CHAPTER I

INTRODUCTION

1.1 Introduction

The subject of clique coverings of graphs has its origins in the problem of representing of a graph by set intersections by Erdős, Goodman and Pósa [5]. Many authors have investigated this topic. They study clique covering numbers of some class of graphs. Varieties of literatures are involved in this section.

In 1983, Caccetta and Pullman [2] studied the problem of determining for each k and n, the set of those integers x where are clique covering numbers of some k-regular, connected graph on n vertices. Roberts [7] presented a variety of results on clique coverings of graphs, and the many applied problems of clique coverings of graphs, in 1985.

In 1988, Erdős, Faudree and Ordman [4] investigated the bounds of clique covering numbers of the cocktail party graph, T_{2n}, such that the lower bound is $(log_2 2n) - 1$ and the upper bound is $2(log_2 2n)$.

In 1990, Brigham and Dutton [1] listed other results that the effect vertex and edge deletion have on the clique covering number of a graph. More recently, Cavers [3] collected the clique covering numbers of graphs and introduced some new results in 2005.

In this thesis, we study clique covering numbers of glued graphs. A glued graph was defined by Uiyyasathain [8]. We introduce a definition of a glued graph in Section 1.2. More details regarding glued graphs can be explored in Promsakon's Thesis [6].

Our purpose in this thesis is to study bounds of clique covering numbers of glued graphs in terms of these clique covering numbers of their original graphs. Also, we investigate values or bounds of clique covering numbers of glued graphs with specified clones such as an induced subgraph of both original graphs and K_n.

In Section 1.2, we introduce a clique covering of any graph, and a glued graph. Moreover, we show examples and some basic properties of these definitions in this section.

In Chapter 2, we study a trivial bound of clique covering numbers of glued graphs, and also introduce a new clique and a new edge for any original graphs in a glued graph.

In Chapter 3, we study clique coverings of glued graphs without new cliques and glued graphs at clone which is an induced subgraph of both original graphs. We also investigate some basic properties of glued graphs without new cliques in this chapter. Lastly, clique coverings of glued graphs at clone K_n is considered in Chapter 4.

All terminologies about graphs follow West [9]. All graphs considered are finite, simple and undirected.

1.2 Definitions

In this section, we introduce a clique covering of a graph G and a glued graph, and give some properties of them.

Definition 1.2.1. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A subgraph H of G is an *induced subgraph*, denoted by $G[V(H)]$, if vertices of $V(H)$ are adjacent in $G[V(H)]$ whenever they are adjacent in G.

Definition 1.2.2. Let H be a subgraph of a graph G. We write $G - H$ for the subgraph of G obtained by deleting the set of edges $E(H)$. A graph G is *H-free* if G does not contain H as a subgraph.

Definition 1.2.3. A *complete graph* is a graph in which each pair of vertices is joined by an edge. The complete graph with n vertices is denoted by K_n.

Example 1.2.4. K_1, K_2, K_3 and K_4 are shown in Figure 1.2.1.

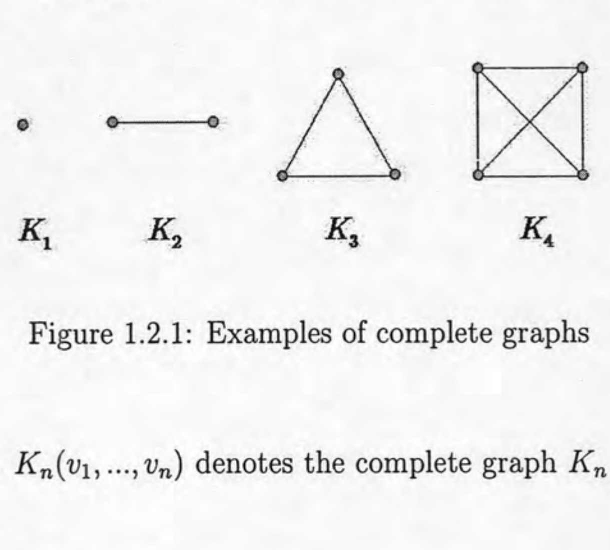


Figure 1.2.1: Examples of complete graphs

Notation 1.2.5. $K_n(v_1, ..., v_n)$ denotes the complete graph K_n on the set of vertices $\{v_1, ..., v_n\}$.

Definition 1.2.6. A *clique* of a graph G is a complete subgraph of G.

Note 1.2.7. A clique of a graph G is not necessarily maximal complete subgraph of G.

Example 1.2.8. Let G be a graph as shown in Figure 1.2.2.

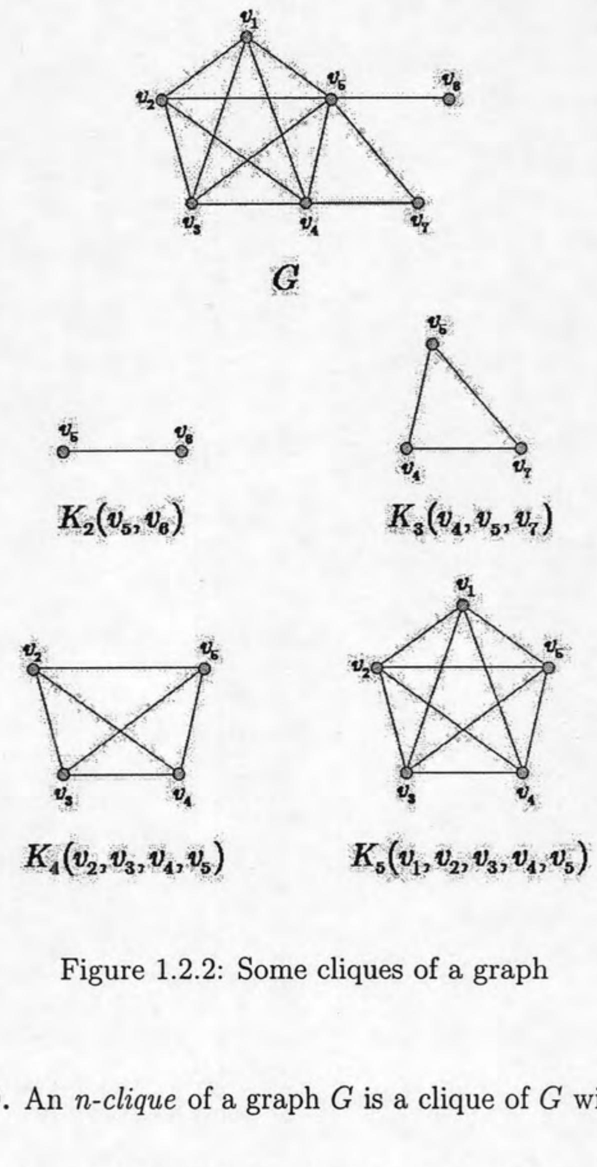


Figure 1.2.2: Some cliques of a graph

Definition 1.2.9. An *n-clique* of a graph G is a clique of G with n vertices.

Example 1.2.10. Consider G illustrated in Figure 1.2.2. We have that $K_2(v_5, v_6)$, $K_3(v_4, v_5, v_7)$, $K_4(v_2, v_3, v_4, v_5)$ and $K_5(v_1, v_2, v_3, v_4, v_5)$ in Figure 1.2.2 are cliques of G. Because $K_4(v_2, v_3, v_4, v_5)$ is a subgraph of $K_5(v_1, v_2, v_3, v_4, v_5)$, $K_4(v_2, v_3, v_4, v_5)$ is not a maximal subgraph of G. But $K_4(v_2, v_3, v_4, v_5)$ is a clique of G. Moreover,

$K_2(v_5, v_6)$, $K_3(v_4, v_5, v_7)$, $K_4(v_2, v_3, v_4, v_5)$ and $K_5(v_1, v_2, v_3, v_4, v_5)$ in Figure 1.2.2 are called that 2-clique, 3-clique, 4-clique and 5-clique, respectively of G.

1.2.1 Definitions of clique coverings of graphs

Definition 1.2.1.1. A *clique covering* of a graph G is a set of cliques of G in which each edge of G is contained in at least one clique. The *clique covering number* of a graph G, denoted by $cc(G)$, is the smallest cardinality of clique coverings of G. A *minimum clique covering* of a graph G is a clique covering of G with cardinality $cc(G)$.

Example 1.2.1.2. Let G be a graph as shown in Figure 1.2.1.1.

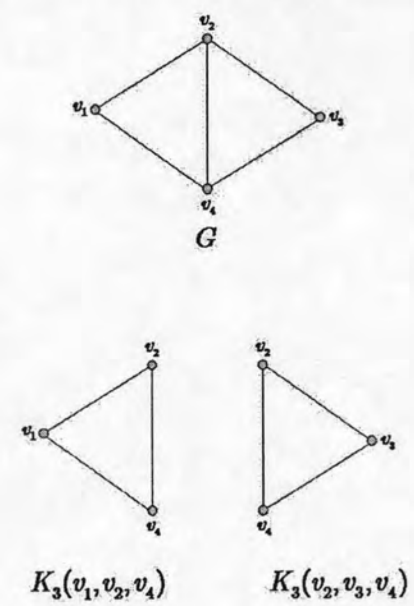
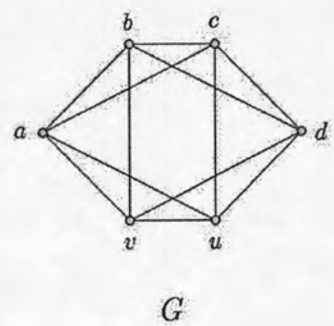


Figure 1.2.1.1: A clique covering of $K_4 - e$

From Figure 1.2.1.1, we have $\{K_3(v_1, v_2, v_4), K_3(v_2, v_3, v_4)\}$ is a clique covering of G. Thus $cc(G) \leq 2$. Since $|V(G)| = 4$ and $G \ncong K_4$, $cc(G) > 1$. Hence $cc(G) = 2$. Moreover, $\{K_3(v_1, v_2, v_4), K_3(v_2, v_3, v_4)\}$ is the minimum clique covering of G. $\square$

Definition 1.2.1.3. Let e and f be any two edges in a graph G. If there is no clique in G containing both e and f, then e and f are *clique-independent edges* of G. A set of pairwise clique-independent edges is called a *clique-independent set.*

Example 1.2.1.4. Consider a graph G illustrated in Figure 1.2.1.2.



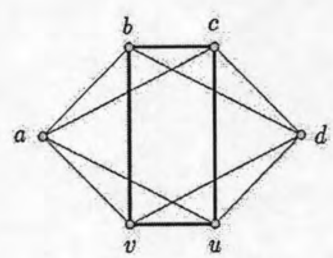


Figure 1.2.1.2: A clique-independent set of G in Example 1.2.1.4

Let $I = \{bc, cu, uv, vb\}$ be a subset of the edge set of G shown as bold edges of Figure 1.2.1.2. Since bu and cv are not edges in G, we have that I is a clique-independent set of G.

$\square$

Remark 1.2.1.5. Let e and f be any two edges in a graph G. If there exist two endpoints of e and f that are not adjacent in G, then e and f are clique-independent edges of G.

Remark 1.2.1.6. Let I be a clique-independent set of a graph G. Since different elements in I must be covered by different cliques of G, $cc(G) \geq |I|$.

Example 1.2.1.7. Consider a graph G illustrated in Figure 1.2.1.2. We have that I is a clique-independent set of G. By Remark 1.2.1.6, $cc(G) \geq |I| = 4$. We can use $K_3(a,b,v)$, $K_3(a,c,u)$, $K_3(b,c,d)$ and $K_3(d,u,v)$ as shown in Figure 1.2.1.3 to cover G. Hence $cc(G) \leq 4$. Therefore $cc(G) = 4$. Moreover, $\{K_3(a,b,v), K_3(a,c,u), K_3(b,c,d), K_3(d,u,v)\}$ is a minimum clique covering of G.

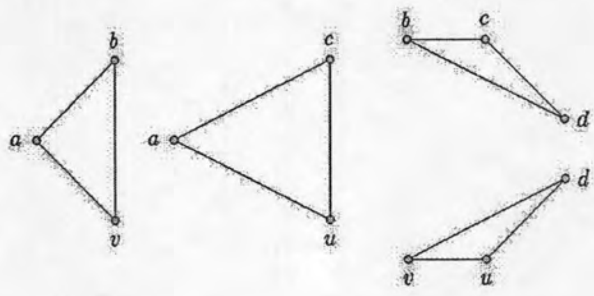


Figure 1.2.1.3: A clique covering of G in Example 1.2.1.7

□

Example 1.2.1.8. Similar to Example 1.2.1.4 and Example 1.2.1.7, we obtain $cc(G_1)$ and $cc(G_2)$ as follows:

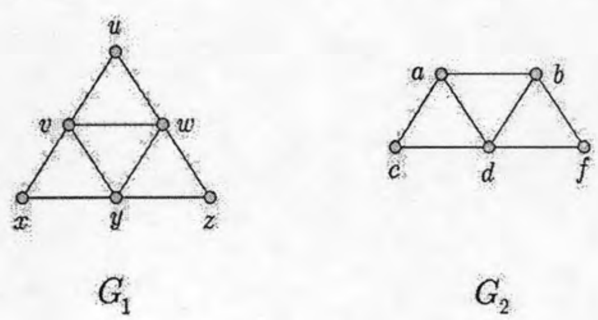


Figure 1.2.1.4: Graphs G_1 and G_2 in Example 1.2.1.8

$\{K_3(u,v,w), K_3(v,x,y), K_3(w,y,z)\}$ is a clique covering of G_1 and $I_1 = \{uv, xv, yz\}$ is a clique-independent set of G_1. Therefore, $cc(G_1) = 3$. Moreover, $\{K_3(u,v,w), K_3(v,x,y), K_3(w,y,z)\}$ is a minimum clique covering of G_1. We can see that $\{K_3(a, c, d), K_2(a, b), K_3(b, d, f)\}$ is a clique covering of G_2 and $I_2 = \{ab, cd, df\}$ is a clique-independent set of G_2. Hence $cc(G_2) = 3$. Moreover, $\{K_3(a, c, d), K_2(a, b), K_3(b, d, f)\}$ is a minimum clique covering of G_2. □

Recall that for any graph G, $cc(G) \geq |I|$ where I is a clique-independent set of G. And we have seen a graph satisfying this equality. However, in general, there exists a graph G with $cc(G) > |I|$ where I is any clique-independent set of G. The cocktail party graphs T_{2n} defined in Definition 1.2.1.9 are examples of such a graph. Theorem 1.2.1.17 reveals this fact.

Definition 1.2.1.9. *The cocktail party graph* T_{2n} denote the graph with $2n$ vertices, u_i and v_i for $i = 1, .., n$ such that u_i and v_i are not adjacent and all other pairs of vertices are adjacent.

Example 1.2.1.10. T_8 is illustrated in Figure 1.2.1.5.

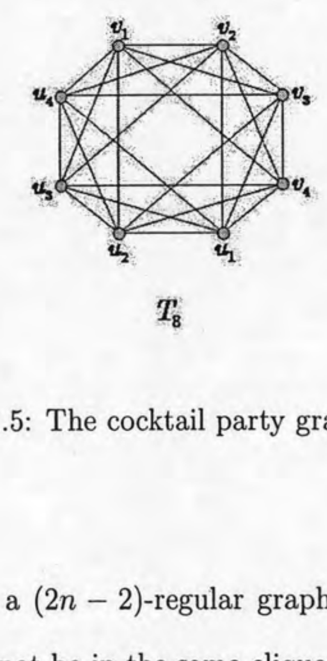


Figure 1.2.1.5: The cocktail party graph of order 8

□

Remark 1.2.1.11. T_{2n} is a $(2n-2)$-regular graph. Thus $e(T_{2n}) = n(2n-2)$. Furthermore, u_i and v_i cannot be in the same clique of T_{2n} for $i = 1, ..., n$.

Remark 1.2.1.12. Let I be a clique-independent set of T_{2n}. Any pair of edges in I must contain an endpoint of a common subscript. In particular, if two edges in a clique-independent set of T_{2n} have a common endpoint, say a then such two edges are av_i and au_i for some $i = 1, ..., n$.

Next, we will show that the cardinality of a clique-independent set of T_{2n} is at most 4. First, we prove two lemmas to consider a clique-independent set of T_{2n}.

Lemma 1.2.1.13. *Any three distinct edges in a clique-independent set of T_{2n} do not have a common endpoint.*

Proof. Let I be a clique-independent set of T_{2n}. Suppose that e_1, e_2 and e_3 are three edges in I sharing a common endpoint, say u_1. All three remaining endpoints of e_1, e_2 and e_3 must be distinct. By Remark 1.2.1.12, they must have the same subscript, say j. However, there are only two vertices, namely u_j and v_j available, a contradiction. □

Lemma 1.2.1.14. *For any clique-independent set I of T_{2n}, if $|I| \geq 4$ then there exist at least two edges in I having a common endpoint.*

Proof. Let I be a clique-independent set of T_{2n} such that $|I| \geq 4$. Suppose that any two edges in I do not have a common endpoint. Thus I is a matching in T_{2n}. We choose one edge in I, say $e_1 = a_i b_j$ where $i \neq j$ and $a, b \in \{u, v\}$. By Remark 1.2.1.12, another edge in I, say e_2, has an endpoint with subscript i or j. Without loss of generality, $e_2 = c_i d_k$ where i, j and k are all distinct, $c \in \{u, v\} \setminus \{a\}$ and $d \in \{u, v\}$. Since I is a matching, u_i and v_i cannot be an endpoint of another edge in $I \setminus \{e_1, e_2\}$, say e_3. By Remark 1.2.1.12 and $e_1, e_2, e_3 \in I$, we have $e_3 = f_j g_k$ where $f \in \{u, v\} \setminus \{b\}$ and $g \in \{u, v\} \setminus \{d\}$. Therefore, another edge in $I \setminus \{e_1, e_2, e_3\}$ does not have an endpoint with subscripts i, j and k, a contradiction to Remark 1.2.1.12. Hence there exist at least two edges in I having a common endpoint. □

Theorem 1.2.1.15. *The cardinality of a clique-independent set of T_{2n} is at most 4.*

Proof. Recall that $V(T_{2n}) = \{u_1, ..., u_n, v_1, ..., v_n\}$ such that u_i and v_i are not adjacent for all $i = 1, ..., n$ and all other pairs of vertices are adjacent. Suppose that there is a clique-independent set of T_{2n} with cardinality 5. Let $I = \{e_1, e_2, e_3, e_4, e_5\}$ be such clique-independent set of T_{2n}. By Lemma 1.2.1.14, there exist at least two edges in I having a common endpoint. Without loss of generality, let $e_1 = x_iu_j$ and $e_2 = x_iv_j$ where $x \in \{u, v\}$. By Lemma 1.2.1.13, e_3, e_4 and e_5 cannot have an endpoint x_i. Since $e_1, e_2, e_3, e_4, e_5 \in I$ and by Remark 1.2.1.12, e_3, e_4 and e_5 must have an endpoint u_j, v_j or x_i' where $x' \in \{u, v\} \setminus \{x\}$.

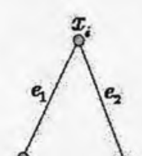


Figure 1.2.1.6: e_1 and e_2 in Theorem 1.2.1.15

Case 1. e_3, e_4 or e_5 have an endpoint u_j or v_j. Without loss of generality, e_3 has an endpoint u_j. We can see that e_1 and e_3 have a common endpoint u_j. By Remark 1.2.1.12, $e_3 = x_i'u_j$. By Lemma 1.2.1.13, u_j cannot be an endpoint of e_4 and e_5. Without loss of generality, by Remark 1.2.1.12, we have $e_4 = x_i'v_j$. By Lemma 1.2.1.13, x_i' and v_j cannot be an endpoint of e_5. By Remark 1.2.1.12, we have $e_5 \notin I$, a contradiction.

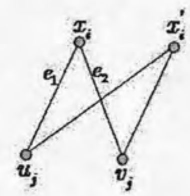


Figure 1.2.1.7: e_1, e_2, e_3 and e_4 in Case 1

Case 2. e_3, e_4 and e_5 do not have an endpoint u_j and v_j. Thus e_3, e_4 and e_5 have an endpoint $x_i^{'}$. Therefore e_3, e_4 and e_5 have a common endpoint $x_i^{'}$. This contradicts Lemma 1.2.1.13.

Hence there is no clique-independent set in T_{2n} with cardinality 5. Therefore, the cardinality of a clique-independent set of T_{2n} is at most 4. □

Theorem 1.2.1.16. [4]*For all n, $cc(T_{2n}) \geq (log_2 2n) - 1$.*

We use Theorem 1.2.1.16 to prove in the next theorem that there exists a graph G without a clique-independent set of cardinality $cc(G)$.

Theorem 1.2.1.17. *There exist infinitely many graphs whose all clique-independent sets have size strictly less than their clique covering numbers.*

Proof. Let I be any clique-independent set of T_{2n} where $n \geq 32$. By Theorem 1.2.1.15, we have $|I| \leq 4$. By Theorem1.2.1.16, we have $cc(T_{2n}) \geq (log_2 2n) - 1 \geq (log_2 64) - 1 = 5$. Therefore, $cc(T_{2n}) > |I|$ where $n \geq 32$. Hence we choose $G = T_{2n}$ where $n \geq 32$. □

The following remark helps us to determine the clique covering number of a graph by considering the clique covering number of its subgraphs.

Remark 1.2.1.18. Let H_1 and H_2 be subgraphs of a graph G such that $E(H_1) \cup E(H_2) = E(G)$ and $E(H_1) \cap E(H_2) = \varnothing$. If there is no clique in G containing both e and f for each pair of edges e in H_1 and f in H_2, then $cc(G) = cc(H_1) + cc(H_2)$.

We can use the previous remark to find the clique covering number of a graph by using blocks of a graph.

Definition 1.2.1.19. A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex.

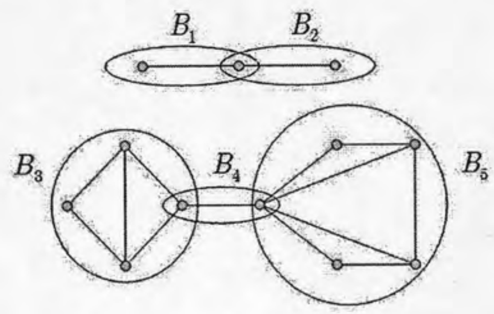


Figure 1.2.1.8: A graph G with 5 blocks

Example 1.2.1.20. Let G be a graph which has k blocks, say $B_1, ..., B_k$. Suppose that there are two distinct blocks B and B' such that edges e and e' in B and B', respectively, are in the same clique of G, say Q. Note that $B \cup B' \cup Q$ is a connected subgraph of G. All vertices in B and B' are not cut-vertices of B and B', so they are not cut-vertices of $B \cup B' \cup Q$. Since Q is a clique, each vertex in Q is not a cut-vertex of $B \cup B' \cup Q$. Hence $B \cup B' \cup Q$ has no cut-vertex. Therefore, $B \cup B' \cup Q$ is a connected subgraph of G that has no cut-vertex and contains B and B'. This contradicts the fact that B and B' are blocks. By Remark 1.2.1.18, we have that $cc(G) = cc(B_1) + ... + cc(B_k)$.

□

The following proposition shows that a set of all maximal cliques containing an element in a minimum clique covering is a minimum clique covering.

Proposition 1.2.1.21. *Let* $\mathscr{C} = \{Q_1, Q_2, ..., Q_k\}$ *be a minimum clique covering of a graph* G *and* $\mathscr{C}^* = \{D_i \mid Q_i \subseteq D_i$ *and* D_i *is a maximal clique in* G, $i = 1, ..., k\}$. *Then* $\mathscr{C}^*$ *is another minimum clique covering of* G.

Proof. Clearly, $|\mathscr{C}^*| \leq k$. But $\mathscr{C}^*$ is also a clique covering of G, so $|\mathscr{C}^*| \geq cc(G) = k$. Hence $\mathscr{C}^*$ is also minimum clique covering of G. □

Example 1.2.1.22. Consider the graph G_2 as shown in Figure 1.2.1.4, $K_3(a,b,d)$ is a maximal clique in G_2 which contains $K_2(a,b)$. We replace $K_2(a,b)$ by $K_3(a,b,d)$ in $\{K_3(a,c,d), K_2(a,b), K_3(b,d,f)\}$. Thus $\{K_3(a,c,d), K_3(a,b,d), K_3(b,d,f)\}$ is another minimum clique covering of G_2. □

Next, we consider the effects of an edge deletion and an n-clique deletion on the clique covering number. Note that we write $G - e$ for the subgraph of G resulting from removing edge e out of G for any edge e of a graph G.

Proposition 1.2.1.23. [1] *For any graph* G *and any edge* e *in* G, $cc(G - e) \geq cc(G) - 1$.

Proof. Let $\mathscr{C}$ be a minimum clique covering of $G - e$ and Q be a subgraph of G containing only edge e. Thus $\mathscr{C} \cup \{Q\}$ is a clique covering of G, it follows that $cc(G) \leq |\mathscr{C} \cup \{Q\}| = |\mathscr{C}| + 1 = cc(G - e) + 1$. □

Proposition 1.2.1.24. *For any graph* G *and clique* Q *of* G, $cc(G-Q) \geq cc(G)-1$.

Proof. Let $\mathscr{C}$ be a minimum clique covering of $G - Q$. Then $\mathscr{C} \cup \{Q\}$ is a clique covering of G, so $cc(G) \leq |\mathscr{C} \cup \{Q\}|$. Since $Q \notin \mathscr{C}$, $|\mathscr{C} \cup \{Q\}| = |\mathscr{C}| + 1 = cc(G - Q) + 1$. Hence $cc(G) - 1 \leq cc(G - Q)$. □

1.2.2 Definition of glued graphs

Next, we will introduce a glued graph.

Definition 1.2.2.1. Let G_1 and G_2 be two nontrivial vertex-disjoint graphs. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f. *The glued graph between* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $\underset{H_1 \cong_f H_2}{G_1 \Leftrightarrow G_2}$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H, H_1 and H_2 as the *clones* of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs.* We use $u \equiv v$ to denote the vertex in the glued graph $\underset{H_1 \cong_f H_2}{G_1 \Leftrightarrow G_2}$ where $u \in V(G_1)$, $v \in V(G_2)$ and $f(u) = v$.

The *glued graph between* G_1 *and* G_2 *at clone* H, written $G_1 \underset{H}{\Leftrightarrow} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $\underset{H_1 \cong_f H_2}{G_1 \Leftrightarrow G_2}$ and H is the copy of H_1 and H_2 in the resulting graph.

We denote $G_1 \Leftrightarrow G_2$ an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

Example 1.2.2.2. Let G_1 and G_2 be graphs in Figure 1.2.2.1.

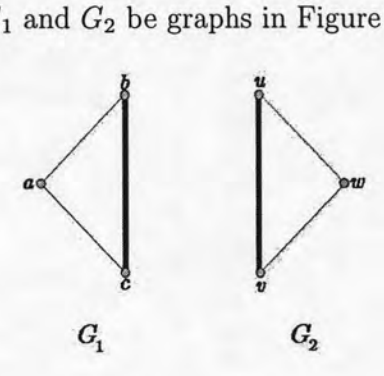
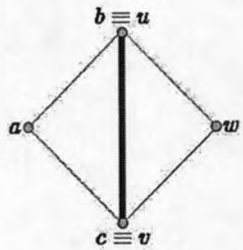


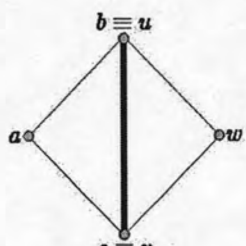
Figure 1.2.2.1: Original graphs of a glued graph

Let $H_1 = K_2(b, c)$ as shown in the bold edge bc and $H_2 = K_2(u, v)$ as shown in the bold edge uv be nontrivial connected subgraphs of G_1 and G_2, respectively, of Figure 1.2.2.1. We consider an isomorphism f between H_1 and H_2, as follows: $f(b) = u$ and $f(c) = v$. We show the glued graph between G_1 and G_2 with respect to f, $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$, in Figure 1.2.2.2. $G_1 \underset{H}{\Diamond} G_2$ as shown in Figure 1.2.2.3, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$ and H is the copy of H_1 and H_2 in the resulting graph. Therefore, H, H_1 and H_2 are clones of $G_1 \underset{H}{\Diamond} G_2$, G_1 and G_2, respectively.



$G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$

Figure 1.2.2.2: A glued graph between G_1 and G_2 with respect to f



$G_1 \underset{H}{\Diamond} G_2$

Figure 1.2.2.3: A glued graph between G_1 and G_2 at clone H

□

The following example shows that different isomorphisms can give the different or the same result.

Example 1.2.2.3. Let G_1 and G_2 be graphs in Figure 1.2.2.4.

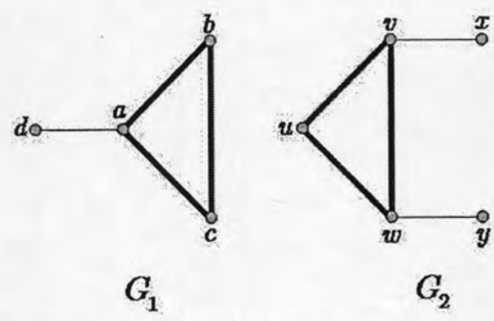


Figure 1.2.2.4: Original graphs illustrating isomorphisms between clones

Let $H_1 = K_3(a, b, c)$ and $H_2 = K_3(u, v, w)$ be nontrivial connected subgraphs of G_1 and G_2, respectively, as shown in the bold edges of Figure 1.2.2.4. Consider three isomorphisms between H_1 and H_2, f, g and h, as follows: $f(a) = u$, $f(b) = v$, $f(c) = w$, $g(a) = v$, $g(b) = u$, $g(c) = w$, $h(a) = v$, $h(b) = w$ and $h(c) = u$. We show glued graphs between G_1 and G_2 with respect to f, g and h in Figure 1.2.2.5. Moreover, we can see that $\underset{H_1 \cong_f H_2}{G_1 \triangleleft\!\triangleright G_2} \not\cong \underset{H_1 \cong_g H_2}{G_1 \triangleleft\!\triangleright G_2}$ and $\underset{H_1 \cong_g H_2}{G_1 \triangleleft\!\triangleright G_2} \cong \underset{H_1 \cong_h H_2}{G_1 \triangleleft\!\triangleright G_2}$.

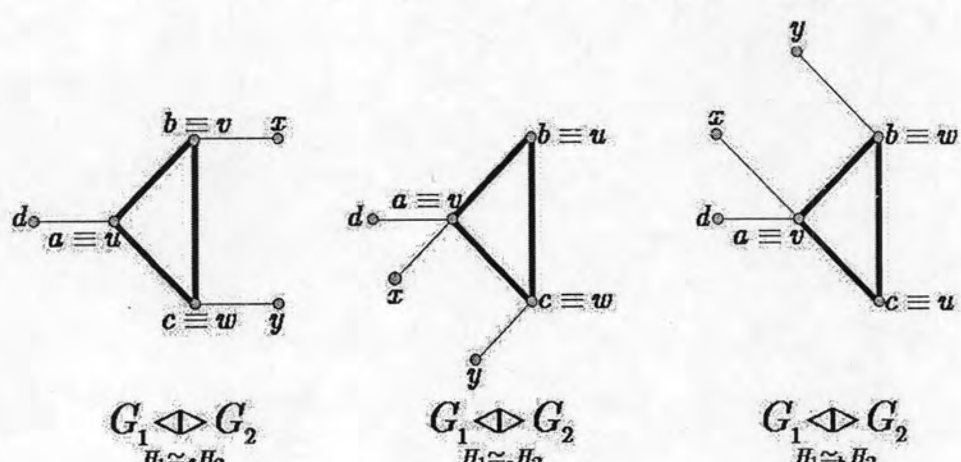


Figure 1.2.2.5: Glued graphs with different isomorphisms

□

1.2.3 Terminologies of clique coverings of glued graphs

First, we show an example such that the correspondence of cliques in G_1 and G_2 lie in $G_1 \triangleleft\!\triangleright G_2$.

Example 1.2.3.1. Let G_1 and G_2 be graphs and $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ be the glued graph whose clone H is shown as bold edges in Figure 1.2.3.1.

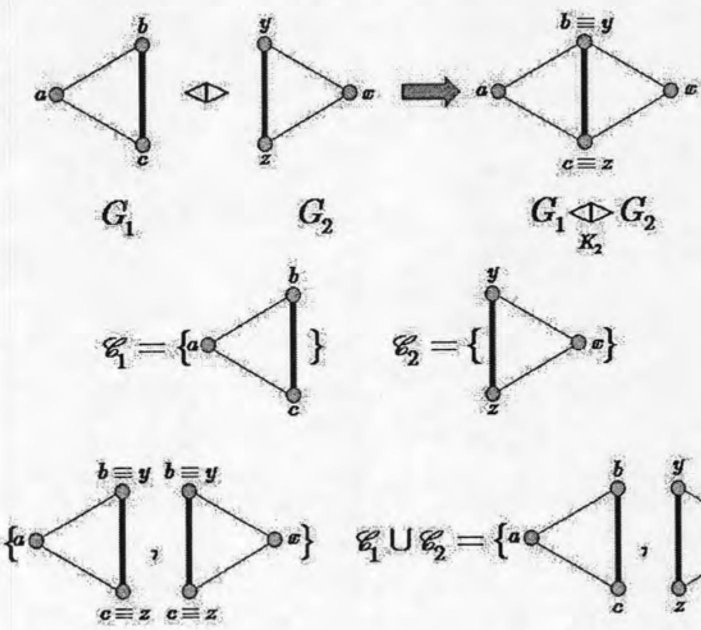


Figure 1.2.3.1: A minimum clique covering of a glued graph

We can see that $\mathscr{C}_1 = \{K_3(a,b,c)\}$, $\mathscr{C}_2 = \{K_3(x,y,z)\}$ and $\mathscr{C} = \{K_3(a, b \equiv y, c \equiv z), K_3(x, b \equiv y, c \equiv z)\}$ as shown in Figure 1.2.3.1 are minimum clique coverings of G_1, G_2 and $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, respectively. Because $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ does not have vertices b, c, y and z, $\mathscr{C}_1 \cup \mathscr{C}_2$ is not a clique covering of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. Note that vertices b and c in G_1 correspond to vertices $b \equiv y$ and $c \equiv z$, respectively, in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. And vertices y and z in G_2 correspond to vertices $b \equiv y$ and $c \equiv z$, respectively, in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. Thus the correspondence of cliques in $\mathscr{C}_1$ and $\mathscr{C}_2$ lie in $\mathscr{C}$. □

In this thesis, we abuse the terminologies by considering subgraphs of original graphs as subgraphs of $G_1 \Leftrightarrow G_2$, and subgraphs in the clone of $G_1 \Leftrightarrow G_2$ are subgraphs in the clones of both original graphs. For example, if Q is a clique in G_1, then Q is also a clique in $G_1 \Leftrightarrow G_2$; or if $\mathscr{C}_1$ and $\mathscr{C}_2$ are clique coverings of G_1 and G_2, respectively, then $\mathscr{C}_1 \cup \mathscr{C}_2$ is a clique covering of $G_1 \Leftrightarrow G_2$; also if e is an edge in the clone of $G_1 \Leftrightarrow G_2$, then e is also an edge in G_1 and G_2.

In Example 1.2.3.1, we can conclude that $K_3(a, b, c) = K_3(a, b \equiv y, c \equiv z)$ and $K_3(x, y, z) = K_3(x, b \equiv y, c \equiv z)$.