CHAPTER IV

SOME OTHER SEMIGROUPS

The following semigroups are considered in this chapter : infinite cyclic semigroups, left zero semigroups, right zero semigroups, Kronecker semigroups, dihedral groups, alternating groups. We characterize when they admit a right nearring structure and a left nearring structure.

Theorem 4.1. *Let S be an infinite cyclic semigroup. Then $S \notin \mathcal{RNR}$ and $S \notin \mathcal{LNR}$.*

Proof. Let S be generated by $a \in S$. Then $S = \{a^n \mid n \in \mathbb{N}\}$ and $a^i \neq a^j$ for all distinct $i, j \in \mathbb{N}$. Suppose that $S \in \mathcal{RNR}$. Since S has no (left) zero, there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a right nearring where $\cdot$ is the operation on S^0. Then $a + a^k = 0$ for some $k \in \mathbb{N}$.

Case 1 : $k > 1$. Then $k - 1 \in \mathbb{N}$ and

$$a^k + a^{2k-1} = (a + a^k)a^{k-1} = 0a^{k-1} = 0 = a + a^k = a^k + a$$

which implies that $a^{2k-1} = a$. This is a contradiction since $2k - 1 > 1$.

Case 2 : $k = 1$. Then $a + a = 0$. Hence $a^i + a^i = 0$ for all $i \in \mathbb{N}$ and $a + a^2 \neq 0$, so $a + a^2 = a^j$ for some $j \in \mathbb{N}$. Since $(S^0, +)$ is cancellative, it follows that $j > 2$. Consequently,

$$a = a^j + (-a^2) = a^j + a^2 = (a^{j-1} + a)a.$$

Since $j - 1 > 1$, we have that $a^{j-1} + a = a^\ell$ for some $\ell \in \mathbb{N}$. Hence

$$a = (a^{j-1} + a)a = a^\ell a = a^{\ell+1}$$

which is a contradiction since $\ell + 1 > 1$.

We can show analogously that $S \notin \mathcal{LNR}$. □

Theorem 4.2. (i) *Every left zero semigroup belongs to* $\mathcal{RNR}$.

(ii) *If S is a left zero semigroup, then $S \in \mathcal{LNR}$ if and only if $|S| = 1$.*

Proof. Let S be a left zero semigroup. Note that every element of S is a left zero of S. By Theorem 2.2, there is an operation $+$ on S such that $(S, +)$ is an abelian group. Since

$$(x + y)z = x + y = xz + yz \quad \text{for all } x, y, z \in S,$$

it follows that $(S, +, \cdot)$ is a right nearring where $\cdot$ is the operation on S. Hence $S \in \mathcal{RNR}$.

Assume that $S \in \mathcal{LNR}$ and $|S| > 1$. Then S has no right zero. Thus there is an operation $\oplus$ on S^0 such that $(S^0, \oplus, \cdot)$ is a left nearring. Let $a, b \in S$ be distinct elements of S. Then $a \oplus b = c$ for some $c \in S^0$. Then

$$a \oplus a = aa \oplus ab = a(a \oplus b) = ac.$$

Case 1 : $c = 0$. Then $a \oplus b = 0 = a \oplus a$. Thus $a = b$, a contradiction.

Case 2 : $c \neq 0$. Then $a \oplus a = a$, so $a = 0$ which is a contradiction.

This proves that $S \in \mathcal{LNR}$ implies $|S| = 1$. The converse is trivial. □

The following theorem is proved dually to Theorem 4.2

Theorem 4.3. (i) *Every right zero semigroup belongs to* $\mathcal{LNR}$.

(ii) *If S is a right zero semigroup, then $S \in \mathcal{RNR}$ if and only if $|S| = 1$.*

Theorem 4.4. *Let S be a Kronecker semigroup.*

(i) *$S \in \mathcal{RNR}$ if and only if $|S| \leq 2$.*

(ii) *$S \in \mathcal{LNR}$ if and only if $|S| \leq 2$.*

Proof. (i) Assume that $S \in \mathcal{RNR}$ and suppose that $|S| > 2$. Let $+$ be the operation on S such that $(S, +, \cdot)$ is a right nearring where $\cdot$ is the operation on S. Let $a, b \in S \smallsetminus \{0\}$ be distinct. Then $a + b = c$ for some $c \in S$.

Case 1 : $c = a$. Then $a + b = a$, so $b = 0$, a contradiction.

Case 2 : $c = b$. Then $a + b = b$, so $a = 0$, a contradiction.

Case 3 : $c \in S \smallsetminus \{a, b\}$. Then $0 = ca = (a + b)a = a^2 + ba = a + 0 = a$, a contradiction.

This proves that if $S \in \mathcal{RNR}$, then $|S| \leq 2$.

Since

$$S = \begin{cases} \{0\} & \text{if } |S| = 1, \\ \{0, a\} \cong (\mathbb{Z}_2, \cdot) & \text{if } |S| = 2, \end{cases}$$

it follows that $S \in \mathcal{R} \subseteq \mathcal{RNR} \cap \mathcal{LNR}$.

(ii) can be proved similarly to (i). □

Theorem 4.4 and its proof provide the following result.

Corollary 4.5. *Let S be a Kronecker semigroup. Then the following statements are equivalent.*

(i) $S \in \mathcal{RNR}$.

(ii) $S \in \mathcal{LNR}$.

(iii) $S \in \mathcal{R}$.

(iv) $|S| \leq 2$.

Theorem 4.6. *Let D_n be the dihedral group. Then $D_n \notin \mathcal{RNR}$ and $D_n \notin \mathcal{LNR}$.*

Proof. By the definition of D_n,

$$D_n = \{1, a, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}$$
$$\text{where } a^n = b^2 = 1 \text{ and } ba = a^{n-1}b.$$

Then $|D_n| = 2n$. Since $(ab)(ba^{n-1})^{-1} = abab = aa^{n-1}bb = 1$, we have that $ab = ba^{n-1}$. Suppose that $D_n \in \mathcal{RNR}$. Since D_n has no left zero, there is an operation $+$ on D_n^0 such that $(D_n^0, +, \cdot)$ is a right nearring. Then $a+b = c$ for some $c \in D_n^0$ which implies that $c \neq a$ and $c \neq b$.

Case 1 : $a + b = a^i$ for some $i \in \{2, \dots, n\}$. Then

$$a^ib = (a+b)b = ab + 1$$
$$= 1 + ab = a^n + ba^{n-1} = (a+b)a^{n-1} = a^ia^{n-1} = a^{i-1},$$

so $ab = 1$ which is a contradiction.

Case 2 : $a + b = a^ib$ for some $i \in \{1, \dots, n-1\}$. Then

$$a^i = (a+b)b = ab + 1$$
$$= 1 + ab = a^n + ba^{n-1} = (a+b)a^{n-1} = a^iba^{n-1} = a^iab = a^{i+1}b$$

which implies that $ab = 1$, a contradiction.

Case 3 : $a + b = 0$. Then $1 + b \neq 0$, $1 + b \neq 1$ and $1 + b \neq b$.

Subcase 3.1 : $1 + b = a^i$ for some $i \in \{1, \dots, n-1\}$. Then

$$a^i = 1 + b = b + 1 = b + b^2 = (1+b)b = a^ib,$$

so $b = 1$, a contradiction.

Subcase 3.2 : $1 + b = a^ib$ for some $i \in \{1, \dots, n-1\}$. Then

$$a^ib = 1 + b = b + b^2 = (1+b)b = a^ib^2 = a^i,$$

thus $b = 1$, a contradiction.

This proves that $D_n \notin \mathcal{RNR}$. We can show similarly that $D_n \notin \mathcal{LNR}$. □

Theorem 4.7. *Let A_n be the alternating group of degree n.*

(i) *$A_n \in \mathcal{RNR}$ if and only if $n \leq 3$.*

(ii) *$A_n \in \mathcal{LNR}$ if and only if $n \leq 3$.*

Proof. (i) Assume that $A_n \in \mathcal{RNR}$ and suppose that $n > 3$. Then $|A_n| = \dfrac{n!}{2} \geq 12$. Thus $(A_n^0, +, \cdot)$ is a right nearring for some operation $+$ on A_n^0 where $\cdot$ is the operation on A_n^0. Since $n \geq 4$, $(1\ 2)(3\ 4)$ and $(1\ 3)(2\ 4)$ belong to A_n. Then $(1) + (1\ 2)(3\ 4) = f$ for some $f \in A_n^0$.

Case 1 : $f \neq 0$. Then

$$\begin{aligned} f(1\ 2)(3\ 4) &= ((1) + (1\ 2)(3\ 4))(1\ 2)(3\ 4) \\ &= (1\ 2)(3\ 4) + (1) = f \end{aligned}$$

which implies that $(1\ 2)(3\ 4) = (1)$, a contradiction.

Case 2 : $f = 0$. Then $(1) + (1\ 3)(2\ 4) = g$ for some $g \in A_n$. Hence

$$\begin{aligned} g(1\ 3)(2\ 4) &= ((1) + (1\ 3)(2\ 4))(1\ 3)(2\ 4) \\ &= (1\ 3)(2\ 4) + (1) = g. \end{aligned}$$

This implies that $(1\ 3)(2\ 4) = (1)$, a contradiction.

This shows that if $A_n \in \mathcal{RNR}$, then $n \leq 3$.

Since $|A_1| = |A_2| = 1$, then $A_1, A_2 \in \mathcal{RNR}$. Observe that

$$A_3^0 = \{0, (1), (1\ 2\ 3), (1\ 3\ 2)\}.$$

Let $a = (1\ 2\ 3)$ and $b = (1\ 3\ 2)$. Define an operation $+$ on A_n^0 by

$+$	0	(1)	a	b
0	0	(1)	a	b
(1)	(1)	0	b	a
a	a	b	0	(1)
b	b	a	(1)	0

It is straightfoward to verify that $(A_3^0, +)$ is an abelian group. Since $|A_3| = 3$, $(A_3, \cdot)$ is an abelian group. Note that $a^2 = b$, $b^2 = a$ and $a^{-1} = b$. Also, we can check directly that $\cdot$ is distributive over $+$. Hence $(A_3^0, +, \cdot)$ is a ring. Therefore we deduce that $A_3 \in \mathcal{R} \subseteq \mathcal{RNR} \cap \mathcal{LNR}$.

(ii) It can be proved similarly to (i). □

As a consequence of Theorem 4.7 and its proof, we have

Corollary 4.8. *The following statements are equivalent.*

(i) $A_n \in \mathcal{RNR}$.

(ii) $A_n \in \mathcal{LNR}$.

(iii) $A_n \in \mathcal{R}$.

(iv) $n \leq 3$.