การลู่เข้าของวิธีไฟไนต์เอลิเมนต์แบบปรับตัวสำหรับสมการเชิงอนุพันธ์ย่อยเชิงวงรีแบบกึ่งเชิงเส้น

นายธนัชยศ จำปาหวาย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

CONVERGENCE OF ADAPTIVE FINITE ELEMENT METHODS FOR SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Mr. Thanatyod Jampawai

A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2014

Thesis Title	CONVERGENCE OF ADAPTIVE FINITE ELEMENT METHODS FOR SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS
By	Mr. Thanatyod Jampawai
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Khamron Mekchay, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

....Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

....Chairman
(Associate Professor Pornchai Satravaha, Ph.D.)

....Thesis Advisor
(Assistant Professor Khamron Mekchay, Ph.D.)

....Examiner
(Ratinan Boonklurb, Ph.D.)

....Examiner
(Sujin Khomrutai, Ph.D.)

....External Examiner
(Associate Professor Montri Maleewong, Ph.D.)

5373892123 : MAJOR MATHEMATICS
KEYWORDS : SEMI-LINEAR / CONTRACTION

THANATYOD JAMPAWAI : CONVERGENCE OF ADAPTIVE FINITE ELEMENT METHODS FOR SEMI-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS. ADVISOR : ASST. PROF. KHAMRON MEKCHAY, Ph.D., 35 pp.

We analyze a standard adaptive finite element method (AFEM) for second order semi-linear elliptic partial differential equations with vanishing boundary over a polygonal domain in $\mathbb{R}^2$. We prove a contraction property for the weighted sum of the energy error and the error estimator between any two consecutive loops, which implies the convergence of AFEM. The result is obtained based on the assumptions that the initial triangulation is sufficiently refined and a Lipschitz constant is suffiently small in order to deal with the nonlinear inhomogeneous term $f(x, u(x))$, which is also assumed to be Lipschitz in the second variable.

Department : Mathematics and Computer Science Student's Signature : ....
Field of Study : Mathematics Advisor's Signature : ....
Academic Year : 2014

ACKNOWLEDGEMENTS

My earnest appreciation goes to Assistant Professor Khamron Mekchay, Ph.D., for his encouragement and merciful supervision. This work would not be completed without his invaluable suggestion and prudent emendation. Besides, I am grateful to Associate Professor Pornchai Satravaha, Ph.D., Ratinan Boonklurb, Ph.D., Sujin Khomrutai, Ph.D. and Associate Professor Montri Maleewong, Ph.D., my thesis committees for their valuable advices and all of my teachers and lecturers for improving my skill and knowledge.

In particular, I would like to thank the Developement and Promotion of Science and Technology Talents Project (DPST).

Last but not least, my sincere gratitude goes to my beloved parents for their unexceptionable encouragement throughout my study.

CONTENTS

NOTATIONS

$H^1(\Omega)$	The Sobolev spaces of functions in $L^2(\Omega)$ whose first derivatives are also in $L^2(\Omega)$
$H_0^1(\Omega)$	The space of functions in $H^1(\Omega)$ with vanishing on boundary $\partial\Omega$
$\mathcal{T}$	A triangulation of Ω
$\mathbb{V}(\mathcal{T})$	The finite element space correspoding to a triangulation $\mathcal{T}$
$\mathbb{P}_n(T)$	The set of all polynomials on T of degree less than or equal to n
$\mathcal{B}(\cdot,\cdot)$	A bilinear form on $H^1(\Omega) \times H^1(\Omega)$
$\langle \cdot,\cdot \rangle$	The standard inner product on $L^2(\Omega)$
$\\| \cdot \\|_1$	The Sobolev norm on the space $H^1(\Omega)$
$\vert \cdot \vert_1$	The seminorm on the space $H^1(\Omega)$
$\\| \cdot \\|_0$	The usual norm on $L^2(\Omega)$
$\\| \cdot \\|_{L^2(\omega)}$	The norm on $L^2(\omega)$, where $\omega \subset \Omega$
$\vert\vert\vert \cdot \vert\vert\vert$	The energy norm on $H^1(\Omega)$

CHAPTER I
INTRODUCTION

The finite element method is a tool widely used for approximating the solution of partial differential equations (PDEs) based on underlying variational formulation. Nowadays, finite element methods can be applied to more complicated problems and becomes the efficient tools for large-scale applications in almost every area of sciences.

Adaptive finite element methods (AFEMs) for the numerical solution of PDEs started in the last 70's and now are standard tools in many areas in sciences and engineering. AFEMs are effective tools to obtain good approximate solutions with low computational costs, especially in the presence of singularities and in the problems with boundary layers.

There exists a vast variety of books about finite elements. Here, we only want to mention the books by Braess [2] and Brenner and Scott [3] as references for this work. Another basic ingredient for an AFEM is known as a posteriori error estimator, described in, for example, Ainsworth and Oden [1], Verf*ü*rth [15] and Nochetto et al. [13], which is the main objective of the analysis of adaptive methods for linear and non-linear problems.

For elliptic PDEs, AFEMs are boiled down to iterations of the form

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}.$$

Given a current mesh (triangulation) of the domain and conditions of the problem, SOLVE finds the approximate solution corresponding to the given mesh; ESTIMATE computes error estimates in a suitable norm based on a posterior error estimators to estimate the error of obtained solution for given mesh; MARK marks the selected element by using a posteriori error estimators; REFINE refines the marked elements to obtain a finer mesh according to the elements with high error estimators. The ultimate purpose is to construct a sequence of approximate solutions that converges to the exact solution.

For linear elliptic partial differential equations, there are several results of AFEMs:

- In 1996, Dörfler [6] introduced a crucial marking stratergy and proved the strict energy error reduction for the Poisson's equation and provided the initial mesh satisfying a fineness assumption.

- Morin et al. [11, 12] studied linear elliptic PDEs,

  $$-\nabla \cdot (A(x)\nabla u(x)) = f(x), \qquad x \in \Omega,$$

  for a piecewise constant function $A(x)$. They proved a convergence without restrictions on the initial mesh and introduced the concepts of data oscillations and the interior node property, which are very important for obtaining the convergence of AFEMs.

- Mekchay and Nochetto [10] extended the idea of [11] to obtain the result for general second order elliptic PDE,

  $$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = f, \qquad \text{in } \Omega,$$

  where A, f, b and c are suitable functions.

- Cascon et al. [4] considered Dirichlet boundary value problem for second order elliptic PDE,

  $$-\mathrm{div}(A\nabla u) + cu = f, \qquad \text{in } \Omega,$$

  for piecewise Lipschitz function $A(x)$. They obtained quasi-optimal convergence rate without the usages of the local lower bound and interior node property, as the new idea for convergence.

For nonlinear elliptic partial differential equations, here are some results of AFEMs:

- Dörfler [7] developed a robust strategy for nonlinear Poisson equation,

$$-\Delta u = f(u), \qquad \text{in } \Omega,$$

  where $\Omega \subset \mathbb{R}^2$ and $f \in C^{0,1}(\mathbb{R})$.

- Veeser [14] proved convergence of AFEM for the nonlinear Laplacian:

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u), \qquad \text{in } \Omega,$$

  for $p \in (1, \infty)$, given that $f \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.

- Diening and Kreuzer [5] proved that the AFEM converges for p-Laplacian with linear rate for

$$-\mathrm{div}(\kappa + |\nabla u|^{p-2}\nabla u) = f, \qquad \text{in } \Omega,$$

  where $p \in (1, \infty)$, $\kappa \geq 0$, and $f \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$.

- Garau et al. [8] showed that the AFEM converges for the quasi-linear problems:

$$-\nabla \cdot [\alpha(\cdot, |\nabla u|^2)\nabla u] = f, \qquad \text{in } \Omega,$$

  where $\alpha : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ and $f \in L^2(\Omega)$. Convergence is based on Kačanov iterations.

In this thesis, we analyze a standard adaptive finite element method for second order semi-linear elliptic partial differential equations of the form

$$-\nabla \cdot (A(x)\nabla u(x)) = f(x, u(x)), \qquad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^2$, for $x \in \Omega$, $f(x, u(x))$ is Lipschitz in the second argument and $A(x)$ is a positive definite matrix with strictly monotonicity property. Our work is developed based on the idea of Cascon et al. [4], who studied the linear model, to obtain contraction property for semi-linear elliptic problem. Our proof is based on the assumptions that the initial triangulation is sufficiently refined in order to deal with the nonlinear function $f(x, u(x))$, which is assumed to be Lipschitz in the second variable. For example, the Poisson-Bolztmann equation that deals with a nonlinear $f(x, u(x)) = \kappa^2 \sinh(u(x))$.

This thesis is organized as follows. In chapter 2, we give some preliminaries, basic definitions and theorems that are important in formulating and obtaining the error estimates in order to obtain the convergence. In chapter 3, we analyze the standard finite element method and AFEM. Here, we construct the crucial lemmas for obtaining the contraction property. In the last chapter the contraction property and the convergence results are presented. Finally, we conclude our finding and provide some ideas for designing AFEM algorithm.

CHAPTER II
PRELIMINALIES

2.1 The Sobolev

We introduce the Sobolev spaces, refer to the book by D. Braess [2]. Let Ω be an open subset of $\mathbb{R}^2$ with piecewise smooth boundary. The Sobolev spaces are built upon the function space $L^2(\Omega)$, which consists of all functions u which are square-integrable over Ω in the sense of Lebesgue measure. The space $L^2(\Omega)$ becomes a Hilbert space with the inner product

$$\langle u, v \rangle := \int_\Omega u(x)v(x)dx$$

and the corresponding norm

$$\|u\|_0 = \sqrt{\langle u, u \rangle}.$$

Definition 2.1. We say that a given function $f \in L^1(\Omega)$, where $L^1(\Omega) := \{g : \Omega \to \mathbb{R} | \int_\Omega |g(x)|\, dx < \infty\}$, has a weak derivative $D_w^\alpha f$, provided there is a function $v \in L^1(\Omega)$ such that

$$\int_\Omega v(x)\varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \varphi(x)\, dx, \qquad \forall \varphi \in C_0^\infty(\Omega),$$

where $\alpha := (\alpha_1, \alpha_2)$ is a multi index with $|\alpha| = \alpha_1 + \alpha_2$ where α_1 and α_2 are nonnegative integers, $D^\alpha \varphi = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \varphi$ and $C_0^\infty(\Omega)$ is the set of smooth functions vanshing on $\partial\Omega$. If such a v exists, we define $D_w^\alpha f = v$. We denote gradient operator $\nabla = (D_w^{(1,0)}, D_w^{(0,1)})$ or $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$.

Definition 2.2. Let $H^1(\Omega)$ be the set of all functions u in $L^2(\Omega)$ which processes weak derivative ∇u. We can define an inner product on $H^1(\Omega)$ by

$$\langle u, v \rangle_1 := \langle u, v \rangle + \langle \nabla u, \nabla v \rangle$$

with the associated norm

$$\|u\|_1 := \sqrt{\langle u, u \rangle_1} = \sqrt{\|u\|_0^2 + \|\nabla u\|_0^2}.$$

The corresponding semi-norm on $H^1(\Omega)$ is defined as

$$|u|_1 := \|\nabla u\|_0.$$

$H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ vanishing on boundary $\partial\Omega$.

Theorem 2.3 (Poincaré inequality)**.** *Suppose $\Omega \subset \mathbb{R}^2$ is an open bounded domain. Then, for all $u \in H_0^1(\Omega)$,*

$$\|u\|_0 \leq C_P \|\nabla u\|_0,$$

where C_P is a constant depending only on Ω.

Proof. See the book of Braess [2], p 30. □

We then obtain that for any $u \in H_0^1(\Omega)$,

$$|u|_1 \leq \|u\|_1 = \|\nabla u\|_0 + \|u\|_0 \leq (1 + C_P)\|\nabla u\|_0 = (1 + C_P)|u|_1.$$

Thus, $|\cdot|_1$ is equivalent to $\|\cdot\|_1$ on $H_0^1(\Omega)$.

Definition 2.4. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. A conforming triangulation (mesh) $\mathcal{T}$ of Ω is a collection $\{T\}$ of triangles (elements) such that:

(i) $\Omega = \bigcup_{T \in \mathcal{T}} T$;

(ii) for $T, T' \in \mathcal{T}$ and $T \neq T'$ the set $T \cap T'$ is empty or consists of a vertex or a common side.

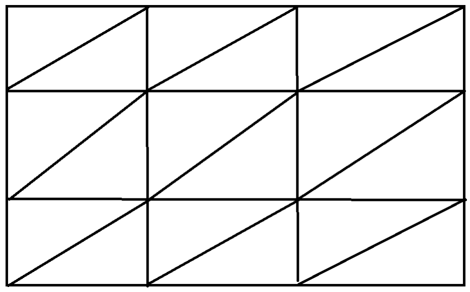

Figure 2.1: An example of conforming triangulation for a rectangular domain.

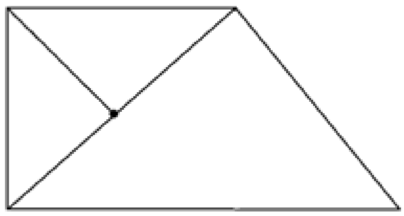

Figure 2.2: A triangulation is nonconforming with hanging node.

Definition 2.5. A family of triangulations $\{\mathcal{T}_i\}$ of Ω is called shape regular provided that there exists a number $\kappa > 0$, shape-regular parameter, such that every $T \in \mathcal{T}_i$ for all $\mathcal{T}_i$ contains a circle of radius ρ_T with

$$\frac{h_T}{\rho_T} \leq \kappa, \tag{2.1.1}$$

where h_T is half of the diameter of T, i.e., $h_T = \frac{1}{2} \max_{x,y \in T} |x - y|$.

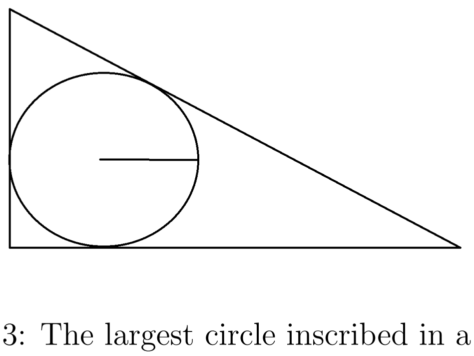

Figure 2.3: The largest circle inscribed in a triangle

Let $\mathcal{T}_0$ be an initial triangulation of Ω. If we decompose a subset of triangles of $\mathcal{T}_0$ into subtriangles such that the resulting set of triangles is again a triangulation of Ω, we call this a refinement of $\mathcal{T}_0$.

Definition 2.6. Let $\mathcal{T}_0$ be an initial triangulation of Ω and $\mathbb{T}$ the class of all shape-regular conforming refinements of $\mathcal{T}_0$. Given any conforming triangulation $\mathcal{T} \in \mathbb{T}$, as define the corresponding finite element space to be the space of continuous piecewise polynomial functions of degree $n \geq 1$,

$$\mathbb{V}^n(\mathcal{T}) := \{v \in H_0^1(\Omega) : v_{|T} \in \mathbb{P}_n(T), \forall T \in \mathcal{T}\},$$

where $\mathbb{P}_n(T)$ is the space of all polynomials on T of degree less than or equal to n. If there is no ambiguity, we will use $\mathbb{V}(\mathcal{T})$ for simplicity.

Next, we introduce the extension of a function defined on $S \subset \partial T$ onto the triangle $T \in \mathcal{T}$. For $v : S \to \mathbb{R}$, let $E(v) : T \to \mathbb{R}$ be the extension of v onto T such that the value of $E(v)$ is constant along a line parallel to side of T. (see Figure 2.4).

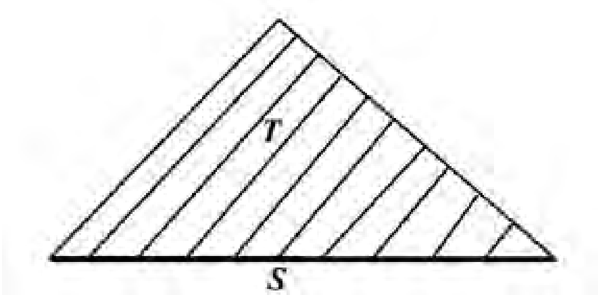

Figure 2.4: The extension from S to T

Theorem 2.7. *Let $\mathcal{T}$ be a shape regular triangulation. Then there exists a constant c which depends only on κ such that, for all $T \in \mathcal{T}$ and all $S \in \partial T$,*

$$\|v\|_{L^2(S)} \leq ch^{-1/2}\|E(v)\|_{L^2(T)}, \quad \forall\, v \in \mathbb{P}_2(T),$$

where $E : L^2(S) \longrightarrow L^2(T)$ is the extension of a function on S onto T.

Proof. See equation (8.26) of Lemma 8.3 in the book of Braess [2], p 174. $\square$

Theorem 2.8 (Inverse estimates)**.** *Let $\mathbb{V}^n(\mathcal{T})$ be a finite element space with a conforming triangulation $\mathcal{T} \in \mathbb{T}$. Then, there exists a constant $c = c(\kappa, n)$ such that*

$$\|v_{\mathcal{T}}\|_1 \leq ch^{-1}\|v_{\mathcal{T}}\|_0, \qquad \forall\, v_{\mathcal{T}} \in \mathbb{V}^n(\mathcal{T}),$$

where $h = \max_{T \in \mathcal{T}} h_T$.

Proof. See the book of Braess [2], p 83. $\square$

2.2 Problem and formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded, polyhedral domain. We consider the second order semi-linear elliptic partial differential equation in divergence form with vanishing boundary condition,

$$-\nabla \cdot (A(x)\nabla u(x)) = f(x, u(x)), \qquad \forall\, x \in \Omega, \tag{2.2.1}$$

$$u(x) = 0, \qquad \forall\, x \in \partial\Omega, \tag{2.2.2}$$

where $f(x, u(x))$ satisfies $\int_\Omega |f(x, u(x))|^2 dx < \infty$ and is Lipschitz in the second argument, i.e., there exists a Lipschitz constant L_f such that

$$|f(x, v) - f(x, w)| \leq L_f \, |v - w| \,, \qquad \forall\, x \in \Omega \;\forall\, v, w \in \mathbb{R},$$

and $A(x)$ is a positive definite matrix having components in $C^1(\Omega)$ and satisfies strictly monotonicity property, i.e., there exists a positive constant θ_* such that

$$[A(x)p(x)] \cdot p(x) \geq \theta_* \, |p(x)|^2 \,, \qquad \forall\, p(x) \in \mathbb{R}^2 \;\; \forall x \in \Omega.$$

A weak solution of (2.2.1)-(2.2.2) is a function $u \in H_0^1(\Omega)$ satisfying

$$\mathcal{B}(u, v) = \mathcal{L}(u; v) \qquad \forall v \in H_0^1(\Omega), \tag{2.2.3}$$

where the bilinear form $\mathcal{B} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{B}(u, v) = \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\, dx. \tag{2.2.4}$$

Note that $\mathcal{B}$ is symmetric since A is positive definite.
The functional $\mathcal{L} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{L}(u; v) = \int_\Omega f(x, u(x))v(x)\, dx.$$

For example, $f(x, u(x)) = e^{-|u(x)|^2}$, we see that $\int_\Omega |f(x, u(x)|^2 dx < \infty$ and $\mathcal{L}$ is well-defined.

Definition 2.9. Let H be a Hilbert space. A bilinear form $a : H \times H \longrightarrow \mathbb{R}$ is called continuous (or bounded) provided there exists $C_a > 0$ such that

$$|a(v,w)| \leq C_a \|v\|_H \|w\|_H, \qquad \forall\, v, w \in H.$$

A symmetric continuous bilinear form a is called coercive on $V \subset H$ provided there exists $c_a > 0$ such that

$$a(v,v) \geq c_a \|v\|_H^2, \qquad \forall\, v \in V.$$

Lemma 2.10. *The bilinear form $\mathcal{B}$ in* (2.2.4) *is coercive on $H_0^1(\Omega)$ and bounded on $H^1(\Omega)$.*

Proof. See Jampawai [9], p 13. □

The bilinear form $\mathcal{B}$ induces the *energy norm* on $H_0^1(\Omega)$, defined as

$$|\!|\!| v |\!|\!| := \sqrt{\mathcal{B}(v,v)}, \qquad \forall\, v \in H_0^1(\Omega).$$

Note that the norm $\|\cdot\|_1$, the semi-norm $|\cdot|_1$, and the energy norm $|\!|\!|\cdot|\!|\!|$ are all equivalent on $H_0^1(\Omega)$. Then there exists the unique approximation of u, called the finite element solution, defined as

$$u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T}); \qquad \mathcal{B}(u_{\mathcal{T}}, v) = \mathcal{L}(u_{\mathcal{T}}; v), \qquad \forall\, v \in \mathbb{V}(\mathcal{T}). \tag{2.2.5}$$

Next, we will construct lemmas for linear elliptic problem. Let L be a second-order elliptic differential operator with divergence structure

$$Lu = -\nabla \cdot (A \nabla u).$$

Consider the second order elliptic boundary-vlaue problem

$$Lu(x) = g(x), \qquad \forall\, x \in \Omega, \tag{2.2.6}$$

$$u(x) = 0, \qquad \forall\, x \in \partial\Omega, \tag{2.2.7}$$

with homogeneous Dirichlet boundary conditions, provided that

$$a(u,v) = \langle g, v \rangle \qquad \forall v \in H_0^1(\Omega),$$

where

$$a(u,v) = \int_\Omega A \nabla u \cdot \nabla v dx.$$

Lemma 2.11 (Céa's lemma)**.** *Suppose the bilinear form a is symmetric and coercive with $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. In addition, suppose $\tilde{u}$ and $\tilde{u}_{\mathcal{T}}$ are the solutions of variational problem in V and $\mathbb{V}(\mathcal{T}) \subset V$ of the problem* (2.2.6)-(2.2.7)*, respectively. Then,*

$$\|\tilde{u} - \tilde{u}_{\mathcal{T}}\|_1 \leq \frac{C_a}{c_a} \inf_{v_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})} \|\tilde{u} - v_{\mathcal{T}}\|_1.$$

Proof. See Theorem 4.2 in Braess [2], p 55. □

Theorem 2.12 (Regularity theorem)**.** *Let Ω be convex. Suppose $\mathcal{T}$ is a family of shape regular triangulations of Ω. Then, the solutions $\tilde{u}$ and $\tilde{u}_{\mathcal{T}}$ in Lemma 2.11 satisfies*

$$\|\tilde{u} - \tilde{u}_{\mathcal{T}}\|_1 \leq ch\|g\|_0,$$

where $c = (\Omega, a)$.

Proof. See Theorem 7.3 in Braess [2], p 90. □

Theorem 2.13 (Duality argument)**.** *Under the hypotheses of Theorem 2.12, if $\tilde{u} \in H^1(\Omega)$ is the solution of the associated variational problem, then*

$$\|\tilde{u} - \tilde{u}_{\mathcal{T}}\|_0 \leq cC_\Omega h\|\tilde{u} - \tilde{u}_{\mathcal{T}}\|_1.$$

If in addition, $g \in L^2(\Omega)$ so that $\tilde{u} \in H^2(\Omega)$, then

$$\|\tilde{u} - \tilde{u}_{\mathcal{T}}\|_0 \leq cC_\Omega^2 h^2\|g\|_0.$$

Proof. See corollary 7.7 in Braess [2], p 92. □

For simplicity, we write $f_{\mathcal{T}} := f(x, u_{\mathcal{T}})$, $f_k := f_{\mathcal{T}_k}$ and $f := f(x, u)$. Based on the results obtained by Jumpawai [9], the L^2 estimates for the error can be given as follows.

Lemma 2.14. *Let u be a weak solution satisfying* (2.2.3) *and $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ be the solution of* (2.2.5)*. Then,*

$$\|u - u_{\mathcal{T}}\|_0 \leq C_1^*\|u - u_{\mathcal{T}}\|_1 \sup_{g \in L^2(\Omega), \|g\|_0 \leq 1} \left(\inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \right) + C_2^*\|f - f_{\mathcal{T}}\|_0,$$

where C_1^* and C_2^* are constants depending only on data. For a given $g \in L^2(\Omega)$, denoted by $\varphi_g \in H_0^1(\Omega)$ the corresponding unique solution of the linear equation

$$\mathcal{B}(\varphi_g, w) = \langle g, w \rangle, \qquad \forall\, w \in H_0^1(\Omega). \tag{2.2.8}$$

Proof. Let $w \in L^2(\Omega)$. Then, $w \in (L^2(\Omega))^*$, the dual space of $L^2(\Omega)$. Ones can easily show that

$$\|w\|_0 = \sup_{g \in L^2(\Omega), \|g\|_0 \le 1} \langle g, w \rangle. \tag{2.2.9}$$

From (2.2.3) and (2.2.5), we have

$$\mathcal{B}(u - u_{\mathcal{T}}, v) = \langle f - f_{\mathcal{T}}, v \rangle, \qquad \forall\, v \in \mathbb{V}(\mathcal{T}). \tag{2.2.10}$$

By setting $w := u - u_{\mathcal{T}} \in H_0^1(\Omega)$ in (2.2.8) and using (2.2.10), for any $\tilde{v} \in \mathbb{V}(\mathcal{T})$ we have

$$\begin{aligned} \langle g, u - u_{\mathcal{T}} \rangle = \mathcal{B}(\varphi_g, u - u_{\mathcal{T}}) &= \mathcal{B}(\varphi_g - \tilde{v}, u - u_{\mathcal{T}}) + \mathcal{B}(\tilde{v}, u - u_{\mathcal{T}}), \\ &= \mathcal{B}(\varphi_g - \tilde{v}, u - u_{\mathcal{T}},) + \langle f - f_{\mathcal{T}}, \tilde{v} \rangle. \end{aligned} \tag{2.2.11}$$

Applying the continuity of $\mathcal{B}$ and the Cauchy-Schwartz inequality to get

$$\langle g, u - u_{\mathcal{T}} \rangle \le C_B \|u - u_{\mathcal{T}}\|_1 \cdot \|\varphi_g - \tilde{v}\|_1 + \|f - f_{\mathcal{T}}\|_0 \|\tilde{v}\|_0. \tag{2.2.12}$$

Let $\varphi_{g,\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ be a finite element solution of φ_g in (2.2.8). By Céa's Lemma 2.11,

$$\|\varphi_g - \varphi_{g,\mathcal{T}}\|_1 \le \frac{C_B}{c_B} \inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1. \tag{2.2.13}$$

Taking $\tilde{v} = \varphi_{g,\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ and using (2.2.13) in (2.2.12), it gives

$$\langle g, u - u_{\mathcal{T}} \rangle \le C_B \|u - u_{\mathcal{T}}\|_1 \cdot \|\varphi_g - \varphi_{g,\mathcal{T}}\|_1 + \|f - f_{\mathcal{T}}\|_0 \|\varphi_{g,\mathcal{T}}\|_0,$$

hence,

$$\langle g, u - u_{\mathcal{T}} \rangle \le \frac{C_B^2}{c_B} \|u - u_{\mathcal{T}}\|_1 \left(\inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \right) + \|f - f_{\mathcal{T}}\|_0 \|\varphi_{g,\mathcal{T}}\|_0. \tag{2.2.14}$$

By triangle inequality, the last term of (2.2.14) becomes

$$\|\varphi_{g,\mathcal{T}}\|_0 = \|\varphi_{g,\mathcal{T}} - \varphi_g + \varphi_g\|_0 \le \|\varphi_{g,\mathcal{T}} - \varphi_g\|_0 + \|\varphi_g\|_0. \tag{2.2.15}$$

By duality technique for linear problem (2.2.8) on a convex polygonal domain and duality argument (Theorem 2.13), the first term on the right hand side of (2.2.15) becomes

$$\|\varphi_{g,\mathcal{T}} - \varphi_g\|_0 \le C_\Omega h \|\varphi_{g,\mathcal{T}} - \varphi_g\|_1 \le c C_\Omega h^2 \|g\|_0, \tag{2.2.16}$$

where C_Ω and c are constants depending on the domain Ω and shape-regularity.

Setting $w = \varphi_g$ in (2.2.8) and applying the coercivity (Lemma (2.10)) and Cauchy-Schwartz inequality, we get

$$c_B \|\varphi_g\|_1^2 \le \mathcal{B}(\varphi_g, \varphi_g) = \langle g, \varphi_g \rangle \le \|g\|_0 \|\varphi_g\|_0.$$

Since $\|v\|_0 \le \|v\|_1$ for all $v \in H^1(\Omega)$, we get $c_B \|\varphi_g\|_0^2 \le \|g\|_0 \|\varphi_g\|_0$. Therefore,

$$\|\varphi_g\|_0 \le \frac{1}{c_B} \|g\|_0. \tag{2.2.17}$$

Combining the previous inequalities into (2.2.15), we have

$$\begin{aligned} \|\varphi_{g,\mathcal{T}}\|_0 &\le \|\varphi_{g,\mathcal{T}} - \varphi_g\|_0 + \|\varphi_g\|_0 \\ &\le c C_\Omega h^2 \|g\|_0 + \frac{1}{c_B} \|g\|_0 \\ &= \left(c C_\Omega h^2 + \frac{1}{c_B} \right) \|g\|_0. \end{aligned}$$

The inequality (2.2.14) becomes

$$\langle g, u - u_\mathcal{T} \rangle \le \frac{C_B}{c_B} \|u - u_\mathcal{T}\|_1 \inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 + \left(c C_\Omega h^2 + \frac{1}{c_B} \right) \|f - f_\mathcal{T}\|_0 \|g\|_0.$$

By setting $C_1^* = \frac{C_B}{c_B}$ and $C_2^* = c C_\Omega + \frac{1}{c_B}$ by assuming that $h < 1$ and taking the suppremum over all $\|g\|_0 \le 1$, we obtain the result

$$\begin{aligned} \|u - u_\mathcal{T}\|_0 &= \sup_{g \in L^2(\Omega), \|g\|_0 \le 1} \langle g, u - u_\mathcal{T} \rangle \\ &\le C_1^* \|u - u_\mathcal{T}\|_1 \sup_{g \in L^2(\Omega), \|g\|_0 \le 1} \left(\inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \right) + C_2^* \|f - f_\mathcal{T}\|_0. \end{aligned}$$

□

Corollary 2.15. *Under the hypotheses of Lemma 2.14 and f satisfies $L_f \le \rho < \frac{1}{C_2^*}$ for some positive ρ. Then,*

$$\|u - u_\mathcal{T}\|_0 \le C_f h \|u - u_\mathcal{T}\|_1,$$

where C_f is a constant depending only on ρ, the shape regularity, and the data (A, Ω).

Proof. By the regularity Theorem 2.12,

$$\inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \leq \|\varphi_g - \varphi_{g,\mathcal{T}}\|_1 \leq ch\|g\|_0.$$

Lemma 2.14 becomes

$$\|u - u_\mathcal{T}\|_0 \leq cC_1^* h\|u - u_\mathcal{T}\|_1 + C_2^*\|f - f_\mathcal{T}\|_0.$$

By the Lipschitz condition, it follows that $\|f - f_\mathcal{T}\|_0 \leq L_f\|u - u_\mathcal{T}\|_0$ and by assumption $L_f \leq \rho$, we get $\|f - f_\mathcal{T}\|_0 \leq \rho\|u - u_\mathcal{T}\|_0$. Hence,

$$\|u - u_\mathcal{T}\|_0 \leq cC_1^* h\|u - u_\mathcal{T}\|_1 + C_2^*\rho\|u - u_\mathcal{T}\|_0.$$

Since $C_2^*\rho < 1$, we can combine terms to get

$$\|u - u_\mathcal{T}\|_0 \leq C_f h\|u - u_\mathcal{T}\|_1,$$

where $C_f := \frac{cC_1^*}{1 - C_2^*\rho}$ is a positive constant. $\square$

CHAPTER III

ADAPTIVE FINITE ELEMENT METHODS

3.1 Adaptive Finite Element Method: AFEM

We analyze here a standard adaptive finite element method (AFEM) as a loop of procedures

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

SOLVE: Given a current triangulation $\mathcal{T}$ of the domain Ω and a finite element space $\mathbb{V}(\mathcal{T})$, it produces the finite element solution $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$,

$$u_{\mathcal{T}} = \text{SOLVE}(\mathcal{T}).$$

We cannot compute integrals involving a nonlinear function f since (2.2.5) is a nonlinear problem. By (2.2.5), we find $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$

$$\mathcal{B}(u_{\mathcal{T}}, v) = \langle f(x, u_{\mathcal{T}}), v \rangle, \qquad \forall v \in \mathbb{V}(\mathcal{T}).$$

Using the finite element basis $\{\psi_i\}_{i=1}^{N_{\mathcal{T}}}$ is formulated as follows: let $u_{\mathcal{T}} = \sum_{i=1}^{N_{\mathcal{T}}} u_i \psi_i$, for $v = \psi_j \in \mathbb{V}(\mathcal{T})$,

$$\langle f(x, u_{\mathcal{T}}), \psi_j \rangle = \mathcal{B}(u_{\mathcal{T}}, \psi_j) = \mathcal{B}(\sum_{i=1}^{N_{\mathcal{T}}} u_i \psi_i, \psi_j) = \sum_{i=1}^{N_{\mathcal{T}}} u_i \mathcal{B}(\psi_i, \psi_j),$$

then

$$\begin{bmatrix} \mathcal{B}(\psi_1, \psi_1) & \mathcal{B}(\psi_1, \psi_2) & \dots & \mathcal{B}(\psi_1, \psi_{N_{\mathcal{T}}}) \\ \vdots & \vdots & \dots & \vdots \\ \mathcal{B}(\psi_{N_{\mathcal{T}}}, \psi_1) & \mathcal{B}(\psi_1, \psi_2) & \dots & \mathcal{B}(\psi_{N_{\mathcal{T}}}, \psi_{N_{\mathcal{T}}}) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \langle f(x, \sum u_i \psi_i(x)), \psi_1 \rangle \\ \vdots \\ \langle f(x, \sum u_i \psi_i(x))), \psi_n \rangle \end{bmatrix}.$$

Set $\mathcal{A} = [b_{ij}]$, where $b_{ij} = \mathcal{B}(\psi_i, \psi_j)$ and $F = [\langle f(x, \sum u_i \psi_i(x))), \psi_j \rangle]^t$ and $U = [u_1\, u_2 \dots u_n]^t$, we have a nonlinear system for U,

$$\mathcal{A}U = F(U).$$

We seek the solution of the nonlinear system

$$\mathcal{A}U - F(U) = 0. \tag{3.1.1}$$

The solution U of (3.1.1) can be estimate by iterative techniques such as the Newton's method or the Kačanov iteration as used in [8] for quasi-linear problem. For the Newton's method, we require that the Jacobian of $\mathcal{A}U - F(U)$ is non-singular, i.e., $\left|\mathcal{A} - \frac{\partial F(U)}{\partial U}\right| \neq 0$, and the initial guest U_0 is chosen approximately in order to the convergence of the method.

ESIMATE: For $\mathcal{T} \in \mathbb{T}$, $T \in \mathcal{T}$ and $v \in H_0^1(\Omega)$, we define the local interior residual

$$\mathcal{R}_T(v) := f(x,v)|_T + \nabla \cdot (A\nabla v)|_T. \tag{3.1.2}$$

The jump residual on side $S \subset \partial T \cap \Omega$

$$J_S(v) := (A\nabla v)|_S \cdot \vec{n}_T + (A\nabla v)|_S \cdot \vec{n}_{T'}, \tag{3.1.3}$$

where $\vec{n}_T$ and $\vec{n}_{T'}$ are the outward unit normal vectors on S corresponding to T and T', respectively (see Figure 3.1).

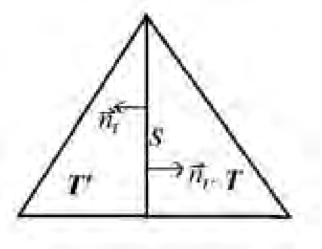

Figure 3.1: The outward unit normal vectors on S corresponding to T and T'

The local error indicator $\eta_{\mathcal{T}}(v,T)$ on T is defined via

$$\eta^2_{\mathcal{T}}(v,T) := h_T^2 \|\mathcal{R}_T(v)\|^2_{L^2(T)} + h_T \|J_S(v)\|^2_{L^2(\partial T \cap \Omega)}, \tag{3.1.4}$$

where we define here that $h_T = |T|^{1/2}$, $|T|$ is the area of T in $\mathbb{R}^2$. We can show that this definition is equivalent to the half diameter of T defined in Definition 2.5.

Let a, b and c be sides of a triangle T such that c is the longest side of T, and contains a circle of radius ρ_T. If θ_c is the angle opposite of side c, then

$$|T|^{1/2} = \left|\frac{1}{2}ab\sin(\theta_c)\right|^{1/2} \leq \left|\frac{1}{2}ab\right|^{1/2} \leq \left|\frac{1}{2}c^2\right|^{1/2} = \sqrt{2}\frac{\mathrm{diam}(T)}{2}.$$

Conversely, by property of triangles, we obtain

$$|T|^{1/2} = \left[\left(\frac{a+b+c}{2}\right)\rho_T\right]^{1/2} = \left[\left(\frac{c}{2}+\frac{a+b}{2}\right)\rho_T\right]^{1/2} \geq [c\rho_T]^{1/2}.$$

By shape-regular (2.1.1),

$$|T|^{1/2} \geq \left[c\frac{c/2}{\kappa}\right]^{1/2} = \sqrt{\frac{2}{\kappa}}\frac{\mathrm{diam}(T)}{2}.$$

Therefore, both definitions of h_T are equivalent.

The global error indicator $\eta_{\mathcal{T}}$ for $\mathcal{T}$ is

$$\eta_{\mathcal{T}}(v) := \left(\sum_{T\in\mathcal{T}} \eta_{\mathcal{T}}^2(v,T)\right)^{1/2},$$

and for any subset $\mathcal{T}' \subset \mathcal{T}$,

$$\eta_{\mathcal{T}}(v,\mathcal{T}') := \left(\sum_{T\in\mathcal{T}'} \eta^2(v,T)\right)^{1/2}.$$

Based on a posteriori error analysis, see [1], Jampawai [9] obtained the upper bound estimate stated as:

Lemma 3.1 (Upper bound)**.** *Let u be the weak solution* (2.2.3) *of the model problem and u_k =SOLVE($\mathcal{T}_k$). Then,*

$$|||u-u_k||| \leq C_1\eta_k(u_k) + C_2 h_k\|f-f_k\|_0, \tag{3.1.5}$$

where C_1, C_2 depend on the shape regularity and the data (A,Ω), h_k is defined to be the maximum of h_T for T in $\mathcal{T}_k$, and denoting $\eta_k(u_k)$ for $\eta_{\mathcal{T}_k}(u_k)$.

Proof. See Jampawai [9], p 20. □

MARK: Given a triangulation $\mathcal{T}$, the set of indicators $\{\eta_{\mathcal{T}}(u_{\mathcal{T}},T)\}_{T\in\mathcal{T}}$, and the marking parameter $\theta \in (0,1]$, the procedure MARK produces a marked subset $\mathcal{M} \subset \mathcal{T}$,

$$\mathcal{M} = \mathrm{MARK}(\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)\}_{T\in\mathcal{T}}, \mathcal{T}, \theta),$$

such that $\mathcal{M}$ satisfies some marking properties in some optimal way. For example, in this paper we use Dörfler's marking [6],

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \geq \theta\eta_{\mathcal{T}}(u_{\mathcal{T}}). \tag{3.1.6}$$

MARK will find an optimal subset $\mathcal{M}$ satisfying the marking property (3.1.6).

REFINE: Given a fixed integer $b \geq 1$, for any $\mathcal{T} \in \mathbb{T}$ and $\mathcal{M} \subset \mathcal{T}$ of marked elements, the procedure produces a finer conforming triangulation

$$\mathcal{T}_* = \mathrm{REFINE}(\mathcal{T}, \mathcal{M})$$

by refining all elements $T \in \mathcal{M}$ for b times, and together with a few more elements surrounding to be conforming. Note that $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}(\mathcal{T}_*)$. For $T' \in \mathcal{T}_*\backslash\mathcal{T}$ obtained by refining $T \in \mathcal{T}$, i.e., by using newest vertex bisection method b times, we have

$$|T'| \leq 2^{-b}|T|. \tag{3.1.7}$$

Note that for T', as a child of T,

$$h_{T'} = 2^{-b/2}h_T. \tag{3.1.8}$$

Adaptive Algorithm.

Given the initial grid $\mathcal{T}_0$, TOL, and marking parameter $0 < \theta \leq 1$, set $k = 0$:

(i) $u_k = \mathrm{SOLVE}(\mathcal{T}_k)$;

(ii) $\{\eta_k(u_k, T)\}_{T\in\mathcal{T}_k} = \mathrm{ESTIMATE}(u_k, \mathcal{T}_k)$; (STOP: if $\eta_k <$ TOL.)

(iii) $\mathcal{M}_k = \mathrm{MARK}(\{\eta_k(u_k, T)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k, \theta)$;

(iv) $\mathcal{T}_{k+1} = \mathrm{REFINE}(\mathcal{M}_k, \mathcal{T}_k)$; set $k = k + 1$, go to Step 1.

Note that from (3.1.8) the algorithm gives the decreasing sequence $\{h_k\}_{k\geq 0}$, namely, $h_k \leq h_0$ for all k.

3.2 Lemmas

In this section we prove lemmas required for obtaining the contraction property and the convergence of AFEM stated in the next section. These lemmas are obtained according to the AFEM algorithm, based on the four main procedures, SOLVE, ESTIMATE, MARK, and REFINE.

Lemma 3.2. *Let u be the weak solution of* (2.2.3), $u_k =$ *SOLVE*$(\mathcal{T}_k)$, *and* $u_{k+1} =$ *SOLVE*$(\mathcal{T}_{k+1})$. *Then,*

$$|||u - u_k|||^2 = |||u - u_{k+1}|||^2 + |||u_{k+1} - u_k|||^2 + 2\left\langle f - f_{k+1}, u_{k+1} - u_k \right\rangle .$$

Proof. By nested property of refinements, we have that $\mathbb{V}_k \subset \mathbb{V}_{k+1} \subset H_0^1(\Omega)$ and $u_{k+1} - u_k \in \mathbb{V}_{k+1} \subseteq H_0^1(\Omega)$. From (2.2.3) and (2.2.5), we get

$$\begin{aligned}
\langle f - f_{k+1}, u_{k+1} - u_k \rangle &= \langle f, u_{k+1} - u_k \rangle - \langle f_{k+1}, u_{k+1} - u_k \rangle \\
&= \mathcal{B}(u, u_{k+1} - u_k) - \mathcal{B}(u_{k+1}, u_{k+1} - u_k) \\
&= \mathcal{B}(u - u_{k+1}, u_{k+1} - u_k).
\end{aligned}$$

By definition of the energy norm, we obtain the followings:

$$\begin{aligned}
\mathcal{B}(u - u_{k+1}, u_{k+1} - u_k) &= \mathcal{B}(u - u_{k+1}, u_{k+1} - u + u - u_k) \\
&= \mathcal{B}(u - u_{k+1}, u_{k+1} - u) + \mathcal{B}(u - u_{k+1}, u - u_k) \\
&= -|||u - u_{k+1}|||^2 + \mathcal{B}(u - u_{k+1}, u - u_k),
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(u - u_{k+1}, u - u_k) &= \mathcal{B}(u - u_k + u_k - u_{k+1}, u - u_k) \\
&= \mathcal{B}(u - u_k, u - u_k) + \mathcal{B}(u_k - u_{k+1}, u - u_k) \\
&= |||u - u_k|||^2 + \mathcal{B}(u_k - u_{k+1}, u - u_k),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}(u_k - u_{k+1}, u - u_k) &= \mathcal{B}(u_k - u_{k+1}, u - u_k + u_{k+1} - u_{k+1}) \\
&= \mathcal{B}(u_k - u_{k+1}, u_{k+1} - u_k) + \mathcal{B}(u_k - u_{k+1}, u - u_{k+1}) \\
&= -|||u_{k+1} - u_k|||^2 - \mathcal{B}(u - u_{k+1}, u_{k+1} - u_k)
\end{aligned}$$

$$= -\|\!| u_{k+1} - u_k \|\!|^2 - \langle f - f_{k+1}, u_{k+1} - u_k \rangle .$$

By combining all these terms together, we obtain

$$\|\!| u - u_k \|\!|^2 = \|\!| u - u_{k+1} \|\!|^2 + \|\!| u_{k+1} - u_k \|\!|^2 + 2 \langle f - f_{k+1}, u_{k+1} - u_k \rangle .$$

□

Lemma 3.3. *Let u satisfies* (2.2.3)*, $u_k = SOLVE(\mathcal{T}_k)$, and $u_{k+1} = SOLVE(\mathcal{T}_{k+1})$. Given that f satisfying the assumption in Corollary 2.15, then*

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq \frac{3}{2} C_e^2 C_f^2 L_f h^2 \|\!| u - u_{k+1} \|\!|^2 + \frac{1}{2} C_e^2 C_f^2 L_f h^2 \|\!| u - u_k \|\!|^2.$$

Proof. By the Cauchy-Schwartz inequality,

$$\begin{aligned} \langle f_{k+1} - f, u_{k+1} - u_k \rangle &= \langle f_{k+1} - f, u_{k+1} - u + u - u_k \rangle \\ &= \langle f_{k+1} - f, u_{k+1} - u \rangle + \langle f_{k+1} - f, u - u_k \rangle \\ &\leq \| f_{k+1} - f \|_0 \| u_{k+1} - u \|_0 + \| f_{k+1} - f \|_0 \| u - u_k \|_0. \end{aligned}$$

Applying the Lipschitz condition for $\| f_{k+1} - f \|_0$, we get

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq L_f \| u_{k+1} - u \|_0^2 + L_f \| u_{k+1} - u \|_0 \| u - u_k \|_0.$$

By Corollary 2.15, we obtain

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq L_f C_f^2 h^2 \| u - u_{k+1} \|_1^2 + L_f C_f^2 h^2 \| u - u_{k+1} \|_1 \| u - u_k \|_1.$$

By the equivalence of norms $\|\!| \cdot \|\!|$ and $\| \cdot \|_1$, i.e., there is a constant $C_e > 0$ such that $\| \cdot \|_1 \leq C_e \|\!| \cdot \|\!|$, we obtain

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq L_f C_e^2 C_f^2 h^2 \|\!| u - u_{k+1} \|\!|^2 + L_f C_e^2 C_f^2 h^2 \|\!| u - u_{k+1} \|\!| \|\!| u - u_k \|\!|,$$

and by applying the inequality, $2ab \leq a^2 + b^2$, we get

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle = \frac{3}{2} C_e^2 C_f^2 L_f h^2 \|\!| u - u_{k+1} \|\!|^2 + \frac{1}{2} C_e^2 C_f^2 L_f h^2 \|\!| u - u_k \|\!|^2.$$

□

Corollary 3.4. *Under the assumption of Lemma 3.3,*

$$\left(1-3C_e^2C_f^2L_fh^2\right)\|u-u_{k+1}\|^2 \le \left(1+C_e^2C_f^2L_fh^2\right)\|u-u_k\|^2-\|u_{k+1}-u_k\|^2.$$

Proof. By Lemma 3.2, we obtain

$$\|u-u_{k+1}\|^2 = \|u-u_k\|^2-\|u_{k+1}-u_k\|^2+2\left\langle f_{k+1}-f,u_{k+1}-u_k\right\rangle. \tag{3.2.1}$$

Applying Lemma 3.3 to the last term of (3.2.1), we get

$$\begin{aligned}\|u-u_{k+1}\|^2 \le \|u-u_k\|^2-\|u_{k+1}-u_k\|^2+3C_e^2C_f^2L_fh^2\|u-u_{k+1}\|^2\\ +C_e^2C_f^2L_fh^2\|u-u_k\|^2,\end{aligned}$$

which leads to

$$\left(1-3C_e^2C_f^2L_fh^2\right)\|u-u_{k+1}\|^2 \le \left(1+C_e^2C_f^2L_fh^2\right)\|u-u_k\|^2-\|u_{k+1}-u_k\|^2.$$

□

Lemma 3.5. *For any $\mathcal{T}\in\mathbb{T}$, there holds for all $v,w\in\mathbb{V}(\mathcal{T})$, and $\delta>0$,*

$$\begin{aligned}\eta_{\mathcal{T}}^2(v,T) &\le (1+\delta)\eta_{\mathcal{T}}^2(w,T)+h_T\left(1+\frac{1}{\delta}\right)\|J_S(v-w)\|^2_{L^2(\partial T\cap\Omega)}\\ &\quad+2h_T^2\left(1+\frac{1}{\delta}\right)\left(\|\nabla\cdot(A\nabla(v-w))\|^2_{L^2(T)}+\|f(v)-f(w)\|^2_{L^2(T)}\right).\end{aligned}$$

Proof. For any $T\in\mathbb{T}$, let $v,w\in\mathbb{V}(\mathcal{T})$. We denote, for simplicity, $f(x,v)$ and $f(x,w)$ by $f(v)$ and $f(w)$, respectively. Consider $T\in\mathcal{T}$ and its sides $S\subset\partial T$, by using (3.1.2) we get,

$$\begin{aligned}\mathcal{R}_T(v) &= \nabla\cdot(A\nabla v)+f(v)\\ &= \nabla\cdot(A\nabla(v-w))+\nabla\cdot(A\nabla w)+f(w)+f(v)-f(w)\\ &= \mathcal{R}_T(w)+\nabla\cdot(A\nabla(v-w))+f(v)-f(w).\end{aligned}$$

By linearity of the jump residual (3.1.3), we have

$$J_S(v)=J_S(v-w)+J_S(w).$$

The local error indicator (3.1.4) leads to

$$\eta_{\mathcal{T}}^2(v,T)=h_T^2\|\mathcal{R}_T(w)+\nabla\cdot(A\nabla(v-w))+f(v)-f(w)\|^2_{L^2(T)}$$

$$+ h_T \left\| J_S(v-w) + J_S(w) \right\|^2_{L^2(\partial T \cap \Omega)}.$$

By triangle inequality,

$$\begin{aligned} \eta^2_{\mathcal{T}}(v,T) \le h_T^2 \left(\|\mathcal{R}_T(w)\|_{L^2(T)} + \|\nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|_{L^2(T)} \right)^2 \\ + h_T \left(\|J_S(v-w)\|_{L^2(\partial T\cap\Omega)} + \|J_S(w)\|_{L^2(\partial T\cap\Omega)} \right)^2. \end{aligned} \tag{3.2.2}$$

For simplicity, let us denote

$$\begin{aligned} a &:= \|\mathcal{R}_T(w)\|_{L^2(T)}, \\ p &:= \|\nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|_{L^2(T)}, \\ q &:= \|J_S(v-w)\|_{L^2(\partial T\cap\Omega)}, \\ t &:= \|J_S(w)\|_{L^2(\partial T\cap\Omega)}. \end{aligned}$$

The inequality (3.2.2) becomes

$$\eta^2_{\mathcal{T}}(v,T) \le h_T^2 \left(a^2 + p^2 + 2ap\right) + h_T \left(q^2 + t^2 + 2qt\right). \tag{3.2.3}$$

Applying the Young's inequality to ap and qt of (3.2.3), we obtain, for $\delta > 0$,

$$\begin{aligned} \eta^2_{\mathcal{T}}(v,T) &\le h_T^2 \left(a^2 + p^2 + \delta a^2 + \frac{1}{\delta} p^2 \right) + h_T \left(q^2 + t^2 + \delta t^2 + \frac{1}{\delta} q^2 \right) \\ &= h_T^2 (1+\delta) a^2 + h_T^2 \left(1 + \frac{1}{\delta}\right) p^2 + h_T \left(1 + \frac{1}{\delta}\right) q^2 + h_T (1+\delta) t^2. \end{aligned}$$

Therefore,

$$\eta^2_{\mathcal{T}}(v,T) \le (1+\delta)\left\{h_T^2 a^2 + h_T t^2\right\} + h_T \left(1 + \frac{1}{\delta}\right) q^2 + h_T^2 \left(1 + \frac{1}{\delta}\right) p^2. \tag{3.2.4}$$

By (3.1.4), the first term of the right hand side of (3.2.4) becomes $\eta^2_{\mathcal{T}}(w,T)$. For the term p^2 we get

$$\begin{aligned} p^2 &= \left(\|\nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|_{L^2(T)} \right)^2 \\ &\le 2 \left(\|\nabla\cdot(A\nabla(v-w))\|^2_{L^2(T)} + \|f(v) - f(w)\|^2_{L^2(T)} \right). \end{aligned}$$

Finally, (3.2.4) becomes

$$\begin{aligned} \eta^2_{\mathcal{T}}(v,T) \le (1+\delta)\eta^2_{\mathcal{T}}(w,T) + h_T \left(1 + \frac{1}{\delta}\right) \|J_S(v-w)\|^2_{L^2(\partial T\cap\Omega)} \\ + 2h_T^2 \left(1 + \frac{1}{\delta}\right) \left(\|\nabla\cdot(A\nabla(v-w))\|^2_{L^2(T)} + \|f(v) - f(w)\|^2_{L^2(T)} \right). \end{aligned}$$

$\square$

Lemma 3.6. *For $\mathcal{T}_k \in \mathbb{T}$, let $\mathcal{M}_k$ =MARK($\{\eta_k(u_k)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k$) and let $\mathcal{T}_{k+1} \in \mathbb{T}$ be defined by $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$ for $\lambda := 1-2^{-b/2} > 0$. Then, for $v \in \mathbb{V}(\mathcal{T}_k)$,*

$$\eta_{k+1}^2(v) \leq \eta_k^2(v) - \lambda\eta_k^2(v, \mathcal{M}_k).$$

Proof. Let $\overline{\mathcal{M}}_k$ be a set of elements in $\mathcal{T}_k$ that are refined to get $\mathcal{T}_{k+1}$ and $\widetilde{\mathcal{M}}_{k+1}$ be a set of newly obtained elements in $\mathcal{T}_{k+1}$ from the refinement of $\mathcal{T}_k$, i.e., $\widetilde{\mathcal{M}}_{k+1} = \mathcal{T}_{k+1}\backslash(\mathcal{T}_{k+1}\cap\mathcal{T}_k)$. Note that the marked set $\mathcal{M}_k \subseteq \overline{\mathcal{M}}_k \subset \mathcal{T}_k$. It is easy to see that $\bigcup_{T\in\overline{\mathcal{M}}_k} T = \bigcup_{T'\in\widetilde{\mathcal{M}}_{k+1}} T'$ and $\overline{\mathcal{M}}_k \cup (\mathcal{T}_k \cap \mathcal{T}_{k+1}) = \mathcal{T}_k$. Since $\mathcal{T}_{k+1}$ is decomposed into two disjoint subsets $\mathcal{T}_k \cap \mathcal{T}_{k+1}$ and $\widetilde{\mathcal{M}}_{k+1}$, we have

$$\eta_{k+1}^2(v) = \sum_{T\in\mathcal{T}_k\cap\mathcal{T}_{k+1}} \eta_{k+1}^2(v,T) + \sum_{T'\in\widetilde{\mathcal{M}}_{k+1}} \eta_{k+1}^2(v,T'). \tag{3.2.5}$$

Similarly, $\mathcal{T}_k$ is the disjoint union of $\mathcal{T}_k \cap \mathcal{T}_{k+1}$ and $\overline{\mathcal{M}}_k$, then

$$\eta_k^2(v) = \sum_{T\in\mathcal{T}_k\cap\mathcal{T}_{k+1}} \eta_k^2(v,T) + \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T).$$

From the definition of indicators (3.1.4), we have that $\eta_k(v,T) = \eta_{k+1}(v,T)$ for all $v \in H_0^1(\Omega)$, $T \in \mathcal{T}_k \cap \mathcal{T}_{k+1}$. Then, (3.2.5) becomes to

$$\eta_{k+1}^2(v) = \eta_k^2(v) + \sum_{T'\in\widetilde{\mathcal{M}}_{k+1}} \eta_{k+1}^2(v,T') - \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T). \tag{3.2.6}$$

For a marked element $T \in \mathcal{M}_k$, we set $\mathcal{P}_{k+1}(T) = \{T' \in \mathcal{T}_{k+1} : T' \subset T\} \subset \widetilde{\mathcal{M}}_{k+1}$, the set of all children of T. By refinement condition, $h_{T'} \leq 2^{-b/2}h_T$, we have

$$\begin{aligned}\sum_{T'\in\mathcal{P}_{k+1}(T)} \eta_{k+1}^2(v,T') &= \sum_{T'\in\mathcal{P}_{k+1}(T)} (h_{T'}^2\|\mathcal{R}_{T'}(v)\|_{L^2(T')}^2 + h_{T'}\|J_S(v)\|_{L^2(\partial T'\cap\Omega)}^2)\\ &\leq \sum_{T'\in\mathcal{P}_{k+1}(T)} 2^{-b}h_T^2\|\mathcal{R}_{T'}(v)\|_{L^2(T')}^2 + \sum_{T'\in\mathcal{P}_{k+1}(T)} 2^{-b/2}h_T\|J_S(v)\|_{L^2(\partial T'\cap\Omega)}^2\end{aligned}$$

For $v \in \mathbb{V}(\mathcal{T}_k)$ and $T \in \mathcal{T}_k$, the restriction $v_{|_T} \in \mathbb{P}_n(T)$ is continuous and has continuously derivative on T. Then, for the interior sides S of T_1 inside T,

$$J_S(v) = A\nabla v \cdot \vec{n}_1 + A\nabla v \cdot \vec{n}_2 = 0,$$

because $\vec{n}_1 = -\vec{n}_2$, where $\vec{n}_1$ and $\vec{n}_2$ are outward unit normal vectors on S corresponding to T_1 and T_2, respectively (see Figure 3.2).

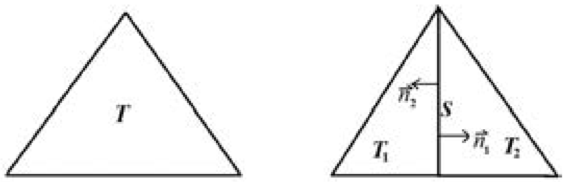

Figure 3.2: The outward unit normal vector for the common side S of T_1 and T_2

Hence,

$$\sum_{T' \in \mathcal{P}_{k+1}(T)} \|J_S(v)\|^2_{L^2(\partial T' \cap \Omega)} = \|J_S(v)\|^2_{L^2(\partial T \cap \Omega)}.$$

By definition of interior residual, since $T = \bigcup_{T' \in \mathcal{P}_{k+1}(T)} T'$,

$$\sum_{T' \in \mathcal{P}_{k+1}(T)} \|\mathcal{R}_{T'}(v)\|^2_{L^2(T')} = \|\mathcal{R}_{T'}(v)\|^2_{L^2(T)}.$$

Thus, we obtain

$$\sum_{T' \in \mathcal{P}_{k+1}(T)} \eta^2_{k+1}(v, T') \leq 2^{-b/2}(h_T^2 \|\mathcal{R}_T(v)\|^2_{L^2(T)} + h_T \|J_S(v)\|^2_{L^2(\partial T \cap \Omega)}) = 2^{-b/2} \eta^2_k(v, T).$$

Thus,

$$\sum_{T \in \mathcal{M}_{k+1}} \eta^2_{k+1}(v, T) = \sum_{T \in \overline{\mathcal{M}}_k} \left(\sum_{T' \in \mathcal{P}_{k+1}(T)} \eta^2_{k+1}(v, T') \right) \leq 2^{-b/2} \sum_{T \in \overline{\mathcal{M}}_k} \eta^2_k(v, T),$$

where the last term comes from the refinement criteria (3.1.8). Therefore,

$$\eta^2_{k+1}(v) \leq \eta^2_k(v) + 2^{-b/2} \sum_{T \in \overline{\mathcal{M}}_k} \eta^2_k(v, T) - \sum_{T \in \overline{\mathcal{M}}_k} \eta^2_k(v, T). \tag{3.2.7}$$

By defining $\lambda = 1 - 2^{-b/2} > 0$, (3.2.7) becomes

$$\eta^2_{k+1}(v) \leq \eta^2_k(v) - \lambda \sum_{T \in \overline{\mathcal{M}}_k} \eta^2_k(v, T).$$

Since $\mathcal{M}_k \subseteq \overline{\mathcal{M}}_k$, $\sum_{T \in \mathcal{M}_k} \eta^2_k(v, T) \leq \sum_{T \in \overline{\mathcal{M}}_k} \eta^2_k(v, T)$, we finally get

$$\eta^2_{k+1}(v) \leq \eta^2_k(v) - \lambda \sum_{T \in \mathcal{M}_k} \eta^2_k(v, T) = \eta^2_k(v) - \lambda \eta^2_k(v, \mathcal{M}_k).$$

$\square$

Lemma 3.7. *For $\mathcal{T}_k \in \mathbb{T}$ and $\mathcal{M}_k$ =MARK$(\{\eta_k(u_k)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k)$, let $\mathcal{T}_{k+1} \in \mathbb{T}$ be defined by $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$. Then, for all $v_k \in \mathbb{V}_k$, $v_{k+1} \in \mathbb{V}_{k+1}$, and $\delta > 0$, there holds*

$$\eta^2_{k+1}(v_{k+1}) \le (1+\delta)\left\{\eta^2_k(v_k) - \lambda\eta^2_k(v_k, \mathcal{M}_k)\right\} + \left(1+\frac{1}{\delta}\right) K_k\|v_{k+1} - v_k\|_1^2,$$

where $K_k := C_A + 2L_f^2 h_k^2 + 2C_{AA}(1+h_k)^2$.

Proof. By setting $\mathcal{T} = \mathcal{T}_{k+1}$, $v = v_{k+1}$ and $w = v_k$ in Lemma 3.5, we get

$$\begin{aligned}
&\eta^2_{k+1}(v_{k+1})\\
&=\sum_{T\in\mathcal{T}_{k+1}} \eta^2_{k+1}(v_{k+1}, T)\\
&\le (1+\delta)\sum_{T\in\mathcal{T}_{k+1}} \eta^2_{k+1}(v_k, T) + \left(1+\frac{1}{\delta}\right)\sum_{T\in\mathcal{T}_{k+1}} h_T\|J_S(v_{k+1}-v_k)\|^2_{L^2(\partial T\cap\Omega)}\\
&+2\left(1+\frac{1}{\delta}\right)\Bigg[\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_{k+1}} h_T^2\|f(v_{k+1}) - f(v_k)\|^2_{L^2(T)}\Bigg].
\end{aligned}$$

To estimate the $\|J_S(v_{k+1} - v_k)\|_{L^2(\partial T\cap\Omega)}$ term, we applied the Theorem 2.7 to obtain

$$\begin{aligned}
\sum_{T\in\mathcal{T}_{k+1}} h_T\|J_S(v_{k+1}-v_k)\|^2_{L^2(\partial T\cap\Omega)} &\le 2\sum_{T\in\mathcal{T}_{k+1}} h_T\|A\nabla(v_{k+1}-v_k)\|^2_{L^2(\partial T\cap\Omega)}\\
&\le C_\partial \sum_{T\in\mathcal{T}_{k+1}} \|A\|_\infty \|\nabla(v_{k+1}-v_k)\|^2_{L^2(T)},
\end{aligned}$$

where C_∂ is a constant depends only on shape-regular parameter, and $\|A\|_\infty$ is a bounded by assumptions. This can be written as

$$\sum_{T\in\mathcal{T}_{k+1}} h_T\|J_S(v_{k+1}-v_k)\|^2_{L^2(\partial T\cap\Omega)} \le C_A\|v_{k+1}-v_k\|_1^2,$$

where $C_A := C_\partial \|A\|_\infty$.

To estimate $\|\nabla\cdot(A\nabla(v_{k+1} - v_k))\|_{L^2(T)}$, we observe that

$$\nabla\cdot(A\nabla(v_{k+1}-v_k)) = (\nabla\cdot A)\cdot(\nabla(v_{k+1}-v_k)) + A : \nabla^2(v_{k+1}-v_k),$$

where $\nabla^2(v_{k+1}-v_k)$ is the Hessian matrix of $v_{k+1}-v_k$ and : denotes the Frobenius inner product, i.e., $A : B = \sum_{i,j} a_{ij}b_{ij}$. This leads to an estimate

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|^2_{L^2(T)}$$

$$\leq \sum_{T\in\mathcal{T}_{k+1}} h_T^2 \Big(\|\nabla\cdot A\|_\infty \|\nabla(v_{k+1}-v_k)\|_{L^2(T)} + \|A\|_\infty \|\nabla^2(v_{k+1}-v_k)\|_{L^2(T)} \Big)^2.$$

Applying the inverse estimates, Theorem 2.8, to the Hessian matrix to get

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2 \|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|^2_{L^2(T)} \leq \sum_{T\in\mathcal{T}_{k+1}} \big(h_T\|\nabla\cdot A\|_\infty + \|A\|_\infty\big)^2 \|\nabla(v_{k+1}-v_k)\|^2_{L^2(T)}.$$

This gives

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2 \|\nabla\cdot(A\nabla(v_k - v_{k+1}))\|^2_{L^2(T)} \leq C_{AA}(1+h_k)^2 \|v_{k+1}-v_k\|_1^2,$$

where $C_{AA} := \max\{\|\nabla\cdot A\|_\infty, \|A\|_\infty\}$.

Finally, by the Lipschitz condition on f, we obtain an estimate

$$\begin{aligned}\sum_{T\in\mathcal{T}_{k+1}} h_T^2 \|f(v_{k+1}) - f(v_k)\|^2_{L^2(T)} &\leq \sum_{T\in\mathcal{T}_{k+1}} h_T^2 L_f^2 \|v_{k+1}-v_k\|^2_{L^2(T)} \\ &\leq h_k^2 L_f^2 \|v_{k+1}-v_k\|_1^2.\end{aligned}$$

After combining all estimates above and applying $v = v_k$ in Lemma 3.6, we get

$$\eta^2_{k+1}(v_{k+1}) = (1+\delta)\left(\eta_k^2(v_k) - \lambda\eta_k^2(v_k,\mathcal{M}_k)\right) + \left(1+\frac{1}{\delta}\right) K_k \|v_{k+1}-v_k\|_1^2,$$

where $K_k := C_A + 2h_k^2 L_f^2 + 2C_{AA}(1+h_k)^2$. □

Note that since $\{h_k\}_{k=0}^\infty$ is non-increasing, i.e, $1 \geq h_0 \geq h_1 \geq ... \geq h_k \geq ...$, therefore the constant K_k is bounded above by K_0, i.e.,

$$K_k \leq C_A + 2L_f^2 + 8C_{AA} := K_0.$$

CHAPTER IV
CONVERGENCE

In this chapter we prove a contraction property for the weighted sum of the energy error and the error estimator for any two consecutive iterations of the AFEM. The convergence of AFEM follows directly from the lemmas in the previous chapter as stated in the corollary. The last section will be the conclusion of this work.

4.1 Contraction property

Theorem 4.1. *Given an initial triangulation $\mathcal{T}_0$ with initial mesh-size h_0, let $\theta \in (0,1]$ and $\{\mathcal{T}_k, \mathbb{V}_k, u_k\}_{k\geq 0}$ be a sequence of triangulations $\mathcal{T}_k$, finite element spaces $\mathbb{V}_k$, and discrete solutions u_k produced by AFEM. Then, there exists a constant K depending only on the data and the Lipschitz constants such that if $h_0 < K$, then there exist constants $\alpha, \gamma > 0$ and $0 < \mu < 1$ such that*

$$\gamma\eta_{k+1}^2(u_{k+1}) + \alpha |\!|\!| u - u_{k+1} |\!|\!|^2 \leq \mu \left(\gamma\eta_k^2(u_k) + \alpha |\!|\!| u - u_k |\!|\!|^2\right).$$

Proof. For simplicity, let us use $\eta_k := \eta_k(u_k)$, $\eta_{k+1} := \eta_{k+1}(u_{k+1})$, $e_{k+1} := |\!|\!| u - u_{k+1} |\!|\!|$, and $|\!|\!| u - u_k |\!|\!| := e_k$. By setting $v_k = u_k$ and $v_{k+1} = u_{k+1}$ in Lemma 3.7, we get

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\eta_k^2(u_k, \mathcal{M}_k)\right\} + \left(1 + \frac{1}{\delta}\right) K_k \| u_{k+1} - u_k \|_1^2. \tag{4.1.1}$$

Using equivalence of norms and setting $E_k = |\!|\!| u_{k+1} - u_k |\!|\!|$, (4.1.1) becomes

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\eta_k^2(u_k, \mathcal{M}_k)\right\} + \left(1 + \frac{1}{\delta}\right) C_e^2 K_k E_k^2,$$

where C_e is a constant for the norm equivalence depending on the data A and Ω. Applying Dörfler Marking (3.1.6) and $\eta_k(u_k, \mathcal{M}_k) \geq \theta\eta_k$, we have

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\theta^2\eta_k^2\right\} + \left(1 + \frac{1}{\delta}\right) C_e^2 K_k E_k^2. \tag{4.1.2}$$

Since $K_k \leq K_0$, (4.1.2) leads to

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\theta^2\eta_k^2\right\} + \left(1+\frac{1}{\delta}\right) C_e^2 K_0 E_k^2. \tag{4.1.3}$$

Multiplying (4.1.3) by $\gamma := \frac{\delta}{C_e^2 K_0(1+\delta)} > 0$ to obtain

$$\gamma\eta_{k+1}^2 \leq \gamma(1+\delta)\eta_k^2 - \gamma\lambda\theta^2(1+\delta)\eta_k^2 + E_k^2.$$

By Corollary 3.4, if $h_0 < \frac{1}{\sqrt{3C_e^2 C_f^2 L_f}}$, then

$$\gamma\eta_{k+1}^2 + (1-3C_e^2C_f^2L_fh_k^2)e_{k+1}^2 \leq \gamma(1+\delta)\eta_k^2 - \gamma\lambda\theta^2(1+\delta)\eta_k^2 + (1+C_e^2C_f^2L_fh_k^2)e_k^2.$$

To balance the η_k term, we can rewrite as, for $\beta > 0$,

$$\begin{aligned}\gamma\eta_{k+1}^2 + \big(1-3C_e^2C_f^2L_fh_k^2\big)e_{k+1}^2 \leq \gamma(1+\delta)\eta_k^2 + \big(1+C_e^2C_f^2L_fh_k^2\big)e_k^2 \\ - \beta\gamma\lambda\theta^2(1+\delta)\eta_k^2 - (1-\beta)\gamma\lambda\theta^2(1+\delta)\eta_k^2.\end{aligned} \tag{4.1.4}$$

Using the upper bound (3.1.5), the Lipschitz condition on f, the Corollary 2.15, and the equivalence of norms, we get

$$\begin{aligned} e_k &\leq C_1\eta_k + C_2h_k\|f-f_k\|_0 \\ &\leq C_1\eta_k + C_2h_kL_f\,\|u-u_k\|_0 \\ &\leq C_1\eta_k + C_2L_fC_fh_k^2\|u-u_k\|_1 \\ &\leq C_1\eta_k + C_eC_2C_fL_fh_k^2e_k. \end{aligned}$$

If $h_0 < \frac{1}{\sqrt{C_eC_2C_fL_f}}$, then we have

$$0 < \left(\frac{1-C_eC_2C_fL_fh_k^2}{C_1}\right)e_k \leq \eta_k. \tag{4.1.5}$$

Combining (4.1.5) to the right hand side of (4.1.4), we have

$$\begin{aligned}&\gamma\eta_{k+1}^2 + (1-3C_e^2C_f^2L_fh_k^2)e_{k+1}^2 \\ &\quad\leq \gamma(1+\delta)\eta_k^2 + \big(1+C_e^2C_f^2L_fh_k^2\big)\,e_k^2 - (1-\beta)\gamma\lambda\theta^2(1+\delta)\eta_k^2 \\ &\quad\quad - \beta\gamma\lambda\theta^2(1+\delta)\left(\frac{1-C_eC_2C_fL_fh_k^2}{C_1}\right)^2 e_k^2.\end{aligned} \tag{4.1.6}$$

For convenience we denote the coefficients as follows;

$$\alpha_1 = 1 - 3C_e^2 C_f^2 L_f h_k^2 > 0,$$
$$\alpha_2 = 1 + C_e^2 C_f^2 L_f h_k^2 - \frac{\delta}{C_e^2 K_0} \beta \lambda \theta^2 \left(\frac{1 - C_e C_2 C_f L_f h_k^2}{C_1} \right)^2,$$
$$\alpha_3 = (1+\delta)\left(1 - (1-\beta)\lambda\theta^2\right).$$

The equation (4.1.6) can be written as

$$\gamma \eta_{k+1}^2 + \alpha_1 e_{k+1}^2 \leq \gamma \alpha_3 \eta_k^2 + \alpha_2 e_k^2 = \gamma \alpha_3 \eta_k^2 + \alpha_1 \left(\frac{\alpha_2}{\alpha_1} \right) e_k^2. \tag{4.1.7}$$

The result follows by setting $\alpha = \alpha_1$ and showing that $\mu := \max\{\alpha_3, \frac{\alpha_2}{\alpha_1}\} < 1$.

Showing that $0 < \alpha_3 < 1$ is equivalent to show that

$$0 < (1+\delta)\left(1 - (1-\beta)\lambda\theta^2\right) < 1.$$

This is the case if we choose $\beta > 0$ such that

$$0 < \beta < 1 - \frac{1}{\lambda\theta^2} \left(\frac{\delta}{1+\delta} \right). \tag{4.1.8}$$

Since λ and θ are known from AFEM and $\lambda\theta^2 < 1$, then we can choose $\beta > 0$ satisfying (4.1.8) provided that $\delta > 0$ is pre-selected so that $\frac{1}{\lambda\theta^2}\left(\frac{\delta}{1+\delta}\right) < 1$, i.e., choosing

$$0 < \delta < \frac{\lambda\theta^2}{1 - \lambda\theta^2}. \tag{4.1.9}$$

In order to arrive at (4.1.7), it requires that $h_0 < \min\left\{ \frac{1}{\sqrt{3C_e^2 C_f^2 L_f}}, \frac{1}{\sqrt{C_e C_2 C_f L_f}} \right\}$, for obtaining (4.1.4) and (4.1.5), thus this gives $\alpha_1 > 0$.

We get $\alpha_2 > 0$ by selecting δ satisfying $\delta \leq \min\left\{ \frac{\lambda\theta^2}{1-\lambda\theta^2}, \frac{C_e^2 K_0 C_1^2}{\lambda\theta^2} \right\}$ so that

$$\beta\delta \frac{\lambda\theta^2}{C_e^2 K_0 C_1^2} \left(1 - C_e C_2 C_f L_f h_k^2\right)^2 < 1.$$

The case $0 < \alpha_2 < \alpha_1$ holds if and only if

$$1 + C_e^2 C_f^2 L_f h_k^2 - \beta\delta \frac{\lambda\theta^2}{C_e^2 K_0 C_1^2} \left(1 - C_e C_2 C_f L_f h_k^2\right)^2 < 1 - 3C_e^2 C_f^2 L_f h_k^2.$$

This is equivalent to

$$h_k^2 < \frac{\delta\beta\lambda\theta^2}{4C_e^4 K_0 C_f^2 L_f C_1^2} (1 - C_e C_2 C_f L_f h_k^2)^2.$$

For convenience, set $r = C_e C_2 C_f L_f$ and $s = \frac{\delta\beta\lambda\theta^2}{4C_e^4 K_0 C_f^2 L_f C_1^2}$. The condition on h_k becomes

$$h_k^2 < s(1 - 2rh_k^2 + r^2 h_k^4).$$

This is the case if $h_0 < \sqrt{\frac{s}{1+2rs}}$ because $sr^2h_k^4 \geq 0$.

By selecting $K := \min\left\{\frac{1}{\sqrt{3C_e^2C_f^2L_f}}, \frac{1}{\sqrt{C_eC_2L_fC_f}}, \sqrt{\frac{s}{1+2rs}}\right\} > 0$, the condition $h_0 < K$ will give us the contraction result for (4.1.7). $\square$

Corollary 4.2 (Convergence)**.** *Under the hypothesis of Theorem 4.1,*

$$\lim_{k\to\infty} \eta_k(u_k) = 0 \textit{ and } \lim_{k\to\infty} |||u - u_k||| = 0.$$

Proof. From Theorem 4.1, it is easy to see that

$$\gamma\eta_{k+1}^2(u_{k+1}) + \alpha|||u - u_{k+1}|||^2 \leq \mu^{k+1}\left(\gamma\eta_0^2(u_0) + \alpha|||u - u_0|||^2\right)$$

Since $\lim_{k\to\infty}\mu^{k+1} = 0$ for $\mu \in (0, 1)$, and $\gamma, \alpha > 0$, thus

$$\lim_{k\to\infty} \eta_k(u_k) = 0 \text{ and } \lim_{k\to\infty} |||u - u_{k+1}||| = 0.$$

$\square$

Remark 4.3.

1. In the contraction result, the reduction factor of the weighted sum $\gamma\eta_{k+1}^2 + |||u - u_{k+1}|||^2$ is $\mu < 1$. By definition μ is controllable by changing parameters $\lambda = 1 - 2^{-b/2}$ and $\theta \in (0, 1]$. To have high reduction rate (μ small), we require that λ and θ are close to 1, i.e., by controlling the marking criterion to have θ close to 1 (nearly uniformly refinement) and/or controlling the refining criterion to have bigger b, the number of bisections for each refinement step.

2. The weights γ and α in the sum of the error estimator η_k and the energy error $|||u - u_k|||$ are chosen to balance these two errors to have the contraction. They are chosen based on the parameter δ that satisfies (4.1.9) and other constants such as C_e, K_0, C_f, L_f and h_0 from the problem and the initial triangulation $\mathcal{T}_0$. Many of these constants are computable, but some of them, such as K_0 and C_f, not computable from the Lemmas . Therefor, γ and α are not computable in general.

3. The constant K in definition is upper bound for initial mesh size h_0 in order to have the contraction. The value of K depends on serveral parameters of the algorithm and the constants of the problem. Similarly, the value of K is not computable due to some constants are not known. Therefore, the control of h_0 by K in order to have convergence result can only be obtained by experiments.

4.2 Examples

In this section, we give examples of semi-linear elliptic partial differential equations satisfying the assumption of the main Theorem 4.1. Let $\Omega = [0,1] \times [0,1]$ and $\mathcal{T}_0$ be an initial triangulation as follows (see Figure 4.1).

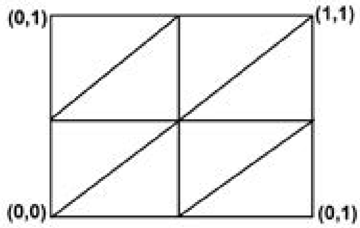

Figure 4.1: An initial triangulation $\mathcal{T}_0$ for the rectangular domain Ω

We consider the problem

$$-\nabla \cdot (A(x)\nabla u(x)) = f(x, u(x)), \qquad \forall\, x \in \Omega, \tag{4.2.1}$$

$$u(x) = 0, \qquad \forall\, x \in \partial\Omega, \tag{4.2.2}$$

where $A(x) = \begin{bmatrix} a(x) & b(x) \\ b(x) & c(x) \end{bmatrix}$, $x \in \Omega$, being positive definite matrix. For convenience, we will omit the variable x. Let λ_1 and λ_2 be eigenvalues of A with $\lambda_2 \geq \lambda_1$. Since A is positive definite, $\lambda_2 \geq \lambda_1 > 0$. Both eigenvalues are positive if $ac > b^2$, where a, b and c are positive. By property of the Rayleigh quotient,

$$\lambda_1(x) \leq \frac{A(x)p \cdot p}{|p|^2} \leq \lambda_2(x), \qquad \forall x \in \Omega \ \ \forall p \in \mathbb{R}^2, p \neq 0,$$

we can choose

$$\theta_* = \inf_{x \in \Omega} \lambda_1(x) > 0.$$

Then, A satisfies strictly monotonicity.

Example 4.4. Let $A(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to see that A is positive definite and has the eigenvalue $\lambda_1 = \lambda_2 = 1$. Then, we can choose $\theta_* = 1$.

Example 4.5. Let $A(x) = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$. Then, the eigenvalues of A are $\lambda_1 = 5 - \sqrt{5}$ and $\lambda_2 = 5 + \sqrt{5}$. Thus, we can choose $\theta_* = 5 - \sqrt{5}$.

Example 4.6. Let $x = (x_1, x_2) \in \Omega$ and $A(x) = \begin{bmatrix} 1 & x_1 \\ x_1 & 2 \end{bmatrix}$. Then, the eigenvalues of A are $\lambda_1(x) = \frac{3-\sqrt{1+4x_1^2}}{2}$ and $\lambda_2(x) = \frac{3+\sqrt{1+4x_1^2}}{2}$. Then, A is positive definite since $\lambda_1(x)$ and $\lambda_2(x)$ are positive for all $x \in \Omega$. Hence, we can choose $\theta_* = \inf_{x\in\Omega} \lambda_1(x) = \frac{3-\sqrt{5}}{2}$.

For the nonlinear function $f(x,u)$, we give examples for those satisfies the assumptions as follows:

Example 4.7. Let $f(x,u) = e^{-\frac{1}{m}u^2}$, where $x \in \Omega$ and a constant $m > 0$. It is clear that

$$\left|\frac{\partial f}{\partial u}\right| = \left|\frac{-2u}{me^{\frac{1}{m}u^2}}\right| = \frac{2}{m} \cdot \frac{|u|}{e^{\frac{1}{m}u^2}}, \qquad \forall u \in \mathbb{R}. \tag{4.2.3}$$

By calculus, $\frac{|u|}{e^{\frac{1}{m}u^2}}$ has absolute maximum $\sqrt{\frac{m}{2e}}$, i.e., $\frac{|u|}{e^{\frac{1}{m}u^2}} \leq \sqrt{\frac{m}{2e}}$, $\forall u \in \mathbb{R}$. Thus,

$$\left|\frac{\partial f}{\partial u}\right| \leq \sqrt{\frac{2}{me}}, \qquad \forall u \in \mathbb{R}.$$

We can choose $L_f = \sqrt{\frac{2}{me}}$. By corollary 2.15, $L_f < \frac{1}{C_2^*}$, i.e.,

$$m > \frac{2}{ec_B^2}(cc_BC_B + 1)^2,$$

where $c_B = \theta_*$, $C_B = \|A\|_\infty$ and $c = \frac{(1+c_1)c_2C_B}{c_B}$. In the case of Example 4.4 where $A = I$, we get that $c_B = C_B = 1$, therefore, and we must $m > \frac{2(c+1)^2}{e}$ in order to satisfy the condition of the Corollary 2.15. Moreover, since $f(x,u)$ is continuous and bounded on Ω,

$$\int_\Omega |f(x,u)|^2 dx = \int_\Omega e^{-\frac{2}{m}u^2}\, dx < \infty.$$

Example 4.8. Let $f((x_1, x_2), u) = x_2 \sin(mx_1 u)$, where $x = (x_1, x_2) \in \Omega$ and a constant $m > 0$. It is easy to see that

$$\left|\frac{\partial f}{\partial u}\right| = |mx_1 x_2 \cos(mx_1 u)| \le m, \qquad \forall u \in \mathbb{R},\ \forall x \in \Omega.$$

Similarly to Example 4.7, we choose $L_f = m$. Then,

$$m < \frac{c_B}{cc_B C_B + 1}.$$

In the case of Example 4.4 where $A = I$, we can compute $c_B = C_B = 1$ and we have $m < \frac{1}{c+1}$. Since $f(x, u)$ is continuous and bounded on Ω,

$$\int_\Omega |f(x, u)|^2 \, dx = \int_\Omega x_2^2 \sin^2(mx_1 u) \, dx < \infty.$$

4.3 Conclusion

In this work we obtain convergence theorem of AFEM for second order semi-linear elliptic PDEs as stated in the main Theorem 4.1. The proof relies on several assumptions as follows. The coefficient matrix A is positive definite and satisfies the strictly monotonicity in order to have the coercivity such that the energy norm is equivalent to the H^1-norm and seminorm on $H_0^1(\Omega)$. Moreover, we require that all components of A are $C^1(\Omega)$ functions in order to have the approximation of the reduction for error estimators as stated in Lemma 3.7. The examples for such A are given in the examples 4.4-4.6. In addition, we require that $f(x, u)$ is Lipschitz in variable u in order to approximate the difference of $\|f(x, v) - f(x, w)\|_0$ in terms of $\|v - w\|_0$ to obtain the upper bound (Lemma 3.1), the estimation of error reduction (Corollary 3.4), and the estimation of error estimator reduction (Lemma 3.7). Examples of $f(x, u)$ satisfying such conditions are shown in Examples 4.7 and 4.8. The key idea for getting the contraction in Theorem 4.1, is to combine the estimators of the error reduction in Corollary 3.4 with the estimator of error estimator reduction in Lemma 3.7, and with the help of the upper bound, the marking property and the refining criterion. Finally, the contraction is obtained under the assumption that the initial meshsize h_0 is sufficiantly refined. The convergence follows easily as stated in Corollary 4.2.

REFERENCES

[1] M. Ainsworth, J.T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, John Wiley & Sons, Inc., New York, 2000.

[2] D. Braess, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, New York, 2001.

[3] S. C. Brenner, L.R. Scott, *The Mathematical theory of Finite Element Methods*, **Texts in Applied Mathematics**, Vol **15**, Springer, 1994.

[4] J. M. Cascon, C. Kreuzer, R. H. Nochetto, K. G. Siebert, *Quasi-optimal convergence rate for adaptive finite element method*, **SIAM J. Numer. Anal.**, **46**(2008), 2524-2550.

[5] L. Diening, C. Kreuzer, *Linear convergence of an adaptive finite element method for the p-Laplacian equation*, **SIAM J. Numer. Anal.**, **46**(2008), 614-638.

[6] W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, **SIAM J. Numer. Anal.**, **33**(1996), 1106-1124.

[7] W. Dörfler, *A robust adaptive strategy for the nonlinear Poisson's equation*, **SIAM J. Numer. Anal.**, **55**(1995), 289-304.

[8] E. M. Garau, P. Morin, C. Zuppa, *Convergence of an adaptive Kačanov FEM for quasi-linear problems*, **Appl. Numer. Math.**, **61**(2011), 512-529.

[9] T. Jampawai, *A posteriori error estimates for semi-linear elliptic partial differential equations*, Thesis of Master Degree, Chulalongkorn University, 2009.

[10] K. Mekchay, R. H. Nochetto, *Convergence of adaptive finite element methods for general second order linear elliptic PDEs*, **SIAM J. Numer. Anal.**, **43**(2005), 1803-1827.

[11] P. Morin, R. H. Nochetto, K. G. Siebert, *Convergence of adaptive finite element methods*, **SIAM Review**, **44**(2002), 631-658.

[12] P. Morin, K. G. Siebert, A. Veeser, *A basic convergence result for conforming adaptive finite elements*, **Math. Mod. Meth. Appl. S.**, **18**(2008), 707-737.

[13] R. H. Nochetto, K. G. Siebert and A. Veeser, *Theory of adptive finite finite element methods: an introduction.* Springer, 2009.

[14] A. Veeser, *Convergent adaptive finite elements for the nonlinear Laplacian*, **Numer. Math.**, **92**(2002), 115-137.

[15] R. Verfürth., *A review of a posteriori error estimation and adaptive mesh refinement tecniques.* B.G. Teubner, 1996.

VITA

Name	Mr. Thanatyod Jampawai
Date of Birth	26 August 1984
Place of Birth	Loei, Thailand
Education	B.Sc.(Mathematics)(Second Class Honors), Chulalongkorn University, 2007
	M.Sc.(Mathematics), Chulalongkorn University, 2010
Scholarship	The Developement and Promotion of Science and Technology Talents Project (DPST)