CHAPTER I

INTRODUCTION

In this thesis we consider only finite undirected graphs without loops and multiple edges. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively, $v = |V(G)|$ and $e = |E(G)|$.

1.1 Definitions

Definition 1.1.1. Let G be a graph. A *total labeling* of G is a bijection $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., v+e\}$. For each vertex x in G, the *weight* of x, denoted by $w_\lambda(x)$, is defined as $w_\lambda(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy)$.

Definition 1.1.2. Let G be a graph and λ a total labeling of G. Then λ is a *vertex-magic total labeling* of G if for every vertex x in G, the *weight*, $w_\lambda(x) = h$ for some constant h which is called *the magic constant* for λ. Moreover λ is a *super vertex-magic total labeling* of G if λ is vertex-magic total labeling of G and $\lambda(V(G)) = \{1, 2, ..., v\}$. G is called a *super vertex-magic graph* if G admits a super vertex-magic total labeling.

Example 1.1.3. The disjoint union of cycles C_3 and C_8, $C_3 + C_8$, is a super vertex-magic graph with the magic constant 40 as shown in Figure 1.1.

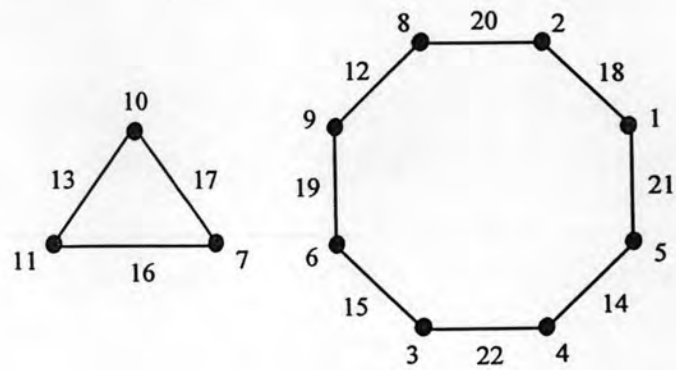

Figure 1.1 : $C_3 + C_8$ is a super vertex-magic graph with the magic constant 40

1.2 Some properties of a super vertex-magic graph

Theorem 1.2.1. ([6]) *The magic constant for any super vertex-magic graph is* $\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}$.

Example 1.2.2. The cycle C_6 has $v=6$ and $e=6$.

Assume that C_6 is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{12\cdot13}{6}-\frac{7}{2}=26-3.5=22.5,$$

which is not an integer, a contradiction.

Hence C_6 is not a super vertex-magic graph.

Example 1.2.3. The super vertex-magic graph C_5+C_6 has 11 vertices and 11 edges with the magic constant 40 is shown in Figure 1.2.

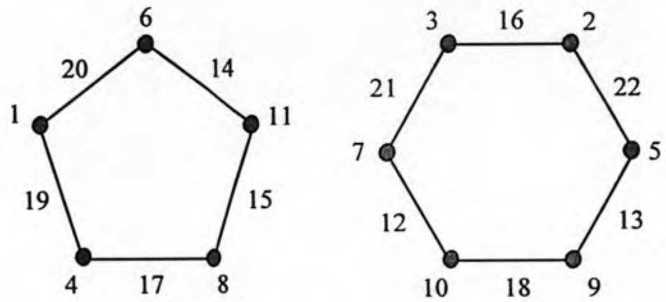

Figure 1.2 : C_5+C_6 is a super vertex-magic graph

Note that $40=\frac{22\cdot23}{11}-\frac{12}{2}=\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}$.

Example 1.2.4. Let *G* be a graph with $v=5$ and $e=5$ as shown in Figure 1.3. Assume that *G* is a super vertex-magic graph with a super vertex-magic total labeling λ.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{10\cdot11}{5}-\frac{6}{2}=22-3=19,$$

which is an integer. We will show that G is not a super vertex-magic graph.

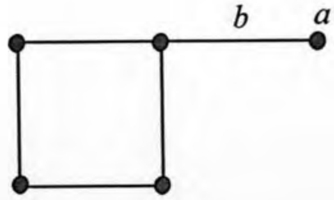

Figure 1.3 : The graph G that is not super vertex-magic but the magic constant is an integer

Let a be the vertex with degree 1 and b be the edge incident with a.
Since $\lambda(V(G)) = \{1,2,3,4,5\}$ and $\lambda(E(G)) = \{6,7,8,9,10\}$, $\lambda(a)+\lambda(b) \leq 5+10=15$.
But the weight of each vertex in G must be equal to the magic constant which is 19, a contradiction.
Hence G is not a super vertex-magic graph.

It is evident that trees and forests are not super vertex-magic. The following theorems shows the necessary conditions to be super vertex-magic.

Theorem 1.2.5. ([6]) *If a nontrivial graph G has a vertex of degree 0 or 1, then G is not a super vertex-magic graph.*

Theorem 1.2.6. ([6]) *Tree is not a super vertex-magic graph.*

Theorem 1.2.7. ([1]) *If G is a super vertex-magic graph, then*

1) *if* $2e \geq \sqrt{10v^2-6v+1}$, *then the minimum degree of G is at least* 3.
2) *if* $2e < \sqrt{10v^2-6v+1}$, *then the maximum degree of G is at most* 6.

Example 1.2.8. Let G be a graph with $v=9$ and $e=12$ as shown in Figure 1.4. Assume that G is a super vertex-magic graph.

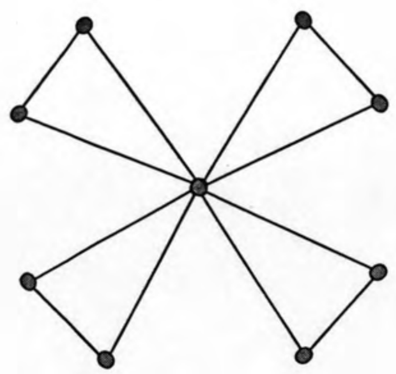

Figure 1.4 : A graph G that is not super vertex-magic

Since $2e = 24 < 27.514 \approx \sqrt{757} = \sqrt{10(9)^2 - 6(9) + 1} = \sqrt{10v^2 - 6v + 1}$,
by Theorem 1.2.7, the maximum degree of G is at most 6, a contradiction.
Hence G is not a super vertex-magic graph.

Theorem 1.2.9. ([1]) *If G be a super vertex-magic graph with the magic constant h, then the degree d of any vertex of G satisfies*

$$v + e + \frac{1}{2} - \sqrt{(v + e + \frac{1}{2})^2 - 2(h - v)} \le d \le -\frac{1}{2} + \sqrt{2(h - 2e) - \frac{7}{4}}\,.$$

Example 1.2.10. Let G be a graph with $v = 8$ and $e = 12$ as shown in Figure 1.5.
Assume that G is a super vertex-magic graph with the magic constant h.
By Theorem 1.2.1,

$$h = \frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{20 \cdot 21}{8} - \frac{9}{2} = \frac{96}{2} = 48\,.$$

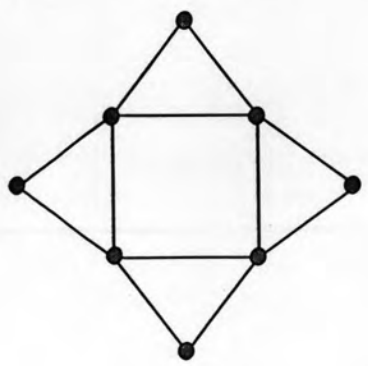

Figure 1.5 : A graph G with $v = 8$ and $e = 12$

Since $v+e+\frac{1}{2}-\sqrt{(v+e+\frac{1}{2})^2-2(h-v)}=20.5-\sqrt{(20.5)^2-2(40)}\approx 2.054>2$ contradicts to the fact that G has 4 vertices of degree 2, by Theorem 1.2.9, G is not a super vertex-magic graph.

1.3 History and Overview

J.A. MacDougall et al. [7] introduced the notion of a vertex-magic total labeling and super vertex-magic total labeling in 1999.

Balbuena et al. [1] obtained upper and lower bounds of any vertex degree in terms of v and e and showed super vertex-magic total labelings of certain families of circulant graphs such as $C_n(1,m)$, for odd $n \geq 5$ and $C_n(1,2,3)$, for odd $n \geq 7$.

J.Gomez [4, 5] showed super vertex-magic total labelings of complete graph K_n, where $n \equiv 0 \pmod 4$, $n \neq 4$ and new method to obtain a super vertex-magic total labelings of graph from an existing one.

J.A. MacDougall et al. [6] investigated the properties of super vertex-magic total labeling such as the magic constant and exhibited some families of graphs that admit super vertex-magic total labelings such as cycle C_n and complete graph K_n, where n is odd and others that do not admit any super vertex-magic total labeling such as wheel graph $W_n \cong K_1 \vee C_n$, ladder graph $L_n \cong P_2 \times P_n$, fan graph $F_n \cong K_1 \vee P_n$, complete bipartite graph $K_{m,n}$, friendship graph f_n, and a graph with a vertex of degree 0 or 1.

J.A. Gallian [3] summarizes the mathematical data on super vertex-magic graphs. It is based on the studies undertaken between 1999 and 2006 performed by many famous mathematicians.

In Chapter II, we show a construction of the new super vertex-magic graph from an existing one. We collect and present super vertex-magic graphs such as certain families of circulant graphs $C_n(1,2,m)$ where $m \in \{3,4,\ldots,\frac{n-1}{2}\}$ and $n \geq 7$,

$C_n(1,3,m)$ where $m \in \{4,5,...,\frac{n-1}{2}\}$ and $n \geq 9$, $C_n(1,2,3,4)$ where $n \geq 9$, $C_n(1,2,3,4,5)$ where $n \geq 11$, n is odd and some graphs that are not super vertex-magic such as prism graph, book graph, and crown graph in Chapter III and Chapter IV, respectively. Lastly, the conclusion and some open problems are summarized in Chapter V.