CHAPTER II

PRELIMINARIES

Throughout this thesis, we assume that G_1 ans G_2 are nontrivial vertex-disjoint graphs and the clone H of $G_1 \underset{H}{\Phi} G_2$ is a nontrivial connected subgraph.

In this chapter, we separate this chapter into two sections. The first section shows a trivial bound of clique covering numbers of glued graphs and the sharpness of this bound. In the last section, we introduce a new clique for any original graphs in the glued graph.

2.1 A trivial bound of clique covering numbers of glued graphs

The following remark shows the upper bound and the lower bound of the clique covering number of $G_1 \Phi G_2$.

Remark 2.1.1. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Phi} G_2$ be any glued graph at clone H. Since H is nontrivial connected, $cc(G_1 \underset{H}{\Phi} G_2) \geq 1$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be minimum clique coverings of G_1 and G_2, respectively. Then $\mathscr{C}_1 \cup \mathscr{C}_2$ is a clique covering of $G_1 \underset{H}{\Phi} G_2$. Thus $cc(G_1 \underset{H}{\Phi} G_2) \leq |\mathscr{C}_1 \cup \mathscr{C}_2| \leq |\mathscr{C}_1| + |\mathscr{C}_2| = cc(G_1) + cc(G_2)$. Hence $1 \leq cc(G_1 \underset{H}{\Phi} G_2) \leq cc(G_1) + cc(G_2)$.

We next give examples of the sharpness of bounds in Remark 2.1.1.

Example 2.1.2. Let G_1 be a Hamiltonian graph on n vertices with a Hamiltonian path P_n as shown in the bold edges and $G_2 = \overline{G}_1 \cup P_n$ of Figure 2.1.1. Then $G_1 \underset{P_n}{\Phi} G_2$

is K_n. It is evident that the graph gluing of original graphs with any arbitrary large clique covering number could yield a resulting glued graph with the clique covering number 1. This circumstance occurs because of the existence of new cliques in the glued graphs. We will introduce a new clique in the glued graph in Section 2.2.

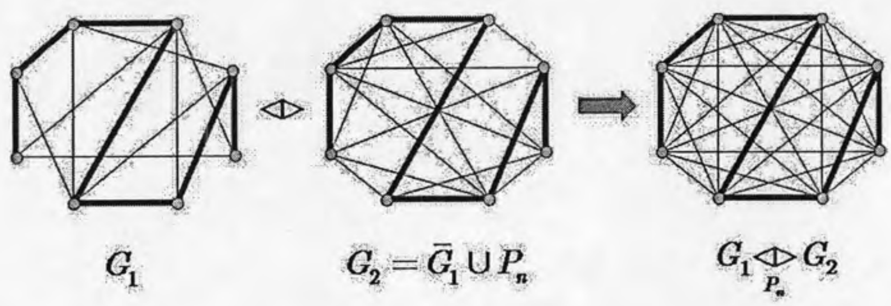

Figure 2.1.1: The sharpness of the lower bound in Remark 2.1.1

□

Example 2.1.3. Let $K_m \underset{H}{\diamond} K_n$ be the glued graph at proper induced subgraph H of both K_m and K_n. The clone H is shown as bold edges illustrated in Figure 2.1.2.

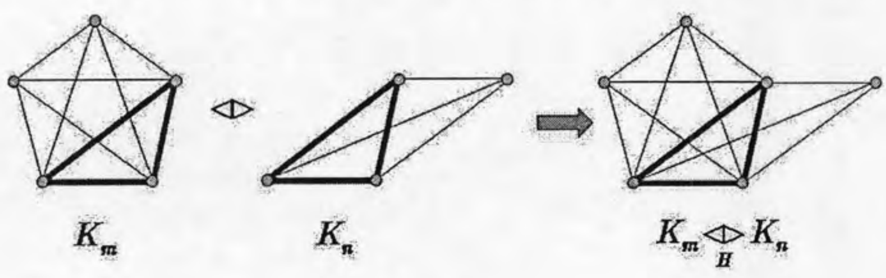

Figure 2.1.2: The sharpness of the upper bound in Remark 2.1.1

By Remark 2.1.1, we have $cc(K_m \underset{H}{\diamond} K_n) \leq cc(K_m) + cc(K_n) = 2$. Since $K_m \underset{H}{\diamond} K_n$ is not complete graph, $cc(K_m \underset{H}{\diamond} K_n) \geq 2$. Hence $cc(K_m \underset{H}{\diamond} K_n) = 2 = cc(K_m) + cc(K_n)$.

□

2.2 New cliques of glued graphs

From Example 2.1.2, we discuss about a new clique in a glued graph. In this section, we give the definition of a new clique and a new edge of a glued graph and properties of the glued graph with them.

Definition 2.2.1. An edge $e = ab$ in any glued graph $G_1 \Leftrightarrow G_2$ is *a new edge for the original graph* G_i, $i = 1$ or 2 if the corresponding vertices of a and b in G_i are not adjacent. An n-clique $Q = K_n(v_1, ..., v_n)$ in any glued graph $G_1 \Leftrightarrow G_2$ is *a new clique for the original graph* G_i, $i = 1$ or 2 if all corresponding vertices of $v_1, ..., v_n$ in G_i do not form an n-clique in G_i.

Example 2.2.2. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\Leftrightarrow} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 2.2.1.

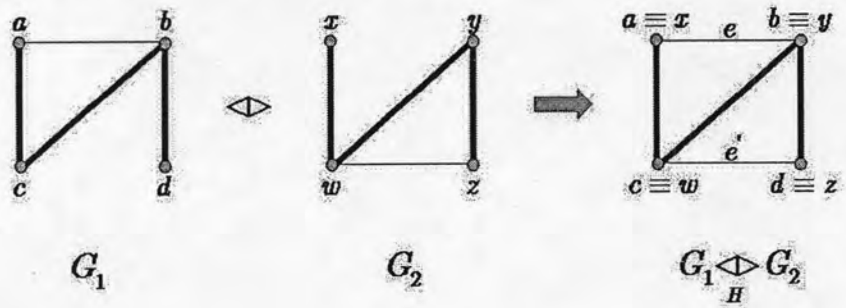

Figure 2.2.1: A glued graph containing new cliques

From Figure 2.2.1, $G_1 \underset{H}{\Leftrightarrow} G_2$ contains the edge e but x and y are not adjacent in G_2. So, e is a new edge for G_2. Similarly, e' is a new edge for G_1. Since $K_3(a \equiv x, b \equiv y, c \equiv w)$ is a clique in $G_1 \underset{H}{\Leftrightarrow} G_2$ and G_2 does not contain a clique $K_3(x, y, w)$, $K_3(a \equiv x, b \equiv y, c \equiv w)$ is a new clique for G_2. Similarly, $K_3(b \equiv y, c \equiv w, d \equiv z)$ is a new clique for G_1.

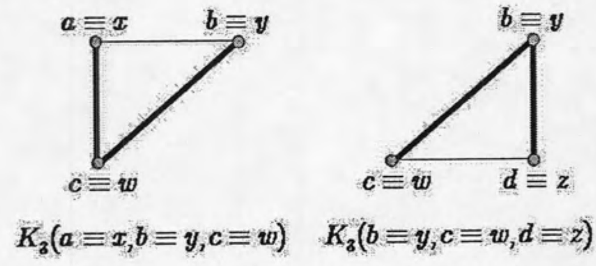

Figure 2.2.2: New cliques in a glued graph in Example2.2.2

□

Example 2.2.2 suggests some basic properties of a new edge and a new clique of a glued graph in the following remark.

Remark 2.2.3.

1. If a glued graph $G_1 \diamond G_2$ has a new clique for G_i, $i = 1$ or 2, then $G_1 \diamond G_2$ has a new edge for G_i.
2. Any new edge of a glued graph cannot be a new edge for both original graphs at the same time.
3. Both endpoints of a new edge of a glued graph must lie in the clone.

A glued graph in the following example contains a new clique for some original graph. However, its clique covering number meets the upper bound of Remark 2.1.1.

Example 2.2.4. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\oplus} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 2.2.3.

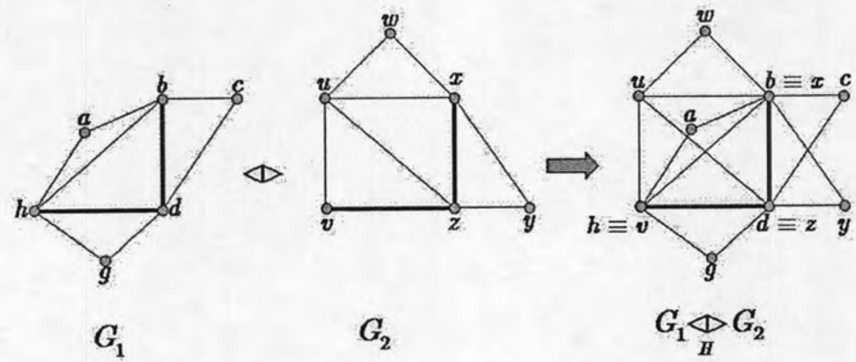

Figure 2.2.3: A glued graph containing new cliques with the clique covering number $cc(G_1) + cc(G_2)$

We can see that $K_4(u, b \equiv x, h \equiv v, d \equiv z)$ in $G_1 \underset{H}{\oplus} G_2$ is a new clique for G_2 as shown in Figure 2.2.4. We have $cc(G_1) = cc(G_2) = 3$. By Remark 2.1.1, $cc(G_1 \underset{H}{\oplus} G_2) \leq cc(G_1) + cc(G_2) = 3 + 3 = 6$.

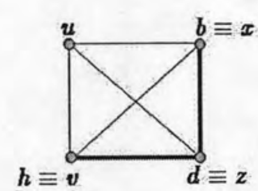

Figure 2.2.4: A new clique for G_2 in a glued graph in Example 2.2.4

Let $I = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the set of dashed edges as shown in Figure 2.2.5. We can see that I is a clique-independent set of $G_1 \underset{H}{\oplus} G_2$. Thus $cc(G_1 \underset{H}{\oplus} G_2) \geq |I| = 6$. Therefore, $cc(G_1 \underset{H}{\oplus} G_2) = 6$. So, $cc(G_1 \underset{H}{\oplus} G_2) = cc(G_1) + cc(G_2)$.

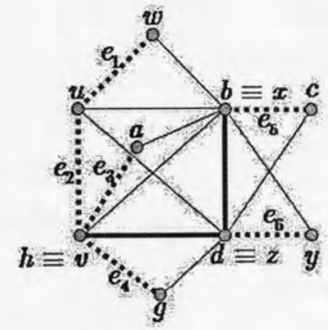

Figure 2.2.5: A clique-independent set of a glued graph in Example 2.2.4

□

We give properties of glued graphs with a new edge in next proposition by using Remark 2.2.3.

Proposition 2.2.5. *If* $G_1 \underset{H}{\Phi} G_2$ *has a new edge for* G_1*(or* G_2*), then* G_2*(or* G_1*) contains a cycle containing such new edge, and the clone* H *has at least 3 vertices.*

Proof. Assume that $G_1 \underset{H}{\Phi} G_2$ has a new edge, $e = ab$, for G_1. By Remark 2.2.3, vertices a and b must lie in the clone H. Then a is not adjacent to b in G_1 while a is adjacent to b in G_2. Since H is connected, particularly in G_1, there is an a, b-path in $H \subseteq G_1$. Hence, there exists the corresponding path of a, b-path in G_2. Together with the edge joining a and b in G_2, this yields a cycle in G_2.

Note that, there is an a, b-path in H while a and b are not adjacent. Thus, H contains another vertex that is not a and b. Hence, H has at least 3 vertices. □

In the rest of the thesis, we consider only glued graphs which do not have a new clique for any original graphs. We show results of clique coverings of glued graphs which do not have a new clique for any original graphs in Chapter 3 and Chapter 4.