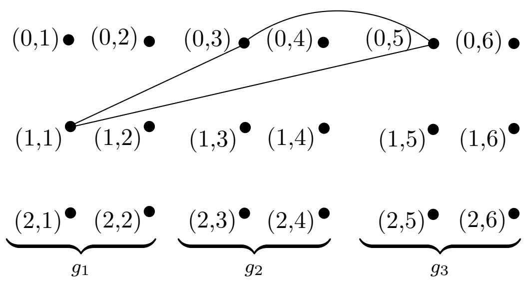


Figure 2.2: A commutative quasigroup of order 6 with holes and one triple from the corresponding C_3-decomposition of $K_{3\times 6}$.

2.4 Decompositions of complete graphs and complete multipartite graphs into 2-regular graphs

The problem of investigating decompositions of complete graphs into 2-regular graphs is one of the more popular problems in the study of G-designs. Perhaps the oldest such problem is the study of C_3-decompositions of K_v. It dates back to 1844 (see [44]) and later became known as the study of Steiner triple systems (see Chapter 2.1). In 1847, T.P. Kirkman [32] proved that there exists a C_3-design of order v if and only if $v \equiv 1$ or 3 (mod 6). It was not until the early 1960's that researchers began investigating other C_n-decompositions of complete graphs. Anton Kotzig and Alex Rosa are credited with publishing some of the earliest such investigations (see for example [33], [38], and [39]). Over the next three decades, several others made significant contributions to the general problem (see for example [29] and [27]). The problem of finding necessary and sufficient conditions for the existence of a C_n-design of order v was settled completely a little over a decade ago by Alspach and Gavlas [8] and by Šajna [40]. Necessary and sufficient conditions for the existence of a G-design of order v are found in [5] when G is a 2-regular graph of order at most 10. For a general 2-regular graph G of order n, the problem of finding necessary and sufficient conditions for the existence of a G-decomposition of K_v is far from settled. It is expected however that for such a G-decomposition will exist for all $v \equiv 1 \pmod{2n}$. This has been confirmed when G is bipartite (see [19] and [10]), when G is almost-bipartite [15], when G is rC_m where m is odd [22], and when G has two components (see [2], [11] and [14]). If in addition n is odd and $(G, v) \notin \{(C_4 \cup C_5, 9), (C_3 \cup C_3 \cup C_5, 11)\}$, then a G-design of order v for all $v \equiv n \pmod{2n}$ is likely to exist. The case $v = n$ is known as the Oberwolfach problem (see Section 2.5).

In recent years, numerous authors have investigated C_n-decompositions of complete multipartite graphs. Particular focus has been placed on C_3-decompositions of complete multipartite graphs. Such decompositions fall under the umbrella of the study of group divisible designs (see [23] for a summary). The problem of C_{2k}-decompositions of the complete bipartite graph $K_{m,n}$ was settled completely by Sotteau in [41]. In [36], Piotrowski settled the problem of G-decompositions of $K_{n,n}$ when G is a 2-regular bipartite graph of order $2n$. In [35], Liu settled the problem of kC_m-decompositions of $K_{r\times s}$ in the case when $km = rs$. We are not aware of any work that has been done on G-decompositions of complete multipartite graphs when G is a 2-regular graph with non-uniform components and the complete graph is not bipartite.

2.5 The Oberwolfach problem

Let t be a positive integer. For $i \in [1, t]$, let $r_i \geq 1$ and $m_i \geq 3$ be integers. Let $n = r_1m_1 + r_2m_2 + \cdots + r_tm_t$. Let G be the 2-regular graph of order n consisting of the vertex-disjoint union $r_1C_{m_1} \cup r_2C_{m_2} \cup \cdots \cup r_tC_{m_t}$. The *Oberwolfach problem* $OP(m_1^{r_1}, m_2^{r_2}, \ldots, m_t^{r_t})$ is a problem of determining whether there exists a G-decomposition of K_n if n is odd or of $K_n - I$, where I is a 1-factor, if n is even. The Oberwolfach problem was posed by G. Ringel in 1967 at a meeting in Oberwolfach, Germany. It was first mentioned in the literature in [24].

Example 2.4. A solution to $OP(3, 4)$ looks as follows, where the vertices of K_7 are labeled $0, 1, \ldots, 6$.

1^{st} 2-factor	2^{nd} 2-factor	3^{rd} 2-factor
$\langle 0, 1, 4\rangle$	$\langle 0, 2, 5\rangle$	$\langle 0, 3, 6\rangle$
$\langle 2, 3, 5, 6\rangle$	$\langle 3, 4, 6, 1\rangle$	$\langle 4, 5, 1, 2\rangle$

It is known that $OP(3, 3)$, $OP(3, 3, 3, 3)$, $OP(4, 5)$ and $OP(3, 3, 5)$ have no solutions (see [13]). The followings are some of the known results on the Oberwolfach problem.

Theorem 2.5. *The following Oberwolfach problems all have solutions.*

(i) $OP(m^t)$ *for all* $t \geq 1$ *and* $m \geq 3$ *(see [9]);*

(ii) $OP(m_1^{r_1}, m_2^{r_2}, \ldots, m_t^{r_t})$ *for* $r_1m_1 + r_2m_2 + \cdots + r_tm_t \leq 17$*;*

(iii) $OP(3^k, 4)$ *for all odd* $k \geq 1$ *(see [16]);*

(iv) $OP(3^k, 5)$ *for all even* $k \geq 4$ *(see [42]);*

(v) $OP(r^k, n - kr)$ *for* $n \geq 6kr - 1, k \geq 1, r \geq 3$*;*

(vi) $OP(r, n - r)$ *for* $3 \leq r \leq 9$ *and* $n \geq r + 3$ *(see [26]);*

(vii) $OP(r, r, n-2r)$ *for* $r = 3, 4$ *and* $n \geq 2r+3$ *(see [26]);*

(viii) $OP(2r_1, 2r_2, \ldots, 2r_k)$ *for all* $r_i \geq 2$ *and* $r_1 + r_2 + \cdots + r_k$ *odd (see [25]);*

(ix) $OP(r, r+1)$ *and* $OP(r, r+2)$ *for* $r \geq 3$*;*

(x) $OP(2r+1, 2r+1, 2r+2)$ *for* $r \geq 1$*;*

(xi) $OP(3, 4r, 4r)$ *for* $r \geq 1$*;*

(xii) $OP(4^k, 2r+1)$ *for* $k \geq 0$ *and* $r \geq 1$*;*

(xiii) $OP((4s)^k, 2r+1)$ *for* $k \geq 0$ *and* $r \geq 1$*;*

Although the general problem is far from settled, Traetta [43] recently settled the case when G has two components.

Theorem 2.6. *Let* $a \geq 2$ *and* $b \geq 1$ *be integers and let* $n = 2a + 2b + 1$*. There exists a* $(C_{2a} \cup C_{2b+1})$*-decomposition of* K_n *if and only if* $(a, b) \neq (2, 2)$*.*

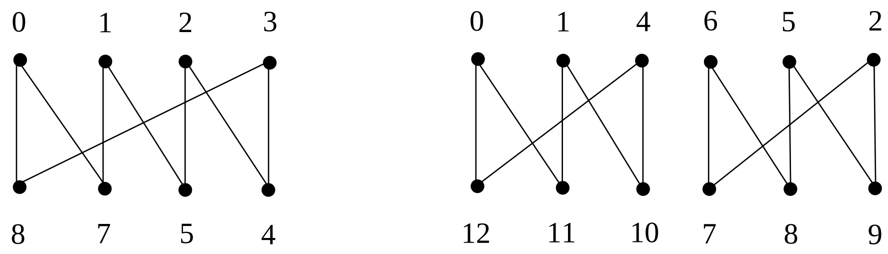
2.6 α-Labelings

In 1967, Rosa [37] introduced a hierarchy of labelings of simple graphs. We use one such labeling in our approach. Let G be a bipartite graph with n edges and vertex bipartition $\{A, B\}$. An *α-labeling* of G is an injection $f : V(G) \to \mathbb{N}$ such that

- $f(a) < f(b) \leq n$ for all $a \in A$ and $b \in B$,
- $\{|f(u) - f(v)| \colon \{u, v\} \in E(G)\} = [1, n]$.

For every such α-labeling, there necessarily exists an integer λ, called the *critical value* of the α-labeling f, such that $\max(A) = \lambda$ and $\min(B) = \lambda + 1$.

Rosa [37] showed that if G has an α-labeling, then there exists a G-decomposition of K_{2nk+1} for all positive integers k. Moreover, α-labelings can be used to obtain decompositions of complete bipartite graphs. For example, if a bipartite graph G of size n admits an α-labeling, then there exists a G-decomposition of $K_{n,n}$ (see



C_8 with $\lambda = 3$ $\qquad$ $C_6 \cup C_6$ with $\lambda = 6$

Figure 2.3: A α-labeling of G where $G = C_8$ or $C_6 \cup C_6$

[28]). In [37], Rosa showed that if a 2-regular bipartite graph G of size n admits an α-labeling, then we must have $n \equiv 0 \pmod 4$.

In [37], Rosa determined when an even cycle admits an α-labeling.

Theorem 2.7. *C_n has an α-labeling if and only if $n \equiv 0 \pmod 4$.*

In [2], Abrham and Kotzig settled the corresponding result for the union of two even cycles.

Theorem 2.8. *$C_{2n} \cup C_{2m}$ has an α-labeling if and only if $2n + 2m \equiv 0 \pmod 4$.*

Because we are concerned with 2-regular graphs, we note the following results on α-labelings.

Theorem 2.9. *The following 2-regular bipartite graphs admit α-labelings.*

(i) *rC_4 if and only if $r \neq 3$ (see [1]).*

(ii) *$C_{2m_1} \cup C_{2m_2} \cup C_{2m_3}$ if and only if $2m_1 + 2m_2 + 2m_3 \equiv 0 \pmod 4$ (see [20]).*

CHAPTER III

MAIN RESULTS

In this chapter, we use novel extensions of the Bose construction for Steiner triple systems to show that there exist a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k and a G-decomposition of $K_{k'\times 2n}$ for every integer $k' \geq 3$ where G is a 2-regular almost-bipartite graph of odd order n. We obtain similar results when G consists of three odd length cycles. In Section 3.1, we focus on the case when G as a single cycle. We also show that there exists a C_n-decomposition of K_v for all $v \equiv n \pmod{2n}$. In Subsection 3.3.1, we concentrate when G has only two components. Additionally, we find a G-decomposition of K_v for all $v \equiv n \pmod{2n}$. In Subsection 3.3.2, we consider the case when G consists of any number of even cycles and one single odd cycle. Finally, in Subsection 3.3.3, we consider the case when G consists of three odd cycles. In the last case, we also obtain a G-decomposition of K_v for all $v \equiv 1 \pmod{2n}$, except when $v = 4n+1$.

3.1 On extensions of the Bose construction

We begin with some sufficient conditions for the existence of a G-decomposition of $K_{(2k+1)\times n}$ and of $K_{k'\times 2n}$ for all integers $k \geq 1$ and $k' \geq 3$. These ideas make use of extensions of the Bose construction for Steiner triple systems.

Let $n \geq 3$ be an odd integer and let k be a positive integer. Let $K_{(2k+1)\times n}$ have vertex set $\mathbb{Z}_n \times [1, 2k+1]$ with the obvious vertex partition. For $i \in [1, k]$, let $h_i = \{2i-1, 2i\}$ and $g_i = \mathbb{Z}_n \times h_i$. Let $H = \{h_1, h_2, \ldots, h_k\}$. Let $V(K_{k\times 2n}) = \mathbb{Z}_n \times [1, 2k]$ with the vertex-set partition $\{g_1, g_2, \ldots, g_k\}$. For $r < s$, if $e = \{(i, r), (j, s)\}$ is an edge in $K_{(2k+1)\times n}$ or in $K_{k\times 2n}$, define the *length* of e to be $j - i$ if $j \geq i$; otherwise the length of e is $n + (j - i)$. Thus, between any two parts, there are edges of lengths $0, 1, \ldots, n-1$. We will often write $-j$ for edge length $n-j$ when

n is understood. Therefore, between any two parts, there are edges of lengths $0, \pm 1, \pm 2, \ldots, \pm \frac{(n-1)}{2}$.

Let K be a subgraph of the graph with vertex set $\mathbb{Z}_n \times [1, 2k+1]$. For a positive integer ℓ, the graph $K + \ell$ has vertex set $\{(i+\ell, z) : (i, z) \in V(K)\}$ and edge set $\{\{(i+\ell, r), (j+\ell, s)\} : \{(i, r), (j, s)\} \in E(K)\}$.

Lemma 3.1. *Let G of odd order n be a 2-regular almost-bipartite graph and let $e \in E(G)$ be such that $G - e$ is bipartite. Let $G' = G - u$ where $u \in V(G)$ is incident to e. Let P of size $\ell \leq n-2$ be the component of G' that is a path. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. Assume that there exists an embedding of G' in $K_{n,n}$ with one edge of each length in $[-(n-1)/2, (n-1)/2] \smallsetminus \{\pm z\}$ for some $z \in [1, (n-1)/2]$ and such that the endpoints of P are $(j, 1)$ and $(j, 2)$ for some $j \in [0, n-1]$. Then there exists a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k.*

Proof. Let k be a positive integer and let $V(K_{(2k+1)\times n}) = \mathbb{Z}_n \times [1, 2k+1]$ with the obvious vertex partition. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k+1$, where $Q = [1, 2k+1]$.

Fix r and s with $1 \leq r < s \leq 2k+1$. Let $G'_{r,s}$ and $P_{r,s}$ be the embeddings (as in the hypothesis of the lemma) of G' and P, respectively, in the subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$ and the obvious vertex partition. Let (j, r) and (j, s) denote the endpoints of $P_{r,s}$ and let z be as in the hypothesis. We construct from $G'_{r,s}$ a graph $G_{r,s}$, isomorphic to G, by adding the edges $\{(j, r), (j+z, r \circ s)\}$ and $\{(j, s), (j+z, r \circ s)\}$ at the endpoints of $P_{r,s}$. Let $G^*_{r,s} = \{G_{r,s} + x : 0 \leq x \leq n-1\}$. Note that $G^*_{r,s}$ contains n distinct copies of G. Moreover, in the subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$, $G^*_{r,s}$ contains all the edges of length i for all $i \in [-(n-1)/2, (n-1)/2] \smallsetminus \{\pm z\}$.

Let $\mathcal{C} = \{G_{r,s} + x : 1 \leq r < s \leq 2k+1, 0 \leq x \leq n-1\}$ and note that $\mathcal{C}$ contains $\binom{2k+1}{2} n$ distinct copies of G. We will show that every edge of $K_{(2k+1)\times n}$ appears on some copy of G in $\mathcal{C}$. Let $e = \{(i, r), (j, s)\}$ with $r < s$ be an arbitrary edge of $K_{(2k+1)\times n}$. Let t' be the unique solution to $r \circ t' = s$ and let $\alpha' = \min\{r, t'\}$ and $\beta' = \max\{r, t'\}$. Let t'' be the unique solution to $s \circ t'' = r$ and let $\alpha'' = \min\{s, t''\}$

and $\beta'' = \max\{s, t''\}$. If $j - i \in [-(n-1)/2, (n-1)/2] \smallsetminus \{\pm z\}$, then e belongs to $G_{r,s} + x$ for some x with $0 \leq x \leq n-1$. If $j - i = z$, then e belongs to $G_{\alpha',\beta'} + x$ where $0 \leq x \leq n-1$. If $j - i = -z$, then e belongs to $G_{\alpha'',\beta''} + x$ where $0 \leq x \leq n-1$. Since every edge of $K_{(2k+1)\times n}$ appears on some copy of G in $\mathcal{C}$ and since $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{(2k+1)\times n}$ into copies of G. □

Lemma 3.2. *Let G of odd order n be a 2-regular almost-bipartite graph and let $e \in E(G)$ be such that $G - e$ is bipartite. Let $G' = G - u$ where $u \in V(G)$ is incident to e. Let P of size $\ell \leq n-2$ be the component of G' that is a path. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1,2]$ with the obvious vertex partition. Assume that there exists an embedding of P in $K_{n,n}$ with one edge of each length in $[-(n-1)/2, (n-1)/2] \smallsetminus \{\pm z\}$ for some $z \in [1, (n-1)/2]$ and such that the endpoints of P are $(j,1)$ and $(j,2)$ for some $j \in [0, n-1]$. Then there exists a G-decomposition of $K_{k\times 2n}$ for every integer $k \geq 3$.*

Proof. Let $k \geq 3$ be an integer and let $Q = [1, 2k]$. For $i \in [1,k]$, let $h_i = \{2i-1, 2i\}$ and $g_i = \mathbb{Z}_n \times h_i$. Let $H = \{h_1, h_2, \ldots, h_k\}$. Let $V(K_{k\times 2n}) = \mathbb{Z}_n \times [1, 2k]$ with the vertex-set partition $\{g_1, g_2, \ldots, g_k\}$. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k$ with holes H.

Fix r and s with $1 \leq r < s \leq 2k$ and $\{r,s\} \notin H$. Let $G'_{r,s}$ and $P_{r,s}$ be the embeddings (as in the hypothesis of the lemma) of G' and P, respectively, in the subgraph of $K_{k\times 2n}$ with vertex set $\mathbb{Z}_n \times \{r,s\}$ and the obvious vertex partition. Let (j,r) and (j,s) denote the endpoints of $P_{r,s}$ and let z be as in the hypothesis. We construct from $G'_{r,s}$ a graph $G_{r,s}$, isomorphic to G, by adding the edges $\{(j,r), (j+z, r\circ s)\}$ and $\{(j,s), (j+z, r\circ s)\}$ at the endpoints of $P_{r,s}$.

We proceed in the same fashion as in the proof of Lemma 3.1. Let $\mathcal{C} = \{G_{r,s} + x : 1 \leq r < s \leq 2k, \{r,s\} \notin H \text{ and } 0 \leq x \leq n-1\}$ and note that $\mathcal{C}$ contains $\binom{2k}{2}n$ distinct copies of G. For the proof that every edge of $K_{k\times 2n}$ appears on some copy of G in $\mathcal{C}$, we proceed in the same fashion as the proof of Lemma 3.1. □

Next, we prove a lemma about the existence of paths with certain edge lengths in $K_{n,n}$.

Lemma 3.3. *Let $n \geq 3$ be an odd integer and let $x \leq n$ be a positive integer. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1,2]$ with the obvious vertex partition. For positive integers $d_1, d_2, \ldots, d_{x-1}$ with $d_1 < d_2 < \cdots < d_{x-1} \leq (n-1)/2$, there exists an embedding of a path P of size $2x-1$ in $K_{n,n}$ whose edges have lengths $0, \pm d_1, \pm d_2, \ldots, \pm d_{x-1}$. Furthermore, $V(P) \subseteq [0, d_{x-1}] \times [1,2]$.*

Proof. If $x = 1$, let P be the path consisting of the edge $\{(0,1),(0,2)\}$. Otherwise, for $k \in [1, x-1]$, define $v_k = \sum_{i=0}^{k-1} (-1)^i d_{x-1-i}$. Note that since $d_1 < d_2 < \cdots < d_{x-1}$, we have that $v_1 > v_3 > \cdots$ and $v_2 < v_4 < \cdots$. Consider the path of size $x-1$ given by P': $(0,1),(v_1,2),(v_2,1),(v_3,2),\ldots$ where P' ends with $(v_{x-1},2)$ if $x-1$ is odd or $(v_{x-1},1)$ if $x-1$ is even. Observe that the lengths of the edges on P', in the order encountered, are $d_{x-1}, d_{x-2}, \ldots, d_1$. Next consider the path P'': $(0,2),(v_1,1),(v_2,2),(v_3,1),\ldots$ where P'' ends with $(v_{x-1},1)$ if $x-1$ is odd or $(v_{x-1},2)$ if $x-1$ is even, and observe that the edges on P'', in the order encountered, are $-d_{x-1}, -d_{x-2}, \ldots, -d_1$. Construct the path P from the paths P' and P'' by adding the edge from $(v_{x-1},1)$ to $(v_{x-1},2)$ in $K_{n,n}$. Note that P has size $2x-1$, the edges of P have lengths $0, \pm d_1, \pm d_2, \ldots, \pm d_{x-1}$, and $V(P) \subseteq [0, d_{x-1}] \times [1,2]$. $\square$

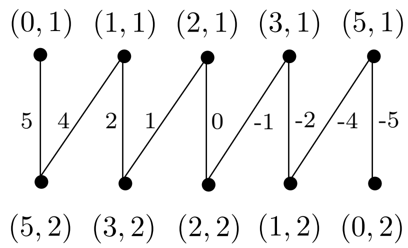


Figure 3.1: A path P of size 9 whose edges have lengths $0, \pm 1, \pm 2, \pm 4, \pm 5$.

Theorem 3.4. *For all odd integers $n \geq 3$, there exists a C_n-decomposition of $K_{(2k+1)\times n}$ for all positive integers k and of $K_{k'\times 2n}$ for all integers $k' \geq 3$.*

Proof. Label the vertex set of $K_{n,n}$ with the elements of the set $\mathbb{Z}_n \times [1,2]$ with the obvious vertex bipartition. It is sufficient to show that there exists an embedding of a path P of size $n-2$ in $K_{n,n}$ with one edge of each length in $[-(n-1)/2,(n-1)/2]\setminus\{\pm z\}$ for some $z\in[1,(n-1)/2]$ and such that the endpoints of P are $(j,1)$ and $(j,2)$ for some $j\in[0,n-1]$. By Lemma 3.3, there exists an such embedding of a path P of size $n-2$ using the edge lengths in $[-(n-3)/2,(n-3)/2]$ with endpoints $(0,1)$ and $(0,2)$. In the lemma we would use $d_1=1,d_2=2,\ldots,d_{(n-3)/2}=(n-3)/2$, so $V(P)\subseteq[0,(n-3)/2]\times[1,2]$. Thus, by Lemma 3.1 and Lemma 3.2, we conclude that there exists a G-decomposition of $K_{(2k+1)\times n}$ for every positive integer k and a G-decomposition of $K_{k'\times 2n}$ for every integer $k'\geq 3$. $\square$

It has long been known that if $n\geq 3$ is odd, then there exists a C_n-decomposition of K_n. This result is often credited to Walecki (see [4] for details).

Theorem 3.5. *For any odd integers $n\geq 3$, there exists a C_n-decomposition of K_n.*

By combining the results from Theorem 3.4 and Theorem 3.5, we obtain the following previously known result (see [29]).

Theorem 3.6. *There exists a C_n-decomposition of K_{2kn+n} for all odd integers $n\geq 3$ and all positive integers k.*

Proof. Observe that $K_{2kn+n}=(2k+1)K_n\cup K_{(2k+1)\times n}$ for all positive integers k. By Theorem 3.5, there exists a C_n-decomposition of K_n and hence of $(2k+1)K_n$ and by Theorem 3.4, there exists a C_n-decomposition of $K_{(2k+1)\times n}$. The result follows. $\square$

Example 3.7. We give an example of a C_5-decomposition of K_{15}.

Let K_{15} have vertex set $\mathbb{Z}_5\times[1,3]$. For each $i\in[1,3]$, there exists a C_5-decomposition of the K_5 with vertex set $\mathbb{Z}_5\times i$ (by Theorem 3.5.) Then for each $i\in[1,3]$, we have two copies of C_5 as follows:

$$\langle(0,i),(1,i),(2,i),(4,i),(3,i)\rangle,\langle(0,i),(2,i),(3,i),(1,i),(4,i)\rangle.$$

Thus we have a C_5-decomposition of $3K_5$.

It remains to find a C_5-decomposition of the complete multipartite subgraph $K_{3\times 5}$. Let $Q=[1,3]$ and let $(Q,\circ)$ denote a commutative idempotent quasigroup of order 3 in Figure 3.2. For fixed r and s with $1 \leq r < s \leq 3$, Let $P_{r,s}$ denote the path $((0,r),(1,s),(1,r),(0,s))$. We construct a 5-cycle $G_{r,s}$ from $P_{r,s}$ by adding the edges $\{(0,r),(2,r\circ s)\}$ and $\{(0,s),(2,r\circ s)\}$. Let $G^*_{r,s} = \{G_{r,s}+x : x \in \mathbb{Z}_5\}$. The 5-cycle $G_{1,2}+1$ is shown in Figure 3.2. Then the cycles in $G^*_{1,2} \cup G^*_{1,3} \cup G^*_{2,3}$ give a C_5-decomposition of $K_{3\times 5}$. The 5 copies of C_5 in $G^*_{1,2}$ are listed below:

$$\langle (0,1),(1,2),(1,1),(0,2),(2,1\circ 2=3)\rangle,$$
$$\langle (1,1),(2,2),(2,1),(1,2),(3,1\circ 2=3)\rangle,$$
$$\langle (2,1),(3,2),(3,1),(2,2),(4,1\circ 2=3)\rangle,$$
$$\langle (3,1),(4,2),(4,1),(3,2),(0,1\circ 2=3)\rangle,$$
$$\langle (4,1),(0,2),(0,1),(4,2),(1,1\circ 2=3)\rangle.$$

Since $K_{15} = 3K_5 \cup K_{5\times 3}$, we have a C_5-decomposition of K_{15}.

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

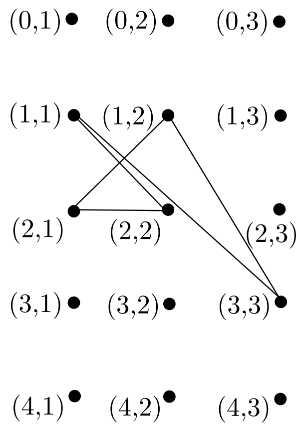


Figure 3.2: An idempotent commutative quasigroup of order 3 and one copy of a C_5 from the corresponding of C_5-decomposition of K_{15}.

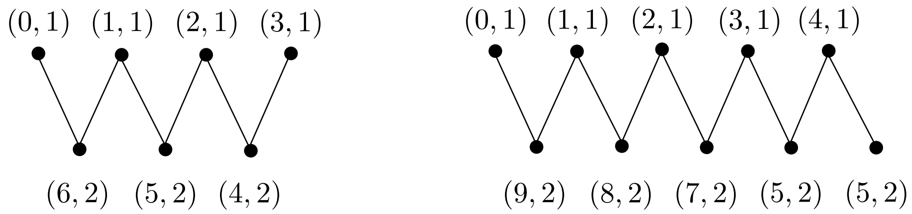
Before proceeding with the remainder of our results, we need some additional notation.

3.2 Additional notation

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

For the remainder of this chapter, we consider only subgraphs of a complete bipartite graphs $K_{m,m}$ with vertex set $[0, m-1] \times [1, 2]$ and the obvious vertex bipartition. Furthermore, if m, n, and i are integers with $m \leq n$, we denote $\{(m,i), (m+1,i), \ldots, (n,i)\}$ by $[(m,i),(n,i)]$

Let $P(k)$ be the path with k edges and $k+1$ vertices given by $\big((0,1), (k,2), (1,1), (k-1,2), (2,1), (k-2,2), \ldots, (\lceil k/2 \rceil, \lceil k/2 \rceil - \lfloor k/2 \rfloor + 1)\big)$. Note that the set of vertices of this graph is $A \cup B$, where $A = \big[(0,1), (\lfloor k/2 \rfloor, 1)\big]$, $B = \big[(\lfloor k/2 \rfloor + 1, 2), (k, 2)\big]$, and every edge joins a vertex of A to one of B. Furthermore, the set of lengths of the edges of $P(k)$ is $[1, k]$.



$P(6)$ $P(9)$

Figure 3.3: Examples of the $P(k)$ notation

Now let a be a nonnegative integer and b be an integer such that $|b| \leq \lfloor k/2 \rfloor + 1$, and let us add $(a, 0)$ to all the vertices of A and $(b, 0)$ to all the vertices of B. We denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1 $P(a, b, k)$ is a path with first vertex $(a, 1)$ and second vertex $(b + k, 2)$. Its last vertex is $(a + k/2, 1)$ if k is even and $(b + (k+1)/2, 2)$ if k is odd.

P2 Each edge of $P(a, b, k)$ joins a vertex of $A' = \big[(a, 1), (\lfloor k/2 \rfloor + a, 1)\big]$ to a vertex of $B' = \big[(\lfloor k/2 \rfloor + 1 + b, 2), (k + b, 2)\big]$.

P3 The set of edge lengths of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Now consider the directed path $Q(k)$ obtained from $P(k)$ replacing each vertex (i, j) with $(k - i, 3 - j)$. The new graph is the path $\big((k, 2), (0, 1), (k -$

$1,2),(1,1),\ldots,(\lfloor k/2\rfloor,\lfloor k/2\rfloor-\lceil k/2\rceil+2)\big)$. The set of vertices of $Q(k)$ is $A\cup B$, where $A=\big[(0,1),(\lceil k/2\rceil-1,1)\big]$ and $B=\big[(\lceil k/2\rceil,2),(k,2)\big]$, and every edge joins a vertex of A to one of B. The set of edge lengths is still $[1,k]$.

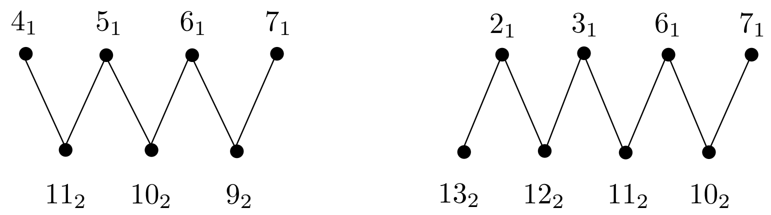
We again add $(a,0)$ to the vertices of A and $(b,0)$ to vertices of B, where a is nonnegative integer and b is an integers with $|b|\leq\lceil k/2\rceil$. We denote the resulting graph by $Q(a,b,k)$. Note that this graph has the following properties.

Q1 $Q(a,b,k)$ is a path with first vertex $(k+b,2)$. Its last vertex is $(b+k/2,2)$ if k is even and $(a+(k-1)/2,1)$ if k is odd.

Q2 Each edge of $Q(a,b,k)$ joins a vertex of $A'=\big[(a,1),(a+\lceil k/2\rceil-1,1)\big]$ to a vertex of $B'=\big[(b+\lceil k/2\rceil,2),(b+k,2)\big]$.

Q3 The set of edge lengths of $Q(a,b,k)$ is $[b-a+1,b-a+k]$.

For ease of notation, we henceforth use i_r and i_s to denote the vertices (i,r) and (i,s), respectively.



$P(4,5,6)$ $Q(2,6,7)$

Figure 3.4: Examples of $P(a,b,k)$ and $Q(a,b,k)$

3.3 A G-decomposition of $K_{(2k+1)\times n}$ and of $K_{k'\times 2n}$

Let A and B be finite subsets of the integers. If $\max(A)\leq\min(B)$, we will write $A\leq B$. We define $A<B$, $A\geq B$, and $A>B$ analogously. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n\times[1,2]$ with the obvious vertex partition. We prove three lemmas about the existence of an embedding of C_m with certain edge lengths in $K_{n,n}$ to use in Subsection 3.3.1 and Subsection 3.3.2. The constructions depend on the congruence class of m modulo 8.

Lemma 3.8. *Let $n \geq 11$ and $m \geq 10$ be integers such that n is odd, $m \equiv 2 \pmod 8$, and $m/2 \leq (n-1)/2$. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1,2]$ with the obvious vertex partition. If $m = 8t+2$, then there exists an embedding of a cycle C of size m in $K_{n,n}$ with one edge of each length in $\pm[2, 4t+2]$. Furthermore, $V(C) \subseteq [0, 4t+2] \times [1,2]$.*

Proof. To embed a cycle C of size m in $K_{n,n}$, let

$$C = G_1 + G_2 + G_3 + G_4 + ((4t+2)_1, (2t+1)_2, 0_1),$$

where

$$\begin{aligned} G_1 &= P(0, 2t+1, 2t+1), \\ G_2 &= Q(t+2, t+3, 2t-1), \\ G_3 &= P(2t+1, 0, 2t-1), \\ G_4 &= Q(3t+2, -(t+1), 2t+1). \end{aligned}$$

We then show that $G_1 + G_2 + G_3 + G_4 + ((4t+2)_1, (2t+1)_2, 0_1)$ is a cycle of size m. Note that by **P1** and **Q1** the first vertex of G_1 is 0_1, and the last vertex is $(3t+2)_2$; the first vertex of G_2 is $(3t+2)_2$, and the last vertex is $(2t+1)_1$; the first vertex of G_3 is $(2t+1)_1$, and the last vertex is t_1; the first vertex of G_4 is t_1, and the last vertex is $(4t+2)_1$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2**, we compute

$$\begin{aligned} A_1 &= [0_1, t_1], & B_1 &= [(3t+2)_2, (4t+2)_2], \\ A_2 &= [(t+2)_1, (2t+1)_1], & B_2 &= [(2t+3)_2, (3t+2)_2], \\ A_3 &= [(2t+1)_1, (3t)_1], & B_3 &= [t_2, (2t-1)_2], \\ A_4 &= [(3t+2)_1, (4t+2)_1], & B_4 &= [0_2, t_2]. \end{aligned}$$

Thus,

$$A_1 < A_2 \leq A_3 < A_4 \text{ and } B_4 \leq B_3 < B_2 \leq B_1.$$

Note that $V(G_1) \cap V(G_2) = \{(3t+2)_2\}$, $V(G_2) \cap V(G_3) = \{(2t+1)_1\}$, and $V(G_3) \cap V(G_4) = \{t_2\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + G_3 + G_4 + ((4t+2)_1, (2t+1)_2, 0_1)$ is a cycle of size m.

Next, let E_i denote the set of edge lengths in G_i for $1 \le i \le 4$. By **P3** and **Q3**, we have edge lengths

$$
\begin{aligned}
E_1 &= [2t+2, 4t+2],\\
E_2 &= [2, 2t],\\
E_3 &= [-2t, -2],\\
E_4 &= [-(4t+2), -(2t+2)].
\end{aligned}
$$

Moreover, the path $((4t+2)_1, (2t+1)_2, 0_1)$ consists of edges lengths $-(2t+1)$ and $2t+1$. Thus, C has edge lengths $\pm[2, 4t+2]$. $\square$

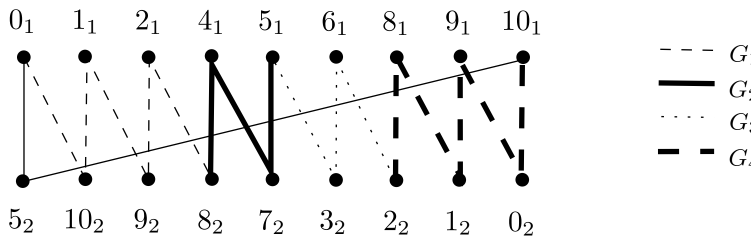


Figure 3.5: The cycle C with paths G_1, G_2, G_3 and G_4 where $t = 2$ in Lemma 3.8

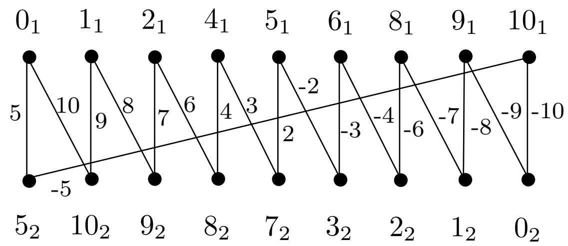


Figure 3.6: The cycle C where $t = 2$ in Lemma 3.8

Lemma 3.9. *Let $n \ge 15$ and $m \ge 14$ be integers such that n is odd, $m \equiv 6 \pmod 8$, and $m/2 \le (n-1)/2$. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1,2]$ with the obvious vertex partition. If $m = 8t+6$, then there exists an embedding of a cycle C of size m in $K_{n,n}$ with one edge of each length in $\pm[1, 4t+4] \setminus \{\pm 2\}$. Furthermore, $V(C) \subseteq [0, 4t+4] \times [1,2]$.*

Proof. To embed a cycle C of size m in $K_{n,n}$, let

$$C = G_1 + G_2 + ((2t+3)_2, (2t+2)_1, (2t+1)_2) + G_3 + G_4 + ((4t+4)_1, (2t+2)_2, 0_1),$$

where

$$\begin{aligned} G_1 &= P(0, 2t+2, 2t+2), \\ G_2 &= P(t+1, t+3, 2t-1), \\ G_3 &= Q(2t+4, 2, 2t-1), \\ G_4 &= P(3t+3, -(t+2), 2t+2). \end{aligned}$$

We then show that $G_1 + G_2 + ((2t+3)_2, (2t+2)_1, (2t+1)_2) + G_3 + G_4 + ((4t+4)_1, (2t+2)_2, 0_1)$ is a cycle of size m. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_1, and the last vertex is $(t+1)_1$; the first vertex of G_2 is $(t+1)_1$, and the last vertex is $(2t+3)_2$; the first vertex of G_3 is $(2t+1)_2$, and the last vertex is $(3t+3)_1$; the first vertex of G_4 is $(3t+3)_1$, and the last vertex is $(4t+4)_1$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2**, we compute

$$\begin{aligned} A_1 &= [0_1, (t+1)_1], & B_1 &= [(3t+4)_2, (4t+4)_2], \\ A_2 &= [(t+1)_1, (2t)_1], & B_2 &= [(2t+3)_2, (3t+2)_2], \\ A_3 &= [(2t+4)_1, (3t+3)_1], & B_3 &= [(t+2)_2, (2t+1)_2], \\ A_4 &= [(3t+3)_1, (4t+4)_1], & B_4 &= [0_2, t_2]. \end{aligned}$$

Thus,

$$A_1 \le A_2 < A_3 \le A_4 \text{ and } B_4 < B_3 < B_2 < B_1.$$

Note that $V(G_1) \cap V(G_2) = \{(t+1)_1\}$, and $V(G_3) \cap V(G_4) = \{(3t+3)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + ((2t+3)_2, (2t+2)_1, (2t+1)_2) + G_3 + G_4 + ((4t+4)_1, (2t+2)_2, 0_1)$ is a cycle of size m.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3** and **Q3**, we have edge lengths

$$E_1 = [2t+3, 4t+4],$$
$$E_2 = [3, 2t+1],$$
$$E_3 = [-(2t+1), -3],$$
$$E_4 = [-(4t+4), -(2t+3)].$$

Moreover, the path $((2t+3)_2, (2t+2)_1, (2t+1)_2)$ consists of edges lengths 1 and -1, and the path $((4t+4)_1, (2t+2)_2, 0_1)$ consists of edges lengths $-(2t+2)$ and $2t+2$. Thus, C has edge lengths $\pm[1, 4t+4] \setminus \{\pm 2\}$. $\square$

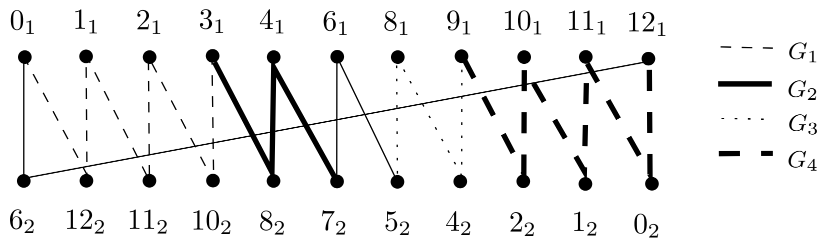


Figure 3.7: The cycle C with paths G_1, G_2, G_3 and G_4 where $t = 2$ in Lemma 3.9

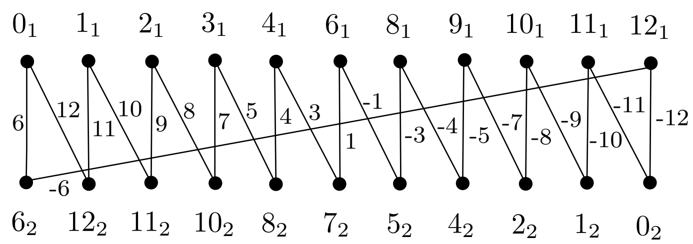


Figure 3.8: The cycle C where $t = 2$ in Lemma 3.9

Lemma 3.10. *Let $n \geq 5$ and $m \geq 4$ be integers such that n is odd, $m \equiv 0 \pmod 4$, and $m/2 \leq (n-1)/2$. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. If $m = 4t$, then there exists an embedding of a cycle C of size m in $K_{n,n}$ with one edge of each length in $\pm[1, 2t]$. Furthermore, $V(C) \subseteq [0, 2t+1] \times [1, 2]$.*

Proof. To embed a cycle C of size m in $K_{n,n}$, let

$$C = G_1 + G_2 + ((2t+1)_1, 1_2, 0_1),$$

where

$$G_1 = P(0, 2t+1, 2t+1),$$
$$G_2 = Q(t+2, t+3, 2t-1).$$

We then show that $G_1 + G_2 + ((2t+1)_1, 1_2, 0_1)$ is a cycle of size m. Note that by **P1** and **Q1** the first vertex of G_1 is 0_1, and the last vertex is $(t+1)_2$; the first vertex of G_2 is $(t+1)_2$, and the last vertex is $(2t+1)_1$. For $i \le i \le 2$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2**, we compute

$$A_1 = [0_1, (t-1)_1], \qquad B_1 = [(t+1)_2, (2t)_2],$$
$$A_2 = [(t+2)_1, (2t+1)_1], \qquad B_2 = [2_2, (t+1)_2].$$

Thus,

$$A_1 < A_2 \text{ and } B_2 \le B_1.$$

Note that $V(G_1) \cap V(G_2) = \{(t+1)_1\}$ otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + ((2t+1)_1, 1_2, 0_1)$ is a cycle of size m.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3** and **Q3**, we have edge lengths

$$E_1 = [2, 2t],$$
$$E_2 = [-(2t-1), -1].$$

Moreover, the path $((2t+1)_1, 1_2, 0_1)$ consists of edges lengths $-2t$ and 1. Thus, C has edge lengths $\pm[1, 2t]$. □

3.3.1 G consisting of one even cycle and one odd cycle

Let G of odd size n be the vertex-disjoint union of one even cycle and one odd cycle. In this section, we will show how to construct a G-decomposition of

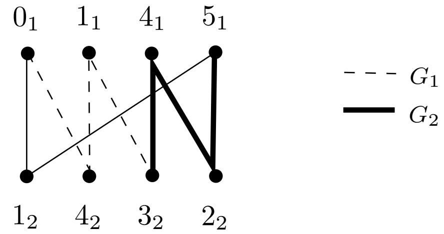


Figure 3.9: The cycle C with paths G_1 and G_2 where $t = 2$ in Lemma 3.10

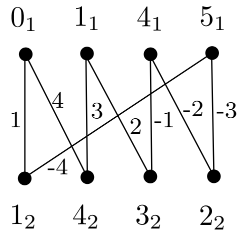


Figure 3.10: The cycle C where $t = 2$ in Lemma 3.10

$K_{(2k+1)\times n}$ for all positive integers k and of $K_{k'\times 2n}$ for all integers $k' \geq 3$. To obtain these results, it suffices to show that there exists an embedding of G that satisfies the statements in Lemma 3.1 and Lemma 3.2. Furthermore, if we combine these results with the results in Theorem 2.6, we obtain a G-decomposition of K_v where $v \equiv n \pmod{2n}$, unless $G = C_4 \cup C_5$ and $v = 9$. Furthermore, we obtain necessary and sufficient conditions for a G-decomposition of K_v when n is a prime power.

Lemma 3.11. *Let G be vertex-disjoint union of a cycle C of size m and a path P of size $2\ell - 1$ where $m, \ell > 0$ are integers and m is even. Let $n = m + 2\ell + 1$ and let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. Then there exists an embedding of G in $K_{n,n}$ with one edge of each length in $[-(n-1)/2, (n-1)/2] \setminus \{\pm z\}$ for some $z \in [1, (n-1)/2]$ and such that the endpoints of P are 0_1 and 0_2.*

Proof. Let $n = m + 2\ell + 1$ and $V(K_{n,n}) = \mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. We proceed by cases depending on the congruence class of m modulo 8.

Case 1. Suppose $m \equiv 2 \pmod 8$. Let $m = 8t + 2$ for some positive integer t. By Lemma 3.8, there exists an embedding of a cycle C of size m with edge lengths $\pm[2, 4t + 4]$ in $K_{n,n}$. Furthermore, $V(C) \subseteq [0, 4t + 2] \times [1, 2]$.

We next give an embedding of P of size $2\ell - 1$ in $K_{n,n}$. If $\ell = 1$, then by Lemma 3.3, there exists an embedding of a path P^* of size 1 using edge length 0 with endpoints 0_1 and 0_2. Let $P = P^* + (4t+3)$ with endpoints $(4t+3)_1$ and $(4t+3)_2$. Note that $4t+3 < 8t+5 = n$. Thus, the edge set of G has one edge of each length $i \in [-(4t+2), 4t+2] \setminus \{\pm 1\} = [-(n-1)/2, (n-1)/2] \setminus \{\pm 1\}$.

Suppose that $\ell \geq 2$. By Lemma 3.3, there exists an embedding of a path P^* of size $2\ell - 1$ using edge lengths $\{-1, 0, 1\} \cup \pm[(4t+3), (n-3)/2]$ with endpoints 0_1 and 0_2. In the lemma we would use $d_1 = 1, d_2 = 4t+3, \ldots, d_{\ell-1} = (n-3)/2$, so $V(P^*) \subseteq [0, (n-3)/2] \times [1, 2]$. Let $P = P^* + (4t+3)$ with endpoints $(4t+3)_1$ and $(4t+3)_2$. Then $V(P) \subseteq [4t+3, (n-3)/2 + (4t+3)] \times [1, 2]$. Note that $(n-3)/2 + (4t+3) = (n+m+1)/2 = (2n-2\ell)/2 < n$ and P is vertex disjoint from C. Thus, the edge set of G has one each of each length $i \in [-(n-3)/2, (n-3)/2]$, except the edge lengths $\pm(n-1)/2$. Figure 3.11 shows an embedding of C and P in $K_{n,n}$ where $t = 1$ and $\ell = 2$.

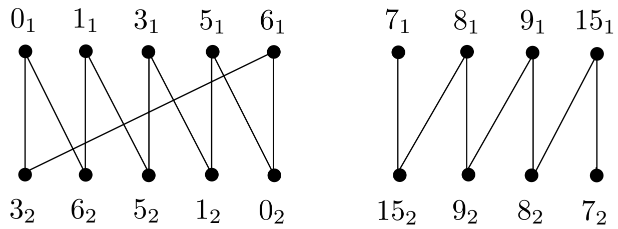


Figure 3.11: An example of C and P in case 1 of Lemma 3.11

Case 2. Suppose $m \equiv 6 \pmod 8$. Let $m = 8t + 6$ where t is nonnegative integer.

Case 2.1. $t = 0$. Let $C^* = \langle 0_1, 3_2, 2_1, 0_2, 3_1, 2_2 \rangle$ be an embedding of C. Its edge lengths are $3, 1, -2, -3, -1, 2$.

We next give an embedding of P of size $2\ell - 1$ in $K_{n,n}$. If $\ell = 1$, then by Lemma 3.3, there exists an embedding of a path P^* of size 1 using edge length 0 with endpoints 0_1 and 0_2. Let $P = P^* + 4$ with endpoints 4_1 and 4_2. Note that $4 < 9 = n$. Thus, the edge set of G has one edge of each length $i \in [-3, 3]$, except the edge lengths ± 4.

Suppose that $\ell \geq 2$. By Lemma 3.3, there exists an embedding of a path P^* of size $2\ell - 1$ using edge lengths $\{0\} \cup \pm[4, (n-3)/2]$ with endpoints 0_1

and 0_2. In the lemma, we would use $d_1 = 4, d_2 = 5, \ldots, d_{\ell-1} = (n-3)/2$, so $V(P^*) \subseteq [0, (n-3)/2] \times [1, 2]$. Let $P = P^* + 4$ with endpoints 4_1 and 4_2. Then $V(P) \subseteq [4, (n-3)/2 + 4] \times [1, 2]$. Note that $(n-3)/2 + 4 = (n+5)/2 = (2n - 2\ell + 2)/2 < n$ since $\ell > 1$, and P is vertex disjoint from C. Thus, the edge set of G has one each of each length $i \in [-(n-3)/2, (n-3)/2]$, except the edge lengths $\pm(n-1)/2$. Figure 3.12 shows an embedding of C and P in $K_{n,n}$ where $t = 0$ and $\ell = 5$.

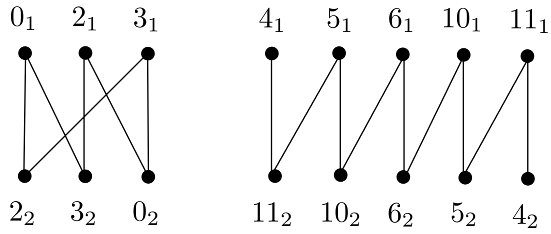


Figure 3.12: An example of C and P in case 2.1 of Lemma 3.11

Case 2.2. $t \geq 1$. By Lemma 3.8, there exists an embedding of a cycle C of size m with edge lengths $\pm[1, 4t+4] \smallsetminus \{\pm 2\}$ in $K_{n,n}$. Furthermore, $V(C) \subseteq [0, 4t+4] \times [1, 2]$.

We next give an embedding of P of size $2\ell - 1$ in $K_{n,n}$. If $\ell = 1$, then by Lemma 3.3, there exists an embedding of a path P^* of size 1 using edge length 0 with endpoints 0_1 and 0_2. Let $P = P^* + (4t+5)$ with endpoints $(4t+5)_1$ and $(4t+5)_2$. Note that $4t + 5 < 8t + 9 = n$. Thus, the edge set of G has one edge of each length $i \in [-(4t+4), 4t+4] \smallsetminus \{\pm 2\} = [-(n-1)/2, (n-1)/2] \smallsetminus \{\pm 2\}$.

Suppose that $\ell \geq 2$. By Lemma 3.3, there exists an embedding of a path P^* of size $2\ell - 1$ using edge lengths $\{-2, 0, 2\} \cup \pm[4t+5, (n-3)/2]$ with endpoints 0_1 and 0_2. In the lemma we would use $d_1 = 2, d_2 = 4t+5, \ldots d_{\ell-1} = (n - 3/2)$, so $V(P^*) \subseteq [0, (n-3)/2] \times [1, 2]$. Let $P = P^* + 4t + 5$ with endpoints $(4t+5)_1$ and $(4t+5)_2$. Then $V(P) \subseteq [4t+5, (n-3)/2 + (4t+5)] \times [1, 2]$. Note that $(n-3)/2 + (4t+5) = (n+m+1)/2 = (2n - 2\ell)/2 < n$ and P is vertex disjoint from C. Thus, the edge set of G has one each of each length $i \in [-(n-3)/2, (n-3)/2]$, except the edge lengths $\pm(n-1)/2$. Figure 3.13 shows an embedding C and P in $K_{n,n}$ where $t = 1$ and $\ell = 3$.

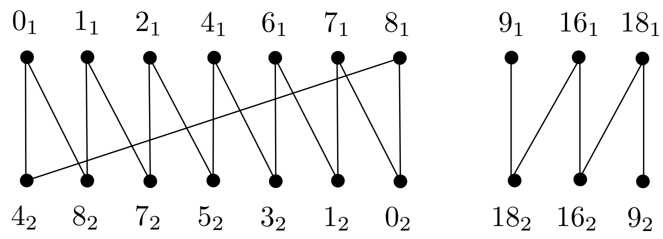


Figure 3.13: An example of C and P in case 2.2 of Lemma 3.11

Case 3. Suppose $m \equiv 0 \pmod 4$. Let $m = 4t$ for some positive integer t. By Lemma 3.10, there exists an embedding of a cycle C of size m with edge lengths $\pm[1, 2t]$ in $K_{n,n}$. Furthermore, $V(C) \subseteq [0, 2t + 1] \times [1, 2]$.

We next give an embedding of P of size $2\ell - 1$ in $K_{n,n}$. If $\ell = 1$, then by Lemma 3.3, there exists an embedding of a path P^* of size 1 using edge length 0 with endpoints 0_1 and 0_2. Let $P = P^* + (2t + 2)$ with endpoints $(2t + 2)_1$ and $(2t + 2)_2$. Note that $2t + 2 < 4t + 3 = n$. Thus, the edge set of G has one edge of each length $i \in [-2t, 2t]$, except the edge lengths $\pm(2t + 1)$.

Suppose that $\ell \geq 2$. By Lemma 3.3, there exists an embedding of a path P^* of size $2\ell - 1$ using edge lengths $\{0\} \cup \pm[2t + 1, (n - 3)/2]$ with endpoints 0_1 and 0_2. In the lemma we would use $d_1 = 2t + 1, d_2 = 2t + 2, \ldots, d_{\ell-1} = (n - 3)/2$, so $V(P^*) \subseteq [0, (n - 3)/2] \times [1, 2]$. Let $P = P^* + (2t + 2)$ with endpoints $(2t + 2)_1$ and $(2t + 2)_2$. Then $V(P) \subseteq [2t + 2, (n - 3)/2 + (2t + 2)] \times [1, 2]$. Note that $(n-3)/2+(2t+2) = (n+m+1)/2 = (2n-2\ell)/2 < n$ and P is vertex disjoint from C. Thus, the edge set of G has one each of each length $i \in -(n-3)/2, (n-3)/2]$, except the edge lengths $\pm(n-1)/2$. Figure 3.14 shows an embedding of C and P in $K_{n,n}$ where $t = 2$ and $\ell = 5$. $\square$

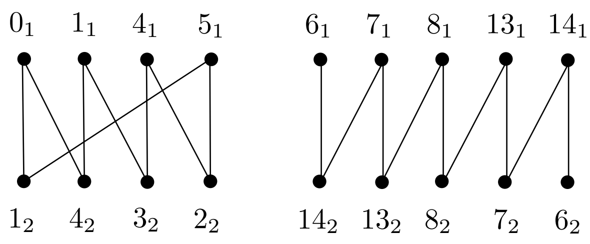


Figure 3.14: An example of C and P in case 3 of Lemma 3.11

Theorem 3.12 is obtained by combining the results from Lemma 3.11 and Lemma 3.1 to show that there exists a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k. Also, by combining the results from Lemma 3.11 and Lemma 3.2 to prove the existence of a G-decomposition of $K_{k'\times 2n}$ for all integers $k' \geq 3$.

Theorem 3.12. *Let G be a 2-regular graph of odd order n consisting of exactly two cycles. Then there exists a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k, and of $K_{k'\times 2n}$ for all integer $k' \geq 3$.*

By combining the results from Theorem 2.6 and Theorem 3.12 we obtain the following theorem.

Theorem 3.13. *Let G be a 2-regular graph of odd order n consisting of exactly two cycles. There exists a G-decomposition of K_v for all $v \equiv n \pmod{2n}$ unless $G = C_4 \cup C_5$ and $v = 9$.*

Proof. In [5], it is shown that there exists a $(C_4 \cup C_5)$-decomposition of K_v if and only if $v \equiv 1$ or $9 \pmod{18}$ and $v \neq 9$. For all other G, let $v = 2kn + n$. Observe that $K_v = (2k+1)K_n \cup K_{(2k+1)\times n}$. By Theorem 2.6, there exists a G-decomposition of K_n and hence of $(2k+1)K_n$ and by Theorem 3.12, there exists a G-decomposition of $K_{(2k+1)\times n}$. The result follows. □

If n in Theorem 3.12 is a power of a prime, then we have the following corollary.

Corollary 3.14. *Let G be a 2-regular graph of odd order n consisting of exactly two cycles. If n is a prime power, then there exists a G-decomposition of K_v if and only if $v \equiv 1$ or $n \pmod{2n}$, unless $G = C_4 \cup C_5$ and $v = 9$.*

Proof. The necessary conditions for the existence of a G-decomposition of K_v are $n|v(v-1)/2$ and $v \geq n$ is odd. If $n = p^k$, where p is prime, then we have $2p^k|v(v-1)$ and $v \geq p^k$ is odd. Since v and $v-1$ are relatively prime, either $p^k|v$ or $p^k|v-1$. Thus, $v \equiv 1$ or $p^k \pmod{2p^k}$.

It is known that there exists a G-decomposition of K_v for all $v \equiv 1 \pmod{2n}$ (see [11] and [5]). By Theorem 3.13, there exists a G-decomposition of K_v for all $v \equiv n \pmod{2n}$ unless $G = C_4 \cup C_5$ and $v = 9$. The result follows. □

3.3.2 G consisting of any number of even cycles and one odd cycle

In this section, we extend the idea of the construction in Subsection 3.3.1 to prove that there exists a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k and of $K_{k'\times 2n}$ for all integers $k' \geq 3$ where G of size n is an almost-bipartite graph consisting of any number of even cycles and one odd cycle. For this construction, we need to use the properties of α-labelings of even cycles to get a new labeling.

Let M_i be a bipartite graph of size m_i, with α-labeling f_i, critical value λ_i, and vertex bipartition $\{A_i, B_i\}$ for all i such that $1 \leq i \leq w$. Let M of size m be a disjoint-union of w bipartite graphs, M_i of size m_i where $i = 1, 2, \ldots, w$. That is, $M = M_1 \cup M_2 \cup \cdots \cup M_w$ with size $m = m_1 + m_2 + \cdots + m_w$.

Lemma 3.15. *For $1 \leq i \leq w$, let M_i be a bipartite graph of size m_i that admits an α-labeling with critical value λ_i. If $\lambda_1 \geq \lambda_{\lceil \frac{w}{2} \rceil+1} \geq \lambda_2 \geq \lambda_{\lceil \frac{w}{2} \rceil+2} \geq \cdots$, then*

$$0 \leq \sum_{i=1}^{\lceil \frac{w}{2} \rceil} \lambda_i \; - \sum_{i=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_i < m_1.$$

Proof. Let $k = w$ if w is even and let $k = (w+1)/2$ if w is odd. Then by the hypothesis, $\lambda_1 \geq \lambda_{\lceil \frac{w}{2} \rceil+1} \geq \lambda_2 \geq \lambda_{\lceil \frac{w}{2} \rceil+2} \geq \cdots \geq \lambda_k$. Hence we have both of the following:

$$0 \leq \lambda_1 - \lambda_{\lceil \frac{w}{2} \rceil+1} + \lambda_2 - \lambda_{\lceil \frac{w}{2} \rceil+2} + \cdots + (-1)^{w-1}\lambda_k = \sum_{i=1}^{\lceil \frac{w}{2} \rceil} \lambda_i \; - \sum_{i=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_i$$

and

$$0 \leq \lambda_{\lceil \frac{w}{2} \rceil+1} - \lambda_2 + \lambda_{\lceil \frac{w}{2} \rceil+2} - \lambda_3 + \cdots + (-1)^{w-2}\lambda_k = \sum_{i=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_i \; - \sum_{i=2}^{\lceil \frac{w}{2} \rceil} \lambda_i.$$

Therefore,

$$0 \leq \sum_{i=1}^{\lceil \frac{w}{2} \rceil} \lambda_i \; - \sum_{i=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_i = \lambda_1 - \left(\sum_{i=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_i \; - \sum_{i=2}^{\lceil \frac{w}{2} \rceil} \lambda_i \right) \leq \lambda_1 < m_1.$$

$\square$

Lemma 3.16. *For $1 \le i \le w$, let M_i be the bipartite graph of size m_i with vertex bipartition $\{A_i, B_i\}$ that admits α-labeling f_i with critical value $\lambda_i = \max\bigl(f_i(A_i)\bigr)$ such that $\lambda_1 \ge \lambda_{\lceil \frac{w}{2} \rceil+1} \ge \lambda_2 \ge \lambda_{\lceil \frac{w}{2} \rceil+2} \ge \cdots$. Let $M = M_1 \cup M_2 \cup \cdots \cup M_w$ and $m = \sum\limits_{i=1}^{w} m_i$. Let $A_L = \bigcup\limits_{i=1}^{\lceil \frac{w}{2} \rceil} A_i$ and $A_R = \bigcup\limits_{i=\lceil \frac{w}{2} \rceil+1}^{w} A_i$ and define B_L and B_R analogously. Let a, b, c, d be integers such that $0 \le a < c$ and $b, d \in [a, c]$, and let $n = m + c + d + 1$. Define a labeling function $f\colon\ V(M) \to [a, n-1]$ by*

$$f(v) = \begin{cases} f_i(v) + \sum\limits_{j=1}^{i-1} (\lambda_j + 1) + a, & v \in A_i \subseteq A_L; \\ f_i(v) + \sum\limits_{j=i+1}^{w} (\lambda_j + 1) + b, & v \in A_i \subseteq A_R; \\ f_i(v) + \sum\limits_{j=1}^{i-1} (\lambda_j + 1) + a + \sum\limits_{j=i+1}^{\lceil \frac{w}{2} \rceil} m_j + \sum\limits_{j=\lceil \frac{w}{2} \rceil+i}^{w} m_j + c, & v \in B_i \subseteq B_L; \\ f_i(v) + \sum\limits_{j=i+1}^{w} (\lambda_j + 1) + b + \sum\limits_{j=1}^{i-\lceil \frac{w}{2} \rceil} m_j + \sum\limits_{j=\lceil \frac{w}{2} \rceil+1}^{i-1} m_j + d, & v \in B_i \subseteq B_R. \end{cases}$$

Then both $f|_{A_L \cup B_L}$ and $f|_{A_R \cup B_R}$ are injective. Furthermore, $f(A_L) \cap f(B_R) = \varnothing = f(A_R) \cap f(B_L)$.

Proof. First, we consider $f|_{(A_L \cup B_L)}$. For $1 \le i \le \lceil \frac{w}{2} \rceil$, we have

$$\min\bigl(f(A_i)\bigr) = 0 + \sum_{j=1}^{i-1} (\lambda_j + 1) + a,$$

$$\max\bigl(f(B_i)\bigr) = m_i + \sum_{j=1}^{i-1} (\lambda_j + 1) + a + \sum_{j=i+1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+i}^{w} m_j + c.$$

Note that

$$\min\bigl(f(A_1)\bigr) = a \text{ and } \max\bigl(f(B_1)\bigr) = m + c + a < m + c + d + 1 = n$$

For $1 \le i \le \lceil \frac{w}{2} \rceil - 1$, we have

$$\max\bigl(f(A_i)\bigr) = \lambda_i + \sum_{j=1}^{i-1}(\lambda_j + 1) + a = \sum_{j=1}^{i}(\lambda_j + 1) + a - 1 = \min\bigl(f(A_{i+1})\bigr) - 1,$$

$$\begin{aligned}
\min\bigl(f(B_i)\bigr) &= (\lambda_i + 1) + \sum_{j=1}^{i-1}(\lambda_j + 1) + a + \sum_{j=i+1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + i}^{w} m_j + c \\
&= \sum_{j=1}^{i}(\lambda_j + 1) + a + \Bigl(\sum_{j=i+2}^{\lceil \frac{w}{2} \rceil} m_j + m_{i+1}\Bigr) + \Bigl(\sum_{j=\lceil \frac{w}{2} \rceil + i + 1}^{w} m_j + m_{\lceil \frac{w}{2} \rceil + i}\Bigr) + c \\
&= \Bigl(m_{i+1} + \sum_{j=1}^{i}(\lambda_i + 1) + a + \sum_{j=i+2}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + i + 1}^{w} m_j + c\Bigr) + m_{\lceil \frac{w}{2} \rceil + i} \\
&= \max\bigl(f(B_{i+1})\bigr) + m_{\lceil \frac{w}{2} \rceil + i}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\max\bigl(f(A_{\lceil \frac{w}{2} \rceil})\bigr) &= \lambda_{\lceil \frac{w}{2} \rceil} + \sum_{j=1}^{\lceil \frac{w}{2} \rceil - 1} (\lambda_j + 1) + a \\
&< (\lambda_{\lceil \frac{w}{2} \rceil} + 1) + \sum_{j=1}^{\lceil \frac{w}{2} \rceil - 1} (\lambda_j + 1) + a + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c \\
&= (\lambda_{\lceil \frac{w}{2} \rceil} + 1) + \sum_{j=1}^{\lceil \frac{w}{2} \rceil - 1} (\lambda_j + 1) + a + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c \\
&= \min\bigl(f(B_{\lceil \frac{w}{2} \rceil})\bigr).
\end{aligned}$$

Hence,

$$\begin{aligned}
a \le f(A_1) < f(A_2) < \cdots < f(A_{\lceil \frac{w}{2} \rceil}) < f(B_{\lceil \frac{w}{2} \rceil}) < f(B_{\lceil \frac{w}{2} \rceil - 1}) < \cdots \\
\cdots < f(B_1) \le n - 1.
\end{aligned}$$

Next, we consider $f|_{A_R \cup B_R}$. Note that for $\lceil \frac{w}{2} \rceil + 1 \le i \le w$, we have

$$\max\bigl(f(A_i)\bigr) = \lambda_i + \sum_{j=i+1}^{w} (\lambda_j + 1) + b,$$

$$\min\bigl(f(B_i)\bigr) = (\lambda_i + 1) + \sum_{j=i+1}^{w} (\lambda_j + 1) + b + \sum_{j=1}^{i - \lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{i-1} m_j + d,$$

and for $\lceil \frac{w}{2} \rceil + 1 \leq i \leq w-1$, we have

$$\begin{aligned}\min\bigl(f(A_i)\bigr) &= 0 + \sum_{j=i+1}^{w} (\lambda_j + 1) + b \\ &= \Bigl(\lambda_{i+1} + \sum_{j=i+2}^{w} (\lambda_j + 1) + b\Bigr) + 1 \\ &= \max\bigl(f(A_{i+1})\bigr) + 1,\end{aligned}$$

$$\begin{aligned}\max\bigl(f(B_i)\bigr) &= m_i + \sum_{j=i+1}^{w} (\lambda_j + 1) + b + \sum_{j=1}^{i-\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{i-1} m_j + d \\ &= m_i + \Bigl(\sum_{j=i+2}^{w} (\lambda_j + 1) + (\lambda_{i+1} + 1)\Bigr) + b + \Bigl(\sum_{j=1}^{i+1-\lceil \frac{w}{2} \rceil} m_j - m_{i+1-\lceil \frac{w}{2} \rceil}\Bigr) + \\ &\quad \Bigl(\sum_{j=\lceil \frac{w}{2} \rceil+1}^{i} m_j - m_i\Bigr) + d \\ &= \Bigl((\lambda_{i+1} + 1) + \sum_{j=i+2}^{w}(\lambda_j + 1) + b + \sum_{j=1}^{i+1-\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{i} m_j + d\Bigr) - m_{i+1-\lceil \frac{w}{2} \rceil} \\ &= \min\bigl(f(B_{i+1})\bigr) - m_{i+1-\lceil \frac{w}{2} \rceil}.\end{aligned}$$

Moreover,

$$\begin{aligned}\max\bigl(f(A_w)\bigr) &= \lambda_w + \sum_{j=i+1}^{w} (\lambda_j + 1) + b \\ &< (\lambda_w + 1) + \sum_{j=i+1}^{w} (\lambda_j + 1) + b + \sum_{j=1}^{w-\lceil \frac{w}{2} \rceil} m_j + \sum_{j=1}^{w-1} m_j + d \\ &= \min\bigl(f(B_w)\bigr).\end{aligned}$$

Also, observe that

$$\min\bigl(f(A_w)\bigr) = b \geq a \text{ and } \max\bigl(f(B_w)\bigr) \leq m + b + d < m + c + d + 1 = n.$$

Hence,

$$\begin{aligned}a \leq f(A_w) < f(A_{w-1}) < \cdots < f(A_{\lceil \frac{w}{2} \rceil+1}) < f(B_{\lceil \frac{w}{2} \rceil+1}) < f(B_{\lceil \frac{w}{2} \rceil+2}) < \cdots \\ \cdots < f(B_w) \leq n-1.\end{aligned}$$

Since for each $i \in [1, w]$ the α-labeling f_i is injective, f is also injective on each A_i and B_i, and the first result follows. Finally, consider

$$\max\bigl(f(A_L)\bigr) = \max\bigl(f(A_{\lceil \frac{w}{2} \rceil})\bigr) = \lambda_{\lceil \frac{w}{2} \rceil} + \sum_{j=1}^{\lceil \frac{w}{2} \rceil - 1} (\lambda_j + 1) + a = \lceil \tfrac{w}{2} \rceil - 1 + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a. \tag{3.1}$$

By Lemma 3.15, we have

$$\begin{aligned}
\max\bigl(f(A_L)\bigr) &< \lceil \tfrac{w}{2} \rceil - 1 + m_1 + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{w} \lambda_j + a \\
&< \lambda_{\lceil \frac{w}{2} \rceil + 1} + m_1 + \Bigl(\lceil \tfrac{w}{2} \rceil - 1 + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{w} \lambda_j\Bigr) + b + d \\
&\leq \lambda_{\lceil \frac{w}{2} \rceil + 1} + m_1 + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{w} (\lambda_j + 1) + b + d \\
&= \lambda_{\lceil \frac{n}{2} \rceil + 1} + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{w} (\lambda_j + 1) + b + \sum_{j=1}^{(\lceil \frac{w}{2} \rceil + 1) - \lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{(\lceil \frac{w}{2} \rceil + 1) - 1} m_j + d \\
&= \min\bigl(f(B_{\lceil \frac{w}{2} \rceil + 1})\bigr) = \min\bigl(f(B_R)\bigr).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\min\bigl(f(B_L)\bigr) &= \min\bigl(f(B_{\lceil \frac{w}{2} \rceil})\bigr) \\
&= (\lambda_{\lceil \frac{w}{2} \rceil} + 1) + \sum_{j=1}^{\lceil \frac{w}{2} \rceil - 1} (\lambda_j + 1) + a + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c \\
&= (\lambda_{\lceil \frac{w}{2} \rceil} + 1) + \Bigl(\lceil \tfrac{w}{2} \rceil - 1 + \sum_{j=1}^{\lceil \frac{w}{2} \rceil - 1} \lambda_j\Bigr) + a + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c \\
&= \lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c.
\end{aligned}$$

Thus,

$$\min\bigl(f(B_L)\bigr) = \min\bigl(f(B_{\lceil \frac{w}{2} \rceil})\bigr) = \lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c, \tag{3.2}$$

and by Lemma 3.15,

$$\begin{aligned}
\min\big(f(B_L)\big) &\geq \lceil \tfrac{w}{2} \rceil + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + a + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c \\
&> \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} (\lambda_j + 1) + a + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil}^{w} m_j + c \\
&\geq \Big(\sum_{j=\lceil \frac{w}{2} \rceil+2}^{w} (\lambda_j + 1) + (\lambda_{\lceil \frac{w}{2} \rceil+1} + 1)\Big) + b \\
&> \lambda_{\lceil \frac{w}{2} \rceil+1} + \sum_{j=\lceil \frac{w}{2} \rceil+2}^{w} (\lambda_j + 1) + b \\
&= \max\big(f(A_{\lceil \frac{w}{2} \rceil+1})\big) = \max\big(f(A_R)\big),
\end{aligned}$$

and thus the second result follows. □

Example 3.17. We illustrate the results from Lemma 3.16 here.

Let $M = M_1 \cup M_2 \cup M_3$ where $M_1 = C_6 \cup C_6$, $M_2 = C_8$ and $M_3 = C_{12}$. In this example, $m = 32$ and $w = 3$. By Theorem 2.7 and Theorem 2.8, each M_i admits an α-labeling f_i with critical value λ_i and vertex bipartition $\{A_i, B_i\}$ shown as in Figure 3.15. Note that $\lambda_1 = 6$, $\lambda_2 = 3$ and $\lambda_3 = 5$ and $\lambda_1 \geq \lambda_3 \geq \lambda_2$. Let

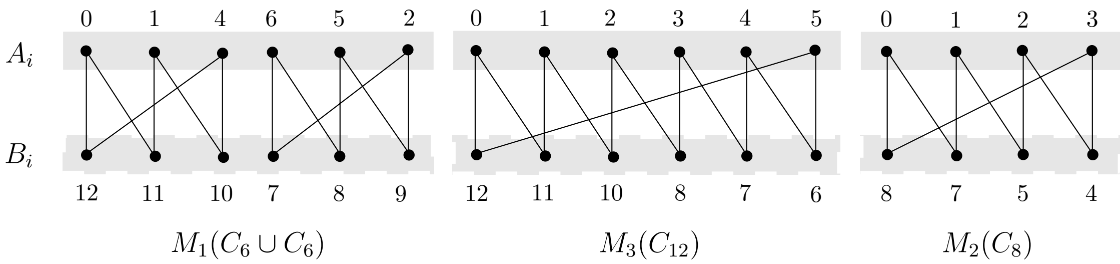


Figure 3.15: A graph $M = M_1 \cup M_2 \cup M_3$ where each of M_i admits an α-labeling

$A_L = A_1 \cup A_2$, $A_R = A_3$, $B_L = B_1 \cup B_2$, and $B_R = B_3$, and let $a = b = 4 = c = d$.

Let $n = 32 + 4 + 4 + 1 = 41$. Define a labeling function $f\colon V(M) \to [4, 40]$ by

$$f(v) = \begin{cases} f_i(v) + \sum\limits_{j=1}^{i-1}(\lambda_j + 1) + 4, & v \in A_1 \cup A_2; \\ f_i(v) + 4, & v \in A_3; \\ f_i(v) + \sum\limits_{j=1}^{i-1}(\lambda_j + 1) + \sum\limits_{j=i+1}^{2} m_j + \sum\limits_{j=2+i}^{3} m_j + 8, & v \in B_1 \cup B_2; \\ f_i(v) + \sum\limits_{j=1}^{i-2} m_j + 8, & v \in B_3. \end{cases}$$

Then we have the graph M that satisfies a new labeling f as shown in Figure 3.16. We can observe that $f|_{A_L \cup B_L}$ and $f|_{A_R \cup B_R}$ are injective. Furthermore, $f(A_L) \cap f(B_R) = \varnothing = f(A_R) \cap f(B_L)$.

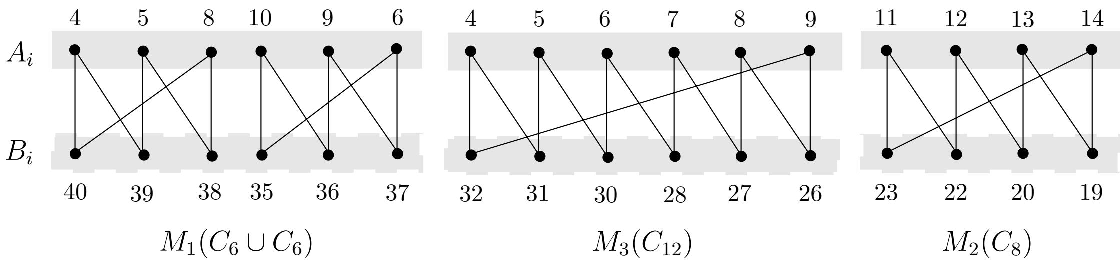


Figure 3.16: A labeling f of $M = M_1 \cup M_2 \cup M_3$

Next, we show how to obtain an embedding of M in $K_{n,n}$.

Lemma 3.18. *Let a, b, c, d, m_i, m, w, M, f, A_L, A_R, B_L and B_R be defined as for Lemma 3.16. Let $n = m+c+d+1$ and let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. Define a labeling function $f'\colon V(M) \to V(K_{n,n})$ by*

$$f'(v) = \begin{cases} f(v)_1, & \text{if } v \in A_L; \\ f(v)_2, & \text{if } v \in A_R; \\ f(v)_2, & \text{if } v \in B_L; \\ f(v)_1, & \text{if } v \in B_R. \end{cases}$$

Then f' is an injective labeling under M. Furthermore, define $\bar{f}' : E(M) \to [0, m + c + d]$ such that if $e = \{v_1, v_2\} \in E(M)$, then $\bar{f}'(e) = f'(v_2) - f'(v_1)$ if $f'(v_2) \geq f'(v_1)$ and $\bar{f}'(e) = n + f'(v_2) - f'(v_1)$, otherwise. Then $\bar{f}'(E(M)) = [c + 1, c + m]$.

Proof. Recall that for $1 \le i \le w$, the set of edge lengths $\bar{f}_i(E(M_i)) = [1, m_i]$, because f_i is an α-labeling of M_i. Also, Lemma 3.16 ensures us that f' is injective. We now consider the set of edge lengths of M under f'. Note that

$$\bar{f}'(E(M_{\lceil \frac{w}{2} \rceil})) = \begin{cases} [c+1, c+m_{\lceil \frac{w}{2} \rceil}], & \text{if } w \text{ odd;} \\ \\ [c+m_w+1, c+m_w+m_{\lceil \frac{w}{2} \rceil}], & \text{if } w \text{ even.} \end{cases}$$

We have edge labels

$$\begin{aligned}
\bar{f}'(E(M_w)) &= n - \left(\bar{f}_w(E(M_w)) + \sum_{j=1}^{w-\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w-1} m_j + d \right) \\
&= \big(\sum_{i=1}^{w} m_j + c + d + 1 \big) - \left([1, m_w] + \sum_{j=1}^{w-\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w-1} m_j + d \right) \\
&= (c+1) + \left(\sum_{j=1}^{w} m_j - \sum_{j=1}^{w-\lceil \frac{w}{2} \rceil} m_j - \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w-1} m_j \right) - [1, m_w] \\
&= (c+1) + \left(\sum_{j=1}^{w} m_j - \sum_{j=1}^{w-\lceil \frac{w}{2} \rceil} m_j - \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w-1} m_j \right) + [0, m_w - 1] - m_w \\
&= c + \left(\sum_{j=w-\lceil \frac{w}{2} \rceil+1}^{w} m_j - \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w-1} m_j \right) + [1, m_w] - m_w \\
&= [1, m_w] + c + \sum_{j=w-\lceil \frac{w}{2} \rceil+1}^{w} m_j - \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} m_j.
\end{aligned}$$

That is

$$\bar{f}'(E(M_w)) = \begin{cases} [c+m_{\lceil \frac{w}{2} \rceil}+1, c+m_{\lceil \frac{w}{2} \rceil}+m_w], & \text{if } w \text{ odd;} \\ \\ [c+1, c+m_w], & \text{if } w \text{ even.} \end{cases}$$

Thus $\bar{f}'(E(M_w)) > \bar{f}'(E(M_{\lceil \frac{w}{2} \rceil}))$ if w is odd and $\bar{f}'(E(M_w)) < \bar{f}'(E(M_{\lceil \frac{w}{2} \rceil}))$ if w is even. Next, for $1 \le i \le \lceil \frac{w}{2} \rceil - 1$, we have edge lengths

$$\begin{aligned}
\bar{f}'(E(M_i)) = \bar{f}_i(E(M_i)) &+ \sum_{j=i+1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+i}^{w} m_j + c \\
&= [1, m_i] + \sum_{j=i+1}^{\lceil \frac{w}{2} \rceil} m_j + \sum_{j=\lceil \frac{w}{2} \rceil+i}^{w} m_j + c.
\end{aligned}$$

Note that $\max\big(\bar{f}'(E(M_1))\big) = c + \sum_{j=1}^{w} m_j = c + m$.

For $1 \le i \le \lceil \frac{w}{2} \rceil - 1$, we have edge labels

$$\begin{aligned}
\bar{f}'(E(M_{\lceil \frac{w}{2}\rceil + i})) &= n - \left(\bar{f}_{\lceil \frac{w}{2}\rceil + i}(E(M_{\lceil \frac{w}{2}\rceil + i})) + \sum_{j=1}^{\lceil \frac{w}{2}\rceil + (i - \lceil \frac{w}{2}\rceil)} m_j + \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i - 1} m_j + d \right) \\
&= \big(\sum_{i=1}^{w} m_j + (c+d) + 1\big) - \left([1, m_{\lceil \frac{w}{2}\rceil + i}] + \sum_{j=1}^{i} m_j + \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i - 1} m_j + d \right) \\
&= (c+1) + \left(\sum_{j=1}^{w} m_j - \sum_{j=1}^{i} m_j - \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i - 1} m_j \right) - [1, m_{\lceil \frac{w}{2}\rceil + i}] \\
&= (c+1) + \left(\sum_{j=1}^{w} m_j - \sum_{j=1}^{i} m_j - \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i - 1} m_j \right) + [0, m_{\lceil \frac{w}{2}\rceil + i} - 1] - m_{\lceil \frac{w}{2}\rceil + i} \\
&= c + \left(\sum_{j=i+1}^{w} m_j - \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i - 1} m_j \right) + [1, m_{\lceil \frac{w}{2}\rceil + i}] - m_{\lceil \frac{w}{2}\rceil + i} \\
&= [1, m_{\lceil \frac{w}{2}\rceil + i}] + c + \sum_{j=i+1}^{w} m_j - \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i} m_j.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{j=i+1}^{\lceil \frac{w}{2}\rceil} m_j + \sum_{j=\lceil \frac{w}{2}\rceil + i}^{w} m_j &= \sum_{j=i+1}^{\lceil \frac{w}{2}\rceil} m_j + \left(\sum_{j=\lceil \frac{w}{2}\rceil + i + 1}^{w} m_j + m_{\lceil \frac{w}{2}\rceil + i} \right) \\
&= \left(\sum_{j=i+1}^{\lceil \frac{w}{2}\rceil} m_j + \sum_{j=\lceil \frac{w}{2}\rceil + i + 1}^{w} m_j \right) + m_{\lceil \frac{w}{2}\rceil + i} \\
&= \sum_{j=i+1}^{w} m_j - \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i} m_j + m_{\lceil \frac{w}{2}\rceil + i},
\end{aligned}$$

we have

$$\bar{f}'(E(M_i)) = [1, m_i] + \max\big(\bar{f}'(E(M_{\lceil \frac{w}{2}\rceil + i}))\big).$$

Since

$$\begin{aligned}
\sum_{j=i+1}^{w} m_j - \sum_{j=\lceil \frac{w}{2}\rceil + 1}^{\lceil \frac{w}{2}\rceil + i} m_j &= \left(\sum_{j=i+1}^{\lceil \frac{w}{2}\rceil} m_j + m_{i+1} \right) + \sum_{j=\lceil \frac{w}{2}\rceil + i + 2}^{w} m_{\lceil \frac{w}{2}\rceil + j} \\
&= \sum_{j=i+1}^{\lceil \frac{w}{2}\rceil} m_j + \sum_{j=\lceil \frac{w}{2}\rceil + i + 2}^{w} m_{\lceil \frac{w}{2}\rceil + j} + m_{i+1},
\end{aligned}$$

we have

$$\bar{f}'(E(M_{\lceil \frac{w}{2} \rceil + i})) = [1, m_{\lceil \frac{w}{2} \rceil + i}] + \max\big(\bar{f}'(E(M_{i+1}))\big).$$

Therefore, all edge lengths of M are distinct because

$$c + m \geq \bar{f}'(E(M_1)) > \bar{f}'(E(M_{\lceil \frac{w}{2} \rceil + 1})) > \bar{f}'(E(M_2)) > \bar{f}'(E(M_{\lceil \frac{w}{2} \rceil + 2})) > \cdots$$
$$\cdots > \bar{f}'(E(M_k)),$$

where $k = \lceil \frac{w}{2} \rceil$ if w is odd and $k = w$ if w is even. Note that

$$\min\big(\bar{f}'(E(M_k))\big) = c + 1.$$

Since $\bar{f}'(E(M_i)) \cap \bar{f}'(E(M_j)) = \varnothing$ for all $i \neq j$, and $|E(M)| = m$, we have $\bar{f}'(E(M)) = [c+1, c+m]$. □

Example 3.19. We illustrate the results from Lemma 3.18 here.

Let $M = M_1 \cup M_2 \cup M_3$ where $M_1 = C_6 \cup C_6$, $M_2 = C_8$ and $M_3 = C_{12}$. Then M is the same graph in Example 3.17 and all vertices of each M_i were labelled as the graph on the top in Figure 3.17. Let a, b, c, d, m_i, m, n, w, f, A_L, A_R, B_L, and B_R be defined as for Example 3.17. Recall that $m = 32$, $w = 3$, $n = 41$, and $a = b = 4 = c = d$. Let $V(K_{41,41}) = \mathbb{Z}_{41} \times [1,2]$ with obvious vertex partition. Define a labeling function $f'\colon V(M) \to V(K_{41,41})$ by

$$f'(v) = \begin{cases} f(v)_1, & \text{if } v \in A_L = A_1 \cup A_2; \\ f(v)_2, & \text{if } v \in A_R = A_3; \\ f(v)_2, & \text{if } v \in B_L = B_1 \cup B_2; \\ f(v)_1, & \text{if } v \in B_R = B_3, \end{cases}$$

By using the labeling f', we can embed M in $K_{41,41}$ as Figure 3.17. Observe that f' is an injective labeling under M. Furthermore, define $\bar{f}' : E(M) \to [0, 40]$ such that if $e = \{v_1, v_2\} \in E(M)$, then $\bar{f}'(e) = f'(v_2) - f'(v_1)$ if $f'(v_2) \geq f'(v_1)$ and $\bar{f}'(e) = n + f'(v_2) - f'(v_1)$, otherwise. Then $\bar{f}'(E(M)) = [5, 36]$.

In Corollary 3.20, we give bounds on the labels of the graph M that is embedded in $K_{n,n}$.

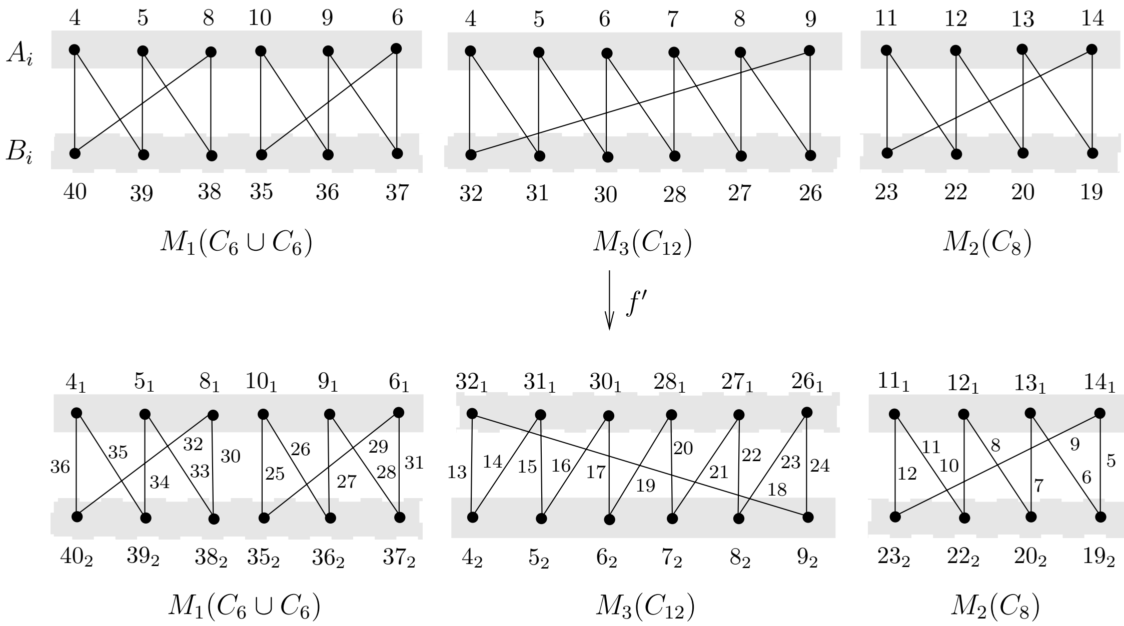


Figure 3.17: An embedding of $M = M_1 \cup M_2 \cup M_3$ in $K_{41,41}$ by using the labeling f'

Corollary 3.20. *Let a, b, c, d, m_i, m, w, M, A_L, A_R, B_L, B_R, f, and f' be defined as for Lemmas 3.16–3.18, and $n = m + c + d + 1$. Let $x = \max\{f'(A_L), f'(A_R)\}$ and $y = \min\{f'(B_L), f'(B_R)\}$. Then $f'(M) \subseteq ([a, x] \cup [y, n-1]) \times [1, 2]$.*

Proof. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. Recall that $f : V(M) \to [a, n-1]$ and $f' : V(M) \to V(K_{n,n})$ such that $f' = f$. Since $f(A_L) < f(B_L)$ and $f(A_R) < f(B_R)$, we have $\max\big(f'(A_L)\big) < \min\big(f'(B_L)\big)$ and $\max\big(f'(A_R)\big) < \min\big(f'(B_R)\big)$. Moreover, in the last part of the proof of Lemma 3.16, we showed that

$$\max\big(f(A_L)\big) < \min\big(f(B_R)\big) \text{ and } \max\big(f(A_R)\big) < \min\big(f(B_L)\big).$$

Thus,

$$\max\big(f'(A_L)\big) = \max\big((f(A_L))_1\big) < \min\big((f(B_R))_1\big) = \min\big(f'(B_R)\big)$$

$$\max\big(f'(A_R)\big) = \max\big((f(A_R))_2\big) < \min\big((f(B_L))_2\big) = \min\big(f'(B_L)\big).$$

We conclude that $x < y$, thus the result follows. $\square$

In the next corollary, we give exact values for the x and y from the the proofs of Lemmas 3.25–3.27. The exactly values of x and y are shown in the next corollary.

Corollary 3.21. *Let a, b, c, d, m_i, m, w, M, A_L, A_R, B_L, B_R, f and f' be defined as for Lemmas 3.16–3.18, and $n = m + c + d + 1$. Let x and y be defined as for Corollary 3.20. Then*

$$x = \max\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a - 1, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + b - 1\},$$

$$y = \begin{cases} \min\{\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + c, \lfloor \frac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + b + d + m_1\}, & \text{if } w \text{ odd;} \\ \min\{\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + c + m_w, \lfloor \frac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + b + d + m_1\}, & \text{if } w \text{ even.} \end{cases}$$

Proof. Since $f' = f$, we can investigate x and y from the function f. From equations (3.1) and (3.2) in the proof of Lemma 3.16, we have

$$\max\big(f(A_L)\big) = \max\big(f(A_{\lceil \frac{w}{2} \rceil})\big) = \lceil \tfrac{w}{2} \rceil - 1 + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a,$$

$$\min\big(f(B_L)\big) = \max\big(f(B_{\lceil \frac{w}{2} \rceil})\big) = \lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + \sum_{j=\lceil \frac{w}{2} \rceil + \lceil \frac{w}{2} \rceil} m_j + c.$$

Note that

$$\min\big(f(B_L)\big) = \begin{cases} \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + c, & \text{if } w \text{ odd;} \\ \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + a + m_w + c, & \text{if } w \text{ even.} \end{cases}$$

In the proof of Lemma 3.16, we have

$$\begin{aligned} \max\big(f(A_R)\big) = \max\big(f(A_{\lceil \frac{w}{2} \rceil+1}\big) &= \lambda_{\lceil \frac{w}{2} \rceil+1} + \sum_{j=\lceil \frac{w}{2} \rceil+2}^{w} (\lambda_j + 1) + b \\ &= \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + b - 1, \end{aligned}$$

$$\begin{aligned}\min\big(f(B_R)\big) &= \min\big(f(B_{\lceil\frac{w}{2}\rceil+1}\big)\\ &= (\lambda_{\lceil\frac{w}{2}\rceil+1}+1)+\sum_{j=\lceil\frac{w}{2}\rceil+2}^{w}(\lambda_j+1)+b+\sum_{j=1}^{(\lceil\frac{w}{2}\rceil+1)-\lceil\frac{w}{2}\rceil} m_j+\sum_{j=\lceil\frac{w}{2}\rceil+1}^{(\lceil\frac{w}{2}\rceil+1)-1} m_j+d\\ &= \lfloor\tfrac{w}{2}\rfloor+\sum_{j=\lceil\frac{w}{2}\rceil+1}^{w}\lambda_j+b+d+m_1.\end{aligned}$$

Thus,

$$x=\max\{\lceil\tfrac{w}{2}\rceil+\sum_{j=1}^{\lceil\frac{w}{2}\rceil}\lambda_j+a-1,\lfloor\tfrac{w}{2}\rfloor+\sum_{j=\lceil\frac{n}{2}\rceil+1}^{w}\lambda_j+b-1\},$$

$$y=\begin{cases}\min\{\lceil\frac{w}{2}\rceil+\sum_{j=1}^{\lceil\frac{w}{2}\rceil}\lambda_j+a+c,\lfloor\frac{w}{2}\rfloor+\sum_{j=\lceil\frac{w}{2}\rceil+1}^{w}\lambda_j+b+d+m_1\}, & \text{if } w \text{ odd;}\\ \min\{\lceil\frac{w}{2}\rceil+\sum_{j=1}^{\lceil\frac{w}{2}\rceil}\lambda_j+a+c+m_w,\lfloor\frac{w}{2}\rfloor+\sum_{j=\lceil\frac{w}{2}\rceil+1}^{w}\lambda_j+b+d+m_1\}, & \text{if } w \text{ even.}\end{cases}$$

This concludes the proof. □

Lemma 3.22. *Let M of size m be the vertex-disjoint union of 2-regular bipartite graphs, each of which admits an α-labeling. Let G be the vertex-disjoint union of the graph M and a path P of size $2\ell-1$, where ℓ is a positive integer. Let $n=m+2\ell+1$ and let $K_{n,n}$ have vertex set $\mathbb{Z}_n\times[1,2]$ with the obvious vertex partition. Then there exists an embedding of G in $K_{n,n}$ with one edge of each length in $[-(n-1)/2,(n-1)/2]\setminus\{\pm z\}$ for some $z\in[1,(n-1)/2]$ and such that the endpoints of P are j_1 and j_2 for some $j\in[0,n-1]$.*

Proof. Let $M=M_1\cup M_2\cup\cdots\cup M_w$ such that each M_i admits an α-labeling f_i with critical value λ_i and $\lambda_1\geq\lambda_{\lceil\frac{w}{2}\rceil+1}\geq\lambda_2\geq\lambda_{\lceil\frac{w}{2}\rceil+2}\geq\cdots$. Let a, b, c, d, f, f', and $\bar{f}'$ be defined as for Lemmas 3.16–3.18 and $n=m+2\ell+1$. Let $V(K_{n,n})=\mathbb{Z}_n\times[1,2]$ with the obvious vertex bipartition, and assume that $a=b=c=d=\ell$. We will embed G in $K_{n,n}$ by giving embeddings of both M and P. To embed M, define a labeling function

$$h\colon V(M)\to[\ell,n-1],\ h'\colon V(M)\to V(K_{n,n}),\ \bar{h}':E(M)\to[0,n-1]$$

by $h=f$, $h'=f'$ and $\bar{h}'=\bar{f}'$. Then by Lemma 3.18, h' is an injective labeling of M and $h'(V(M))\subseteq[\ell,n-1]\times[1,2]$. Furthermore $\bar{h}'(E(M))=[\ell+1,\ell+m]=\pm[l+1,m/2+\ell]$.

By Lemma 3.3, there exists an embedding of P of size $2\ell-1$ by using the edge lengths in $\{0\}\cup\pm[1,\ell-1]$ with endpoints 0_1 and 0_2. In the lemma, we would use $d_1=1, d_2=2,\ldots,d_{\ell-1}=\ell-1$, so $V(P)\subseteq[0,\ell-1]\times[1,2]$.

Thus, M and P are vertex disjoint and the edge set of G has one edge of each length $i\in[-(m/2+\ell),m/2+\ell]\setminus\{\pm\ell\}$. $\square$

Example 3.23. We illustrate the results from Lemma 3.22 here.

Let G be the vertex-disjoint union of C_{12}, C_6, C_6, C_8, and P_8. Let $M_1=C_6\cup C_6$, $M_2=C_8$ and $M_3=C_{12}$. Then by Theorem 2.7 and Theorem 2.8, each M_i admits an α-labeling f_i with critical value λ_i and vertex bipartition $\{A_i,B_i\}$ (see Figure 3.15). Then $\lambda_1\geq\lambda_3\geq\lambda_2$. Let a, b, c, d, f, f' and $\bar{f}'$ be defined as for Lemmas 3.16–3.18. Let $M=M_1\cup M_1\cup M_3$. Then $G=M\cup P_8$. Since $\ell=2$, we have $a=b=2=c=d$ and $n=m+2\ell+1=41$. The vertex labeling f' of M is shown in Figure 3.17. Let $V(K_{41,41})=\mathbb{Z}_{41}\times[1,2]$ with obvious vertex partition. To embed M in $K_{41,41}$, define labeling functions

$$h\colon\ V(M)\to[4,40],\ h'\colon\ V(M)\to V(K_{41,41}),\ \bar{h}':E(M)\to[0,40]$$

by $h=f$, $h'=f'$ and $\bar{h}'=\bar{f}'$. Note that the set of edge lengths of M is $\bar{h}'(E(M))=[\ell+1,\ell+m]=\pm[l+1,m/2+\ell]=\pm[5,20]$.

By Lemma 3.3, there exists an embedding P of P_8 such that $V(P)\subseteq[0,3]\times[1,2]$ and with edge lengths set $\{0\}\cup\pm[1,3]$. Thus G can be embedded in $K_{41,41}$ with the edge set of G having one edge of each length $i\in[-20,20]\setminus\{\pm4\}$ (see Figure 3.18).

Theorem 3.24. *Let M be a 2-regular bipartite graph of order $m\equiv 0 \pmod 4$. Let G be the disjoint union of M and a cycle C of size $2\ell+1$ where ℓ is a positive integer and $n=m+2\ell+1$. Then there exists a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k and of $K_{k'\times 2n}$ for all integers $k'\geq 3$.*

Proof. Since $m\equiv 0\pmod 4$, the graph M is the union of graphs that admit α-labelings. Combine with Lemma 3.22 and Lemma 3.1, we obtain a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k. By Lemma 3.22 and Lemma 3.2, a G-decomposition of $K_{k'\times n}$ exists for all integers $k'\geq 3$. $\square$

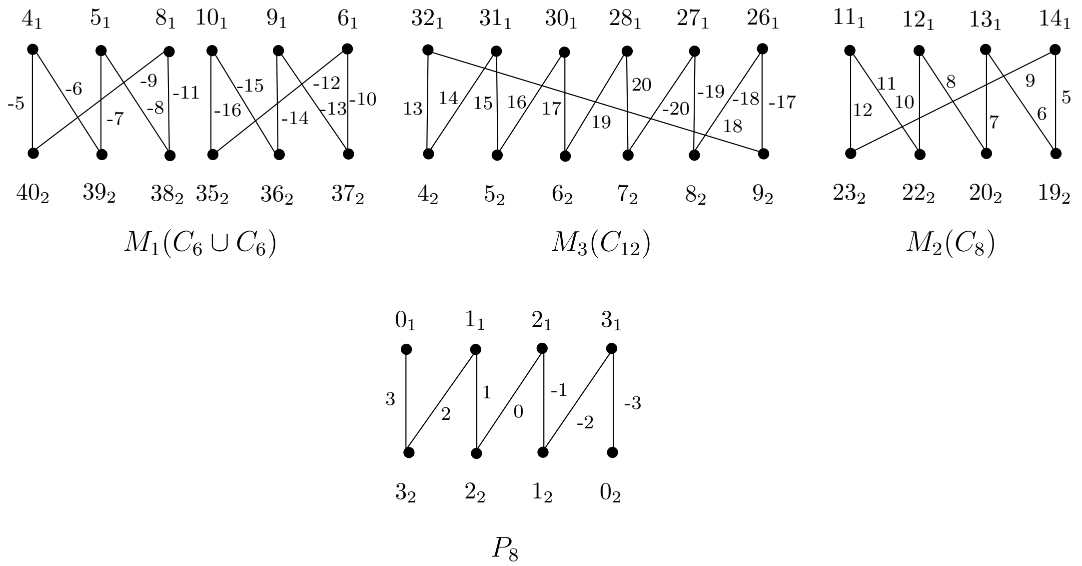


Figure 3.18: An embedding of $G = C_{12} \cup C_6 \cup C_6 \cup C_8 \cup P_8$ in $K_{41,41}$

Next, we focus on the case when the number of cycles of order 2 (mod 4) in G is odd.

Lemma 3.25. *Let M of size m be the vertex-disjoint union of 2-regular bipartite graphs that admit α-labeling. Let G be the vertex-disjoint union of M, a cycle C of size $m' \equiv 2 \pmod 4$ and a path P of size 1. Let $n = m + m' + 3$ and let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1,2]$ with the obvious vertex partition. Then there exists an embedding of G in $K_{n,n}$ with one edge of each length in $[-(n-1)/2, (n-1)/2] \setminus \{\pm z\}$ for some $z \in [1, n-1]$ and such that the endpoints of P are j_1 and j_2 for some $j \in [0, n-1]$.*

Proof. Let $M = M_1 \cup M_2 \cup \cdots \cup M_w$ such that M_i admits an α-labeling f_i with critical value λ_i and vertex bipartition $\{A_i, B_i\}$ where $\lambda_1 \geq \lambda_{\lceil \frac{w}{2} \rceil + 1} \geq \lambda_2 \geq \lambda_{\lceil \frac{w}{2} \rceil + 2} \geq \cdots$. Let a, b, c, d, f, f', $\bar{f}'$, A_L, A_R, B_L and B_R be defined as for Lemmas 3.16–3.18 and $n = m + m' + 3$. Let $V(K_{n,n}) = \mathbb{Z}_n \times [1,2]$ with the obvious vertex bipartition. We will embed the graph G in $K_{n,n}$, consisting of the graph M of size m, the cycle C of size m' and the path P of size 1.

Case 1. Suppose that w is even. Assume that $a = 1 = b$ and $c = \frac{m'}{2} + 1 = d$.

Let x and y be defined as for Corollary 3.21. Then

$$x = \max\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j\} = \lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j,$$

$$y = \min\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 2 + \tfrac{m'}{2} + m_w, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 2 + \tfrac{m'}{2} + m_1\}.$$

Define a labeling function

$$h\colon V(M) \to [1, n-1], h'\colon V(M) \to V(K_{n,n}), \bar{h}' : E(M) \to [0, n-1]$$

by $h = f$, $h' = f'$ and $\bar{h}' = \bar{f}'$. Then by Lemma 3.18, h' is a injective labeling under M and $h'(M) \subseteq ([1, x] \cup [y, n-1]) \times [1, 2]$. Furthermore M has a set of edge lengths

$$\bar{h}'(E(M)) = [\tfrac{m'}{2} + 2, m + \tfrac{m'}{2} + 1] = \pm[\tfrac{m'}{2} + 2, \tfrac{m}{2} + \tfrac{m'}{2} + 1].$$

By Lemma 3.3, there exists an embedding of the path P of size 1 using edge length 0 with endpoints 0_1 and 0_2. Next, we will embed the graph M of size m and the cycle C of size m' by considering the congruence class of m modulo 8.

Case 1.1. Suppose that $m' \equiv 2 \pmod 8$. Let $m' = 8t + 2$ for some positive integer t. Recall that M has a set of edge lengths

$$\bar{h}'(E(M)) = \pm[\tfrac{m'}{2} + 2, \tfrac{m}{2} + \tfrac{m'}{2} + 1],$$

$$y = \min\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 4t + 3 + m_w, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 4t + m_1 + 3\}.$$

By Lemma 3.8, there exists an embedding of a cycle C^* of size m' with edge lengths

$$\pm[2, 4t + 2] = \pm[2, \tfrac{m'}{2} + 1].$$

Furthermore, $V(C^*) \subseteq [0, 4t + 2] \times [1, 2]$. Let $C = C^* + (x + 1)$. Then $V(C) \subseteq [x + 1, 4t + 2 + (x + 1)] \times [1, 2]$. By using Lemma 3.15, note that $4t + 2 + (x + 1) = 4t + 3 + \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j < y$.

Thus the edge set of G has one edge of each length $i \in [-(\frac{m}{2} + \frac{m'}{2} + 1), \frac{m}{2} + \frac{m'}{2} + 1] \setminus \{\pm 1\}$.

Case 1.2. Suppose that $m' \equiv 6 \pmod 8$. Let $m' = 8t+6$ for some nonnegative integer t.

Subcase 1.2(a). $t = 0$. Then M has a set of edge lengths

$$\bar{h}'(E(M)) = \pm[5, \tfrac{m}{2} + 4].$$

Recall that

$$y = \min\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + m_w + 5, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + m_1 + 5\}.$$

To embed C of size m' in $K_{n,n}$, let $C^* = \langle 0_1, 3_2, 2_1, 0_2, 3_1, 2_2 \rangle$. Its lengths are $3, -1, 2, -3, 1, -2$. Let $C = C^* + (x+1)$. Then $V(C) \subseteq [x+1, 3+(x+1)] \times [1, 2]$. By Lemma 3.15, note that $3 + (x+1) = \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 4 < y$.

Thus, M, C and P are vertex-disjoint, and the edge set of G has one edge of each length $i \in [-(\frac{m}{2} + 4), \frac{m}{2} + 4] \setminus \{\pm 4\}$.

Subcase 1.2(b). $t \geq 1$. Then M has a set of edge lengths

$$\bar{h}'(E(M)) = \pm[\tfrac{m'}{2} + 2, \tfrac{m}{2} + \tfrac{m'}{2} + 1],$$

$$y = \min\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 4t + 5 + m_w, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 4t + 5 + m_1\}.$$

By Lemma 3.9, there exists an embedding of a cycle C^* of size m' with edge lengths is

$$\pm[1, 4t+4)] \setminus \{\pm 2\}] = \pm[1, \tfrac{m'}{2} + 1] \setminus \{\pm 2\}.$$

Furthermore, $V(C^*) \subseteq [0, 4t+4] \times [1, 2]$. Let $C = C^* + 1$. Then $V(C) \subseteq [x+1, 4t+4+(x+1)] \times [1, 2]$. By Lemma 3.15, note that $4t + 4 + (x+1) = 4t + 5 + \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j < y$.

Thus the edge set of G has one edge of each length $i \in [-(\frac{m}{2} + \frac{m'}{2} + 1), \frac{m}{2} + \frac{m'}{2} + 1] \setminus \{\pm 2\}$.

Case 2. Suppose that w is odd. Let $m' = 4t + 2$ for some positive integer t, and assume that $a = 2 = d$, $b = 2t + 2$ and $c = 4t + 2$. By Corollary 3.21, we have

$$x = \max\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 1, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{n}{2} \rceil+1}^{w} \lambda_j + 2t + 1\},$$

$$y = \min\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 4t + 4, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil + 1}^{w} \lambda_j + 2t + 4 + m_1\}$$

.

Define a labeling function

$$h\colon V(M) \to [2, n-1], h'\colon V(M) \to V(K_{n,n}), \bar{h}' : E(M) \to [0, n-1]$$

by $h = f$, $h' = f'$ and $\bar{h}' = \bar{f}'$. Then by Lemma 3.18, h' is a injective labeling under M. Furthermore, M has a set of edge lengths

$$\bar{h}'(E(M)) = [4t+3, 4t+2+m] = [m'+1, \tfrac{m}{2} + \tfrac{m'}{2} + 1] \cup [-(\tfrac{m}{2} + \tfrac{m'}{2} + 1), -3].$$

To embed C of size $4t+2$ in $K_{n,n}$, let

$$C = G_1 + G_2 + \big(2_2, 1_1, 0_2, (2t+4+m)_1\big)$$

where

$$G_1 = P\big(m + 2t + 4, 0, 2t + 1\big),$$
$$G_2 = Q\big(m + 3t + 6, -(t-3), 2t-2\big).$$

We then show that $G_1 + G_2 + \big(2_2, 1_1, 0_2, (m+2t+4)_1\big)$ is a cycle of size $4t+2$. Note that by **P1** and **Q1**, the first vertex of G_1 is $(m+2t+4)_1$, and the last vertex is $(t+1)_2$; the first vertex of G_2 is $(t+1)_2$, and the last vertex is 2_2. For $1 \le i \le 2$, let $A_i(C)$ and $B_i(C)$ denote the sets labeled A' and B' in **P2** and **Q2**, we compute

$$A_1(C) = [(m+2t+4)_1, (m+3t+4)_1], \qquad B_1(C) = [(t+1)_2, (2t+1)_2],$$
$$A_2(C) = [(m+3t+6)_1, (m+4t+4)_1], \qquad B_2(C) = [2_2, (t+1)_2].$$

Thus,

$$A_1(C) < A_2(C) \text{ and } B_2(C) \le B_1(C).$$

Note that $V(G_1) \cap V(G_2) = \{(t+1)_2\}$; otherwise, G_1 and G_2 are vertex disjoint. Therefore, $G_1 + G_2 + \big(2_2, 1_1, 0_2, (m+2t+4)_1\big)$ is a cycle of size $4t+2$.

Next, let $E_i(C)$ denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3** and **Q3**, we have edge lengths

$$E_1(C) = [-(m+2t+3), -(m+3)] = [\tfrac{m'}{2}+1, m'],$$
$$E_2(C) = [-(m+4t+2), -(m+2t+5)] = [3, \tfrac{m'}{2}-1].$$

Moreover, the path $\big(2_2, 1_1, 0_2, (m+2t+4)_1\big)$ consists of edges lengths 1, -1, and $-(m+2t+4) = 2t+1 = \frac{m'}{2}$.

Note that in this case

$$\min\big(h'(A_L)\big) = \min\big(h'(A_1)\big) = 2$$

$$\max\big(h'(B_w)\big) = m_w + b + \sum_{j=1}^{\lfloor \frac{w}{2} \rfloor} m_j + \sum_{j=\lceil \frac{w}{2}+1 \rceil}^{w-1} m_j + d < b + d + m = 2t+4+m.$$

Also,

$$\min\big(h'(A_R)\big) = \min\big(h'(A_w)\big) = b = 2t+2,$$
$$\max\big(h'(B_L)\big) = \max\big(h'(B_1)\big) = m+a+c = m+4t+4 = n-1.$$

Since

$$2 = \min\big(h'(A_L)\big) < \max\big(h'(B_R)\big) < 2t+4+m = \min\big(A_1(C) \cup A_2(C)\big),$$

$$n-1 = \max\big(h'(B_L)\big) > \min\big(h'(A_R)\big) = 2t+2 > 2t+1 = \max\big(B_1(C) \cup B_2(C)\big),$$

we have M and C are vertex disjoint.

By Lemma 3.3, there exists an embedding of a path P^* of size 1 using edge length 0 with endpoints 0_1 and 0_2. Let $P = P^* + (x+1)$ with endpoints $(x+1)_1$ and $(x+1)_2$. Note that $2t+2 < 4t+3 = n$. Thus, the edge set of G has one edge of each length $i \in [-2t, 2t]$, except the edge lengths $\pm(2t+1)$. Since $x+1 < y$, we have P is vertex disjoint from M and C. Thus, the edge set of G has one edge of each length $i \in [-(\frac{m}{2}+\frac{m'}{2}+1), \frac{m}{2}+\frac{m'}{2}+1] \setminus \{\pm 2\}$. $\square$

Example 3.26. We illustrate the results from Lemma 3.25 here.

Let G be the vertex-disjoint union of C_{10}, C_8, C_8, C_4, C_4, C_4 and P_2. Let $M_1 = C_8$, $M_2 = C_4$, $M_3 = C_4$, $M_4 = C_8$ and $M_5 = C_4$. Then by Theorem 2.7, each

M_i admits an α-labeling f_i with critical value λ_i and vertex bipartition $\{A_i, B_i\}$. Figure 3.19 illustrates the α-labeling of M_i. Then $\lambda_1 \geq \lambda_4 \geq \lambda_2 \geq \lambda_5 \geq \lambda_3$. Let a, b, c, d, f, f', $\bar{f}'$, A_L, A_R, B_L and B_R be defined as for Lemmas 3.16–3.18. Let $M = M_1 \cup M_2 \cup \cdots \cup M_5$. Then $G = M \cup C_{10} \cup P_2$. Note that $m' = 10$, $m = 28$ and $t = 2$. Then $n = m + m' + 3 = 41$. Since $10 \equiv 2 \pmod 8$ and M consists of five subgraphs M_i, we need to use Case 2 in Lemma 3.25 to embed G in $K_{41,41}$. In this case, we assume that $a = 2 = d$, $b = 2t + 2 = 6$ and $c = 4t + 2 = 10$.

To embed M in $K_{41,41}$, define a labeling function

$$h\colon\ V(M) \to [2, 40],\ \ h'\colon\ V(M) \to V(K_{41,41}),\ \ \bar{h}' : E(M) \to [0, 40]$$

by $h = f$, $h' = f'$ and $\bar{h}' = \bar{f}'$. Then the vertices of M are labelled as in Figure 3.20. Note that $V(M) \subseteq ([2, 11] \cup [20, 40]) \times [1, 2]$ and the set of edge lengths of M is $\bar{h}'(E(M)) = [11, 20] \cup [-20, -3]$.

To embed C_{10} in $K_{41,41}$, let $C = G_1 + G_2 + \big(2_2, 1_1, 0_2, 36_1\big)$ be an embedding of C_{10} where $G_1 = P(36, 0, 5)$ and $G_2 = Q(40, 1, 2)$. For $1 \leq i \leq 2$, let $A_i(C)$ and $B_i(C)$ denote the sets labeled A' and B' in **P2** and **Q2**, we compute

$$\begin{aligned}
A_1(C) &= [(m + 2t + 4)_1, (m + 3t + 4)_1] = [36_1, 38_1],\\
B_1(C) &= [(t + 1)_2, (2t + 1)_2] = [3_2, 5_2],\\
A_2(C) &= [(m + 3t + 6)_1, (m + 4t + 4)_1] = \{40_1\},\\
B_2(C) &= [2_2, (t + 1)_2] = [2_2, 3_2].
\end{aligned}$$

Then $V(C) = (([36, 40] \cup \{1\}) \times \{1\}) \cup (([2, 5] \cup \{0\}) \times \{2\})$ and the set of edge lengths of C is $[6, 10] \cup [3, 4] \cup \{-1, 1, 5\}$. Thus, C is vertex disjoint from M.

Let x and y be defined as for Corollary 3.20. Then we can note that

$$x = \max\Big(\bigcup_{i=1}^{5} h'(A_i)\Big) = 11 \text{ and } y = \min\Big(\bigcup_{i=1}^{5} h'(B_i)\Big) = 20$$

By Lemma 3.3, there exists an embedding P^* of P_2 in $K_{41,41}$ with endpoints 0_1 and 0_2, and its edge length 0. Let $P = P^* + (x + 1) = P^* + 12$ with endpoints 12_1 and 12_2. Hence, G can be embedded in $K_{41,41}$ so that the edge set of G has one edge of each length $i \in [-20, 20] \setminus \{\pm 2\}$.

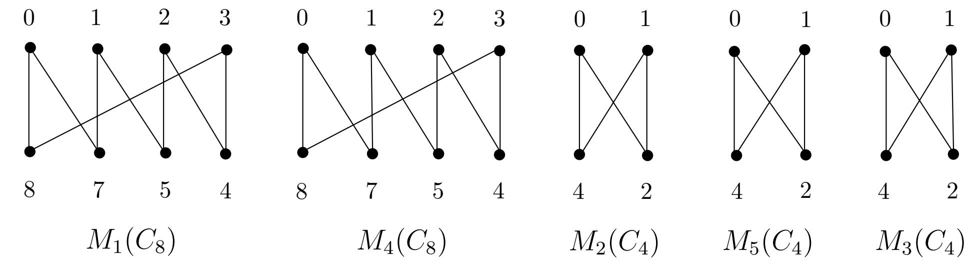


Figure 3.19: A graph $M = M_1 \cup M_2 \cup \cdots \cup M_5$ where each M_i admits an α-labeling.

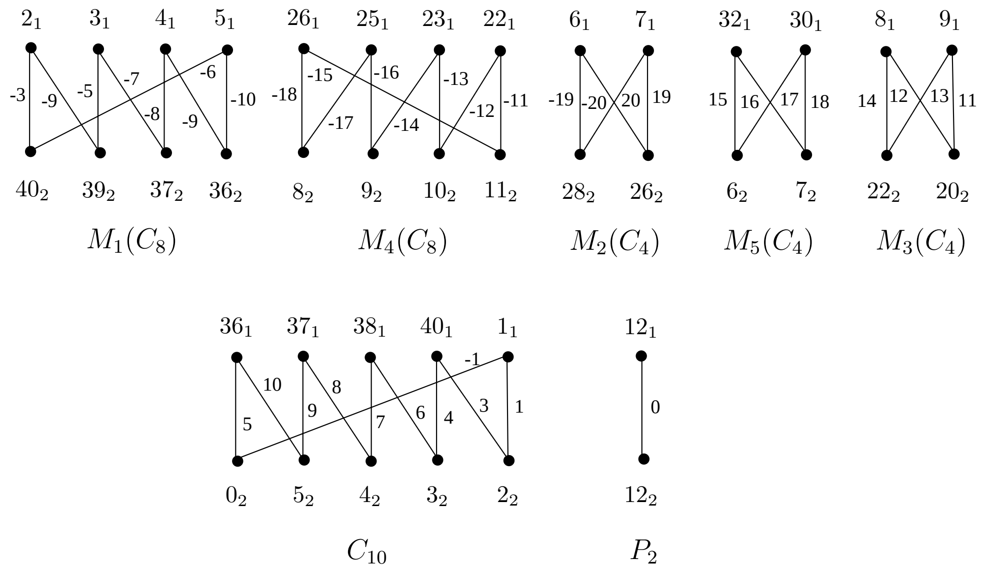


Figure 3.20: An embedding of $G = C_8 \cup C_8 \cup C_4 \cup C_4 \cup C_4 \cup C_{10} \cup P_2$ in $K_{41,41}$

Lemma 3.27. *Let M of size m be the vertex-disjoint union of any 2-regular bipartite graphs, each of which admits an α-labeling. Let G be the vertex-disjoint union of M, a cycle C of size m' and a path P of size $2\ell - 1$ where $m' \equiv 2 (mod\ 4)$ and $\ell \geq 2$ is an integer. Let $n = m + m' + 2\ell + 1$ and let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times [1, 2]$ with the obvious vertex partition. Then there exists an embedding of G in $K_{n,n}$ with one edge of each length in $[-(n-1)/2, (n-1)/2] \setminus \{\pm z\}$ for some $z \in [1, (n-1)/2]$ and such that the endpoints of P are j_1 and j_2 for some $j \in [0, n-1]$.*

Proof. Let $M = M_1 \cup M_2 \cup \cdots \cup M_w$ such that M_i admits an α-labeling f_i with critical value λ_i and vertex bipartition $\{A_i, B_i\}$ where $\lambda_1 \geq \lambda_{\lceil \frac{w}{2} \rceil + 1} \geq \lambda_2 \geq \lambda_{\lceil \frac{w}{2} \rceil + 2} \geq \cdots$. Let a, b, c, d, f, f', $\bar{f}'$, A_L, A_R, B_L and B_R be defined as for Lemmas 3.16-

3.18 and $n = m + m' + 2\ell + 1$. Let $V(K_{n,n}) = \mathbb{Z}_n \times [1,2]$ with the obvious vertex bipartition. We will embed the graph G, consisting of the graph M of size m, the cycle C of size m' and the path P of size $2\ell - 1$.

Case 1. $m' = 6$. To embed M of size m in $K_{n,n}$, assume that $a = b = c = d = \ell + 3$. Define a labeling function

$$h\colon\ V(M) \to [\ell+3, n-1], h'\colon\ V(M) \to V(K_{n,n}), \bar{h}' : E(M) \to [0, n-1]$$

by $h = f$, $h' = f'$ and $\bar{h}' = \bar{f}'$. Then by Lemma 3.18, h' is a injective labeling under M and $h'(M) \subseteq ([\ell+3, x] \cup [y, n-1]) \times [1,2]$. Furthermore M has a set of edge lengths

$$\bar{h}'(E(M)) = [\ell+4, m+\ell+3] = \pm[\ell+4, \tfrac{m}{2}+\ell+3].$$

Let x and y be defined as for Corollary 3.21. Then

$$x = \max\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + 2, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + \ell + 2\} = \lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + 2,$$

$$y = \begin{cases} \min\{\lceil \frac{w}{2} \rceil + \sum\limits_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 2\ell + 6, \lfloor \frac{w}{2} \rfloor + \sum\limits_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 2\ell + 6 + m_1\}, & \text{if } w \text{ odd;} \\ \min\{\lceil \frac{w}{2} \rceil + \sum\limits_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 2\ell + 6 + m_w, \lfloor \frac{w}{2} \rfloor + \sum\limits_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 2\ell + 6 + m_1\}, & \text{if } w \text{ even.} \end{cases}$$

To embed C of size 6 $K_{n,n}$, let $C^* = \langle 0_1, 3_2, 2_1, 0_2, 3_1, 2_2 \rangle$. Its edge lengths are $3, -1, 2, -3, 1, -2$. Let $C = C^* + (x+1)$. Then $V(C) \subseteq [x+1, 3+(x+1)] \times [1,2]$. By Lemma 3.15 and $\ell \geq 2$, we have $3 + (\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + 3) = \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + 6 < y$.

By Lemma 3.3, there exists an embedding of P of size $2\ell - 1$ using the edge lengths in $\{0\} \cup \pm\{4, \ell+2\}$ with endpoints 0_1 and 0_2. In the lemma we would use $d_1 = 4, d_2 = 5, \ldots, d_{\ell-1} = \ell + 2$, so $V(P) \subseteq [0, \ell+2] \times [1,2]$.

Thus the edge set of G has one edge of each length $i \in [-(\frac{m}{2} + \ell + 3), \frac{m}{2} + \ell + 3] \setminus \{\pm(\ell+3)\}$

Case 2. $m' \geq 10$. First we will embed a graph M in $K_{n,n}$, assume that $a = b = c = d = \ell + \frac{m'}{2}$.

Let x and y be defined as for Corollary 3.21. Then

$$\begin{aligned} x &= \max\{\lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + \tfrac{m'}{2} - 1, \lfloor \tfrac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + \ell + \tfrac{m'}{2} - 1\} \\ &= \lceil \tfrac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + \tfrac{m'}{2} - 1, \end{aligned}$$

$$y = \begin{cases} \min\{\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 2\ell + m', \lfloor \frac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 2\ell + m' + m_1\}, & \text{if } w \text{ odd;} \\ \min\{\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + 2\ell + m' + m_w, \lfloor \frac{w}{2} \rfloor + \sum_{j=\lceil \frac{w}{2} \rceil+1}^{w} \lambda_j + 2\ell + m' + m_1\}, & \text{if } w \text{ even.} \end{cases}$$

Define a labeling function

$$h\colon\ V(M) \to [\ell + \tfrac{m'}{2}, n-1], h'\colon\ V(M) \to V(K_{n,n}), \bar{h}' : E(M) \to [0, n-1]$$

by $h = f$, $h' = f'$ and $\bar{h}' = \bar{f}'$. Then by Lemma 3.18, h' is a injective labeling under M and $h'(M) \subseteq ([\ell + \frac{m'}{2}, x] \cup [y, n-1]) \times [1, 2]$. Furthermore M has a set of edge lengths

$$\bar{h}'(E(M)) = [\ell + \tfrac{m'}{2} + 1, \ell + \tfrac{m'}{2} + m] = \pm[\ell + \tfrac{m'}{2} + 1, \ell + \tfrac{m'}{2} + \tfrac{m}{2}].$$

For an embedding of the remaining graphs C of size m' and P of size $2\ell - 1$ in $K_{n,n}$, we will consider 2 cases.

Case 2.1. $m' \equiv 2 \pmod 8$. Let $m' = 8t + 2$ for some positive integer t. By Lemma 3.8, there exists an embedding of a cycle C^* of size m' with edge lengths

$$\pm[1, 4t + 2] = \pm[1, \tfrac{m'}{2} + 1]$$

and $V(C^*) \subseteq [0, 4t + 2] \times [1, 2]$. Let $C = C^* + (x + 1)$. Then $V(C) \subseteq [x + 1, 4t + 2 + (x + 1)] \times [1, 2]$. Note that by using Lemma 3.15, we have $4t + 2 + (x + 1) = \frac{m'}{2} + 1 + (\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + \frac{m'}{2}) = \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + m' + 1 < y$. Thus, C is vertex disjoint from M.

By Lemma 3.3, there exists an embedding of P of size $2\ell - 1$ using the edge lengths in $\{-1, 0, 1\} \cup \pm[\frac{m'}{2} + 2, \ell + \frac{m'}{2} - 1]$ with endpoints 0_1 and 0_2. In the lemma we would use $d_1 = 1, d_2 = \frac{m'}{2} + 2, \ldots, d_{\ell-1} = \ell + \frac{m'}{2} - 1$, so $V(P) \subseteq [0, \ell + \frac{m'}{2} - 1] \times [1, 2]$.

Thus, M, P and C are vertex disjoint, and the edge set of G has one edge of each length $i \in [-(\frac{m}{2} + \frac{m'}{2} + \ell), \frac{m}{2} + \frac{m'}{2} + \ell] \setminus \{\pm(\frac{m'}{2} + \ell)\}$.

Case 2.2. Suppose that $m' \equiv 6 (\mathrm{mod}\ 8)$. Let $m' = 8t + 6$ for some positive integer t. By Lemma 3.9, there exists an embedding of a cycle C^* of length m' with edge lengths

$$\pm[1, 4t + 4] \setminus \{\pm 2\} = \pm[1, \frac{m'}{2} + 1] \setminus \{\pm 2\}$$

and $V(C^*) \subseteq [0, 4t + 4] \times [1, 2]$. Let $C = C^* + (x + 1)$. Then $V(C) \subseteq [x + 1, 4t + 4 + (x + 1)] \times [1, 2]$. Note that by using Lemma 3.15, we have $4t + 4 + (x + 1) = \frac{m'}{2} + 1 + (\lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + \frac{m'}{2}) = \lceil \frac{w}{2} \rceil + \sum_{j=1}^{\lceil \frac{w}{2} \rceil} \lambda_j + \ell + m' + 1 < y$.

By Lemma 3.3, there exists an embedding of P of size $2\ell - 1$ using the edge lengths in $\{0\} \cup \{\pm 2\} \cup \pm[\frac{m'}{2} + 2, \frac{m'}{2} + \ell - 1]$ with endpoints 0_1 and 0_2. In the lemma we would use $d_1 = 2, d_2 = \frac{m'}{2} + 2, \ldots, d_{\ell-1} = \ell + \frac{m'}{2} - 1$, so $V(P) \subseteq [-(\ell + \frac{m'}{2} - 1), \ell + \frac{m'}{2} - 1] \times [1, 2]$.

Thus the edge set of G has one edge of each length $i \in \pm[-(\frac{m}{2} + \frac{m'}{2} + \ell), \frac{m}{2} + \frac{m'}{2} + \ell] \setminus \{\pm(\frac{m'}{2} + \ell)\}$. $\square$

Example 3.28. We illustrate the results from Lemma 3.27 here.

Let G be the vertex-disjoint union of C_{14}, C_{12}, C_8, C_6, C_6, C_4 and P_4. Let $M_1 = C_6 \cup C_6$, $M_2 = C_8$, $M_3 = C_{12}$ and $M_4 = C_4$. Then by Theorem 2.7 and Theorem 2.8, M_i admits an α-labeling f_i with critical value λ_i and vertex bipartition $\{A_i, B_i\}$. Figure 3.21 illustrates the α-labeling of M_i. Then $\lambda_1 \geq \lambda_3 \geq \lambda_2 \geq \lambda_4$. Let a, b, c, d, f, f', $\bar{f}'$, A_L, A_R, B_L and B_R be defined as for Lemmas 3.16–3.18. Let $M = M_1 \cup M_2 \cup M_3 \cup M_4$. Then $G = M \cup C_{14} \cup P_4$. Note that $m' = 14$, $m = 36$ and $\ell = 2$. Then $n = m + m' + 2\ell + 1 = 55$. Since $14 \equiv 6$ (mod 8), we need to use Case 2.2 in Lemma 3.27 to embed G in $K_{55,55}$. In this case, we assume that $a = b = c = d = 9$.

To embed M in $K_{55,55}$, define a labeling function

$$h\colon V(M) \to [9, 54],\ h'\colon V(M) \to V(K_{55,55}),\ \bar{h}' : E(M) \to [0, 54]$$

by $h = f$, $h' = f'$ and $\bar{h}' = \bar{f}'$. Then the vertices of M are labelled as in Figure 3.21. Note that the set of edge lengths of M is $\bar{h}'(E(M)) = \pm[10, 27]$.

Let x and y be be defined as for Corollary 3.20. Then we can observe that

$$x = \max\Big(\bigcup_{i=1}^{4} h'(A_i)\Big) = 19 \text{ and } y = \min\Big(\bigcup_{i=1}^{4} h'(B_i)\Big) = 32.$$

Note that $V(M) \subseteq ([9, 19] \cup [32, 54] \times [1, 2]$.

By Lemma 3.9, there exists an embedding of a cycle C^* of size 14 with edge lengths $\pm[1, 8] \smallsetminus \{\pm 2\}$, and $V(C^*) \subseteq [0, 8] \times [1, 2]$. Let $C = C^* + (x+1) = C^* + 20$ be an embedding of C_{14}. Note that $V(C) \subseteq [20, 28] \times [1, 2]$. Thus C is vertex disjoint from M.

Finally, by Lemma 3.3, there exists an embedding P of P_4 using the edge lengths in $\{-2, 0, 2\}$ with endpoints 0_1 and 0_2, and $V(P) \subseteq [0, 2] \times [1, 2]$.

Hence, G can be embedded in $K_{n,n}$ with edge set of G has one edge of each length $i \in [-27, 27] \smallsetminus \{\pm 9\}$.

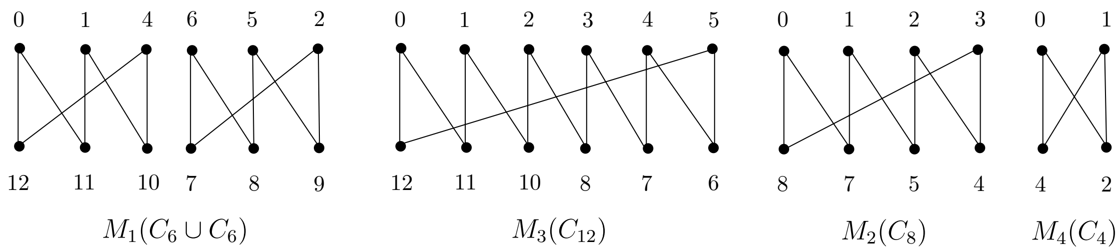


Figure 3.21: A graph $M = M_1 \cup M_2 \cup M_3 \cup M_4$ where each M_i admits an α-labeling

By using the results of Lemmas 3.1–3.2, we obtain the following theorem.

Theorem 3.29. *Let G be a 2-regular graph of odd order n consisting of any number of even cycles and only one odd cycle. There exists a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k and of $K_{k'\times 2n}$ for all integers $k' \geq 3$.*

Proof. Let $M = M_1 \cup M_2 \cup \cdots \cup M_w$ where M_i is an even cycle of size m_i, and let $m = m_1 + m_2 + \cdots + m_w$. Let $G = M \cup C_{m'}$ of size n where $m' \geq 3$ is an odd integer. If $w = 1$, then G has only two components; G consists of one even cycle and one odd cycle. Then the results follow from Theorem 3.12. Assume that

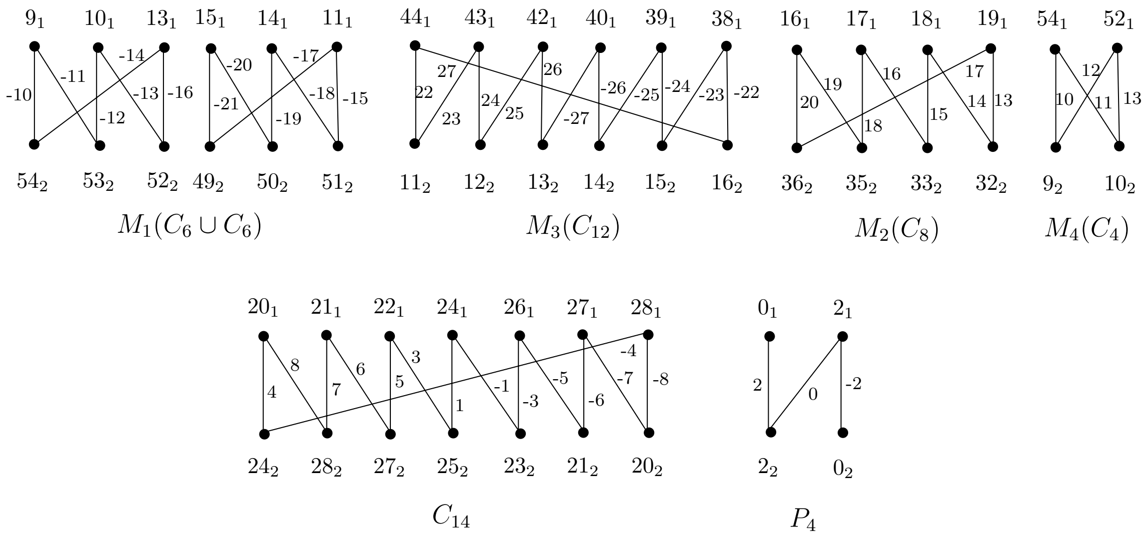


Figure 3.22: An embedding of $G = C_{12} \cup C_6 \cup C_6 \cup C_8 \cup C_4 \cup C_{14} \cup P_4$ in $K_{55,55}$

$w \geq 2$. If $m \equiv 0 \pmod 4$, we are done (see Theorem 3.24). Suppose that $m \equiv 2 \pmod 4$. Let $M^* = \{M_i : M_i \subseteq M \text{ and } m_i \equiv 2 \pmod 4\}$. Then $|M^*|$ is odd. Let $C_{m'}$ be one of the cycles in M^*. The cycles in $M^* \setminus \{C_{m'}\}$ can be partitioned into pairs of graphs that admits α-labelings. Also note that the cycles in $M \setminus M^*$ all have lengths $0 \pmod 4$ and thus admit α-labelings. By combining the results of Lemma 3.1 and Lemmas 3.25–3.27, we obtain a G-decomposition of $K_{(2k+1)\times n}$ for all positive integers k. By combining Lemma 3.2 and Lemmas 3.25–3.27, a G-decomposition of $K_{k'\times 2n}$ exists for all integers $k' \geq 3$. $\square$

If a G-decomposition of K_n exists (i.e., if the Oberwolfach problem has a solution in this case), then a G-decomposition of K_{2kn+n} will also exist.

Theorem 3.30. *Let G of order n be a 2-regular almost bipartite graph. If a G-decomposition of K_n exists, then there exists a G-decomposition of K_{2kn+k} for all positive integers k.*

Proof. Observe that $K_{2kn+n} = (2k+1)K_n \cup K_{(2k+1)\times n}$. Since a G-decomposition of K_n exists, a G-decomposition of $(2k+1)K_n$ will also exist. By Theorem 3.29, there exists a G-decomposition of $K_{(2k+1)\times n}$. The result follows. $\square$

3.3.3 G consisting of three odd cycles

Finally we consider the case where G consists of three cycles of odd length.

Lemma 3.31. *Let $n \geq 3$ be an odd integer and let $m \leq (n-1)/2$ be a positive integer. Let $K_{n,n}$ have vertex set $\mathbb{Z}_n \times \{1,2\}$ with the obvious vertex partition. Let $d_1, d_2, \ldots, d_{m-1}$ be an increasing sequence of consecutive positive integers with $d_{m-1} \leq (n-1)/2$. There exists a path P in $K_{n,n}$ of size $2m-1$ whose edges have lengths $0, \pm d_1, \pm d_2, \ldots, \pm d_{m-1}$ with endpoints 0_1 and 0_2. Furthermore, $V(P) \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1,2]$.*

Proof. If $m = 1$, let P be the path consisting of the edge $\{0_1, 0_2\}$. Otherwise, for $k \in [1, m-1]$, define $v_k = \displaystyle\sum_{i=0}^{k-1} (-1)^i d_{m-1-i}$. Note that since $d_{i+1} - d_i = 1$, we have $v_{2j} = j$ and $v_{2j+1} = d_{m-1} - j$. Thus, $v_{m-1} = \lceil \frac{m}{2} \rceil - 1$ if $m-1$ is even and $v_{m-1} = d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1$ if $m-1$ is odd. Similarly, $v_{m-2} = \lceil \frac{m}{2} \rceil - 1$ or $d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1$ if $m-1$ is odd or even, respectively.

Consider the path of size $m-1$ given by $P'\colon 0_1, (v_1)_2, (v_2)_1, (v_3)_2, \ldots$ where P' ends with $(v_{m-1})_2$ if $m-1$ is odd or $(v_{m-1})_1$ if $m-1$ is even. Thus, $V(P') \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1,2]$. Also, observe that the lengths of the edges of P', in the order encountered, are $d_{m-1}, d_{m-2}, \ldots, d_1$.

Next consider the path $P''\colon 0_2, (v_1)_1, (v_2)_2, (v_3)_1, \ldots$ where P'' ends with $(v_{m-1})_1$ if $m-1$ is odd or $(v_{m-1})_2$ if $m-1$ is even, and observe that the edges on P'', in the order encountered, are $-d_{m-1}, -d_{m-2}, \ldots, -d_1$. Since P'' is constructed in the same way as P' with the corresponding vertices lying in the opposite parts of $V(K_{n,n})$, we have $V(P'') \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1,2]$, and $V(P') \cap V(P'') = \varnothing$.

Construct the path P from the paths P' and P'' by adding the edge from $(v_{m-1})_1$ to $(v_{m-1})_2$. Note that P has size $2m-1$, the edges of P have lengths $0, \pm d_1, \pm d_2, \ldots, \pm d_{m-1}$, and $V(P) \subseteq \big([0, \lceil \frac{m}{2} \rceil - 1] \cup [d_{m-1} - \lfloor \frac{m}{2} \rfloor + 1, d_{m-1}]\big) \times [1,2]$. $\square$

Theorem 3.32. *Let G be a 2-regular graph of order n consisting of exactly three odd cycles. For every positive integer k, there exists a G-decomposition* *of* $K_{(2k+1)\times n}$.

Proof. Let $G = C_{2x+1} \cup C_{2y+1} \cup C_{2z+1}$ where x, y, and z are positive integers and let $n = 2x + 2y + 2z + 3$. Let $k \geq 1$ be an integer. Label the vertex set of $K_{(2k+1)\times n}$ with the elements of the group $\mathbb{Z}_n \times [1, 2k + 1]$ with the obvious vertex partition. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2k + 1$, where $Q = [1, 2k + 1]$.

Fix r and s with $1 \leq r < s \leq 2k+1$. We will construct a graph $G_{r,s}$ consisting of the vertex-disjoint union of the following three cycles: $C_{r,s}$ of size $2x + 1$, $C^{'}_{r,s}$ of size $2y + 1$, and $C^{''}_{r,s}$ of size $2z + 1$. We will consider two cases.

Case 1. G has at least two cycles of size 3. Without loss of generality, we may assume that $x = y = 1$. Then the vertex sets of $C_{r,s}$ and $C^{'}_{r,s}$ can be given by $\langle 0_r, 1_s, 3_{r\circ s}\rangle$ and $\langle 3_r, 2_s, 5_{r\circ s}\rangle$, respectively. If $z = 1$, then the vertex set of $C''_{r,s}$ can be given by $\langle 4_r, 4_s, 8_{r\circ s}\rangle$. Suppose that $z \geq 2$. By Lemma 3.31, there exists a path $P^*_{r,s}$ of size $2z - 1$ whose edges have lengths $\{0\} \cup \pm[5, z + 3]$. In the lemma, we would use $d_1 = 5$, $d_2 = 6$, $\ldots$, $d_{z-1} = z + 3$, so $V(P^*_{r,s}) \subseteq [0, z + 3] \times \{r, s\}$ with endpoints 0_r and 0_s. Let $P''_{r,s} = P^*_{r,s} + 4$. Thus $P''_{r,s}$ has endpoints 4_r and 4_s. Then $V(P''_{r,s}) \subseteq [4, z + 7] \times \{r, s\}$. Thus, $P''_{r,s}$ is vertex disjoint from $C_{r,s}$ and $C'_{r,s}$. Construct the cycle $C''_{r,s}$ of length $2z + 1$ from the path $P''_{r,s}$ by adding the edges $\{4_r, 8_{r\circ s}\}$ and $\{4_s, 8_{r\circ s}\}$.

Note that in the subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$, $G_{r,s}$ contains one edge of each length $i \in [-1, 1] \cup \pm[5, z+3]$ (if $z = 1$, the $G_{r,s}$ contains one edge of each length $i \in [-1, 1]$). Moreover, the three edges of $G_{r,s}$ that are incident with vertices in $\mathbb{Z}_n \times \{r, r \circ s\}$ are all of different lengths. For instance, the edges $\{0_r, 3_{r\circ s}\}$ in $C_{r,s}$, $\{3_r, 5_{r\circ s}\}$ in $C'_{r,s}$, and $\{4_r, 8_{r\circ s}\}$ in $C''_{r,s}$, have lengths 3, 2, and 4, respectively, if $r < r \circ s$, and lengths -3, -2, and -4, respectively, otherwise. Similarly, the three edges of $G_{r,s}$ that are incident only with vertices in $\mathbb{Z}_n \times \{s, r \circ s\}$ are all of different lengths. For instance, the edges $\{1_s, 3_{r\circ s}\}$ in $C_{r,s}$, $\{2_s, 5_{r\circ s}\}$ in $C'_{r,s}$, and $\{4_s, 8_{r\circ s}\}$ in $C''_{r,s}$, have lengths 2, 3, and 4, respectively, if $s < r \circ s$, and lengths -2, -3, and -4, respectively, otherwise. Figure 3.23 shows an example of $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = y = 1$ and $z = 4$.

Next, let $G^*_{r,s} = \{G_{r,s} + \ell : 0 \le \ell < n-1\}$. Thus $G^*_{r,s}$ contains n distinct copies of G. Moreover, in the subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r,s\}$, G^* contains all edges of length i for all $i \in [-(n-1)/2, (n-1)/2] \setminus \pm[2,4]$. Let $\mathcal{C} = \{G_{r,s} + \ell : 1 \le r < s \le 2k+1, 0 \le \ell \le n-1\}$ and note that $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G. We will show that every edge of $K_{(2k+1)\times n}$ appears on some copy of G in $\mathcal{C}$. Let $e = \{i_r, j_s\}$ with $r < s$ be an arbitrary edge of $K_{(2k+1)\times n}$. Let t' be the unique solution to $r \circ t' = s$ and let $\alpha' = \min\{r, t'\}$ and $\beta' = \max\{r, t'\}$. Let t'' be the unique solution to $s \circ t'' = r$ and let $\alpha'' = \min\{s, t''\}$ and $\beta'' = \max\{s, t''\}$. If $j - i \in [-(n-1)/2, (n-2)/2] \setminus \pm[2,4]$ then e belongs to $G_{r,s} + \ell$ where $0 \le \ell \le n-1$.

Note that if $j - i = 2$, then e belongs to the triple $\{(i,r), (i-1,t'), (j,s)\}$ which is a copy of $C_{t',r}$ if $t' < r$, or a copy of $C'_{r,t'}$ if $r < t'$. If $j - i = 3$, then e belongs to the triple $\{(i,r), (i+1,t'), (j,s)\}$ which is a copy of $C'_{t',r}$ if $t' < r$, and a copy of $C_{r,t'}$ if $r < t'$. Also, if $j - i = 4$, then e belongs to some copy of $C''_{\alpha',\beta'}$. Thus, if $j - i \in [2,4]$, then e belongs to $G_{\alpha',\beta'} + \ell$ where $0 \le \ell \le n-1$.

Observe that if $j - i = -2$, then e belongs to the triple $\{(j,s), (j-1,t''), (i,r)\}$ which is a copy of $C_{t'',s}$ if $t'' < s$, or a copy of $C'_{s,t''}$ if $s < t''$. If $j - i = -3$, then e belongs to the triple $\{(j,s), (j+1,t''), (i,r)\}$ which is a copy of $C'_{t'',s}$ if $t'' < s$, or a copy of $C_{s,t''}$ if $s < t''$. Also, if $i - j = -4$, then e belongs to some copy of $C''_{\alpha'',\beta''}$. Thus, if $j - i \in [-4,-2]$, then e belongs to $G_{\alpha'',\beta''} + \ell$ where $0 \le \ell \le n-1$. Since every edge of $K_{(2k+1)\times n}$ appears on some copy of H in $\mathcal{C}$ and since $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{(2k+1)\times n}$ into copies of G.

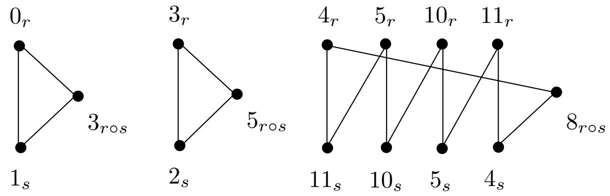


Figure 3.23: $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = y = 1$ and $z = 4$

Case 2. G has at most one cycle of size 3. Suppose $y \ge 2$ and $z \ge 2$. By

Lemma 3.31, there exist a path $P_{r,s}$ of size $2x-1$ using the edge lengths in $\{0\} \cup \pm[y+z+3, x+y+z+1]$ with endpoints 0_r and 0_s. In the lemma, we would use $d_1 = y+z+3$, $d_2 = y+z+4$, $\ldots$, $d_{x-1} = x+y+z+1$, so $V(P_{r,s}) \subseteq \big([0, \lceil \frac{x}{2} \rceil - 1] \cup [\lceil \frac{x}{2} \rceil + y+z+2, x+y+z+1]\big) \times \{r,s\}$. We construct the cycle $C_{r,s}$ of size $2x+1$ from $P_{r,s}$ by adding the edges $\{0_r, (y+z)_{ros}\}$ and $\{0_s, (y+z)_{ros}\}$.

Next, we will construct the cycle $C'_{r,s}$ of size $2y+1$. Let $P'_{r,s} = G'_1 + G'_2 + G'_3$ where

$$G'_1 = P(\lceil \tfrac{x}{2} \rceil, \lceil \tfrac{x}{2} \rceil + 3, y-2)$$

$$G'_2 = \begin{cases} \big((\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_s, (\lceil \frac{x}{2} \rceil + \frac{y+1}{2})_r, (\lceil \frac{x}{2} \rceil + \frac{y-1}{2})_s, \lceil \frac{x}{2} \rceil + \frac{y+5}{2})_r\big), & \text{if } y-2 \text{ odd;} \\ \big((\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_r, (\lceil \frac{x}{2} \rceil + \frac{y+2}{2})_s, (\lceil \frac{x}{2} \rceil + \frac{y+4}{2})_r, \lceil \frac{x}{2} \rceil + \frac{y-2}{2})_s\big), & \text{if } y-2 \text{ even,} \end{cases}$$

$$G'_3 = \begin{cases} P\big(\lceil \frac{x}{2} \rceil + \frac{y+5}{2}, \lceil \frac{x}{2} \rceil - \frac{y-1}{2}, y-2\big), & \text{if } y-2 \text{ odd;} \\ Q\big(\lceil \frac{x}{2} \rceil + \frac{y+6}{2}, \lceil \frac{x}{2} \rceil - \frac{y-2}{2}, y-2\big), & \text{if } y-2 \text{ even.} \end{cases}$$

If $y=2$, then $P'_{r,s} = G'_2 = \big(\lceil \frac{x}{2} \rceil_r, (\lceil \frac{x}{2} \rceil + 2)_s, (\lceil \frac{x}{2} \rceil + 3)_r, \lceil \frac{x}{2} \rceil_s\big)$.

Note that by **P1**, the first vertex of G'_1 is $\lceil \frac{x}{2} \rceil_r$, and the last vertex is $(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_s$ if $y-2$ is odd and $(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_r$ if $y-2$ is even; the first vertex of G'_3 is $(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_r$ and the last vertex is $\lceil \frac{x}{2} \rceil_s$ if $y-2$ is odd. By **Q1**, the first vertex of G'_3 is $(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_s$ and the last vertex is $\lceil \frac{x}{2} \rceil_s$ if $y-2$ is even.

For $i = 1$ or 3, let A'_i and B'_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the graph G_i. Then using **P2** and **Q2**, we compute

$$A'_1 = [\lceil \tfrac{x}{2} \rceil_r, (\lceil \tfrac{x}{2} \rceil + \lfloor \tfrac{y-2}{2} \rfloor)_r],$$

$$B'_1 = [(\lceil \tfrac{x}{2} \rceil + \lceil \tfrac{y+5}{2} \rceil)_s, (\lceil \tfrac{x}{2} \rceil + y + 1)_s],$$

$$A'_3 = [(\lceil \tfrac{x}{2} \rceil + \lceil \tfrac{y+5}{2} \rceil)_r, (\lceil \tfrac{x}{2} \rceil + y + 1)_r],$$

$$B'_3 = [\lceil \tfrac{x}{2} \rceil_s, (\lceil \tfrac{x}{2} \rceil + \lfloor \tfrac{y-2}{2} \rfloor)_s].$$

Thus,

$$A'_1 < A'_3 \text{ and } B'_1 < B'_3.$$

Note that $V(G'_1) \cap V(G'_2) = \{(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_s\}$ if $y-2$ is odd and $V(G'_1) \cap V(G'_2) = \{(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_r\}$ if $y-2$ is even and, $V(G'_2) \cap V(G'_3) = \{(\lceil \frac{x}{2} \rceil + \frac{y+5}{2})_r\}$ if $y-2$ is odd and $V(G'_2) \cap V(G'_3) = \{(\lceil \frac{x}{2} \rceil + \frac{y-2}{2})_s\}$ if $y-2$ is even; otherwise, G'_1, G'_2 and G'_3 are vertex disjoint. Therefore, $G'_1 + G'_2 + G'_3$ is a path of size $2y-1$ with the endpoints $\lceil \frac{x}{2} \rceil_r$ and $\lceil \frac{x}{2} \rceil_s$. Since $V(P'_{r,s}) \subseteq [\lceil \frac{x}{2} \rceil, \lceil \frac{x}{2} \rceil + y + 1] \times \{r, s\}$, $P'_{r,s}$ is vertex disjoint from $P_{r,s}$.

Next, let E'_i denote the set of edge lengths in G'_i for $i = 1$ or 3. By **P3** and **Q3**, we have edge lengths

$$E'_1 = [4, y+1],$$
$$E'_3 = [-(y+1), -4].$$

Notice that the set of edge lengths in G'_2 is $\{2, -1, -3\}$. Then construct the cycle $C'_{r,s}$ of size $2y+1$ from the path $P'_{r,s}$ by adding the edges $\{\lceil \frac{x}{2} \rceil_r, (\lceil \frac{x}{2} \rceil + y + z + 1)_{ros}\}$ and $\{\lceil \frac{x}{2} \rceil_s, (\lceil \frac{x}{2} \rceil + y + z + 1)_{ros}\}$.

Finally we will construct the cycle $C''_{r,s}$ of size $2z+1$. Let $P''_{r,s} = G''_1 + G''_2 + G''_3$ where

$$G''_1 = P(x+y+z+2, x+2y+z+3, z-2),$$

$$G''_2 = \begin{cases} \left((\frac{2x+4y+3z+5}{2})_s, (\frac{2x+4y+3z-1}{2})_r, (\frac{2x+4y+3z+1}{2})_s, (\frac{2x+4y+3z+5}{2})_r\right), & \text{if } z-2 \text{ odd;} \\ \left((\frac{2x+2y+3z+2}{2})_r, (\frac{2x+2y+3z+8}{2})_s, (\frac{2x+2y+3z+6}{2})_r, (\frac{2x+2y+3z+2}{2})_s\right), & \text{if } z-2 \text{ even,} \end{cases}$$

$$G''_3 = \begin{cases} P\left(\frac{2x+4y+3z+5}{2}, \frac{2x+2y+z+5}{2}, z-2\right), & \text{if } z-2 \text{ odd;} \\ Q\left(\frac{2x+4y+3z+6}{2}, \frac{2x+2y+z+6}{2}, z-2\right), & \text{if } z-2 \text{ even.} \end{cases}$$

If $z = 2$, then $P''_{r,s} = G''_2 = \left((x+y+4)_r, (x+y+7)_s, (x+y+6)_r, (x+y+4)_s\right)$.

Note that by **P1**, the first vertex of G''_1 is $(x+y+z+2)_r$, and the last vertex is $(\frac{2x+4y+3z+5}{2})_s$ if $z-2$ is odd and $(\frac{2x+2y+3z+2}{2})_r$ if $z-2$ is even; the first vertex of G_3'' is $(\frac{2x+4y+3z+5}{2})_r$ and the last vertex is $(x+y+z+2)_s$ if $z-2$ is odd. By **Q1**, the first vertex of G_3'' is $(\frac{2x+2y+3z+2}{2})_s$ and the last vertex is $(x+y+z+2)_r$ if $z-2$ is even.

For $i = 1$ or 3, let A_i'' and B_i'' denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the graph G_i''. Then using **P2** and **Q2**, we compute

$$A_1'' = [(x+y+z+2)_r, (x+y+\lfloor \tfrac{3z}{2} \rfloor + 1)_r],$$

$$B_1'' = [(x+2y+\lceil \tfrac{3z+5}{2} \rceil)_s, (x+2y+2z+1)_s],$$

$$A_3'' = [(x+2y+\lceil \tfrac{3z+5}{2} \rceil)_r, (x+2y+2z+1)_r],$$

$$B_3'' = [(x+y+z+2)_s, (x+y+\lfloor \tfrac{3z}{2} \rfloor + 1)_s].$$

Thus,

$$A_1'' < A_3'' \text{ and } B_1'' < B_3''.$$

Note that $V(G_1'') \cap V(G_2'') = \{(x+2y+\lceil \frac{3z+5}{2} \rceil)_s\}$ if $z-2$ is odd and $V(G_1'') \cap V(G_2'') = \{(x+y+\lfloor \frac{3z}{2} \rfloor + 1)_r\}$ if $z-2$ is even and, $V(G_2'') \cap V(G_3'') = \{(x+2y+\lceil \frac{3z+5}{2} \rceil)_r\}$ if $z-2$ is odd and $V(G_2'') \cap V(G_3'') = \{(x+y+\lfloor \frac{3z}{2} \rfloor + 1)_s\}$ if $z-2$ is even; otherwise, G_1'', G_2'' and G_3'' are vertex disjoint. Therefore, $G_1'' + G_2'' + G_3''$ is a path of size $2z-1$ with the endpoints $(x+y+z+2)_r$ and $(x+y+z+2)_s$. Since $V(P_{r,s}'') \subseteq [x+y+z+2, x+2y+2z+1] \times \{r, s\}$, $P_{r,s}''$ is vertex disjoint from $P_{r,s}$ and $P_{r,s}'$.

Next, let E_i'' denote the set of edge lengths in G_i'' for $i = 1$ or 3. By **P3** and **Q3**, we have edge lengths

$$E_1'' = [y+2, y+z-1]$$

$$E_3'' = [-(y+z-1), -(y+2)].$$

Notice that the set of edge lengths in G_2'' is $\{3, 1, -2\}$. Then, construct the cycle $C_{r,s}^{''}$ of size $2z+1$ from the path $P_{r,s}^{''}$ by adding the edges $\{(x+y+z+2)_r, (x+2y+2z+4)_{ros}\}$ and $\{(x+y+z+2)_s, (x+2y+2z+4)_{ros}\}$.

Since $(y+z)_{ros}$,$(\lceil \frac{x}{2} \rceil + y + z + 1)_{ros}$ and $(x+2y+2z+4)_{ros}$ are different vertices, and $P_{r,s}$, $P_{r,s}'$ and $P_{r,s}''$ are vertex disjoint, we have $C_{r,s}$, $C_{r,s}^{'}$ and $C_{r,s}^{''}$ are also vertex disjoint. Figure 3.24 shows an example of $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = 4, y = 2$ and $z = 5$.

Let $G^*_{r,s} = \{G_{r,s} + \ell : 0 \leq \ell \leq n-1\}$. Then $G^*_{r,s}$ contains n distinct copies of G and all the edges of each length $i \in [-(n-1)/2, (n-1)/2] \setminus \pm[y+z, y+z+2]$ in the subgraph of $K_{(2k+1)\times n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$. Let $\mathcal{C} = \{G_{r,s} + \ell : 1 \leq r < s \leq 2k+1, 0 \leq \ell \leq n-1\}$ and note that $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G. We will show that every edge of $K_{(2k+1)\times n}$ appears on some copy of G in $\mathcal{C}$. Let $e = \{i_r, j_s\}$ with $r < s$ be an arbitrary edge of $K_{(2k+1)\times n}$. Let t' be the unique solution to $r \circ t' = s$ and let $\alpha' = \min\{r, t'\}$ and $\beta' = \max\{r, t'\}$. Let t'' be the unique solution to $s \circ t'' = r$ and let $\alpha'' = \min\{s, t''\}$ and $\beta'' = \max\{s, t''\}$. If $j - i \in [-(n-1)/2, (n-1)/2] \setminus \pm[y+z, y+z+2]$, then e belongs to $G_{r,s} + \ell$ for some ℓ with $0 \leq \ell \leq n-1$. If $j - i \in [y+z, y+z+2]$, then e belongs to $G_{\alpha',\beta'} + \ell$ where $0 \leq \ell \leq n-1$. If $j - i \in [-(y+z+2), -(y+z)]$, then e belongs to $G_{\alpha'',\beta''} + \ell$ where $0 \leq \ell \leq n-1$. Since every edge of $K_{(2k+1)\times n}$ appears on some copy of $G_{r,s}$ in $\mathcal{C}$ and since $\mathcal{C}$ contains $\binom{2k+1}{2}n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{(2k+1)\times n}$ into copies of G. $\square$

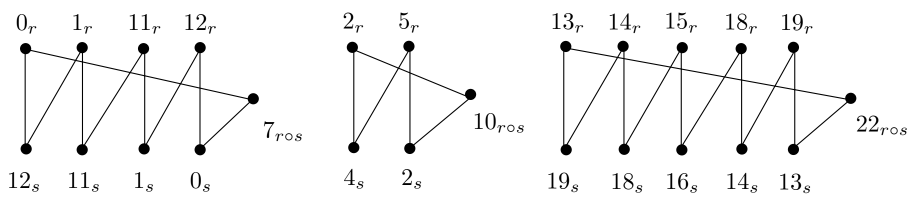


Figure 3.24: $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ where $x = 4$, $y = 2$ and $z = 5$

In the proof of Theorem 3.32, if we replace idempotent symmetric quasigroups with symmetric quasigroups with holes, then we obtain a G-decomposition of $K_{k\times 2n}$ for every integer $k \geq 3$.

Theorem 3.33. *Let G be a 2-regular graph of order n consisting of exactly three odd cycles. For every integer $k \geq 3$, there exists a G-decomposition of $K_{k\times 2n}$.*

Proof. Let $G = C_{2x+1} \cup C_{2y+1} \cup C_{2z+1}$, where $x, y, z \geq 1$. Let $k \geq 3$ be an integer and let $Q = [1, 2k]$. For $i \in [1, k]$, let $h_i = \{2i-1, 2i\}$ and $g_i = \mathbb{Z}_n \times h_i$. Let

$n = 2x + 2y + 2z + 3$ and let $V(K_{k\times 2n}) = \mathbb{Z}_n \times [1, 2k]$ with the vertex-set partition $\{g_1, g_2, \ldots, g_k\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $2k$ with holes H.

Fix r and s with $1 \le r < s \le 2k$ and $\{r, s\} \notin H$. We proceed in the same fashion as in the proof of Theorem 3.32 producing the graph $G_{r,s}$ consisting of a cycle $C_{r,s}$ of size $2x + 1$, a cycle $C'_{r,s}$ of size $2y + 1$ and a cycle $C''_{r,s}$ of size $2z + 1$ such that $C_{r,s}$, $C'_{r,s}$ and $C''_{r,s}$ are vertex disjoint.

Note that for fixed r and s with $1 \le r < s \le 2k$ and with $\{r, s\} \notin H$, the set $\{G_{r,s} + \ell\colon 0 \le \ell \le n-1\}$ contains n distinct copies of G and all the edges of lengths $i \in [-(x + y + z + 1), x + y + z + 1] \smallsetminus \pm[y + z, y + z + 2]$ in the subgraph of $K_{k\times 2n}$ with vertex set $\mathbb{Z}_n \times \{r, s\}$. Let $\mathcal{C} = \{G_{r,s} + \ell\colon 1 \le r < s \le 2k, \{r, s\} \notin H, 0 \le \ell \le n-1\}$ and note that $\mathcal{C}$ contains $\binom{2k}{2}n$ distinct copies of G. We wish to show that every edge of $K_{k\times 2n}$ appears on some copy of G in $\mathcal{C}$. Let $e = \{i_r, j_s\}$ be an arbitrary edge of $K_{k\times 2n}$. Without loss of generality, we may assume $r < s$. If $j - i \in [0, x + y + z + 1] \smallsetminus [y + z, y + z + 2]$, then e belongs to $G_{r,s} + \ell$ for some ℓ with $0 \le \ell \le n-1$. If $j - i = [y + z, y + z + 2]$, then e belongs to $G_{r,t} + \ell$ where t is the unique solution to $r \circ t = s$ and $0 \le \ell \le n-1$. If $j - i = [-(y + z + 2), -(y + z)]$, then e belongs to $G_{s,t} + \ell$ where t is the unique solution to $s \circ t = r$ and $0 \le \ell \le n-1$. Since every edge of $K_{k\times 2n}$ appears on some copy of G in $\mathcal{C}$ and since $\mathcal{C}$ contains $\binom{2k}{2}n$ distinct copies of G, it follows that $\mathcal{C}$ is a decomposition of $K_{k\times 2n}$ into copies of G. □

Let G of order n be the vertex-disjoint union of three odd cycles. It is shown in [7] and [6] that there exists a G-decomposition of K_{2n+1}. It was not known whether a G-decomposition of K_{2kn+1} exists for every positive integer k. Using the decomposition of K_{2n+1} and the result from Theorem 3.33, we can answer this question in the affirmative for $k \ge 3$.

Theorem 3.34. *Let G of order n be the vertex-disjoint union of three odd cycles. There exists a G-decomposition of K_{2kn+1} for every positive integer $k \neq 2$.*

Proof. Since there exists a G-decomposition of K_{2n+1}, we can assume that $k \ge 3$. For $i \in [1, k]$, let S_i be a set with $2n$ elements and let H_i be a complete graph of order $2n+1$ with vertex set $S_i \cup \{\infty\}$. Let $V(K_{2kn+1}) = S_1 \cup S_2 \cup \ldots \cup S_k \cup \{\infty\}$. Thus, $K_{2kn+1} = H_1 \cup H_2 \cup \ldots \cup H_k \cup K_{k\times 2n}$. Since there is a G-decomposition of H_i for $i \in [1,k]$ and there is a G-decomposition of $K_{k\times 2n}$, the result follows. □

If a G-decomposition of K_n exists (i.e., if the Oberwolfach problem has a solution in this case), then a G-decomposition of K_{2kn+n} will also exist.

Theorem 3.35. *Let G of order n be the vertex-disjoint union of three odd cycles. If a G-decomposition of K_n exists, then there exists a G-decomposition of K_{2kn+k} for every positive integer k.*

Proof. Observe that $K_{2kn+n} = (2k+1)K_n \cup K_{(2k+1)\times n}$. Since a G-decomposition of K_n exists, a G-decomposition of $(2k+1)K_n$ will also exist. By Theorem 3.32, there exists a G-decomposition of $K_{(2k+1)\times n}$. The result follows. □

CHAPTER IV
SUMMARY AND OPEN PROBLEMS

4.1 Summary

Let G be a 2-regular graph of odd order n such that either G is almost bipartite or G consists of three cycles of odd lengths. By using novel extensions of the Bose construction for Steiner triple systems, we proved the existence of G-decompositions of several classes of complete multipartite graphs as well as of some complete graphs. Our results are summarized below.

(i) If G is C_n, then there exist G-decompositions of $K_{(2k+1)\times n}$ and of K_{2kn+n} for every positive integer k, and of $K_{k'\times 2n}$ for every integer $k' \geq 3$.

(ii) If G is almost-bipartite, then there exist G-decompositions of $K_{(2k+1)\times n}$ and of $K_{k'\times 2n}$ for all positive integers k and $k' \geq 3$.

(iii) If G is the vertex-disjoint union of one even cycle and one odd cycle, then there exist G-decompositions of K_v for all $v \equiv n \pmod{2n}$, unless $(G, v) = (C_4 \cup C_5, 9)$.

(iv) If G consists of three odd cycles, then there exist G-decompositions of $K_{(2k+1)\times n}$ and of $K_{k'\times 2n}$ for all positive integers k and $k' \geq 3$. We also found G-decompositions of K_v for all $v \equiv 1 \pmod{2n}$, $v \neq 4n+1$.

4.2 Open Problems

Several open problems related to the results in this dissertation warrant further investigation.

(i) If G is almost-bipartite of order n, find G-decompositions of K_v for all $v \equiv n \pmod{2n}$.

(ii) If G of order n consists (or contains) of an odd number of odd cycles, find G-decompositions of $K_{(2k+1)\times n}$ and of $K_{k'\times 2n}$ for all positive integers k and $k' \geq 3$. Also, find G-decompositions of K_v for all $v \equiv 1$ or $n \pmod{2n}$.

(iii) If G of order n is the vertex-disjoint union of one even cycle and one odd cycle, find G-decompositions of K_v for all odd v that satisfy $v(v-1) \equiv 0 \pmod{2n}$.

(iv) Investigate the Oberwolfach problem for three odd cycles and for almost-bipartite 2-regular graphs.

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